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A note on coalgebras and presheaves

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We show that the category of coalgebras of a wide-pullback preserving endofunctor on a category of presheaves is itself a category of presheaves. This illustrates a connection between Jacobs' temporal logic of coalgebras and Ghilardi and Meloni's presheaf semantics for temporal modalities.

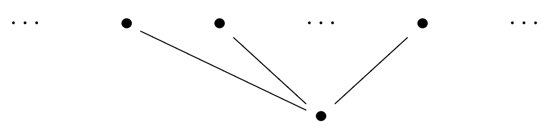
1. Introduction

Recall that a presheaf category is one that is equivalent to a functor category $\mathsf{Set}^{\mathscr{C}^{op}}$ for some small category $\mathscr{C}$. We show that the category of coalgebras of a wide-pullback preserving endofunctor T on a presheaf category is itself a presheaf category. In fact, we construct a freely generated path category $\mathscr{C}$ from the functor T such that T-coalgebras correspond to presheaves on $\mathscr{C}$. This construction is an adaptation of one used by Carboni and Johnstone (Carboni and Johnstone 1995) in showing that the category obtained by Artin gluing along a limit preserving functor between presheaf categories is also a presheaf category.

Coalgebras and presheaves have both been shown to yield Galois algebras – the algebraic structures required to model Computation Tree Logic. We show that the Galois algebra generated by a given coalgebra is isomorphic to the Galois algebra generated by the corresponding presheaf.

2. Wide pullbacks

Definition 2.1. (Carboni and Johnstone 1995) A *wide pullback* is the limit of a diagram indexed by a poset P, where P arises by adjoining a greatest element to an anti-chain.



The class of wide-pullback preserving endofunctors on Set is the smallest class containing the constant functors which is closed under arbitrary products and coproducts of functors (Carboni and Johnstone 1995). The free monad and the cofree comonad generated by a

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wide-pullback preserving set functor also preserve wide pullbacks (Johnstone *et al.* 2001). As a running example, we consider the finite list functor

$$T(X) = X^* = \coprod_{n\in\mathbb{N}} X^n.$$

Example 2.1. Let $T : \mathsf{Set} \to \mathsf{Set}$ be the subfunctor of the exponential $(-)^{\mathbb{N}}$ consisting of the functions with finite range. Since T preserves the final object, preservation of wide pullbacks amounts to the preservation of all products. However, T does not preserve the countably infinite product

$$P = \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N} \times \cdots,$$

since there is no map $f : \mathbb{N} \to P$ in TP that corresponds to the tuple $\langle f_n : \mathbb{N} \to \mathbb{N}\rangle_{n\in\mathbb{N}}$, where $f_n(x) = \min(x, n)$. On the other hand, T preserves ordinary pullbacks.

Proposition 2.1. If $\mathscr{E}$ is a complete category, every wide-pullback preserving endofunctor $T : \mathscr{E} \to \mathscr{E}$ has a final coalgebra.

Proof. A wide-pullback preserving functor whose domain is a complete category preserves all connected limits (Carboni and Johnstone 1995, Lemma 2.1). It follows that T preserves limits indexed by the chain ω^{op}. Thus the final coalgebra of T may be constructed as the limit of the ω^{op}-chain

$$1 \leftarrow T1 \leftarrow T^2 1 \leftarrow \ldots$$

in the standard manner (Rutten 2000). □

Let $\mathscr{E}$ be a complete category, suppose $T : \mathscr{E} \to \mathscr{E}$ preserves wide pullbacks, and let $\alpha : A \to TA$ be given. We define a *reduction* of T to a limit preserving endofunctor on the slice category $\mathscr{E}/A$ as follows. T has an obvious lifting to a functor $T_A : \mathscr{E}/A \to \mathscr{E}/TA$, and, composing this with the pullback functor $\alpha^* : \mathscr{E}/TA \to \mathscr{E}/A$, we obtain an endofunctor $T_\alpha : \mathscr{E}/A \to \mathscr{E}/A$. Thus, for an object $f : B \to A$ of $\mathscr{E}/A$, $T_\alpha f$ is defined by the pullback

$$\begin{array}{ccc} \bullet & \longrightarrow & TB \\ {\scriptstyle T_\alpha f}\downarrow & & \downarrow{\scriptstyle Tf} \\ A & \xrightarrow[\alpha]{} & TA \end{array} \tag{1}$$

Proposition 2.2.

