Proceedings of the Edinburgh Mathematical Society (2015) **58**, 787–806
DOI:10.1017/S0013091515000024

SMOOTHINGS OF FANO VARIETIES WITH NORMAL CROSSING SINGULARITIES

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(Received 26 January 2012)

Abstract This paper obtains criteria for a Fano variety X defined over an algebraically closed field of characteristic zero with normal crossing singularities to be smoothable. In particular, we show that X is smoothable by a flat deformation $\mathcal{X} \to \Delta$ with smooth total space $\mathcal{X}$ if and only if $T^1_X \cong \mathcal{O}_D$, where D is the singular locus of X.

Keywords: algebraic geometry; Fano varieties; smoothings; normal crossing

2010 *Mathematics subject classification:* Primary 14D15; 14D06; 14J45

1. Introduction

This paper studies the deformation theory of a Fano variety defined over an algebraically closed field of characteristic zero with normal crossing singularities. In particular, it is investigated when such a variety is smoothable. This means that there is a flat projective morphism $f\colon \mathcal{X} \to \Delta$, where Δ is the spectrum of a discrete valuation ring (R, m_R), such that $\mathcal{X} \otimes_R (R/m_R) \cong X$ and $\mathcal{X} \otimes_R K(R)$ is smooth over the function field $K(R)$ of R. Moreover, it is investigated when such a smoothing exists with smooth total space $\mathcal{X}$. In this case X is said to be totally smoothable.

Normal crossing singularities appear quite naturally in any degeneration problem. Let $f\colon \mathcal{X} \to C$ be a flat projective morphism from a variety $\mathcal{X}$ to a curve C. Then, according to Mumford's semi-stable reduction theorem [**6**], after a finite base change and a birational modification the family can be brought to the standard form $f'\colon \mathcal{X}' \to C'$, where $\mathcal{X}'$ is smooth and the special fibres are simple normal crossing varieties.

Smoothings of Fano varieties play a fundamental role in higher-dimensional birational geometry as well. The outcome of the minimal model program starting with a smooth n-dimensional projective variety X is a $\mathbb{Q}$-factorial terminal projective variety Y such that either K_Y is numerically effective or Y has a Mori fibre space structure. This means that there is a projective morphism $f\colon Y \to Z$ such that $-K_Y$ is f-ample, Z is normal and $\dim Z \leqslant \dim X - 1$. Suppose that the second case happens and $\dim Z = 1$. Let $z \in Z$ and $Y_z = f^{-1}(z)$. Then Y_z is a Fano variety of dimension $n-1$ and Y is a smoothing of Y_z. The singularities of the special fibres are difficult to describe but normal crossing singularities naturally occur and are the simplest possible non-normal singularities.

Moreover, the study of smoothings $f\colon \mathcal{X} \to \Delta$ such that $\mathcal{X}$ is smooth, $-K_{\mathcal{X}}$ is f-ample and the special fibre is a simple normal crossing divisor has a central role in the classification of smooth Fano varieties [**5**]. In dimension 2 Fujita [**5**] has described all the possible degenerations of smooth Del Pezzo surfaces to simple normal crossing Del Pezzo surfaces and Kachi showed that all of these actually occur [**7**]. As far as the author knows, this problem is completely open in higher dimensions.

It is therefore of interest to study which Fano varieties with normal crossing singularities are smoothable and, in particular, which are totally smoothable.

The paper is organized as follows.

In § 3 we review the basic properties of the theory of logarithmic structures and logarithmic deformations developed in [**8**, **9**] in the algebraic case and in [**10**] in the complex analytic case. These notions are essential in the investigation of when a Fano variety with normal crossing singularities is totally smoothable. The point is that sometimes singular varieties admit logarithmic structures in such a way that they become smooth in the log category. Moreover, deformations of varieties with smooth log structures behave like deformations of smooth varieties and therefore have very good deformation theory. Of course, not all varieties admit smooth logarithmic structures. However, a variety X with normal crossing singularities admits a semi-stable logarithmic structure and becomes log smooth if and only if $T^1_X = \mathcal{O}_D$, where D is the singular locus of X (see Theorem 3.9), which is exactly the case when X is totally smoothable and the reason why the theory of logarithmic deformations is so useful in the investigation of when a variety with normal crossing singularities is totally smoothable.

