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A continuum theory for one-dimensional self-similar elasticity and applications to wave propagation and diffusion

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We analyse some fundamental problems of linear elasticity in one-dimensional (1D) continua where the material points of the medium interact in a self-similar manner. This continuum with 'self-similar' elastic properties is obtained as the continuum limit of a linear chain with self-similar harmonic interactions (harmonic springs) which was introduced in [19] and (Michelitsch T.M. (2011) The self-similar field and its application to a diffusion problem. *J. Phys. A Math. Theor*. **44**, 465206). We deduce a continuous field approach where the self-similar elasticity is reflected by self-similar Laplacian-generating equations of motion which are spatially non-local convolutions with power-function kernels (fractional integrals). We obtain closed-form expressions for the static displacement Green's function due to a unit δ-force. In the dynamic framework we derive the solution of the *Cauchy problem* and the retarded Green's function. We deduce the distributions of a self-similar variant of diffusion problem with Lévi-stable distributions as solutions with infinite mean fluctuations. In both dynamic cases we obtain a hierarchy of solutions for the self-similar Poisson's equation, which we call 'self-similar potentials'. These non-local singular potentials are in a sense self-similar analogues to Newtonian potentials and to the 1D Dirac's δ-function. The approach can be a point of departure for a theory of self-similar elasticity in 2D and 3D and for other field theories (e.g. in electrodynamics) of systems with scale invariant interactions.

Key words: Mechanics of fractal structures; Self-similar elasticity; Fractional calculus

1 Introduction

The discovery of fractal geometry in nature by Mandelbrot in the seventies of the last century had a great impact on science and opened us a new view on nature [14–16, 22]. In the last decade also an increasing interest has risen in the mechanics of materials having a microstructure which is exactly or statistically invariant over a wide range of scales. Such materials can be in a good approximation idealised as fractal materials. Such an idealisation is to be understood in the same way as the notion of the infinite medium. Although there is no infinite medium in nature, this notion has proven to be a highly appropriate idealisation. Some interesting results to describe the mechanical properties in fractal materials have been achieved recently [3,6,10,20]. Fractal properties of crack patterns have been investigated [4], mechanical properties of self-similar and self-affine crack have been investigated in [24] and some fractal aspects of bifractal surfaces have been studied in [13]. In spite of these progresses a generally accepted theory of the mechanics of fractal and scale invariant (self-similar) structures is not yet established. Therefore, simple, even simplistic, models capturing only a few aspects of the characteristic fractal behaviour in materials are highly desirable.

The point of departure in the present paper is a continuum limit in the one-dimensional (1D) medium, which has been introduced as *quasi-continuous linear chain* in [19]. This linear chain consists of a spatially homogeneous distribution of material points: Each mass point is non-locally connected with other mass points by a spatially self-similar distribution of interparticle springs. The linear chain of [19] is characterised by a *discrete* spatially self-similar set of harmonic interparticle springs with a self-similar elastic energy (relation (2.2) in Section 2). In the present paper we analyse this chain in its *continuum limit*, where each material point is connected with others by a *continuous* self-similar set of interparticle springs (where the elastic energy takes relation (2.5)). In the discrete case treated in [19], the dispersion relation exhibits fractal features. In the continuum limit, which is subject of the present paper, all fractal features of the dispersion relation (equation (3.11)) are wiped out. However, the property of self-similarity is conserved in the continuum limit.

Before we start with our analysis, we should like to specify our notion of 'self-similarity': The notion self-similarity used in the present paper as well as in [17,19] corresponds to the notion of *self-similarity at a point* commonly used in the mathematical literature [21]. An object is self-similar in the strict sense if it *consists* of parts which are exact re-scaled copies of the entire object. In contrast is the notion of self-similarity at a point (with point[1] we mean a space-point which is contained by or attached to the object, where we place, for example, the origin of a coordinate system): An object which is 'self-similar at a point' *contains only a single part* which is a re-scaled copy of the entire object [21]. The notion of self-similarity at a point implies that the scaling invariance repeats over an infinity of scales. In this paper as well as in [17,19] we call an object just *self-similar* without always mentioning that we actually mean self-similar at a point. To be more precise, when we call a function $\Lambda(h)$ self-similar with respect to h, which we define by $\Lambda(Nh) = N^{\delta}\Lambda(h)$ for a *prescribed* scaling factor N, then strictly speaking the function $\Lambda(h)$ is self-similar at the point $h = 0$. The simplest self-similar functions of this type are power-functions h^{δ}.

The paper is organised as follows: In Section 3 we deduce the *static* Green's function of displacements due to a δ-unit force in closed form. In Section 4 we construct the basic solution of the Cauchy problem in the form of two integral kernels which give the solutions due to prescribed initial displacement- and velocity-field, respectively, in

[1] Where that point is a fixed-point of that scaling operation.

analogy to the classical d'Alembert's solution. One of these two kernels also determines the retarded time-domain dynamic Green's function.

In Section 5 we analyse a self-similar diffusion problem which we formulate as an initial value problem for a given initial distribution by utilising our self-similar Laplacian. The solutions of this problem turn out to be Lévi-stable distributions with infinite mean fluctuations. This problem has also been analysed in [17]; however, the asymptotic regime of large-scaled times was not considered. As a spin-off result we obtain a hierarchy of functions which we call 'self-similar potentials' as they play the analogue role for our self-similar Laplacian as the Newtonian potentials do in the case of a 'traditional' Poisson's equation. This hierarchy of potentials includes also the *static Green's function* (inverse operator of the self-similar Laplacian) as a solution of a δ-type source.

2 The self-similar elastic continuum

We consider a purely elastic medium in 1D with a spatial continuous constant mass distribution of density 1 where any spatial point x represents a material point which interacts harmonically through interconnecting springs (spring constants $N^{-\delta s}$) with an ensemble of other material points located at $x \pm hN^s$ ($s \in \mathbb{Z}_0$) such that the elastic energy is a self-similar function with respect to h at $h = 0$. This is expressed by relation (2.3). The fractal properties of the dispersion relation of this linear chain have been analysed in our recent paper [19]. As a point of departure for our continuum approach, we consider this quasi-continuous chain with self-similar harmonic springs which has the Hamiltonian [19]

$$H = \frac{1}{2}\int_{-\infty}^{\infty} \left(\dot{u}^2(x,t) + \mathscr{V}(x,t,h)\right) \mathrm{d}x, \tag{2.1}$$

where x denotes the space- and t the time coordinates. $\frac{1}{2}\mathscr{V}(x,t,h)$ indicates the elastic energy density,[2]

$$\mathscr{V}(x,t,h) = \frac{1}{2}\sum_{s=-\infty}^{\infty} N^{-\delta s}\left[(u(x,t) - u(x+hN^s,t))^2 + (u(x,t) - u(x-hN^s,t))^2\right], \tag{2.2}$$

which converges in the range $0 < \delta < 2$ and where we assume $h > 0$ and N being a prescribed scaling factor. Without loss of generality we can restrict ourselves to $N > 1$ ($N \in \mathbb{R}$). In (2.1) and (2.2) u and $\dot{u} = \frac{\partial}{\partial t}u$ stand for the displacement field and the velocity field respectively. Equation (2.2) has the property of being self-similar with respect to h at point $h = 0$, namely

$$\mathscr{V}(x,t,Nh) = N^{\delta}\mathscr{V}(x,t,h). \tag{2.3}$$

As a point of departure for the approach to be developed, we evoke the continuum limit of (2.2) and the resulting equation of motion [17]. An important point is that variable h does not have the meaning of an internal length scale. Due to self-similarity relation (2.3) there is no characteristic length scale of range for the interparticle interactions. We define the continuum limit as $N = 1 + \zeta$ ($0 < \zeta \ll 1$) so $\tau = hN^s$ becomes a continuous variable and we can write a self-similar function $\Lambda(h)$ which fulfils a self-similarity condition (2.3)

[2] The additional factor 1/2 in the elastic energy avoids double counting.

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asymptotically as [19],

$$\Lambda(h) = \sum_{s=-\infty}^{\infty} N^{-\delta s} f(N^s h) \approx \frac{h^\delta}{\zeta} \int_0^\infty \frac{f(\tau)}{\tau^{\delta+1}} \mathrm{d}\tau \tag{2.4}$$

having the form of a power function $\Lambda(h) = const\, h^\delta$, i.e. the self-similarity with respect to h is conserved in the continuum limit. Both the discrete as well as the continuous representation of (2.4) converge for the same sufficiently good functions (see details in [19]). From (2.4) follows that in that continuum limit we can write the elastic energy density defined by (2.2) as a function of the displacement field $u(x,t)$ in the form

$$\mathscr{V}(x,t,h) = \frac{h^\delta}{2\zeta} \int_0^\infty \frac{\{(u(x,t) - u(x+\tau,t))^2 + (u(x,t) - u(x-\tau,t))^2\}}{\tau^{\delta+1}} \mathrm{d}\tau, \tag{2.5}$$

which exists as in the discrete case (2.2) in the band $0 < \delta < 2$. The equation of motion (self-similar wave equation) has then the form [19]

$$\frac{\partial^2}{\partial t^2} u(x,t) = \Delta_{(\delta,h,\zeta)} u(x,t). \tag{2.6}$$

The right-hand side of (2.6) can be conceived as the *self-similar Laplacian* of the 1D medium. We can further conceive (2.6) as the *wave equation* governing the 1D *self-similar elastic medium*. There were many models that suggested involving of fractional non-local wave operators, e.g. [2, 5]. In many of these models, however, the choice of the fractional operator (e.g. the Caputo fractional derivative in [2]) appears rather arbitrary. In contrast, the spatially non-local wave equation (2.6) is a *deduced* equation of motion in the self-similar elastic medium with the elastic energy density determined by (2.5) leading to (2.6) without any further assumptions. The self-similar Laplacian is necessarily a non-local and *self-adjoint* negative definite operator and is obtained as

$$\Delta_{(\delta,h,\zeta)} u(x) = \frac{h^\delta}{\zeta} \int_0^\infty \frac{(u(x-\tau) + u(x+\tau) - 2u(x))}{\tau^{1+\delta}} \,\mathrm{d}\tau, \qquad 0 < \delta < 2 \tag{2.7}$$

and exists in the interval $0 < \delta < 2$. It might be sometimes convenient to rewrite (2.7) in the equivalent form,

$$\Delta_{(\delta,h,\zeta)} u(x) = \frac{h^\delta}{\zeta\delta} \frac{d}{dx} \int_0^\infty \frac{(u(x+\tau) - u(x-\tau))}{\tau^\delta} \,\mathrm{d}\tau, \qquad 0 < \delta < 2. \tag{2.8}$$

