A DERIVATION OF CERTAIN VARIATIONAL PRINCIPLES FOR MIXED BOUNDARY VALUE PROBLEMS IN POTENTIAL THEORY

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1. Introduction

We consider the following problem: A potential function ϕ satisfies Laplace's equation $\nabla^2\phi = \phi_{xx}+\phi_{yy} = 0$ in a region R bounded by a closed curve C on which mixed boundary conditions are specified, i.e. $\phi = f(s)$ on a part A of the boundary and $\partial\phi/\partial n = g(s)$ on a part B, where $C = A+B$ and distance along C is denoted by s. Electrostatic problems of this type have been solved approximately in (**1**) and (**2**) by formulating them in terms of integral equations and then applying variational principles to the integral equations. In that approach, attention is concentrated on integrals over the boundary of the region R. The most common type of variational principle for potential problems involves integrals over the region R rather than integrals over the boundary of R. An example is given by the Rayleigh-Ritz method which depends on the stationary character of Dirichlet's integral

$$\iint_R \left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right\} dxdy. \qquad \text{.............................(1)}$$

In this paper we show that the variational principles used in (**1**), (**2**), are closely connected with the more usual type of variational principles, by deriving the principles used in (**1**), (**2**) from inequalities deduced by considering integrals of type (1) over the region R.

We shall use the notation

$$(f, g) = \iint_R f(x, y)g(x, y)dxdy,$$

$$(\nabla f, \nabla g) = \iint_R \left\{\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y}\right\} dxdy.$$

Green's theorem states that

$$(\nabla f, \nabla g) = -(f, \nabla^2 g) + \int_C f\frac{\partial g}{\partial n}\, ds, \qquad \text{........................(2)}$$

where $\partial/\partial n$ denotes differentiation normal to the boundary in an outward direction.

2. Upper and Lower Limits

Consider the potential function ϕ satisfying the mixed boundary conditions specified at the beginning of this paper. Then using (2), since $\nabla^2\phi = 0$,

$$(\nabla\phi, \nabla\phi) = -(\phi, \nabla^2\phi) + \int_C \phi \frac{\partial\phi}{\partial n} ds$$

$$= \int_A f(s) \frac{\partial\phi}{\partial n} ds + \int_B g(s)\phi(s) ds \quad \text{.........................(3)}$$

Suppose that Φ is a function which approximates to ϕ, so that we can set $\Phi = \phi + \delta$ where δ is an error function and we try to choose Φ so that δ is small. Let Φ satisfy the following conditions:

(i) $\partial\Phi/\partial n = g(s)$ on B, so that $\partial\delta/\partial n = 0$ on B,

(ii) $\nabla^2\Phi = 0$ in R, so that $\nabla^2\delta = 0$ in R.

Then

$$(\nabla\Phi, \nabla\Phi) = (\nabla\phi, \nabla\phi) + 2(\nabla\delta, \nabla\phi) + (\nabla\delta, \nabla\delta), \quad \text{..................(4)}$$

$$(\nabla\delta, \nabla\phi) = -(\phi, \nabla^2\delta) + \int_C \phi \frac{\partial\delta}{\partial n} ds$$

$$= \int_A f(s) \frac{\partial\Phi}{\partial n} ds - \int_A f(s) \frac{\partial\phi}{\partial n} ds. \quad \text{........................(5)}$$

On substituting (5) in (4) we find that

$$(\nabla\Phi, \nabla\Phi) - 2\int_A f(s) \frac{\partial\Phi}{\partial n} ds = (\nabla\phi, \nabla\phi) - 2\int_A f(s) \frac{\partial\phi}{\partial n} ds + (\nabla\delta, \nabla\delta). \quad \text{.........(6)}$$

If we define

$$I(\chi) = \int_B g(s)\chi(s) ds - \int_A f(s) \frac{\partial\chi}{\partial n} ds \quad \text{..........................(7)}$$

then (6) gives, on using (3), since $(\nabla\delta, \nabla\delta) \geqq 0$,

$$(\nabla\Phi, \nabla\Phi) - 2\int_A f(s) \frac{\partial\Phi}{\partial n} ds \geqq I(\phi). \quad \text{..........................(8)}$$