In § 4 we study the obstruction spaces to deform a Fano variety X with normal crossing singularities defined over an algebraically closed field of characteristic zero. It is well known that $H^2(T_X)$ and $H^1(T^1_X)$ are obstruction spaces to deformations of X. If X has a semi-stable logarithmic structure, then Theorem 3.5 shows that $H^2(\mathcal{H}\mathrm{om}_X(\Omega_X(\log), \mathcal{O}_X))$ is an obstruction space to logarithmic deformations. If X is a simple normal crossing, which means that X has smooth irreducible components, then its obstruction theory is deeply clarified by the work of Friedman [**4**]. However, for the general case in which X is not necessarily reducible, Friedman's theory does not directly apply. In Theorem 4.8 we show that if X is a Fano variety with normal crossing singularities, then $H^2(T_X) = 0$. Moreover, if X admits a semi-stable logarithmic structure, then $H^2(\mathcal{H}\mathrm{om}_X(\Omega_X(\log), \mathcal{O}_X)) = 0$ and hence X has unobstructed logarithmic deformations. However, usual deformations can be obstructed since the other obstruction space $H^1(T^1_X)$ may not vanish. This is the case in Example 6.2. However, T^1_X is a line bundle on the singular locus D of X and in order for X to be smoothable one has to impose some positivity conditions on T^1_X that will force it to vanish. If X has at worst double points, then in Theorem 4.11 we show that $H^1(T^1_X) = 0$ and hence X has unobstructed deformations in this case.

In § 5 we apply the results of the previous sections to obtain criteria for the existence of a smoothing of a Fano variety X with normal crossing singularities. We also study the problem of when X is totally smoothable. Proposition 5.1 shows that if X is totally smoothable, then $T^1_X \cong \mathcal{O}_D$, where D is the singular locus of X. Therefore, by Proposition 3.9, X has a logarithmic structure and the theory of logarithmic deformations applies in this case. The main result of § 5 is the following theorem.

Theorem 1.1. *Let X be a Fano variety defined over an algebraically closed field of characteristic zero with normal crossing singularities. Assume that one of the following conditions hold.*

(1) *T^1_X is finitely generated by global sections and $H^1(T^1_X) = 0$.*

(2) *X has at worst double point normal crossing singularities and T^1_X is finitely generated by global sections.*

(3) *X is d-semi-stable, i.e. $T^1_X \cong \mathcal{O}_D$, where D is the singular locus of X.*

Then X is smoothable. Moreover, X is smoothable by a flat deformation $f\colon \mathcal{X} \to \Delta$ such that $\mathcal{X}$ is smooth if and only if X is d-semi-stable.

The author does not know if the condition that T^1_X is finitely generated by its global sections is a necessary condition, too, for X to be smoothable. X is certainly not smoothable if $H^0(T^1_X) = 0$ [**15**]. In all the cases of the previous theorem, the condition finitely generated by global sections implies that $\mathrm{Def}(X)$ is smooth. If it is true that $\mathrm{Def}(X)$ is smooth for any X, then X smoothable implies that T^1_X is finitely generated by its global sections, too.

In § 6 we give one example of a smoothable and one example of a non-smoothable Fano threefold.

Finally, the requirement that we work over an algebraically closed field is more technical than essential. In the general case the author believes that the definition of normal crossing singularities must be modified to allow singularities like $x_0^2 + x_1^2 = 0$ in $\mathbb{R}^2$. This would make the arguments more complicated without adding anything of essence to the proofs. However, the characteristic-zero assumption is essential since we make repeated use of the Akizuki–Kodaira–Nakano vanishing theorem.

2. Terminology and notation

All schemes in this paper are defined over an algebraically closed field k.