(i) T_α preserves all (small) limits.
(ii) Coalg T_α is isomorphic to the slice category Coalg $T/(A, \alpha)$.

Proof.

(i) Observing that wide pullbacks in a slice category $\mathscr{E}/A$ are created by the domain functor $\mathscr{E}/A \to \mathscr{E}$, it is easy to see that they are preserved by T_A. Furthermore, T_A clearly preserves final objects. Thus T_A preserves all limits, since any limit can be constructed from final objects and wide pullbacks. The functor α^* is a right adjoint, and thus preserves all limits. It follows that $T_\alpha = \alpha^* \cdot T_A$ is continuous.

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(ii) If $f : B \to A$ is a map in $\mathscr{E}$, then a coalgebra structure $f \to T_\alpha f$ clearly corresponds to a map $\beta : B \to TB$ such that f is a coalgebra map $(B, \beta) \to (A, \alpha)$. This extends to an isomorphism of categories, acting as the identity on homsets. □

Example 2.2. Consider the finite list functor $T(X) = X^*$. The final T-coalgebra (A, α) is obtained by setting A to be the set of rooted, finitely branching trees, such that the set of children of each node is equipped with a total ordering. The structure map $\alpha : A \to A^*$ maps each tree $t \in A$ to the list of the subtrees originating from the children of the root of t.

Regarding objects of Set/A as A-indexed sets, the functor $T_\alpha : \mathsf{Set}/A \to \mathsf{Set}/A$ is given by

$$T_\alpha(X)_t = X_{t_1} \times \cdots \times X_{t_n}, \text{ where } \alpha(t) = \langle t_1, \ldots, t_n \rangle.$$

3. Bimodules and continuous functors

In this short section we recall an equivalence between continuous functors on presheaf categories and bimodules (Carboni and Johnstone 1995).

Definition 3.1. (Carboni and Johnstone 1995; Street 1980) A *bimodule* (also called a profunctor or distributor) from a category $\mathscr{A}$ to a category $\mathscr{B}$, written $\phi : \mathscr{A} \nrightarrow \mathscr{B}$, is a functor

$$\phi : \mathscr{B}^{\mathrm{op}} \times \mathscr{A} \to \mathsf{Set}.$$

When $\mathscr{A}$ and $\mathscr{B}$ are small, there is a category of bimodules $\mathscr{A} \nrightarrow \mathscr{B}$ and natural transformations, which we denote $\mathrm{Mod}(\mathscr{A}, \mathscr{B})$.

There is an equivalence

$$\mathrm{Mod}(\mathscr{A}, \mathscr{B}) \simeq \mathrm{Cocont}(\mathsf{Set}^{\mathscr{A}^{\mathrm{op}}}, \mathsf{Set}^{\mathscr{B}^{\mathrm{op}}}) \tag{2}$$

between the category of bimodules between two small categories and the category of cocontinuous functors between the respective categories of presheaves. To see this, first observe that a bimodule $\phi : \mathscr{A} \nrightarrow \mathscr{B}$ may be regarded as a functor $\mathscr{A} \to \mathsf{Set}^{\mathscr{B}^{\mathrm{op}}}$. Then the two components of the equivalence (2) are given, respectively, by restriction and left Kan extension along the Yoneda embedding $\mathsf{y}_{\mathscr{A}} : \mathscr{A} \to \mathsf{Set}^{\mathscr{A}^{\mathrm{op}}}$.

From the Adjoint Functor Theorem, each continuous functor between presheaf categories has a left adjoint, and each cocontinuous functor between presheaf categories has a right adjoint. This yields an equivalence:

$$\mathrm{Cocont}(\mathsf{Set}^{\mathscr{A}^{\mathrm{op}}}, \mathsf{Set}^{\mathscr{B}^{\mathrm{op}}})^{\mathrm{op}} \simeq \mathrm{Cont}(\mathsf{Set}^{\mathscr{B}^{\mathrm{op}}}, \mathsf{Set}^{\mathscr{A}^{\mathrm{op}}}). \tag{3}$$

Composing (2) and (3), we get the desired equivalence:

$$\mathrm{Mod}(\mathscr{A}, \mathscr{B})^{\mathrm{op}} \simeq \mathrm{Cont}(\mathsf{Set}^{\mathscr{B}^{\mathrm{op}}}, \mathsf{Set}^{\mathscr{A}^{\mathrm{op}}}). \tag{4}$$

Next we give an explicit calculation of the image of the bimodule ϕ under the above equivalence, which we denote $[\phi, -]^{\mathscr{B}}$. This notation is explained in Lawvere (1973) and Street (1980).