With (2.8) the equation of motion (2.6) takes the form

$$\frac{\partial^2}{\partial t^2} u(x,t) = \frac{\partial}{\partial x} \sigma(x,t), \tag{2.9}$$

where $\frac{\partial}{\partial x}$ indicates the traditional partial derivative with respect to x. The function $\sigma(x,t)$ can be conceived as the 1×1 stress tensor at space-point x having the form

$$\sigma(x) = \frac{h^\delta}{\zeta\delta} \int_0^\infty \frac{(u(x+\tau) - u(x-\tau))}{\tau^\delta} \,\mathrm{d}\tau, \qquad 0 < \delta < 2, \tag{2.10}$$

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where the integration constant can be put to zero. We can express the Laplacian (2.7), (2.8) in terms of the *Weyl–Marchaud fractional derivatives,* which are defined as [23]

$$D_l^{\delta}u(x) = \frac{\delta}{\Gamma(1-\delta)} \int_0^{\infty} \frac{(u(x) - u(x-\tau))}{\tau^{\delta+1}} \mathrm{d}\tau, \qquad \text{left} - \text{sided}, \qquad 0 < \delta < 1, \quad (2.11)$$

and

$$D_r^{\delta}u(x) = \frac{(-1)^{\delta}\delta}{\Gamma(1-\delta)} \int_0^{\infty} \frac{(u(x) - u(x+\tau))}{\tau^{\delta+1}} \mathrm{d}\tau, \qquad \text{right} - \text{sided}, \qquad 0 < \delta < 1, \quad (2.12)$$

where $0 < \delta < 1$ indicates the range of existence of each fractional integral $D_{l,r}^{\delta}u(x)$ respectively. $\Gamma(z)$ indicates the Γ-function (faculty-function) [1][3]

$$\Gamma(z+1) =: z! = \int_0^{\infty} e^{-\tau}\tau^z \mathrm{d}\tau, \qquad Re(z) > -1. \quad (2.13)$$

The condition $Re(z) > -1$ is required that integral (2.13) exists. Then the Laplacian (2.7) has the representation[4]

$$\Delta_{(\delta,h,\zeta)}u(x) = -\frac{h^{\delta}\Gamma(1-\delta)}{\zeta\delta}\left(D_l^{\delta} + (-1)^{-\delta}D_r^{\delta}\right)u(x), \qquad 0 < \delta < 1, \quad (2.14)$$

where the actual range of existence of (2.14) is $0 < \delta < 2$ when we write both terms as one single integral like in (2.7). We should like to mention that the representation (2.14) in terms of standard fractional integrals is not crucially helpful in order to solve problems. Therefore, we should rather conceive the Laplacian (2.7), (2.8) itself as definition of a self-adjoint combination of fractional derivatives.

The goal of this paper is to analyse some static and dynamic problems which are governed by the self-similar Laplacian (2.7), (2.8). The following section is devoted to deduce an explicit expression for the static displacement Green's function.

3 Static Green's function

This section is devoted to deduce the *static* Green's function for the displacement field due to a δ-point force. We define this Green's function $g(x)$ by

$$\Delta_{(\delta,h,\zeta)}g(x) + \delta(x) = 0, \quad (3.1)$$

where $\delta(x)$ denotes Dirac's δ-function. The static displacement field $u(x)$ due to a force density $f(x)$ is then defined by

$$\Delta_{(\delta,h,\zeta)}u(x) + f(x) = 0, \quad (3.2)$$

[3] Re(..) denotes the real part and Im(..) the imaginary part of a complex number (..).

[4] We have $-1 = e^{i\pi}$ with $(-1)^{-\delta} = e^{-\pi i\delta}$ as we use throughout this paper for any complex number $z = |z|e^{i\varphi}$ the *principal value* $-\pi < \varphi = Arg(z) \leqslant \pi$ for its argument φ.

and can be represented as a convolution

$$u(x) = \int_{-\infty}^{\infty} g(x-\tau)f(\tau)\mathrm{d}\tau \tag{3.3}$$

with the static Green's function $g(x)$ of (3.1) as convolution kernel. By introducing the Fourier transforms

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{g}(k)e^{ikx}\mathrm{d}k \tag{3.4}$$

and

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\mathrm{d}k \tag{3.5}$$

it follows from (3.1) that

$$\tilde{g}(k) = \frac{1}{\omega^2(k)}. \tag{3.6}$$

By taking into account that e^{ikx} are the eigenfunctions of the $\Delta_{(\delta,h,\zeta)}$-operator (2.7) with $\Delta_{(\delta,h,\zeta)}e^{ikx} = -\omega^2(k)e^{ikx}$, the dispersion relation is obtained from

$$\omega^2(k) = -\frac{h^\delta}{\zeta}\int_0^{\infty}\left(e^{ik\tau}+e^{-ik\tau}-2\right)\tau^{-\delta-1}\mathrm{d}\tau, \tag{3.7}$$

which was also given in [19] in the form $\omega^2(k) = A_\delta|k|^\delta$ with the coefficient

$$A_\delta = \frac{2h^\delta}{\zeta}\int_0^{\infty}\frac{(1-\cos(s))}{s^{1+\delta}}\mathrm{d}s, \qquad 0<\delta<2. \tag{3.8}$$

This integral exists in the same interval $0<\delta<2$ just as the Laplacian (2.7). We use now the identity

$$|\tau|^{-\delta-1} = \frac{1}{\cos\frac{\pi(\delta+1)}{2}}Re\{\lim_{\epsilon\to 0+}(\epsilon-i\tau)^{-\delta-1}\} = \lim_{\epsilon\to 0+}\frac{1}{\delta!\cos\frac{\pi(\delta+1)}{2}}Re\{\int_0^{\infty}e^{-s(\epsilon-i\tau)}s^\delta\mathrm{d}s\}, \tag{3.9}$$

which is well defined in the range $0<\delta<2$ of existence of the Laplacian. Replacing identity (3.9) in integral (3.7) yields[5]

$$\omega^2(k) = -\frac{h^\delta\pi}{\zeta\delta!\cos\frac{\pi(\delta+1)}{2}}\int_0^{\infty}s^\delta\left\{\delta(s-|k|)-2\delta(s)\right\}\mathrm{d}s = \frac{h^\delta\pi}{\zeta\delta!\sin\frac{\pi\delta}{2}}|k|^\delta, \qquad 0<\delta<2 \tag{3.10}$$

which indeed has the form [19]

$$\omega^2(k) = A_\delta|k|^\delta, \qquad 0<\delta<2, \tag{3.11}$$

where coefficient A_δ is obtained as

$$A_\delta = \frac{h^\delta}{\zeta}\frac{\pi}{\delta!\sin\left(\frac{\pi\delta}{2}\right)} > 0, \qquad 0<\delta<2. \tag{3.12}$$

[5] See details on integrals of this type in Appendix D.

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We conclude that the positiveness of the constant $A_\delta > 0$ (which is determined by $\sin\frac{\pi\delta}{2} > 0$ within $0 < \delta < 2$) of (3.12) due to the positiveness of integral (3.8) reflects nothing but the condition of elastic stability, i.e. $\omega^2(k) > 0$ for $k \neq 0$, which is a *physically necessary* condition. This condition together with the condition of existence of Laplacian (2.7) constitute a *physically sufficient condition* which gives the δ-range of *physical consistency*, namely the range $0 < \delta < 2$. We emphasise that the regularisation performed in (3.9) would allow mathematically to obtain converging integrals for almost all $\delta \in \mathbb{R}$ (i.e. except for the zeros of the cosine). However, among these *mathematically admissible* δ-values only those within the range $0 < \delta < 2$ lead to *physically meaningful results*. This is true for all *physical* problems formulated with the Laplacian (2.7).

3.1 Explicit Green's function

Integral (3.4) assumes with (3.6) and (3.11) the form

$$g(x) = \frac{1}{2\pi A_\delta}\int_{-\infty}^{\infty}|k|^{-\delta}e^{ikx}\mathrm{d}k, \tag{3.13}$$

where we make use of the relation holding for $k \neq 0$

$$|k|^{-\delta} = \frac{1}{(\delta-1)!\cos\frac{\delta\pi}{2}}\lim_{\epsilon\to 0+} Re\{\int_0^{\infty} e^{-\tau(\epsilon-ik)}\tau^{\delta-1}\mathrm{d}\tau\}, \qquad \delta \neq 1, \tag{3.14}$$

which is well defined in the range of existence $0 < \delta < 2$ (except for the zero of the cosine in this interval, i.e. $\delta \neq 1$) of the Laplacian (2.7). Using this identity to evaluate (3.13), we finally arrive at ($x \neq 0$ if $\delta < 1$)

$$g(x) = g_\delta\,|x|^{\delta-1}, \qquad 0 < \delta < 2, \qquad \delta \neq 1 \tag{3.15}$$

with the pre-factor

$$g_\delta = \frac{1}{2A_\delta(\delta-1)!\cos\frac{\pi\delta}{2}} = \frac{\zeta\delta}{2\pi h^\delta}\frac{\sin\frac{\delta\pi}{2}}{\cos\frac{\delta\pi}{2}}, \qquad 0 < \delta < 2, \qquad \delta \neq 1. \tag{3.16}$$

For $\delta = 1$, (3.13) diverges logarithically. However, we can evaluate its spatial derivative

$$\frac{d}{dx}g_{\delta=1}(x) = -\frac{1}{\pi A_1}\lim_{\epsilon\to 0+}\int_0^{\infty} e^{-k\epsilon}\sin kx\,\mathrm{d}k = -\frac{1}{\pi A_1}\lim_{\epsilon\to 0+} Im\frac{1}{(\epsilon - ix)} \tag{3.17}$$

and obtain for $x \neq 0$

$$\frac{d}{dx}g_1(x) = -\frac{1}{\pi A_1}\lim_{\epsilon\to 0+}\frac{x}{x^2+\epsilon^2} = -\frac{1}{\pi A_1 x}. \tag{3.18}$$

Neglecting an unimportant integration constant, we obtain for the Green's function for $x \neq 0$ the expression

$$g_1(x) = -\frac{1}{\pi A_1}\lim_{\epsilon\to 0+}\ln\sqrt{x^2+\epsilon^2} = -\frac{1}{\pi A_1}\ln|x|. \tag{3.19}$$