A reduced scheme X of finite type over k is said to have normal crossing (n.c.) singularities at a point $P \in X$ if $\hat{\mathcal{O}}_{X,P} \cong k(P)[\![x_0, \dots, x_n]\!]/(x_0 \cdots x_r)$, for some $r = r(P)$, where $k(P)$ is the residue field of $\mathcal{O}_{X,P}$ and $\hat{\mathcal{O}}_{X,P}$ is the completion of $\mathcal{O}_{X,P}$ at its maximal ideal. If $r = 2$, then we say that $P \in X$ is a double point normal crossing singularity. X is called a normal crossing variety if it has normal crossing singularities at every point. In addition, if X has smooth irreducible components then it is called a simple normal crossing variety.

A reduced projective scheme X with normal crossing singularities is called a Fano variety if and only if ω_X^{-1} is an ample invertible sheaf on X.

For any scheme X we denote by T^1_X the sheaf of infinitesimal first order deformations of X [**12**]. If X is reduced, then $T^1_X = \mathcal{E}\mathrm{xt}^1_X(\Omega_X, \mathcal{O}_X)$. If X has n.c. singularities then a straightforward local calculation shows that T^1_X is a line bundle on the singular locus D of X. Moreover, if $X = \bigcup_{i=1}^N X_i$ is a simple normal crossings variety, then [**4**]

$$T^1_X = \mathcal{H}\mathrm{om}_D((I_{X_1}/I_{X_1}I_D) \otimes \cdots \otimes (I_{X_k}/I_{X_k}I_D), \mathcal{O}_D).$$

A variety X with normal crossing singularities is called d-semi-stable if and only if $T^1_X \cong \mathcal{O}_D$, where D is the singular locus of X.

We say that X is smoothable if there is a flat morphism of finite type $f\colon \mathcal{X} \to \Delta$, $\Delta = \mathrm{Spec}(R)$, where R is a discrete valuation ring, such that the central fibre $\mathcal{X}_0$ is isomorphic to X and the general fibre $\mathcal{X}_g$ is smooth over the function field $K(R)$ of R.

Finally, we will repeatedly make use of the Akizuki–Kodaira–Nakano vanishing theorem and its logarithmic version, which we state next.

Theorem 2.1 (Akizuki–Kodaira–Nakano [1, 3]). *Let X be a smooth variety defined over an algebraically closed field of characteristic zero and let $\mathcal{L}$ be an ample invertible sheaf on X. The following conditions then hold.*

(1)
$$H^b(X, \Omega^a_X \otimes \mathcal{L}^{-1}) = 0$$
for all a and b such that $a + b < \dim X$.

(2) *Moreover, if D is a reduced simple normal crossings divisor of X, then*
$$H^b(X, \Omega^a_X(\log(D)) \otimes \mathcal{L}^{-1}) = 0$$
for all a and b such that $a + b < \dim X$.

3. Logarithmic Structures.

In order to study the deformation theory of certain Fano varieties we will use the theory of logarithmic structures and deformations that was developed in [**8, 9**] in the algebraic case and in [**10**] in the complex analytic case. For the convenience of the reader, we include a short review of basic properties and results that will be used in this paper and refer the reader to the aforementioned papers for more details.

Definition 3.1. Let X be a scheme. A pre-logarithmic structure on X is a sheaf of monoids $\mathcal{M}$ on the étale site X_{et} together with a sheaf of monoids homomorphism $\alpha\colon \mathcal{M} \to \mathcal{O}_X$ with respect to the multiplication of $\mathcal{O}_X$. A pre-logarithmic structure is called a logarithmic structure if $\alpha^{-1}(\mathcal{O}^*_X) \cong \mathcal{O}^*_X$.

For simplicity, from now on pre-logarithmic structures will be called pre-log structures and logarithmic structures will be called log structures.