Given a bimodule $\phi : \mathscr{A} \nrightarrow \mathscr{B}$, let the functor $\lambda a\, \lambda b\; \phi(b,a) : \mathscr{A} \to \mathsf{Set}^{\mathscr{B}^{\mathrm{op}}}$ be denoted $\overline{\phi}$, and consider $\mathrm{Lan}_{\mathsf{y}_{\mathscr{A}}}\overline{\phi}$, the left Kan extension of $\overline{\phi}$ along the Yoneda embedding $\mathsf{y}_{\mathscr{A}} : \mathscr{A} \to \mathsf{Set}^{\mathscr{A}^{\mathrm{op}}}$. This is a cocontinuous functor and is given by the formula

$$\mathrm{Lan}_{\mathsf{y}_{\mathscr{A}}}\overline{\phi}(P) = \mathrm{Colim}(\mathrm{Elts}(P) \xrightarrow{U} \mathscr{A} \xrightarrow{\overline{\phi}} \mathsf{Set}^{\mathscr{B}^{\mathrm{op}}})$$

for a presheaf $P : \mathscr{A}^{\mathrm{op}} \to \mathsf{Set}$, where $\mathrm{Elts}(P)$ is the comma category $(1 \downarrow P)$.

Fixing a presheaf $Q : \mathscr{B}^{\mathrm{op}} \to \mathsf{Set}$, a morphism $\mathrm{Lan}_{\mathsf{y}_{\mathscr{A}}}\overline{\phi}(P) \Rightarrow Q$ corresponds to a cocone from the diagram $\overline{\phi} \circ U$ to Q. The data for such a cocone is, for each pair $(a,x) \in \mathrm{Elts}(P)$, a choice of a natural transformation $\alpha_{(a,x)} : \overline{\phi}(a) \Rightarrow Q$ – this choice being natural in (a,x). This amounts to a natural transformation $P \Rightarrow [\phi, Q]^{\mathscr{B}}$, where the functor

$$[\phi, -]^{\mathscr{B}} : \mathsf{Set}^{\mathscr{B}^{\mathrm{op}}} \to \mathsf{Set}^{\mathscr{A}^{\mathrm{op}}}$$

is defined by

$$[\phi, Q]^{\mathscr{B}}(a) = \mathsf{Set}^{\mathscr{B}^{\mathrm{op}}}(\phi(-,a), Q),$$

and is, by this definition, right adjoint to $\mathrm{Lan}_{\mathsf{y}_{\mathscr{A}}}\overline{\phi}$.

Example 3.1. The slice category Set/A in Example 2.2 is a presheaf category (over A regarded as a discrete category). Here we calculate the bimodule corresponding to the continuous functor $T_\alpha : \mathsf{Set}/A \to \mathsf{Set}/A$ occurring in that example.

Given $s, t \in A$, write $t \overset{n}{\rightsquigarrow} s$ if s occurs exactly n times in the list $\alpha(t)$. Now the left adjoint L to T_α is given by

$$L(X)_s = \coprod_{t \overset{n}{\rightsquigarrow} s} n \cdot X_t.$$

Identifying $t \in A$ with the object $t : 1 \to A$ of the slice category Set/A, the bimodule $\phi : A \times A \to \mathsf{Set}$ corresponding to L is given by

$$\phi(s,t) = L(t)_s = \{1, \ldots, n\}, \text{ where } t \overset{n}{\rightsquigarrow} s.$$

4. A path category construction

Suppose $\mathscr{A}$ is a small category and $\phi : \mathscr{A} \nrightarrow \mathscr{A}$. We construct a category $\mathscr{C}(\phi)$ such that the category of presheaves $\mathsf{Set}^{\mathscr{C}(\phi)^{\mathrm{op}}}$ is equivalent to the category of T-coalgebras, where $T = [\phi, -]^{\mathscr{A}} : \mathsf{Set}^{\mathscr{A}^{\mathrm{op}}} \to \mathsf{Set}^{\mathscr{A}^{\mathrm{op}}}$ is the continuous endofunctor corresponding to ϕ.