Expression (3.15) holds for $0 < \delta < 2$ ($\delta \neq 1$) and yields together with $g_{\delta=1}(x)$ physical consistent behaviour for the deformation $\frac{d}{dx}g(x)$ having the exponent $\delta - 2 < 0$: $|\frac{d}{dx}g(x \to 0)| \to \infty$ being singular at $x = 0$ and $|\frac{d}{dx}g(x \to \pm\infty)| \to 0$ vanishing for $|x| \to \infty$. The consistence of expressions (3.15) and (3.16) with (3.6) can be directly verified by performing the back transformation

$$\tilde{g}(k) = \int_{-\infty}^{\infty} g(x)e^{-ikx}\,\mathrm{d}x = 2g_\delta \lim_{\epsilon\to 0+} Re\{\int_0^{\infty} |x|^{\delta-1}e^{-x(\epsilon-ik)}\,\mathrm{d}x\}, \tag{3.20}$$

which can be rewritten as

$$\tilde{g}(k) = 2g_\delta \Gamma(\delta)\cos\frac{\pi\delta}{2}|k|^{-\delta}. \tag{3.21}$$

This expression must coincide with the expression $\tilde{g}(k) = \frac{1}{\omega^2(k)}$ of (3.6) with the dispersion relation (3.11). This leads to the condition that

$$\frac{1}{A_\delta} = 2g_\delta \Gamma(\delta)\cos\frac{\pi\delta}{2}, \tag{3.22}$$

which is fulfilled by (3.16). It is possible to evaluate (3.13) in the range $0 < \delta < 1$ directly to arrive at

$$g(x) = \frac{1}{\pi A_\delta}\lim_{\epsilon\to 0+} Re\left\{\int_0^{\infty} e^{-k(\epsilon-ix)}|k|^{-\delta}\mathrm{d}k\right\} = \frac{(-\delta)!}{\pi A_\delta}Re(\epsilon - ix)^{\delta-1} = \frac{\Gamma(1-\delta)\sin\frac{\pi\delta}{2}}{\pi A_\delta}|x|^{\delta-1}. \tag{3.23}$$

Comparision of g_δ of (3.23) and (3.16) leads to the condition that

$$\Gamma(\delta)\Gamma(1-\delta) = \frac{\pi}{\sin\pi\delta}, \tag{3.24}$$

which is known as *Euler's reflection formula* [1]. Consequently the correctness of expressions (3.15) and (3.16) for the Green's function and (3.6) is proven. For the analysis to follow, it is useful to generalise the definition of the Γ-function extending it to $\alpha < -1$ in such a way that for the extended definition of $\alpha!$ Euler-relation (3.24) is fulfilled[6]

$$\alpha! = \Gamma(\alpha+1) = -\frac{\pi}{\Gamma(-\alpha)\sin\pi\alpha}, \qquad \alpha < -1 \notin \mathbb{Z}, \tag{3.25}$$

where $\Gamma(-\alpha)$ is well defined by (2.13) for $\alpha < -1$. Definition (3.25) is motivated in Appendix D for $\alpha < -1$. Equation (3.25) complements the usual definition (2.13) of the Γ-function by extending it to the 'forbidden' α-range of (2.13) $\alpha < -1$.

We again emphasize our previous observation that the Green's function (3.15)ff. is *physically meaningful* only in the range $0 < \delta < 2$ with consistent behaviour for the *deformation* $\frac{d}{dx}g(x)$ behaving as $|x|^{\delta-2}$, such as its vanishing for $|x| \to \infty$ and its singular behaviour for $x \to 0$.

[6] This definition of $\alpha!$ maintains the representations of regularised integrals of the form (A16) for all admissible values $\alpha \in \mathbb{R}$.

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4 Cauchy problem and dynamic Green's functions

In this section we construct some fundamental solutions of dynamic problems of wave propagation. First in Section 4.1 we construct the solution of the Cauchy problem and in Section 4.2, the causal (retarded) time-domain Green's function. Further, we make some brief remarks on the frequency domain representation of the dynamic Green's function representing the solution of the self-similar Helmholtz equation.

4.1 Cauchy problem

The *Cauchy problem* is defined as follows: Construct the displacement field $u(x,t)$ solving (2.6), which we now rewrite for our convenience in the form

$$\frac{\partial^2}{\partial t^2}u(x,t) = -\mathscr{L}u(x,t) \tag{4.1}$$

with $\Delta_{(\delta,h,\zeta)} = -\mathscr{L}$ denoting the self-similar Laplacian of (2.7). The operator $\mathscr{L}$ is necessarily self-adjoint[7] and positive definite with $\mathscr{L}e^{ikx} = \omega^2(k)e^{ikx}$ and the dispersion relation $\omega^2(k) > 0$ for $k \neq 0$ (equation (3.11) of Section 3). Since $\mathscr{L}$ is a linear operator, we can take advantage of this and employ in the analysis to follow the calculus of linear operators. The displacement field $u(x,t)$ is required to fulfil the following initial condition at $t = 0$:

$$u(x, t = 0) = u_0(x) \tag{4.2}$$

with the prescribed initial displacement field $u_0(x)$ at $t = 0$. The velocity field $\frac{\partial}{\partial t}u(x,t)$ is required to fulfil the following initial condition at $t = 0$:

$$\frac{\partial}{\partial t}u(x, t = 0) = v_0(x), \tag{4.3}$$

where $v_0(x)$ denotes the prescribed initial velocity field at $t = 0$. Equation (4.1) with the initial conditions (4.2) and (4.3) represents the Cauchy problem which is defined in the domains $-\infty < x < \infty$ and $-\infty < t < \infty$. To construct the unique solution of the Cauchy problem, it is convenient to write the displacement field $u(x,t)$ at time t in the form

$$u(x,t) = \cos(\mathscr{L}^{\frac{1}{2}}t)\,u_0(x) + \mathscr{L}^{-\frac{1}{2}}\sin(\mathscr{L}^{\frac{1}{2}}t)\,v_0(x), \tag{4.4}$$

and as a consequence of (4.4) the velocity field $v(x,t) = \frac{\partial}{\partial t}u(x,t)$ at time t

$$v(x,t) = -\mathscr{L}^{\frac{1}{2}}\sin(\mathscr{L}^{\frac{1}{2}}t)\,u_0(x) + \cos(\mathscr{L}^{\frac{1}{2}}t)\,v_0(x). \tag{4.5}$$

Expressions (4.4) and (4.5) are defined by their power-series involving only entire powers $\mathscr{L}^n$ with $n = 0, 1, 2, ... \in \mathbb{N}_0$ of operator $\mathscr{L} = -\Delta_{\delta,h,\zeta}$ with the identical operator $\mathscr{L}^0 = 1$.

To evaluate expression (4.4) it is convenient to write it in its spectral representation, which in our case is the Fourier representation. Introducing the Fourier transformations

$$u_0(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{u}_0(k)e^{ikx}\mathrm{d}k \tag{4.6}$$

[7] i.e. $\mathscr{L} = \mathscr{L}^+$.

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and

$$v_0(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{v}_0(k)e^{ikx}\mathrm{d}k \tag{4.7}$$

and taking into account that

$$\mathscr{L}^{\frac{1}{2}}e^{ikx} = \omega(k)e^{ikx} \tag{4.8}$$

with

$$f(\mathscr{L}^{\frac{1}{2}}t)e^{ikx} = f(\omega(k)t)e^{ikx} \tag{4.9}$$

for any sufficiently smooth function $f(\omega t) = \sum_{m=0}^{\infty} a_m\omega^m t^m$ and where $\omega(k)$ denotes the (positive) square-root of dispersion relation (3.11). It is now convenient to rewrite (4.4) in rather trivial manner

$$u(x,t) = \cos(\mathscr{L}^{\frac{1}{2}}t)\int_{-\infty}^{\infty}\delta(x-\xi)\,u_0(\xi)\mathrm{d}\xi + \mathscr{L}^{-\frac{1}{2}}\sin(\mathscr{L}^{\frac{1}{2}}t)\int_{-\infty}^{\infty}\delta(x-\xi)\,v_0(\xi)\mathrm{d}\xi, \tag{4.10}$$

where the operators act only on the dependence on x. By using the Fourier representation of the δ-function (3.5) we have

$$Q(x,t) = \mathscr{L}^{-\frac{1}{2}}\sin(\mathscr{L}^{\frac{1}{2}}t)\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\frac{\sin(\omega(k)t)}{\omega(k)}\mathrm{d}k \tag{4.11}$$

and

$$\frac{\partial}{\partial t}Q(x,t) = \cos(\mathscr{L}^{\frac{1}{2}}t)\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\cos(\omega(k)t)\mathrm{d}k. \tag{4.12}$$

We can then write (4.10) in terms of the kernels $Q(x,t)$ and $\frac{\partial}{\partial t}Q(x,t)$, namely

$$u(x,t) = \int_{-\infty}^{\infty}\frac{\partial}{\partial t}Q(x-\xi,t)u_0(\xi)\mathrm{d}\xi + \int_{-\infty}^{\infty}Q(x-\xi,t)v_0(\xi)\mathrm{d}\xi, \tag{4.13}$$

and the velocity field is written as

$$v(x,t) = \int_{-\infty}^{\infty}\frac{\partial^2}{\partial t^2}Q(x-\xi,t)u_0(\xi)\mathrm{d}\xi + \int_{-\infty}^{\infty}\frac{\partial}{\partial t}Q(x-\xi,t)v_0(\xi)\mathrm{d}\xi. \tag{4.14}$$

From relations (4.11) and (4.12) we observe the initial conditions

$$Q(x,t=0) = 0\,, \qquad \frac{\partial}{\partial t}Q(x,t=0) = \delta(x)\,, \qquad \frac{\partial^2}{\partial t^2}Q(x,t=0) = 0, \tag{4.15}$$

which guarantee that (4.13) and (4.14) indeed solve the Cauchy problem. The problem is solved for our medium by the explicit determination of relations (4.11) and (4.12). To this end we evaluate relations (4.11) and (4.12), which can be written as[8]

$$Q(x,t) = \sum_{n=0}^{\infty}(-1)^n\frac{t^{2n+1}}{(2n+1)!}\,q_n(x) \tag{4.16}$$

[8] Where the convergence of these series will be verified in Appendix B.