If δ is a first-order quantity then the difference between the two sides is second-order. We have

$$(\nabla\Phi, \nabla\Phi) = -(\Phi, \nabla^2\Phi) + \int_C \Phi \frac{\partial\Phi}{\partial n} ds$$

$$= \int_A \Phi(s) \frac{\partial\Phi}{\partial n} ds + \int_B g(s)\Phi(s) ds.$$

Hence (8) gives

$$\int_B g(s)\Phi(s) ds + \int_A \Phi(s) \frac{\partial\Phi}{\partial n} ds - 2\int_A f(s) \frac{\partial\Phi}{\partial n} ds \geqq I(\phi). \quad \text{..............(9)}$$

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Suppose next that we define a function Ψ such that $\Psi = \phi+\varepsilon$ with ε small, and

(i)′ $\Psi = f(s)$ on A, so that $\varepsilon = 0$ on A.

Then

$$(\nabla\Psi, \nabla\Psi) = (\nabla\phi, \nabla\phi)+2(\nabla\varepsilon, \nabla\phi)+(\nabla\varepsilon, \nabla\varepsilon),$$

$$(\nabla\varepsilon, \nabla\phi) = -(\varepsilon, \nabla^2\phi)+\int_C \varepsilon \frac{\partial\phi}{\partial n}\, ds$$

$$= \int_B g(s)\Psi(s)ds - \int_B g(s)\phi(s)ds.$$

On combining these equations and using (3), (7) we find that

$$-(\nabla\Psi, \nabla\Psi)+2\int_B g(s)\Psi(s)ds \leqq I(\phi). \quad \text{......................(10)}$$

We now make the assumption that

(ii)′ $\nabla^2\Psi = 0$.

Then

$$(\nabla\Psi, \nabla\Psi) = \int_C \Psi \frac{\partial\Psi}{\partial n}\, ds = \int_A f(s) \frac{\partial\Psi}{\partial n}\, ds + \int_B \Psi \frac{\partial\Psi}{\partial n}\, ds,$$

and (10) gives

$$-\int_A f(s) \frac{\partial\Psi}{\partial n}\, ds - \int_B \Psi(s) \frac{\partial\Psi}{\partial n}\, ds + 2\int_B g(s)\Psi(s)ds \leqq I(\phi). \quad \text{...........(11)}$$

Equations (9) and (11) give upper and lower limits for $I(\phi)$ in terms of line integrals along the boundary of the region. However the expressions on the left of (9) and (11) are not very useful as they stand. In (9) for example we have assumed that $\partial\Phi/\partial n = g(s)$ on B; this means that Φ cannot be chosen arbitrarily on A and B. A suitable method for specifying Φ on A and B is given in the next section.

3. Derivation of the Variational Principles

In addition to assuming that Φ satisfies conditions (i) and (ii) suppose that

(iii) $\partial\Phi/\partial n = G(s)$ on A, where $G(s)$ is known and has been chosen as an approximation to the unknown value of $\partial\phi/\partial n$ on A.

Then $\partial\Phi/\partial n$ is known on the whole of the boundary C and by Green's theorem, using suitable Green's functions, we can determine expressions for the unknown function $\Phi(s)$ on B in terms of $g(s)$ and $G(s)$:

$$\Phi(s) = \int_A K(s, t)G(t)dt + \int_B K(s, t)g(t)dt, \quad \text{....................(12)}$$

where $K(s, t)$ is assumed to be a known function so that all functions on the right-hand side of this equation are known. Then substituting in (9) we find

$$\int_B\int_B K(s, t)g(s)g(t)dsdt - J(G) \geqq I(\phi), \quad \text{.....................(13)}$$

where

$$J(G) = 2\left\{\int_A f(s)G(s)ds + \int_A\int_B K(s, t)G(s)g(t)dsdt\right\}$$
$$-\int_A\int_A K(s, t)G(s)G(t)dsdt, \quad \text{..................(14)}$$

and in the derivation we have used the fact that $K(s, t) = K(t, s)$ since K is derived from a Green's function.