Let $\mathscr{G}(\phi)$ be the graph with:

(i) Nodes: the set of objects of $\mathscr{A}$;

(ii) Edges: for each arrow $f : a \to b$ of $\mathscr{A}$ an edge $f : a \to b$ of $\mathscr{G}(\phi)$, and, for each pair of objects a, b of $\mathscr{A}$ and each $e \in \phi(b,a)$, an edge $e : b \to a$ of $\mathscr{G}(\phi)$.

From the graph $\mathscr{G}(\phi)$ we freely generate a category, which we denote $\mathscr{C}(\phi)$, subject to the following equations on composition in $\mathscr{C}(\phi)$ (written as $\cdot_{\mathscr{C}(\phi)}$).

(a) For composable morphisms f, g of $\mathscr{A}$, $f \cdot_{\mathscr{C}(\phi)} g = f \cdot g$ (that is, we include all the indentities holding in $\mathscr{A}$).

(b) If $e \in \phi(b,a)$ and $f : b' \to b$ is an arrow of $\mathscr{A}$, then $e \cdot_{\mathscr{C}(\phi)} f = \phi(f,a)e$.

(c) If $e \in \phi(b, a)$ and $f : a \to a'$ is an arrow of $\mathscr{A}$, then $f \cdot_{\mathscr{C}(\phi)} e = \phi(b, f)e$.

A graph homomorphism $P : \mathscr{G}(\phi)^{op} \to \mathsf{Set}$ is a graph homomorphism $P_0 : \mathscr{A}^{op} \to \mathsf{Set}$ together with a family of mappings, indexed over pairs of objects $a, b \in \mathscr{A}$,

$$\alpha(b, a) : \phi(b, a) \to P_0(b)^{P_0(a)}.$$

By exponential transposition, this last datum amounts to a family of mappings,

$$\overline{\alpha}(b, a) : P_0(a) \to P_0(b)^{\phi(b,a)}.$$

The graph homomorphism P will be a functor if it preserves identities in $\mathscr{C}(\phi)$ and the three types of composition (a)–(c) above. The preservation of identities and composites of type (a) is equivalent to P_0 being a functor $\mathscr{A}^{op} \to \mathsf{Set}$. Given this, P preserves composites of type (b) precisely when, for each $x \in P_0(a)$, it holds that $\overline{\alpha}(-, a)x$ is a natural transformation $\phi(-, a) \Rightarrow P_0$, that is,

$$\overline{\alpha}(-, a) : P_0(a) \to [\phi, P_0]^{\mathscr{A}}(a). \tag{5}$$

In addition, preservation of composites of type (c) is the same as requiring that the above family of maps is natural in $a \in \mathscr{A}$. Thus we have shown that a presheaf P on $\mathscr{C}(\phi)$ amounts to a pair $(P_0, \overline{\alpha})$, where P_0 is a presheaf on $\mathscr{A}$ and

$$\overline{\alpha} : P_0 \to [\phi, P_0]^{\mathscr{A}} \tag{6}$$

is a natural transformation.

Let us suppose we have another presheaf Q on $\mathscr{C}(\phi)$, consisting of a presheaf Q_0 on $\mathscr{A}$, and a family of maps $\beta(b, a) : \phi(b, a) \to Q_0(b)^{Q_0(a)}$ indexed over $b, a \in \mathscr{A}$. A natural transformation $\Xi : P \Rightarrow Q$ is precisely a natural transformation $\Xi_0 : P_0 \Rightarrow Q_0$ such that the left-hand diagram below commutes for each pair of objects $a, b \in \mathscr{A}$. But, by exponential transposition (and the naturality properties of the transposes, expressed in (5) and (6)), this is just the same as requiring that the right-hand diagram commutes.