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and

$$\frac{\partial}{\partial t}Q(x,t) = \sum_{n=0}^{\infty}(-1)^n \frac{t^{2n}}{(2n)!}\, q_n(x), \tag{4.17}$$

with the function $q_n(x)$ defined by

$$\begin{aligned} q_n(x) &= \mathscr{L}^n\delta(x) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,\omega^{2n}(k)\,\mathrm{d}k =: \lim_{\epsilon\to 0+}\frac{1}{\pi}\int_0^{\infty} e^{-k\epsilon}\omega^{2n}(k)\cos kx\,\mathrm{d}k\,, n = 0,1,2,... \in \mathbb{N}_0. \end{aligned} \tag{4.18}$$

The integral $\int_0^{\infty} e^{-k\epsilon}k^{\alpha}\cos kx\,\mathrm{d}k$ exists for $\alpha > -1$ and contains the entire admissible range $0 < \delta < 2$ of the Laplacian (2.7). Hence, we have

$$q_n(x) = \lim_{\epsilon\to 0+}\frac{A_\delta^n}{\pi}\int_0^{\infty} e^{-k\epsilon}k^{\delta n}\cos kx\,\mathrm{d}k = \frac{A_\delta^n}{\pi}Re(J_{n\delta})\,, n = 0,1,2,.. \in \mathbb{N}_0\,, 0<\delta<2 \tag{4.19}$$

with

$$\begin{aligned} J_{n\delta} &= \lim_{\epsilon\to 0+}\frac{1}{(\epsilon - i|x|)^{(n\delta+1)}}\int_0^{\infty} e^{-s}s^{n\delta}\,\mathrm{d}s \\ &= \lim_{\epsilon\to 0+}\frac{(n\delta)!}{(\epsilon - i|x|)^{(n\delta+1)}} \begin{cases} i\,e^{i\frac{\pi\delta n}{2}}\,\dfrac{(n\delta)!}{|x|^{n\delta+1}}\,, x \neq 0, \quad n = 1,2,.. \in \mathbb{N} \\ \displaystyle\lim_{\epsilon\to 0+}\frac{1}{\epsilon - i|x|} = \pi\delta(x) + i\mathscr{P}\left(\frac{1}{|x|}\right), \quad n = 0 \end{cases}, \end{aligned} \tag{4.20}$$

where $(n\delta)! = \Gamma(n\delta+1)$ and $\mathscr{P}(..)$ denotes the principal value of (..). We then obtain

$$q_n(x) = q_n(|x|) = \begin{cases} \frac{A_\delta^n}{\pi}Re(J_{n\delta}) = -\frac{A_\delta^n}{\pi}\sin\left(\frac{\pi\delta n}{2}\right)\frac{(n\delta)!}{|x|^{n\delta+1}}\,, \quad x \neq 0\,, n = 1,2,.. \in \mathbb{N}, \quad 0<\delta<2 \\ q_0(x) = \frac{1}{\pi}Re(J_0) = \delta(x) = 0\,, \quad x \neq 0 \end{cases}, \tag{4.21}$$

where A_δ is determined in (3.12). On the right-hand sides of (4.20) and (4.21), singular terms occur at $x = 0$, which are vanishing for $x \neq 0$ and are only important when we integrate these functions over $x = 0$. We come back to this important point in Section 6 by considering integrals of the type J_α for the entire range of their existence ($\alpha > -1$).

The series (4.16) and (4.17) together with functions (4.21) constitute the solution of the Cauchy problem being defined in the entire range $0 < \delta < 2$ of existence of the Laplacian (2.7). It is shown in Appendix B that the series (4.16) and (4.17) converge uniformly absolutely for all x and t.

4.2 Retarded time domain Green's function

In view of the results of the last section it is only a small step to determine the *retarded* (or causal) dynamic Green's function. The dynamic Green's function is defined as the solution of the self-similar wave equation

$$\left(\left(\frac{\partial}{\partial t} + \epsilon\right)^2 - \Delta_{(\delta,h,\zeta)}\right) g(x,t) = \delta(x)\delta(t) \tag{4.22}$$

with the self-similar Laplacian $\Delta_{(\delta,h,\zeta)}$ defined by (2.7). *Causality* means that $g(x,t)$ must be zero before and non-zero only after the impact of the pulse $\delta(x)\delta(t)$, i.e. $g(x,t<0)=0$ and $g(x,t>0) \neq 0$ [8, 18]. We introduced in (4.22) an infinitesimal *positive* damping term $\epsilon \to 0+$ which breaks the time-inversion symmetry of the wave operator and guarantees in this way $g(x,t)$ being the *causal* solution of (4.22). The physical interpretation of g is the displacement field due to a δ-type force density of the form on the right-hand side of (4.22). The Green's function defined by (4.22) solves the dynamic problem

$$\left(\left(\frac{\partial}{\partial t}+\epsilon\right)^2 - \Delta_{(\delta,h,\zeta)}\right) u(x,t) = f(x,t), \tag{4.23}$$

where $f(x,t)$ is the density of external forces and $u(x,t)$ is the corresponding displacement field by the convolution

$$u(x,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{t} g(x-\xi,t-\tau)f(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau, \tag{4.24}$$

where $g(x,t)$ indicates the Green's function defined by (4.22). The upper limit in the time-integration in (4.24) is due to the causality of $g(x,t)$. With the Fourier transformations

$$g(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{g}(k,t)e^{ikx}\,\mathrm{d}k \tag{4.25}$$

and

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,\mathrm{d}k \tag{4.26}$$

yields (4.22) the evolution equation for the normal amplitudes $g(k,t)$ in the form

$$\left(\left(\frac{\partial}{\partial t}+\epsilon\right)^2 + \omega^2(k)\right)\tilde{g}(k,t) = \delta(t), \tag{4.27}$$

which is nothing but the equation for the causal Green's function of a damped harmonic oscillator with a damping constant[9] $\epsilon > 0$ and eigenfrequency $\omega(k)$, which has the (causal) solution of the well-known form[10]

$$\tilde{g}(k,t) = e^{-\epsilon t}\Theta(t)\frac{\sin\omega(k)t}{\omega(k)}, \tag{4.28}$$

where $\Theta(t)$ denotes the Heaviside unit-step function being defined as $\Theta(t)=1$ for $t>0$ and $\Theta(t)=0$ for $t<0$. Equation (4.28) holds for any (also non-infinitesimal) damping $\epsilon > 0$ where the dissipative factor $e^{-\epsilon t}$ can be skipped in the limit of an infinitesimal $\epsilon > 0$. Frequency $\omega(k) = \sqrt{A_\delta}|k|^{\frac{\delta}{2}}$ denotes the positive root of the dispersion relation (3.11) holding for $0<\delta<2$. In view of the relation

$$\mathscr{L}^{-\frac{1}{2}}\sin\mathscr{L}^{\frac{1}{2}}t\,e^{ikx} = \frac{\sin\omega(k)t}{\omega(k)}e^{ikx} \tag{4.29}$$

[9] $\epsilon \to 0+$ being positive and infinitesimal being sufficient to obtain the causal solution.
[10] See any textbook of theoretical physics, or e.g. [18].

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with $\mathscr{L} = -\Delta_{\delta,h,\zeta}$, we have the retarded the space-time Green's function $g(x,t)$ to the form

$$g(x,t) = e^{-\epsilon t}\,\Theta(t)\mathscr{L}^{-\frac{1}{2}}\,\sin(\mathscr{L}^{\frac{1}{2}}t)\,\delta(x) = e^{-\epsilon t}\,\Theta(t)\,Q(x,t), \tag{4.30}$$

where $Q(x,t)$ has been evaluated in relations of (4.16)ff.

4.3 Some remarks on the dynamic Green's function in the space-frequency domain

For the sake of completeness, we give a brief representation of the frequency-space Green's function, i.e. Green's function of the self-similar Helmholtz equation. The Helmholtz Green's function is defined as

$$\hat{g}(x,\omega) = \int_0^\infty e^{i\omega t} g(x,t)\,\mathrm{d}t, \tag{4.31}$$

where $g(x,t)$ is the retarded space-time-domain Green's function (4.30) and the lower integration limit is due to causality. Equation (4.31) solves the self-similar Helmholtz equation,

$$\left((\omega + i\epsilon)^2 + \Delta_{(\delta,h,\zeta)}\right)\hat{g}(x,\omega) = -\delta(x), \tag{4.32}$$

where $\epsilon > 0$ is a positive infinitesimal damping constant which guarantees existence of the Helmholtz Green's operator defined in (4.33) and causality in the time domain. We can invert this Helmholtz equation with $\Delta_{(\delta,h,\zeta)} = -\mathscr{L}$ where the Helmholtz Green's function can be written as

$$\hat{g}(x,\omega) = \left(\mathscr{L} - (\omega + i\epsilon)^2\right)^{-1}\delta(x) \tag{4.33}$$

with the Green's operator (resolvent operator) of the frequency-domain $(\mathscr{L} - (\omega + i\epsilon)^2)^{-1}$.

5 Self-similar diffusion problem

This section is devoted to the analysis of the diffusion problem governed by the self-similar Laplacian (2.7). We consider an ensemble of particles with local concentration $\rho(x,t)$. We can also conceive this problem to be defined below as a fractional differential equation for a (probability) distribution $\rho(x,t)$. The idea to represent statistical distributions as solutions of fractional differential equations is not new, we refer, for instance, to the paper by Li *et al.* [12] where 'stable' distributions are generated by fractional differential equations. In order to allow stochastic description of *physical processes*, it is important that these generating fractional operators have a physical basis and justification. However, so far the fractional differential equations used to generate stochastic distributions were in many cases based on arbitrary assumptions and guesswork rather than derived from physical 'first principles'.

The goal of this section is to analyse the distributions generated by a diffusion equation employing the self-similar Laplacian (2.7),

$$\frac{\partial}{\partial t}\rho(x,t) = -\mathscr{L}\rho(x,t), \tag{5.1}$$

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where $-\mathscr{L} = \Delta_{(\delta,h,\epsilon)}$ indicates the self-similar Laplacian (2.7). We consider this problem in the domain $-\infty < x < \infty$ and $t > 0$. It is important that we restrict on $t > 0$, as diffusion is a time-irreversible process[11] which is expressed by the odd-order of the time derivative in (5.1). $\rho(x,t)$ denotes the density of the diffusing particles but can be also conceived as a probability-density distribution function. Equation (5.1) recovers the traditional diffusion equation by replacing the self-similar Laplacian $\Delta_{(\delta,h,\zeta)}$ by the traditional 1D Laplacian $\frac{\partial^2}{\partial x^2}$ leading to a Gaussian distribution.