Similarly, in addition to assuming that Ψ satisfies (i)′ and (ii)′, suppose that

(iii)′ $\Psi = F(s)$ on B, where $F(s)$ is known and has been chosen as an approximation to the unknown value of ϕ on B.

Then Ψ is known on the whole of C and by Green's theorem, using suitable Green's functions, we can deduce an expression for the unknown function $\partial\Psi/\partial n$ on A in terms of $f(s)$ and $F(s)$:

$$\partial\Psi/\partial n = \int_A L(s, t)f(t)dt + \int_B L(s, t)F(t)dt,$$

where $L(s, t)$ is assumed known. On substituting in (11) we find that

$$H(F) - \int_A\int_A L(s, t)f(s)f(t)dsdt \leqq I(\phi), \text{........................(15)}$$

where

$$H(F) = 2\left\{\int_B g(s)F(s)ds - \int_A\int_B L(s, t)f(s)F(t)dsdt\right\}$$
$$-\int_B\int_B L(s, t)F(s)F(t)dsdt. \text{...........................(16)}$$

Expressions (13) and (15) yield the required variational principles. By choosing a form for $G(s)$ which contains arbitrary parameters and by minimising the left-hand side of (13) with respect to these parameters we can find an upper bound for $I(\phi)$. Similarly by choosing $F(s)$ so as to maximise the left-hand side of (15) we can find a lower bound for $I(\phi)$.

Equations (13), (15) are generalisations of the variational expressions derived in (**1**), (**2**). The method used in these references, due essentially to J. Schwinger, starts from integral equations from which the variational principles are derived. Here we have derived the variational principles directly from Dirichlet's integral.

For completeness we briefly derive integral equations from our variational expressions. Suppose that in (13), (14) we have $G(s) = \Gamma(s)+\eta\gamma(s)$ where $\Gamma(s)$ is the exact value of $\partial\phi/\partial n$ on A and $\eta\gamma(s)$ is the error, where η is a small parameter. Then (14) gives

$$J(G) = J(\Gamma)+2\eta P(\Gamma, \gamma)-\eta^2\int_A\int_A K(s, t)\gamma(s)\gamma(t)dsdt, \quad \text{............(17)}$$

where, on using the symmetry of $K(s, t)$,

$$P(\Gamma, \gamma) = \int_A \gamma(s)\left\{f(s)-\int_B K(s, t)g(t)dt-\int_A K(s, t)\Gamma(t)dt\right\}ds.$$

If the inequality (13) is to be true for any choice of $G(s)$ this implies that $P(\Gamma, \gamma) = 0$ for any $\gamma(s)$ and hence that

$$\int_A K(s, t)\Gamma(t)dt = f(s)-\int_B K(s, t)g(t)dt, \ (s \text{ on } A). \quad \text{...............(18)}$$

This is an integral equation for $\Gamma(t)$.

Similarly from (15), (16), if $F(s) = \Theta(s)+\eta\theta(s)$ where $\Theta(s)$ is the exact value of ϕ on B, we find the following integral equation for Θ:

$$\int_B L(s, t)\Theta(t)dt = g(s)-\int_A L(s, t)f(t)dt, \ (s \text{ on } B)\text{................(19)}$$

Special cases of the integral equations (18), (19) form the starting point for the analysis in (**1**), (**2**). In the last paragraph we have reversed the procedure used in these references since, having derived the variational principles by an independent method, we can deduce the integral equations from the variational principles.

REFERENCES

(**1**) J. F. Carlson and T. J. Hendrickson, *J. Appl. Phys.*, **24** (1953), 1462-1465.

(**2**) B. Noble, *Proc. Edin. Math. Soc.*, **11** (1958), 115-126.

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