$$\begin{array}{ccc} \phi(b,a) & \xrightarrow{\alpha(b,a)} & P_0(b)^{P_0(a)} \\ {\scriptstyle \beta(b,a)}\downarrow & & \downarrow{\scriptstyle \Xi_{0,b}{}^{P_0(a)}} \\ Q_0(b)^{Q_0(a)} & \xrightarrow[Q_0(b)^{\Xi_{0,a}}]{} & Q_0(b)^{P_0(a)} \end{array} \qquad \begin{array}{ccc} P_0 & \xrightarrow{\overline{\alpha}} & [\phi, P_0]^{\mathscr{A}} \\ {\scriptstyle \Xi_0}\downarrow & & \downarrow{\scriptstyle [\phi,\Xi_0]^{\mathscr{A}}} \\ Q_0 & \xrightarrow[\overline{\beta}]{} & [\phi, Q_0]^{\mathscr{A}} \end{array}$$

It follows that there is an isomorphism of categories between $\mathsf{Set}^{\mathscr{C}(\phi)^{\mathrm{op}}}$ and the category of coalgebras of $[\phi, -]^{\mathscr{A}}$. Since any continuous endofunctor on $\mathsf{Set}^{\mathscr{A}^{\mathrm{op}}}$ is isomorphic to $[\phi, -]^{\mathscr{A}}$, for some ϕ, we obtain the following result.

Theorem 4.1. If T is a continuous endofunctor on a presheaf category, Coalg T is itself a presheaf category.

Using the reduction of wide-pullback preserving functors to continuous functors from Section 2, we can weaken the hypothesis 'T is continuous' to 'T preserves wide pullbacks'.

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Corollary 4.1. If T is a wide-pullback preserving endofunctor on a presheaf category, then Coalg T is itself a presheaf category.

Proof. Proposition 2.1 tells us that there is a final T-coalgebra (A, α). From Proposition 2.2 it follows that Coalg $T \simeq$ Coalg $T/(A, \alpha) \simeq$ Coalg T_α. Since T_α is continuous, and the property of being a presheaf category is preserved by taking slices (Johnstone 1977, Corollary 2.18), the result follows. □

Example 4.1. Recall our running example: the finite list functor $T : \mathsf{Set} \to \mathsf{Set}$. Then Coalg $T \simeq \mathsf{Set}^{\mathscr{C}^{op}}$, where $\mathscr{C}$ is the free category over a certain graph $\mathscr{G}$. The set of nodes of $\mathscr{G}$ is the carrier of the final T-coalgebra (A, α), that is, the set of trees described in Example 2.2. The number of edges $t \to s$ in $\mathscr{G}$ is the number of times t occurs in the list $\alpha(s)$.

The proof of Theorem 4.1 followed an idea in Carboni and Johnstone (1995). They showed that, for a continuous functor $T : \mathscr{E} \to \mathscr{F}$ between presheaf categories, the comma category $(\mathscr{F} \downarrow T)$ is again a presheaf category. The precise relationship is best understood from a bicategorical perspective (Street 1980), as we now explain.

Given an endofunctor $T : \mathscr{E} \to \mathscr{E}$, Coalg T has a universal property in the 2-category CAT of large categories, functors and natural transformations: it is the oplax limit of a diagram with shape

(7)

where the node is labelled $\mathscr{E}$ and the edge T (Adámek and Rosický 1994, Definition 2.69). Given a bimodule $\phi : \mathscr{A} \nrightarrow \mathscr{A}$, $\mathscr{C}(\phi)$ has the same universal property in the bicategory Mod of small categories, bimodules and natural transformations: it is the oplax limit of a diagram whose shape is given by (7), but where the node is labelled $\mathscr{A}$ and the edge ϕ. On the other hand, for $T : \mathscr{E} \to \mathscr{F}$, the comma category $(\mathscr{F} \downarrow T)$ is the oplax limit in CAT of a diagram of shape $\bullet \to \bullet$. Now, Carboni and Johnstone (1995) shows that $(\mathscr{F} \downarrow T)$ is a category of presheaves over a small category obtained as the oplax limit in Mod of a diagram of the same shape.

5. Galois algebras from presheaves and coalgebras

Coalgebras and presheaves have both been used to model temporal logic. For coalgebras this is described by Jacobs (Jacobs 2002), and for presheaves by Ghilardi and Meloni (Ghilardi and Meloni 1988). Given that coalgebras can be seen as presheaves, a natural question arises as to how these different semantics are related.