We give an intuitive physical picture of the diffusion processes described by (5.1) as follows: Let us assume we have $\rho(x,t)\mathrm{d}x$ particles at time t at location x. Any particle is allowed to jump at time t from position x to arrive at $t + dt$ at any location $x \pm \tau$ different from x (with $0 < \tau < \infty$). The jump rate of this process is measured by $(\rho(x+\tau,t) + \rho(x-\tau,t) - 2\rho(x,t))/\tau^{-(1+\delta)}$, which is given by the integrand of (2.7) where $0 < \delta < 2$. We observe the following: The jump rate from x to $x \pm \tau$ is zero if $(\rho(x+\tau,t) + \rho(x-\tau,t) - 2\rho(x,t)) = 0$, i.e. there are no jumps for an equal distribution $\rho = const$. Jumps to 'far' locations are more seldom compared to jumps to 'close" locations for equal $(\rho(x+\tau,t)+\rho(x-\tau,t)-2\rho(x,t)) \neq 0$. The possibility of a particle to make jumps of arbitrary distances is expressed by the non-locality and continuity of the Laplacian (2.7). Particle jumps to any location are possible in (5.1), whereas non-local jumps are suppressed in the traditional (Gaussian) case which is reflected by the locality of the traditional Laplacian $\frac{\partial^2}{\partial x^2}$. The non-local Laplacian (2.7) sums up over all possible jump events by which particles can reach to or escape from a space-point x. For $\rho = const$ the jumping rate is zero (since $\mathscr{L}1 = 0$) and, as we will show, the equal-distribution is a stationary solution and $\rho = 0$ an *attractor* for $t \to \infty$ for any initial distribution $\rho_0(x)$ just as in the case of Gaussian distribution.

If we assume $\rho(x,t)$ to be the probability distribution of a single particle and $\rho(x,t)\mathrm{d}x$ to be the probability to find a particle in the interval within $[x, x + \mathrm{d}x]$ at time t, then the trajectory of the particle propagating due to probability distribution $\rho(x,t)$, which evolves according to (5.1), is discontinuous and erratic, whereas in the Gaussian case the particle motion describes a continuous trajectory. The discontinuous characteristics of the particle trajectory associated with (5.1) becomes more and more pronounced as $\delta \to 0$, which is due to the weaker decay of the kernel $\tau^{-(\delta+1)}$ for $\tau \to \infty$. We would like to mention that motions characterised by jump-probability distributions of the form of a power law as $\tau^{-(\delta+1)}$ are known in the literature as *Lévi flights*, where δ is called Lévi parameter. Lévi flights correspond to Lévi-stable probability distributions for $0 < \delta \leqslant 2$ [15]. Typical features of Lévi distributions are their infinite variances in the range $0 < \delta < 2$. The present analysis will end up in probability distributions of exactly this type.

In order to define diffusion problem (5.1) as an initial value problem, we prescribe an initial distribution $\rho_0(x)$ at $t = 0$ as

$$\rho(x, t = 0) = \rho_0(x). \tag{5.2}$$

We assume that the particle number is conserved in time, there is neither particle generation nor annihilation. The diffusion equation (5.1) then describes the continuity

[11] That means the time-inverse process is physically inadmissible.

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equation accounting for the particle (probability) balance at space-point x and time t having the form

$$\frac{\partial}{\partial t}\rho(x,t) + \nabla \cdot j(x,t) = 0, \tag{5.3}$$

where $\nabla = \frac{\partial}{\partial x}$ denotes the usual 1D gradient operator. Unlike in the Gaussian case where the traditional Laplacian comes into play by assuming Fick's law as a consequence of a gradient dynamics as constitutive law, in our case the constitutive law is, instead of a (local) gradient, a non-local constitutive law for the particle (probability) flux. An explicit form of the particle (probability) flux is obtained in view of (2.8) as

$$j(x,t) = -\frac{h^{\delta}}{\zeta\delta}\int_0^{\infty} \frac{(\rho(x+\tau,t) - \rho(x-\tau,t))}{\tau^{\delta}}\, \mathrm{d}\tau, \qquad 0 < \delta < 2, \tag{5.4}$$

which is our self-similar constitutive law and can be conceived as the self-similar analogue to Fick's law $j_{\text{gauss}}(x,t) = -\nabla\rho(x,t)$ of the Gaussian case. In (5.4), an irrelevant integration constant has been neglected. We can consider (5.4) as definition of the self-adjoint part of the self-similar gradient operator acting on $\rho(x,t)$. The distribution function $\rho(x,t)$ is required to fulfil the normalisation condition

$$\int_{-\infty}^{\infty} \rho(x,t)\mathrm{d}x = \int_{-\infty}^{\infty} \rho_0(x)\mathrm{d}x = 1, \qquad \forall t > 0. \tag{5.5}$$

Equation (5.5) indicates that $\rho(x,t)$ has to be a normalised *distribution*. $\rho(x,t)$ is uniquely determined by (5.1) and (5.2) and can be written in the form

$$\rho(x,t) = e^{-\mathscr{L}t}\rho_0(x), \tag{5.6}$$

where we restrict us to $t > 0$. The question to be answered in the following is as follows: Does (5.6) represent a normalised distribution, i.e. fulfils (5.5) under the condition that $\rho_0(x)$ represents a normalised distribution? We can answer this question by 'yes' by giving the following formal proof by integrating (5.6)

$$\int_{-\infty}^{\infty} \rho(x,t)\,\mathrm{d}x = e^{-\mathscr{L}t}\int_{-\infty}^{\infty} \rho_0(x)\,\mathrm{d}x = e^{-\mathscr{L}t}\,1 = 1, \tag{5.7}$$

where we take into account that $\mathscr{L}^n 1 = \delta_{n0}$ since the Laplacian (2.7) applied to a constant yields zero. Let us evaluate (5.6) and deduce the propagator (kernel) that represents $\rho(x,t)$ in terms of a convolution

$$\rho(x,t) = \int_{-\infty}^{\infty} W(x-\xi,t)\rho_0(\xi)\mathrm{d}\xi = e^{-\mathscr{L}t}\int_{-\infty}^{\infty} \delta(x-\xi)\rho_0(\xi)\,\mathrm{d}\xi. \tag{5.8}$$

The propagator $W(x,t)$ is obtained by

$$W(x,t) = e^{-\mathscr{L}t}\delta(x), \tag{5.9}$$

and fulfils the initial condition

$$W(x,t=0) = \delta(x). \tag{5.10}$$

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The necessary and sufficient condition that $W(x,t)$ describes a normalised probability distribution is

$$\int_{-\infty}^{\infty} W(x,t)\,\mathrm{d}x = 1, \tag{5.11}$$

which is indeed fulfilled as can be seen by putting $\rho_0(x) = \delta(x)$ in (5.7). If $\rho(x,t)$ is conceived as a probability distribution, then $W(x,t)$ is the *conditional probability density* to find a particle which was at $t = 0$ located at $x = 0$ at time $t > 0$ at point x. The propagator $W(x,t)$ can be evaluated by using $\mathscr{L}e^{ikx} = \omega^2(k)e^{ikx}$ in the form

$$W(x,t) = e^{-\mathscr{L}t}\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}e^{-\omega^2(k)t}\,\mathrm{d}k = \frac{1}{\pi}\int_0^{\infty} e^{-A_\delta k^\delta t}\cos(kx)\,\mathrm{d}k, \tag{5.12}$$

where $\omega^2(k) = A|k|^\delta$ is the dispersion relation (3.11), which is defined in $0 < \delta < 2$. Integral (5.12) exists in the whole range of $0 < \delta < 2$ where $A_\delta > 0$ for $t > 0$. We observe further because of $\omega^2(k) \geqslant 0$ that $e^{-\omega^2(k)t} \to 0$ at $t \to \infty$, thus (5.12) tends to zero as t tends to infinity,

$$\lim_{t\to\infty} W(x,t) = 0, \tag{5.13}$$

so the stationary solution $\rho_\infty = 0$ is always an attractor independent of the initial distribution $\rho_0(x)$ in the physical admissible interval $0 < \delta < 2$. It is important to note that for any *finite* $0 < t < \infty$, (5.13) maintains in some regions non-zero (positive) values to fulfil (5.11). The goal is now to evaluate (5.12).

It can be further seen in (5.12), since $e^{-A_\delta k^\delta t}$ is a monotonous decreasing function in k, that the positive contributions due to $\cos kx$ dominate the negative contributions in the last integral of (5.12) so that $W(x,t)$ fulfils another necessary characteristics of a probability distribution, namely

$$W(x,t) \geqslant 0 \qquad \forall x \in \mathbb{R} \quad \text{and} \quad t > 0. \tag{5.14}$$

So we can infer that the initial value problem defined by (5.1) and (5.2) with the self-similar Laplacian (2.7) indeed describes a diffusion problem (transport problem) under the condition that (5.2) is a normalised distribution. It remains to us now to evaluate (5.12) in more explicit form. By putting $u^\delta = k^\delta A_\delta t$ (5.12) takes the form

$$W(x,t) = \frac{1}{(A_\delta t)^{\frac{1}{\delta}}} P\left(\frac{|x|}{(A_\delta t)^{\frac{1}{\delta}}}\right), \qquad P(\lambda^{-1}) = \frac{1}{\pi}\int_0^{\infty} e^{-u^\delta}\cos\frac{u}{\lambda}\mathrm{d}u \qquad 0 < \delta < 2, \tag{5.15}$$

where $0 < \delta < 2$ is the range of existence of A_δ. The exact relation (5.15) is especially useful when we consider the regime of large 'scaled times' $A_\delta t/|x|^\delta \gg 1$: Distribution (5.15) then behaves as $W \sim P(0)(A_\delta t)^{-\frac{1}{\delta}}$ (being the dominating term in the λ^{-1} expansion in this regime) indicating a spatial quasi-equal distribution located in the region $|x| \ll (A_\delta t)^{\frac{1}{\delta}}$, which is expanding in time.

Distributions of the form (5.12) and (5.15) have already been proposed by Lévi [11,16] and refers to the category of a 'Lévi-stable' or 'L-stable' distribution (for further details on Lévi-stable distribution, we refer e.g. to [16], p. 106ff. and the references therein). For

$\delta = 1$ we can directly evaluate (5.15) and obtain

$$W_{\delta=1}(x,t) = Re\left\{\frac{1}{\pi}\int_0^\infty e^{-k(A_1 t - ix)}\,\mathrm{d}k\right\}, \tag{5.16}$$

which yields

$$W_{\delta=1}(x,t) = \frac{1}{\pi}\frac{A_1 t}{(x^2 + (A_1 t)^2)}, \tag{5.17}$$

where $A_1 = \frac{h\pi}{\zeta}$. Relation (5.17) represents spatially a distribution of the Cauchy-type with $A_1 t$ being a 'parameter'. One verifies directly that (5.17) fulfils the conditions (5.10), (5.11) and (5.13). Lévi already gave static spatial probability distributions of the general form (5.15) [15] and found that in general these are admissible for $0 < \delta \leqslant 2$, where $\delta = 1$ corresponds to the Cauchy distribution and $\delta = 2$ to the Gaussian distribution. For $\delta = 2$ the Laplacian (2.7) does not exist and hence A_2 is not defined (diverging) in our approach.