A morphism $(X, \mathcal{M}) \to (Y, \mathcal{N})$ of schemes with log structures is a pair (f, g), where $f\colon X \to Y$ is a scheme morphism and $g\colon f^{-1}(\mathcal{N}) \to \mathcal{M}$ is a sheaf of monoids map, such that the following diagram commutes:

$$\begin{array}{ccc} f^{-1}(\mathcal{N}) & \xrightarrow{g} & \mathcal{M} \\ \downarrow & & \downarrow \\ f^{-1}(\mathcal{O}_Y) & \longrightarrow & \mathcal{O}_X \end{array}$$

To any pre-log structure $(\mathcal{M}, \alpha)$ on a scheme X there is a naturally defined log structure (M^a, α) that is universal for homomorphisms of pre-log structures from $\mathcal{M}$ to log structures of X. Moreover, given a scheme morphism $f\colon X \to Y$ and log structures $\mathcal{M}$ and $\mathcal{N}$ on X and Y, respectively, the preimage and direct image log structures $f^*\mathcal{N}$ and $f_*\mathcal{M}$ are also naturally defined.

Let $\mathcal{M}$ be a log structure on a scheme X. The log structure is called integral if $\mathcal{M}$ is a sheaf of integral monoids and fine if étale locally on X there is a finitely generated monoid P and a sheaf of monoids map $P_X \to \mathcal{O}_X$, where P_X is the constant sheaf associated with P, such that the log structure associated with P_X is $\mathcal{M}$.

A morphism $(X, \mathcal{M}) \to (Y, \mathcal{N})$ of schemes with fine log structures is called smooth if it satisfies a logarithmic version of the infinitesimal criterion of smoothness. It is a natural extension of the usual notion of smoothness in the category of schemes with fine log structures. An interesting part of the theory is that morphisms that are not smooth in the category of schemes become smooth in the log-category with suitably chosen log structures. Example 3.7 exhibits such a case.

3.1. Log differentials, log derivations and log deformations.

There is a natural extension of differentials, derivations and deformations in the log category.

Definition 3.2.

(1) Let $f\colon (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a morphism of schemes with fine log structures and let $\mathcal{E}$ be an $\mathcal{O}_X$-module. A log derivation from $(X, \mathcal{M})$ to $\mathcal{E}$ over $(Y, \mathcal{N})$ is a pair $(D, D\log)$, where $D \in \mathrm{Der}_Y(X, \mathcal{E})$ is a usual derivation and $D\log\colon \mathcal{M} \to \mathcal{E}$ is a map such that:

(a) $D\log(ab) = D\log(a) + D\log(b)$ for $a, b \in \mathcal{M}$;
(b) $D(\alpha(a)) = \alpha(a) D\log(a)$ for $a \in \mathcal{M}$;
(c) $D\log(\phi(c)) = 0$ for all $c \in f^{-1}\mathcal{N}$, where $\phi\colon f^{-1} \to \mathcal{M}$ is the sheaf of monoids map associated with the morphism f.

(2) The sheaf of log differentials of $(X, \mathcal{M})$ over $(Y, \mathcal{N})$ is the $\mathcal{O}_X$-module

$$\Omega_{X/Y}(\log(\mathcal{M}/\mathcal{N}))$$

defined by

$$\Omega_{X/Y}(\log(\mathcal{M}/\mathcal{N})) = \frac{\Omega_{X/Y} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}^{gp})}{K},$$

where K is the $\mathcal{O}_X$-submodule generated by $(d\alpha(a), 0) - (0, \alpha(a) \otimes a)$ and $(0, 1 \otimes \phi(b))$ for all $a \in \mathcal{M}$, $b \in \mathcal{N}$.

Next, we define the notion of a log deformation. Let Λ be a complete Noetherian local ring with maximal ideal m_Λ and residue field k. Let Q be a fine saturated monoid having no invertible element other than 1. Q defines a log structure $(\operatorname{Spec} k, Q)$. Let $\Lambda[\![Q]\!]$ be the completion of the monoid ring $\Lambda[Q]$ along the maximal ideal $m_\Lambda + \Lambda[Q - \{1\}]$ (if $Q = \mathbb{N}$, $\Lambda[\![Q]\!] = \Lambda[\![t]\!]$). The map $Q \to \Lambda[\![Q]\!]$ defines a log structure on $\Lambda[\![Q]\!]$ and on any Artin local Λ-algebra A via the $\Lambda[\![Q]\!]$-algebra map $\Lambda[\![Q]\!] \to A$. Let $\mathrm{Art}_{\Lambda[\![Q]\!]}(k)$ be the category of local Artin $\Lambda[\![Q]\!]$-algebras.