Definition 5.1. (Karger 1998) A *Galois algebra* is a complete Boolean algebra B, together with a 'henceforth' operator $[\mathrm{F}] : B \to B$ preserving all meets. A morphism of Galois algebras $f : (B, [\mathrm{F}]) \to (B', [\mathrm{F}]')$ is a map $f : B \to B'$ preserving all meets and joins, such that $f \circ [\mathrm{F}] = [\mathrm{F}]' \circ f$. This yields a category, which we denote GA.

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Galois algebras provide an algebraic semantics for Computation Tree Logic (CTL) in that all the axioms and rules of CTL are valid in an arbitrary Galois algebra (see Karger (1998)). We read [F]S as 'in all future states S'. Furthermore, we denote the left adjoint of [F] by ⟨P⟩, and read ⟨P⟩S as 'in some past state S'. Next we recall from Jacobs (2002) and Rutten (2000) how coalgebras give rise to Galois algebras.

Assume that $T : \mathsf{Set} \to \mathsf{Set}$ preserves wide pullbacks. Then the forgetful functor from Coalg T to Set also preserves wide pullbacks, and hence preserves monos. Thus, given a T-coalgebra (B, β), the carrier of a subcoalgebra may be assumed to be a subset of B. On the other hand, given $S \subseteq B$, there is at most one coalgebra structure $S \to TS$ making the inclusion $S \subseteq B$ a subcoalgebra. So the class $\mathrm{Sub}((B, \beta))$ of subcoalgebras of (B, β) may be identified with a certain class of subsets of B. Moreover, this class is closed under all unions and intersections in $\mathrm{Sub}(B)$; see Rutten (2000, Section 6). Thus we get an embedding of complete lattices

$$\mathrm{Sub}((B, \beta)) \hookrightarrow \mathrm{Sub}(B).$$

This map has a right adjoint, sending $S \subseteq B$ to the largest subcoalgebra of (B, β) whose carrier is contained in S: the subcoalgebra *cogenerated* by S. The interior operator corresponding to this adjunction is denoted

$$[\mathrm{F}] : \mathrm{Sub}(B) \to \mathrm{Sub}(B).$$

Thus [F]S is the largest subset of S that carries a subcoalgebra structure. The left adjoint to [F] is ⟨P⟩, where ⟨P⟩S is the carrier of the smallest subcoalgebra containing S: the subcoalgebra *generated* by S.

In this way the coalgebra (B, β) yields a Galois algebra $(\mathrm{Sub}(B), [\mathrm{F}])$. In fact, we get an *indexed Galois algebra*, that is, a functor

$$\mathrm{Coalg}\,(T) \longrightarrow \mathsf{GA}^{\mathrm{op}}, \tag{8}$$

where the coalgebra map $f : (B, \beta) \to (B', \beta')$ is sent to the pullback map $f^* : \mathrm{Sub}(B') \to \mathrm{Sub}(B)$.

There is also a natural way to obtain a Galois algebra from a presheaf. The idea, due to Ghilardi and Meloni (1988), is to interpret the temporal modalities [F] and ⟨P⟩ as *cogenerated subpresheaf* and *generated subpresheaf*, respectively.

Suppose $\mathscr{A}$ is a small category with $\mathrm{obj}(\mathscr{A}) = A$. Then we have a forgetful functor $\mathsf{Set}^{\mathscr{A}^{\mathrm{op}}} \to \mathsf{Set}/A$ sending a presheaf $Q : \mathscr{A}^{\mathrm{op}} \to \mathsf{Set}$ to the A-indexed set Q_0, where $(Q_0)_a = Q(a)$ for $a \in A$. Since meets and joins in $\mathrm{Sub}(Q)$ are computed pointwise, we get an embedding of complete lattices

$$\mathrm{Sub}(Q) \hookrightarrow \mathrm{Sub}(Q_0).$$

This map has a right adjoint, sending an A-indexed subset S of Q_0 to the cogenerated subpresheaf – the largest subfunctor of Q contained in S. The corresponding interior operator

$$[\mathrm{F}] : \mathrm{Sub}(Q_0) \to \mathrm{Sub}(Q_0)$$

is given by the formula

$$([\mathrm{F}]S)_a = \{x \in Q(a) \mid (\forall b \in A)(\forall f : b \to a)\ Q(f)(x) \in S_b\}.$$

We also have the generated subpresheaf $\langle \mathrm{P} \rangle S$, that is, the least subset of Q containing S that is also a subpresheaf of Q. This is given by the formula

$$(\langle \mathrm{P} \rangle S)_a = \{y \in Q(a) \mid (\exists b \in A)(\exists g : a \to b)(\exists x \in Q(b))\ y = Q(g)(x)\}.$$