Mandelbrot showed that Lévi-stable distributions play a crucial role in the description of economic processes and stock courses and describe the statistics of fractal irregular trajectories. The irregularity of these trajectories decreases from $\delta = 0$ to $\delta \to 2$ [15], which is consistent with our above-given physical picture. The irregularity is due to the fact that (5.15) produces in the admissible range of existence of the Laplacian (2.7) $0 < \delta < 2$ infinite fluctuations due to discontinuous trajectories. The Cauchy distribution ($\delta = 1$) itself refers also to this category. Only the Gaussian case corresponds to a diffusion equation with a traditional Laplacian describing continuous trajectories (Brownian motion) with *finite* mean fluctuation (variance). A consequence is that the unjustified use of Gaussian statistics instead of Lévi statistics underestimates fluctuations and risks due to the disregard of the possibility of spontaneously occurring discontinuities (non-local particle jumps)! In this context we refer to [14–16, 22] and the references therein. We will briefly reconsider this important point at the end of this section.

It still remains to evaluate (5.15), which we do by expanding $e^{-\omega^2(k)t}$ in a series and arrive at the expression

$$W(x,t) = \sum_{n=0}^{\infty}(-1)^n \frac{t^n}{n!}\, q_n(x), \qquad 0 < \delta < 1, \tag{5.18}$$

where this series converges only in the range $0 < \delta < 1$, which is shown in Appendix C. In (5.18) appear the functions

$$q_n(x) = \mathscr{L}^n \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,\omega^{2n}(k)\,\mathrm{d}k, \tag{5.19}$$

which were already determined in the last section by (4.18)–(4.21). Due to their importance for our approach, we devote Section 6 to their thorough analysis. The series (5.18) is written as

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$$W(x,t) = \frac{1}{\pi}\lim_{\epsilon\to 0+} Re\left\{\sum_{n=0}^{\infty}\frac{(-1)^n((\delta n)!(A_\delta t)^n)}{n!(\epsilon - i|x|)^{n\delta+1}}\right\}$$
$$= \delta(x) + \frac{1}{\pi}\lim_{\epsilon\to 0+} Re\left\{\sum_{n=1}^{\infty}\frac{(-1)^n((\delta n)!)(A_\delta t)^n}{n!(\epsilon - i|x|)^{n\delta+1}}\right\}, \qquad (5.20)$$

which assumes for $x \neq 0$

$$W(x,t) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(n\delta)!}{n!}\sin\left(\frac{\pi n\delta}{2}\right)\frac{A_\delta^n t^n}{|x|^{n\delta+1}}, \qquad 0 < \delta < 1, \qquad (5.21)$$

where the singular parts at $x = 0$ vanishing for $x \neq 0$ have been omitted. Relation (5.21) has the form

$$W(x,t) = -\frac{1}{\pi|x|} Im\{w(\xi(x,t))\} \qquad (5.22)$$

when we introduce the variable

$$\xi(x,t) = \frac{A_\delta t}{|x|^\delta} e^{\frac{\pi i\delta}{2}} \qquad (5.23)$$

with the function $w(\xi)$

$$w(\xi) = \sum_{n=1}^{\infty}(-1)^n\frac{(n\delta)!}{n!}\xi^n, \qquad 0 < \delta < 1, \qquad (5.24)$$

where series representations (5.18) and (5.20) converge $\forall\xi$ in the interval $0 < \delta < 1$ (Appendix C).

Let us now discuss some of the main characteristics of the distribution $W(x,t)$ where representation (5.15) is defined in the whole range $0 < \delta < 2$. In view of (5.12) and (4.21) we observe that $W(x,t) = W(-x,t) = W(|x|,t)$ is a symmetric distribution with respect to x and as a consequence all moments of odd order

$$< x^{2n+1} >= \int_{-\infty}^{\infty} W(x,t)\, x^{2n+1}\,\mathrm{d}x = 0, \qquad n = 0, 1, .. \in \mathbb{N}_0 \qquad (5.25)$$

are vanishing, especially the mean-value $< x >= 0$. The moments of even order obtained from

$$< x^{2n} >= \int_{-\infty}^{\infty} W(x,t)\, x^{2n}\,\mathrm{d}x \to \infty, \qquad n = 1, .. \in \mathbb{N} \qquad (5.26)$$

are diverging. We can conclude without any further calculation that (5.26) is divergent since any *real-valued power function* (except a constant) is not admissible in (2.7). The existence of (2.7) however is achieved for certain complex valued power functions as we show in Section 6.

6 Self-similar potentials

Finally, we analyse more closely functions of the type $q_n(x)$ of (5.19), which were determined in (4.18)–(4.21). To this end let usevoke the definition of the $q_n(x) = \mathscr{L}^n\delta(x)$, which

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was defined by using the Laplacian (2.7), (2.8). We should emphasize that $\Delta_{(\delta,h,\zeta)} = -\mathscr{L}$ depends on the continuous parameter δ, which was restricted in the Laplacian (2.7), (2.8) to the interval $0 < \delta < 2$ where the upper limit $\delta < 2$ came into play since $2u(x) - u(x+\tau) - u(x-\tau) \to C(x)\tau^2$ is a quadratic function of τ for small τ. In this section we will show that for a certain class of *singular* functions, namely functions of the form $b_{n\delta}(x) = \mathscr{L}^n\delta(x)/A_\delta^n$, the upper limit $\delta < 2$ *does not exist* and for which the condition $n\delta > -1$ is sufficient. We call these functions due to their importance as solutions of the Laplace equation (2.7) as 'self-similar potentials'. Their singular behaviour at $x = 0$ becomes especially important when we consider integrals or convolutions of these functions. Whereas any *real-valued power-functions* x^β are not admissible in (2.7), self-similar potentials to be analysed, however, are admissible and give the only way to an admissible definition of powers $\mathscr{L}^n$. Let us consider the Fourier transformation of $|k|^\alpha e^{-|k|\epsilon}$ with $\epsilon > 0$, namely[12]

$$b_\alpha(x) = \lim_{\epsilon\to 0+} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-|k|\epsilon}|k|^\alpha \mathrm{d}k = \lim_{\epsilon\to 0+}\frac{1}{\pi}\int_0^\infty e^{-\epsilon k}k^\alpha \cos kx\,\mathrm{d}k, \qquad \alpha > -1 \in \mathbb{R}, \tag{6.1}$$

which exists for $\alpha > -1$. This function can be identified with $q_n(x)$ of (5.19) for $\alpha = n\delta$. We can rewrite (6.1) by introducing the integration variable $s = k(\epsilon - i|x|)$ in the form

$$b_\alpha(x) = Re\left\{\lim_{\epsilon\to 0+}\frac{1}{\pi}\frac{1}{(\epsilon - i|x|)^{\alpha+1}}\int_0^\infty e^{-s}s^\alpha \mathrm{d}s\right\} = \frac{\alpha!}{\pi}Re\left\{\lim_{\epsilon\to 0+}\frac{i^{\alpha+1}}{(x+i\epsilon)^{\alpha+1}}\right\}, \quad \alpha > -1 \in \mathbb{R} \tag{6.2}$$

which holds for $\alpha > -1$ and where $\alpha! = \Gamma(\alpha+1)$. We note that $(\epsilon - ix)^{-\alpha-1}$ assumes a complex conjugate value when replacing $x \leftrightarrow -x$. So its real-part depends only on $|x|$ and we can rewrite

$$b_\alpha(x) = b_\alpha(|x|) = \frac{\alpha!}{\pi}Re\left\{\lim_{\epsilon\to 0+}\frac{i^{\alpha+1}}{(|x|+i\epsilon)^{\alpha+1}}\right\}, \qquad \alpha > -1 \in \mathbb{R}. \tag{6.3}$$

Expressions (6.2) and (6.3) are in accordance with the result given by Gel'fand and Shilov for the Fourier transform of $|k|^\alpha$ ([7], Equation (13) p. 447).

We summarise the following important observations:

(i) $b_\alpha(x) = b_\alpha(-x) = b_\alpha(|x|)$ is a symmetric function in x for all $\alpha > -1$.

(ii) $b_{\alpha=0}(x) = \delta(x)$ represents the usual Dirac-δ-function, i.e. is *localised* at $x = 0$.

(iii) For $\alpha = 2n$ with $n = 0, 1, 2, ... \in \mathbb{N}_0$ (6.2) takes the form of even derivatives of the δ-function

$$b_{2n}(x) = (-1)^n \frac{d^{2n}}{dx^{2n}}\delta(x), \tag{6.4}$$

which are *localised* at $x = 0$.

(iv) $\alpha > -1 \in \mathbb{R}$.
For $x \neq 0$ (6.2), b_α has the explicit form (for $x \neq 0$ we can put directly $\epsilon = 0$ in (6.2))

[12] For further details on the Fourier transformation of $|k|^\alpha$, see [7].

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$$b_\alpha(x) = -\frac{\alpha!}{\pi|x|^{\alpha+1}}\sin\left(\frac{\alpha\pi}{2}\right), \tag{6.5}$$

where we directly verify that this expression is zero for $\alpha = 2n$ (in accordance with case (iii) for $x \neq 0$). For $\alpha > -1 \in \mathbb{R} \notin 0, 2, 4, ...$, expression (6.5) is non-zero for $x \neq 0$ and hence in contrast to the Dirac's δ-function, *non-local*. Expression (6.5) includes also the case of odd integers $\alpha = \alpha_n = 2n + 1$ where $n = 0, 1, 2, ... \in \mathbb{N}_0$.

We observe that $b_\alpha(x)$ defined by (6.1) can be conceived as the Fourier transform of $|k|^\alpha$, which exists only for $\alpha > -1$. For exponents $\alpha < -1$, the integral (6.1) diverges. However, we can define a *regularised Fourier transform* in the spirit of generalised functions as given in [7], which exists also for $\alpha < -1$. We devote Appendix D to this case.