Definition 3.3. Let $f\colon (X, \mathcal{M}) \to (\operatorname{Spec} k, Q)$ be a log-smooth morphism and $A \in \mathrm{Art}_{\Lambda[\![Q]\!]}(k)$. A log-smooth deformation of f over A is a Cartesian diagram

$$\begin{array}{ccc} (X, \mathcal{M}) & \longrightarrow & (X_A, \mathcal{M}_A) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_A} \\ (\operatorname{Spec} k, Q) & \longrightarrow & (\operatorname{Spec}(A), Q) \end{array}$$

where $f_A\colon (X_A, \mathcal{M}_A) \to (\operatorname{Spec}(A), Q)$ is log smooth. In particular, if $Q = \mathbb{N}$, then the underlying scheme morphisms are flat and hence we have a usual deformation.

Having defined log deformations, the log deformation functor

$$\mathrm{LD}(X, \mathcal{M})\colon \mathrm{Art}_{\Lambda[\![Q]\!]}(k) \to (\mathrm{Sets})$$

is naturally defined.

Theorem 3.4 (Kato [9, Theorem 8.7]). *If $f\colon (X\mathcal{M}) \to (\operatorname{Spec} k, Q)$ is integral and X is proper, then $\mathrm{LD}(X, \mathcal{M})$ has a hull.*

The log deformation theory of log-smooth maps is very similar to the deformation theory of smooth varieties. The next theorem describes the obstructions to lift log-smooth deformations.

Theorem 3.5 (Kato [8, Theorem 3.14]). *Let $f_A\colon (X_A, \mathcal{M}_A) \to (\operatorname{Spec}(A), Q)$ be a log deformation of the log-smooth map $f\colon (X, \mathcal{M}) \to (\operatorname{Spec} k, Q)$. Let*

$$0 \to I \to B \to A \to 0$$

be a square zero extension in $\mathrm{Art}_{\Lambda[\![Q]\!]}(k)$. Then the obstructions for lifting f_A to B are in

$$H^2(X_A, \mathcal{H}\mathrm{om}_{X_A}(\Omega_{X_A/A}(\log(\mathcal{M}_A/Q)))) \otimes_A I.$$

3.2. Logarithmic structures on varieties with normal crossing singularities

Next we present some logarithmic structures on schemes with normal crossing singularities that will be needed for the study of the smoothability of Fano varieties with normal crossing singularities.

Example 3.6. Let $D \subset X$ be a reduced divisor with normal crossings in a smooth scheme X. Let $M \subset \mathcal{O}_X$ be the subsheaf of $\mathcal{O}_X$ of regular functions that are invertible outside of D. M is a log structure on X. Moreover, if $i\colon D \to X$ is the closed immersion, then i^*M is a log structure on D. This log structure is said to be 'of embedding type', and it is fine because it is étale locally around D, $D \subset X$ is just $(x_1 \cdots x_r = 0) \subset \mathbb{A}^d_k$ and the log structure is induced by the monoid map

$$\alpha\colon \mathbb{N}^d \to \frac{k[x_1, \ldots, x_d]}{(x_1 \cdots x_r)}$$

given by $\alpha(e_i) = x_i$ if $i \leqslant r$, and 1 for $r < i \leqslant d$.

Example 3.7. Let

$$X = \operatorname{Spec} \frac{k[x_1, \ldots, x_d]}{(x_1 \cdots x_r)}$$

be a simple normal crossing variety and $\mathbb{N}^d \to \mathcal{O}_X$ the log structure defined in the previous example. The map $\beta\colon \mathbb{N} \to k$ such that $\beta(0) = 1$ and $\beta(n) = 0$ for $n \neq 0$ defines a log structure on $\operatorname{Spec} k$. Let $\delta\colon \mathbb{N} \to \mathbb{N}^d$ be the diagonal map. Then the corresponding map of log schemes $(X, \mathbb{N}^d) \to (\operatorname{Spec} k, \mathbb{N})$ is log smooth and is called a logarithmic semi-stable map.