The map sending a presheaf Q to the Galois algebra $(\mathrm{Sub}(Q_0), [\mathrm{F}])$ yields an indexed Galois algebra (that is, a functor)

$$\mathsf{Set}^{\mathscr{A}^{\mathrm{op}}} \longrightarrow \mathsf{GA}^{\mathrm{op}}. \tag{9}$$

5.1. *An isomorphism of Galois algebras*

Let T be a wide-pullback preserving set functor, and suppose (A, α) is the final T-coalgebra. Then Theorem 4.1 constructs an equivalence between $\mathrm{Coalg}\, T$ and $\mathsf{Set}^{\mathscr{A}^{\mathrm{op}}}$ for some category $\mathscr{A}$ with $\mathrm{obj}(\mathscr{A}) = A$.

Theorem 5.1. The indexed Galois algebra $\mathsf{Set}^{\mathscr{A}^{\mathrm{op}}} \longrightarrow \mathsf{GA}^{\mathrm{op}}$ in (9) is naturally isomorphic to the composition of the indexed Galois algebra $\mathrm{Coalg}\, T \to \mathsf{GA}^{\mathrm{op}}$ in (8) with the equivalence $\mathsf{Set}^{\mathscr{A}^{\mathrm{op}}} \simeq \mathrm{Coalg}\, T$.

Proof. Suppose that the presheaf $Q : \mathscr{A}^{\mathrm{op}} \to \mathsf{Set}$ is mapped to the T-coalgebra (B, β) under the equivalence $\mathsf{Set}^{\mathscr{A}^{\mathrm{op}}} \simeq \mathrm{Coalg}\, T$. We show that the associated Galois algebras are isomorphic. (We omit the straightforward verification that this isomorphism in natural in Q.)

Observe that the following diagram commutes:

$$\begin{array}{ccccc} \mathsf{Set}^{\mathscr{A}^{\mathrm{op}}} & \xrightarrow{\simeq} & \mathrm{Coalg}\, T_\alpha & \xrightarrow{\simeq} & \mathrm{Coalg}\, T \\ & \searrow & \downarrow & & \downarrow \\ & & \mathsf{Set}/A & \xrightarrow[\sum_{\mathrm{A}}]{} & \mathsf{Set} \end{array} \tag{10}$$

The functor $\sum_{\mathrm{A}} : \mathsf{Set}/A \to \mathsf{Set}$ sends an object of the slice category to its domain. The two horizontal arrows represent the equivalences constructed in Proposition 2.2 and Theorem 4.1. The three remaining arrows are the relevant 'forgetful functors' described above.

Diagram (10) cuts down to the following commuting diagram of complete lattices and maps preserving all meets and joins:

$$\begin{array}{ccc} \mathrm{Sub}(Q) & \xrightarrow{\simeq} & \mathrm{Sub}((B, \beta)) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Sub}(Q_0) & \xrightarrow{\simeq} & \mathrm{Sub}(B) \end{array} \tag{11}$$

The bottom leg in (11) is an isomorphism of complete lattices. Moreover, it is readily verified from the commutativity of (11) and the corresponding diagram of right adjoints that this isomorphism respects the Galois algebra structures on Sub(B) and Sub(Q_0), respectively. □

Jacobs (2002) presents a result closely related to Theorem 5.1: under the representation of a given category of presheaves as a category of coalgebras for a comonad, generated and cogenerated subpresheaves agree with generated and cogenerated subcoalgebras.

6. Future work

It is possible to generalise the ideas of Ghilardi and Meloni to sheaves on a site. That is, for a predicate S on a sheaf Q we have a generated subsheaf $\langle \mathrm{P} \rangle S$ and a cogenerated subsheaf $[\mathrm{F}]S$. We would like to see if these correspond to generated and cogenerated subcoalgebras under the coalgebras-as-sheaves correspondence presented in Johnstone *et al.* (2001) for coalgebras of weak-pullback preserving functors. In general, a Grothendieck topos is equivalent to a category of sheaves on many different sites, and it seems to us that the key to solving this problem is to find the 'right' sites for the toposes considered in Johnstone *et al.* (2001).

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