(v) Let us now consider the following integral:

$$\int_{-\infty}^{\infty} b_\alpha(x)\mathrm{d}x = \int_{-\infty}^{\infty} e^{-|k|\epsilon}|k|^\alpha\delta(k)\mathrm{d}k = \begin{cases} 0, & \alpha > 0 \\ 1, & \alpha = 0 \\ \infty, & \alpha < 0 \end{cases}. \tag{6.6}$$

For $\alpha > 0$ we observe that integral (6.6) is vanishing. On the other hand, we observe that $b_\alpha(x) \neq 0$ for $x \neq 0$ and $sign(b_\alpha(x)) = -sign(\sin(\frac{\alpha\pi}{2}))$ is the *the same* for $\forall x \neq 0$.
Let us raise the following question: How is it possible that the integral (6.6) is vanishing for $\alpha > 0$ whereas $b_\alpha \neq 0$ having the *same sign* as $-sign(\sin(\frac{\alpha\pi}{2}))$ almost everywhere, i.e. $\forall x \neq 0$? To give the answer, let us consider the integral

$$\mathscr{I}_\alpha(a) = \int_{a>0}^{\infty} b_\alpha(x)\mathrm{d}x = -\frac{(\alpha-1)!}{\pi a^\alpha}\sin\left(\frac{\pi\alpha}{2}\right), \qquad \alpha > 0, \tag{6.7}$$

where we have the property $\lim_{a\to 0+}|\mathscr{I}_\alpha(a)| \to \infty$. Let us again ask, why does (6.7) not converge in the limiting case $a \to 0+$ towards zero for $\alpha > 0$ as indicated in (6.6) since $b_\alpha(x) = b_\alpha(-x)$ is an even function? If integral (6.6) is indeed vanishing, then (6.7) should be compensated by the integral

$$\mathscr{J}_\alpha(a) = \int_0^a b_\alpha(x)\mathrm{d}x = \frac{1}{2}\int_{-a}^{a} b_\alpha(x)\mathrm{d}x, \qquad \alpha > 0 \tag{6.8}$$

since $\int_0^a b_\alpha(x)\mathrm{d}x + \int_a^\infty b_\alpha(x)\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{\infty} b_\alpha(x)\mathrm{d}x = 0$ $(b(x) = b(|x|))$.

To evaluate this integral we have to use (6.1) or directly (6.3), which hold $\forall x$, *including* $x = 0$ to arrive at

$$\mathscr{J}_\alpha(a) = \lim_{\epsilon\to 0+}\frac{1}{\pi}\int_0^\infty e^{-\epsilon k}k^\alpha\left(\int_0^a \cos kx\,\mathrm{d}x\right)\mathrm{d}k, \tag{6.9}$$

which yields

$$\mathscr{J}_\alpha(a) = \lim_{\epsilon\to 0+}\frac{1}{\pi}\int_0^\infty e^{-\epsilon k}k^{\alpha-1}\sin(|k|a)\,\mathrm{d}k = \frac{(\alpha-1)!}{\pi}Im\left\{\frac{1}{(\epsilon - ia)^\alpha}\right\}, \tag{6.10}$$

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since $a > 0$ being non-zero we can further evaluate (6.10) by putting $\epsilon = 0$ to arrive at

$$\mathscr{J}_{\alpha}(a) = \int_0^a b_{\alpha}(x)\mathrm{d}x = \frac{(\alpha-1)!}{\pi a^{\alpha}} \sin\left(\frac{\pi\alpha}{2}\right), \tag{6.11}$$

which indeed compensates (6.7) so that $\mathscr{J}_{\alpha}(a) + \mathscr{I}_{\alpha}(a) = 0$, thus (6.6) indeed is fulfilled with the remarkable property

$$\lim_{a\to 0+} \mathscr{J}_{\alpha}(a) = \lim_{a\to 0+} \int_0^a b_{\alpha}(x)\mathrm{d}x = sign\left(\sin\left(\frac{\pi\alpha}{2}\right)\right) \times \infty, \qquad \alpha > 0 \tag{6.12}$$

is diverging for $\alpha > 0$ as $a^{-\alpha}$ (except in the cases of even integers $\alpha \in 2, 4, ..., 2n, ...$)! That means if we imagine for a moment $|b_{\alpha}|$ being a mass density, then the total mass concentrated in the interval $0 \leqslant x \leqslant a$ is the same as the total mass in the interval $a < x < \infty$. We conclude that $b_{\alpha}(x)$ has, in dependence on α, as many as $\sin\frac{\pi\alpha}{2}$ in $[0, \alpha]$ oscillatoric peaks of alternate signs localised for $\epsilon \to 0+$ infinitely close to $x = 0$. The integral over these localised oscillations up to a finite limit $a > 0$ compensates integral (6.7). The strength of these oscillatoric peaks are 'infinitely stronger' as that one of δ-function. The closed form expression of b_{α}, which takes into account this singularity, is therefore (6.2) which fulfils for instance the 'desired' property of a fractional derivative of order $\alpha > 0$ (e.g. [9]) that it yields zero when applied to a constant. This property requires that

$$\int_0^a f(x)\mathrm{d}x = Const \times a^{-\alpha}, \tag{6.13}$$

which is obviously not solved for $\alpha > 0$ by any real-valued power-function $f(x) = const\, x^{-\alpha-1} \in \mathbb{R}$, since its primitive in (6.13) diverges at $x = 0$. In contrast, $b_{\alpha}(x)$ solves (6.13) and

$$Re\left\{\int_0^{\infty} \frac{\mathrm{d}x}{(\epsilon - ix)^{\alpha+1}}\right\} = Re\left\{\frac{(-i)}{\alpha}\frac{1}{(\epsilon - ix)^{\alpha}}|_0^{\infty}\right\} = 0, \qquad \alpha > 0 \tag{6.14}$$

for any $\epsilon > 0$ since the lower limit of the integral is purely imaginary and the upper limit vanishing for $\alpha > 0$. Hence, (6.2) defines for $\alpha > 0$ an integration kernel for a fractional derivative which yields zero when applied to a constant which is due to the vanishing of (6.6) which we can indeed verify directly. We analyse further in Appendix D the properties of b_{α}.

Let us now return to the functions $q_n(x)$ of (5.19). Taking into account (6.2) have with $\alpha = n\delta > -1$ and with

$$b_{n\delta}(x) = \frac{\mathscr{L}^n}{A_{\delta}^n}\delta(x) = \frac{(n\delta)!}{\pi} \lim_{\epsilon\to 0+} Re\left\{\frac{i^{n\delta+1}}{(|x| + i\epsilon)^{n\delta+1}}\right\}, \qquad n\delta > -1, \tag{6.15}$$

where A_{δ} is given in (3.12). For $n = 1$ we obtain

$$b_{\delta}(x) = \frac{1}{A_{\delta}}\mathscr{L}\delta(x) = \frac{\delta!}{\pi} \lim_{\epsilon\to 0+} Re\left\{\frac{i^{\delta+1}}{(|x| + i\epsilon)^{\delta+1}}\right\} \tag{6.16}$$

being in accordance with $q_1(x) = \mathscr{L}\delta(x) = -\Delta_{(\delta,h,\zeta)}\delta(x)$ obtained by evaluation of (2.7)

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or (2.8) for $x \neq 0$. From (6.15) one can recover the *static* Green's function by (3.1)

$$q_{-1}(x) = \frac{b_{-\delta}(x)}{A_\delta} = \mathscr{L}^{-1}\delta(x) = \frac{(-\delta)!}{\pi A_\delta} Re\left\{\frac{i^{1-\delta}}{(|x|+i\epsilon)^{1-\delta}}\right\}, \qquad 0 < \delta < 2, \quad (6.17)$$

which indeed recovers expression (3.15) for this Green's function $q_{-1}(x) = g(x)$. Expression (6.17) takes into account in a 'correct' way the singular behaviour at $x = 0$ and is therefore more general as (3.15) holding only for $x \neq 0$. Whereas the $b_{n\delta}(x)$ holds for $n\delta > -1$, the functions $q_n(x) = A_\delta^n b_{n\delta}(x)$ are only physically meaningful where $A_\delta > 0$ and where (2.7) exists, namely $0 < \delta < 2$. The range of validity of the $b_\alpha(x)$ can be further extended *mathematically* to all $\alpha < -1$ if one utilises instead of (2.13) the regularised definition of the Γ-function (3.25).

The performed analysis gives reason to the following observation: There is a *general* link between non-integer α, including odd integers α, and the non-locality and self-similarity of physical properties. This behaviour is also reflected by the non-locality of our Laplacian (2.7). Intuitively, in a wider sense one is tempted to state that *even* integer-exponents α (case **(iii)**) corresponds to local ('regular'), non-integer and odd exponents α (case **(iv)**) to non-local, self-similar ('irregular')[13] physical behaviour.

7 Conclusions

We have analysed an elastic medium with self-similar elastic energy density. In this way we defined a continuum with 'self-similar elastic properties'. The notion of 'self-similar elasticity' indicates here the scaling invariance as a symmetry property of the elastic energy density in analogy to 'transversely isotropic elasticity' or 'cubic elasticity' reflecting certain crystal symmetries. The source of the physics in this continuum is the self-similar Laplacian (2.7) having the form of a non-local convolution with a power-law convolution kernel. This Laplacian can be represented in terms of fractional derivatives and is obtained by the continuous limiting case of the discrete self-similar Laplacian introduced in [19]. The resulting equations of motion are self-adjoint partial *fractional* differential equations which are only in the range $0 < \delta < 2$ physically meaningful.

We deduced closed form solutions **(i)** for the *static* Green's function (displacement field due to a unit δ-force), **(ii)** for the Cauchy problem (two convolution kernels 'propagators') and **(iii)** for the solution of a self-similar diffusion problem. The solution was obtained in terms of a convolution kernel solving the initial value problem. It has been found that the obtained convolution kernel represents normalised Lévi-stable probablity distributions with infinite variances (mean fluctuations) in their range of definition $0 < \delta < 2$.

As a spin-off result, we have obtained functions $b_\alpha(x)$, which we denoted as *self-similar potentials* appearing as regularised Fourier transforms of $|k|^\alpha$. These functions are to be conceived as *generalised functions* in the distributional sense of Gel'fand and Shilov [7]. Self-similar potentials are special solutions of the self-similar Poisson's equation and play in statics the analogous role as Newtonian potentials for the traditional Poisson's equation.

[13] The meaning of the term 'irregular' is here used in the sense of 'erratic', it does *not* mean the absence of rules.

It has been demonstrated that the concept of self-similar functions introduced in [19] is useful to describe material systems with self-similar interactions. Our approach opens the door to a continuum description of a rich fond of new physical, especially dynamic, phenomena in such systems. Especially interesting are the probability distributions obtained as solutions of the self-similar (fractional) diffusion equation (5.1). These probability distributions are Lévi-stable, which are known to be a characteristic feature of the statistics of irregular and erratic motions characterised by discontinuous trajectories (Lévi flights) [14, 15]. It is worth mentioning that our diffusion model recovers the range $0 < \delta < 2$ of admissible Lévi parameter δ given previously in the literature (e.g. [14, 15]).