Definition 3.8. Let $\alpha\colon \mathcal{M} \to \mathcal{O}_X$ be a log structure on a scheme X with normal crossing singularities.

(1) The log structure is said to be 'of embedding type' if locally in the étale topology it is equivalent to the log structure of embedding type defined in Example 3.6.

(2) The log structure is said to be 'of semi-stable type' if there is a map of log schemes $f\colon (X, \mathcal{M}) \to (\operatorname{Spec} k, \mathbb{N})$ that is locally in the étale topology equivalent to the logarithmic semi-stable map defined in Example 3.7.

In the case of a scheme with semi-stable log structure as above, for simplicity we denote by $\Omega_X(\log)$ the sheaf of logarithmic differentials of $(X, \mathcal{M})$ over $(\operatorname{Spec} k, \mathbb{N})$. $\Omega_X(\log)$ is a free $\mathcal{O}_X$-module locally generated by the logarithmic differentials

$$\frac{\mathrm{d}x_1}{x_1}, \ldots, \frac{\mathrm{d}x_r}{x_r}, \mathrm{d}x_{r+1}, \ldots, \mathrm{d}x_n$$

with the relation

$$\frac{\mathrm{d}x_1}{x_1} + \cdots + \frac{\mathrm{d}x_r}{x_r} = 0.$$

The existence of logarithmic structures of embedded type or semi-stable type is very closely related to the deformation theory of X.

Theorem 3.9 (Kato [9, Theorem 11.7]). *Let X be a scheme with normal crossing singularities and D its singular locus.*

(1) *A log structure of embedding type exists on X if and only if there exists a line bundle L on X such that $L \otimes_{\mathcal{O}_X} \mathcal{O}_D \cong T^1_X$.*

(2) *A log structure of semi-stable type exists on X if and only if $T^1_X \cong \mathcal{O}_D$, i.e. if X is d-semi-stable.*

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4. Obstructions

Let X be a variety with normal crossing singularities. It is well known that $H^2(T_X)$ and $H^1(T^1_X)$ are obstruction spaces to deformations of X. The purpose of this section is to describe these spaces and, moreover, to describe the obstruction space $H^2(\mathcal{H}\mathrm{om}(\Omega_X(\log), \mathcal{O}_X))$ to logarithmic deformations of an n.c. variety X with a semi-stable logarithmic structure. We begin with some preliminary results.

Proposition 4.1 (Friedman [4]). *Let X be a scheme with only normal crossing singularities. Let $\tau_X \subset \Omega_X$ be the torsion subsheaf of Ω_X. Then, for all $i \geqslant 0$,*

$$\Omega_X/\tau_X \cong \Omega_X^{**},$$
$$\mathrm{Ext}^i_X(\Omega_X/\tau_X, \mathcal{O}_X) \cong H^i(T_X),$$
$$\mathrm{Ext}^i_X(\tau_X, \mathcal{O}_X) = H^{i-1}(T^1_X).$$

Corollary 4.2. *Let X be a projective scheme with only normal crossing singularities. Then,*

$$H^2(T_X) = H^{n-2}((\Omega_X/\tau_X) \otimes \omega_X),$$

where $\dim X = n$.

Proof. By Proposition 4.1, $H^2(T_X) = \mathrm{Ext}^2_X(\Omega_X/\tau_X, \mathcal{O}_X) = H^{n-2}((\Omega_X/\tau_X) \otimes \omega_X)$, by Serre duality. □

Definition 4.3. Let X be a scheme with normal crossing singularities defined over a field k. We denote by $X_{[k]} \subset X$, $k \geqslant 0$, the subschemes of X defined inductively by $X_{[0]} = X$ and by $X_{[k]}$ the singular locus of $X_{[k-1]}$ with reduced structure. We also denote by $\pi_i \colon \tilde{X}_{[i]} \to X_{[i]}$ the normalization of $X_{[i]}$, $i \geqslant 0$.