We have further formulated the 1D continuity equation for the self-similar diffusion problem and formulated the self-similar probability flux (equation (5.4)). An interesting point which can be treated by the present approach remains the formulation of the non-local self-similar Gauss-theorem in dimensions higher than one, and more general to formulate continuity equations and related problems in self-similar continua embedded into the 2D or 3D space. The present approach can be a point of departure to answer such questions in a rather rigorous way. We hope that the present paper stimulates further work in such directions.

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Appendix A Fractional derivative

In view of the analysis of the last section we can easily recall a *fractional derivative* $D_x^\alpha f(x)$ which fulfils for $\alpha > 0$ all desirable properties of a derivative. We call positive orders (fractional) *derivatives* and negative orders (fractional) *integrals*. Let us consider the case $\alpha > -1$ which can however be extended to $\alpha < -1$ in the spirit of generalised functions [7] as outlined in Appendix D.

Let us first of all determine

$$y_\alpha(x) = D_x^\alpha \delta(x), \qquad \alpha > -1, \tag{A 1}$$

where we call $y_\alpha(x)$ a fractional *derivative*. For $\alpha = 0$, the solution should be $y_{\alpha=0}(x) = \delta(x)$.

From (A1) follows then for the definition of this fractional derivative applied on a function $f(x)$ the convolution

$$D_x^\alpha f(x) = \int_{-\infty}^{\infty} y_\alpha(x-\tau) f(\tau) \mathrm{d}\tau, \qquad \alpha > -1 \tag{A 2}$$

with the kernel $y_\alpha(x)$ (A5) to be determined. Further, we assume that $D_x^\alpha e^{ikx} = (ik)^\alpha e^{ikx}$, so we can write $y_\alpha(x)$ as a Fourier-integral

$$y_\alpha(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} (ik)^\alpha \mathrm{d}k, \tag{A 3}$$

which we can rewrite

$$y_\alpha(x) = \frac{1}{2\pi} \lim_{\epsilon \to 0+} \int_0^\infty e^{-\epsilon k} k^\alpha \left(e^{i(kx + \frac{\alpha\pi}{2})} + e^{-i(kx + \frac{\alpha\pi}{2})} \right) \mathrm{d}k, \tag{A 4}$$

which can be further written as

$$y_\alpha(x) = \frac{\alpha!}{\pi} \lim_{\epsilon \to 0+} Re \left\{ \frac{i^{2\alpha+1}}{(x + i\epsilon)^{\alpha+1}} \right\}, \qquad \alpha > -1. \tag{A 5}$$

For $x \neq 0$ (A5) assumes the form

$$y_\alpha(x) = -\frac{\Theta(x)\alpha!}{\pi x^{\alpha+1}} \sin \pi\alpha, \tag{A 6}$$

where $\Theta(x)$ denotes the Heaviside unit-step function.[14] We verify that for integer $\alpha = 0, 1, 2, ... \in \mathbb{N}_0$ (A6) is vanishing for $x \neq 0$ and (A5) yields

$$y_n(x) = \frac{(-1)^n n!}{\pi} \lim_{\epsilon \to 0+} Re \left\{ \frac{i}{(x + i\epsilon)^{n+1}} \right\} = \frac{d^n}{dx^n} \frac{1}{\pi} \lim_{\epsilon \to 0+} Re \left\{ \frac{i}{(x + i\epsilon)} \right\} = \frac{d^n}{dx^n} \delta(x) \tag{A 7}$$

reproducing in the integer-case the correct localised integer-order derivatives.

Appendix B Convergence of series (4.16), (4.17)

It still remains to verify the convergence of the series (4.16) and (4.17). To this end we consider the series (4.17), which has the form

$$\frac{\partial}{\partial t} Q(x,t) = -\frac{1}{\pi |x|} Im \left\{ P \left(\frac{A_\delta t^2 e^{\frac{\pi i \delta}{2}}}{|x|^\delta} \right) \right\}, \qquad x \neq 0, \tag{B 1}$$

where $q_0(x) = \delta(x)$ and P is only a function of $\xi = \frac{A_\delta t^2 e^{\frac{\pi i \delta}{2}}}{|x|^\delta}$, and P is given by

$$P(\xi) = \sum_{n=1}^{\infty} \frac{(n\delta)!}{(2n)!} \xi^n, \qquad 0 < \delta < 2. \tag{B 2}$$

In order to judge the convergence of this series, the asymptotic representation of $\frac{(n\delta)!}{(2n)!}$ for $n \gg 1$ will be useful. To this end we put $s(n) = 2n(1 - \frac{\delta}{2}))$ with $n\delta = 2n - s$

$$\frac{(n\delta)!}{(2n)!} = \frac{(2n - s(n))!}{2n!} \to \frac{1}{(2n)^{2n(1-\frac{\delta}{2})}}, \qquad 0 < \delta < 2. \tag{B 3}$$

From (B3) we can determine the radius of convergence ρ_c of the series (B2), where the series $\sum_{n_0}^{\infty} a_n \xi^n$ converges absolutely for $|\xi| < \rho_c$ with

$$\rho_c = \lim_{n \to \infty} |\frac{a_n}{a_{n+1}}| = \lim_{n \to \infty} (2ne)^{1-\frac{\delta}{2}} = \infty, \qquad 0 < \delta < 2 \tag{B 4}$$

[14] $\Theta(x) = 1$ for $x > 0$ and $\Theta(x) = 0$, $x < 0$.

since $1-\frac{\delta}{2} > 0$. Hence, series (B2) converges absolutely for all $\xi \neq 0$ which guarantees the convergence of (4.16) and (4.17) for all $x \neq 0$ and t for δ being in the interval $0 < \delta < 2$.

Appendix C Convergence of series (5.18)

We consider the convergence of (5.18),

$$W(x,t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!} q_n(x), \qquad 0 < \delta < 1, \tag{C 1}$$

which we can rewrite for $x \neq 0$ in the form

$$W(x,t) = -\frac{1}{\pi |x|} Im \left\{ w\left(\frac{A_\delta t e^{\frac{\pi i \delta}{2}}}{|x|^\delta} \right) \right\}, \qquad x \neq 0, \tag{C 2}$$

where $w(\xi)$ with $\xi = \frac{A_\delta t e^{\frac{\pi i \delta}{2}}}{|x|^\delta}$ has the representation

$$w(\xi) = \sum_{n=1}^{\infty} (-1)^n \frac{(n\delta)!}{n!} \xi^n, \qquad 0 < \delta < 1. \tag{C 3}$$

The radius of convergence r_c of this series is

$$r_c = \lim_{n\to\infty} \frac{(n\delta)!}{n!} \frac{(n+1)!}{((n+1)\delta)!} = \lim_{n\to\infty} (n+1) \frac{(n\delta)!}{((n+1)\delta)!} \to n^{1-\delta} \delta^{-\delta} \to \infty, \qquad 0 < \delta < 1. \tag{C 4}$$

Hence (C3) converges for all ξ absolutely uniformly. Hence, (5.18) converges in the domain $-\infty < x < \infty$ (where $x \neq 0$) and $t > 0$ for δ being in the interval $0 < \delta < 1$.

Appendix D Some remarks on the Fourier-integral (6.1) for $\alpha < -1$

A further observation is important in $b_\alpha(x)$ of (6.1), which is the Fourier transform of $|k|^\alpha$ exists for $\alpha > -1$. For exponents $\alpha < -1$, the integral (6.1) diverges. However, we can define a Fourier transform of $Re\{(-i)^\alpha (|k| + i\epsilon)^\alpha\}$ in the limiting case $\epsilon \to 0+$, which exists also for $\alpha < -1$.

We generalise the definition of $b_\alpha(x)$ (6.1) for $\alpha < -1$ by replacing $|k|^\alpha \to \frac{1}{\cos\frac{\pi\alpha}{2}} Re(\epsilon - i|k|)^\alpha)$ in the form

$$b_\alpha(x) = \lim_{\epsilon\to 0+} \frac{1}{\pi \cos\frac{\pi\alpha}{2}} Re\left\{ (-i)^\alpha \int_0^\infty (k + i\epsilon)^\alpha \cos kx \, dk \right\}, \qquad \alpha \in \mathbb{R} \notin \pm 1, \pm 3, ... \pm(2n+1), ..., \tag{D 1}$$

which exists for $\alpha \in \mathbb{R}$ where the zeros of the cosine have to be excluded. For $\alpha > -1$ (D1) is identical with (6.1),[15] and for $\alpha < -1$ it complements the representation (6.1), which diverges in this case. Let us especially evaluate (D1) in the 'forbidden' case, $\alpha < -1$.

[15] Expression (6.1) also holds for positive odd integers $\alpha = 1, 3, ..., (2n+1), ...$

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Taking into account

$$(\epsilon - i|k|)^{\alpha} = \frac{1}{\Gamma(-\alpha)} \int_0^{\infty} e^{-\tau(\epsilon - i|k|)} \tau^{-\alpha-1} \mathrm{d}\tau \tag{D 2}$$

existing for $\alpha < -1$, we obtain the representation

$$b_{\alpha}(x) = \lim_{\epsilon \to 0+} \frac{1}{\pi \cos \frac{\pi\alpha}{2} \Gamma(-\alpha)} Re \left\{ \int_0^{\infty} \mathrm{d}\tau \, e^{-\epsilon\tau} \tau^{-\alpha-1} \int_0^{\infty} \cos kx \, e^{ik\tau} \mathrm{d}k \right\}, \tag{D 3}$$

which we evaluate by using

$$Re \int_0^{\infty} \cos kx \, e^{ik\tau} \mathrm{d}k = \frac{\pi}{2} \delta(\tau - |x|) \tag{D 4}$$

and arrive at

$$b_{\alpha}(x) = \frac{1}{2 \cos \frac{\pi\alpha}{2}} \frac{|x|^{-\alpha-1}}{\Gamma(-\alpha)}, \qquad \alpha < -1, \notin -1, -3, .. - (2n+1), ..., \tag{D 5}$$

where $\Gamma(-\alpha)$ is for $\alpha < -1$ well defined in (2.13).

By taking into account the Euler relation (3.25), we arrive at

$$b_{\alpha}(x) = -\frac{\alpha!}{\pi |x|^{\alpha+1}} \sin \frac{\alpha\pi}{2}, \qquad \alpha < -1, \notin -3, .. - (2n+1), ..., \tag{D 6}$$

which has the same form as expression (6.5), which holds for $\alpha > -1$ but with the generalised definition of $\alpha!$ given in (3.25). In conclusion (D6) holds for $\alpha \in \mathbb{R} \notin -1, -3, ... - (2n+1), ...$ For cases α being negative odd integers, expression (D6) is not valid, we refer in this context further to [7] where the Fourier transforms $1/|k|^{2n+1}$ $(n = 0, 1, 2...)$ are further analysed.

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