Theorem 4.4. *Let X be a scheme with normal crossing singularities defined over a field k. Then the following hold.*

(1) *There are exact sequences*

$$0 \to \Omega_X/\tau_X \to (\pi_0)_*\Omega_{\tilde{X}} \xrightarrow{\delta_1} (\pi_1)_*(\Omega_{\tilde{X}_{[1]}} \otimes L_1) \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_N} (\pi_N)_*(\Omega_{\tilde{X}_{[N]}} \otimes L_N) \to 0, \quad (4.1)$$

$$0 \to \mathcal{O}_X \to (\pi_0)_*\mathcal{O}_{\tilde{X}} \to (\pi_1)_*Q_1 \to \cdots \to (\pi_N)_*Q_N \to 0. \quad (4.2)$$

(2) *Suppose that X has a semi-stable logarithmic structure. There is then an exact sequence*

$$0 \to \Omega_X/\tau_X \to \Omega_X(log) \xrightarrow{\lambda_1} (\pi_1)_*(\mathcal{O}_{\tilde{X}_{[1]}} \otimes M_1) \xrightarrow{\lambda_2} \cdots \xrightarrow{\lambda_m} (\pi_m)_*(\mathcal{O}_{\tilde{X}_{[m]}} \otimes M_m) \to 0, \quad (4.3)$$

where $m, N \leqslant \dim X$ and Q_i, L_i and M_i are 2-torsion invertible sheaves on $\tilde{X}_{[i]}$, i.e. $L_i^{\otimes 2} \cong \mathcal{O}_{\tilde{X}_{[i]}}$, $M_i^{\otimes 2} \cong \mathcal{O}_{\tilde{X}_{[i]}}$ and $Q_i^{\otimes 2} \cong \mathcal{O}_{\tilde{X}_{[i]}}$ for all i.

Remark 4.5. In the case of simple normal crossing complex analytic spaces, Theorem 4.4 was proved in [**4**].

The following result is needed for the proof of the theorem.

Lemma 4.6. *Let $f\colon Y \to X$ be an étale morphism of schemes. Let $\pi\colon \tilde{X} \to X$ be the normalization of X. Then $p_Y\colon \tilde{X}\times_X Y \to Y$ is the normalization of Y, where $\tilde{X}\times_X Y \to Y$ is the fibre product of $\tilde{X}$ and Y over X and p_Y is the projection to Y.*

Proof. From the fibre square diagram

$$\begin{array}{ccc} \tilde{X} \times_X Y & \xrightarrow{p_Y} & Y \\ \downarrow{\scriptstyle p_{\tilde{X}}} & & \downarrow{\scriptstyle f} \\ \tilde{X} & \xrightarrow{\pi} & X \end{array}$$

it follows that $p_{\tilde{X}}$ is étale and p_Y is finite. Hence, $\tilde{X} \times_X Y$ is normal. Moreover, p_Y is generically isomorphism. Therefore, there is a factorization $g\colon \tilde{X} \times_X Y \to \tilde{Y}$ of p_Y through the normalization $\tilde{Y}$ of Y. But then, since both $\tilde{X} \times_X Y$ and $\tilde{Y}$ are normal, g is finite and generically isomorphism, g is in fact an isomorphism. □

Proof of Theorem 4.4. We will only prove the existence of the exact sequence (4.1) in detail. The proof for the others is similar so we only sketch it and leave the details to the reader.

The proof of the first part is in two steps. First we show the existence of the exact sequence (4.1) for an affine simple normal crossing scheme and then we prove the general case. The proof of this part is similar to the one exhibited by Friedman [**4**] in the case of a simple normal crossing complex analytic space. For the sake of completeness, and since the explicit local construction of the sequence is needed for the general case, we present a short proof here following the lines of Friedman's proof.

Step 1. Suppose that

$$X = \operatorname{Spec} \frac{k[x_1, \dots, x_n]}{(x_1 \cdots x_r)}.$$

Then $X = \bigcup_{i=1}^{r} X_i$, where $X_i \subset X$ is the component given by $x_i = 0$, $1 \leqslant i \leqslant r$. Then, for $i \geqslant 1$,

$$X_{[i]} = \bigcup_{k_0 < \dots < k_i} (X_{k_0} \cap \dots \cap X_{k_i}).$$

Moreover, $\tilde{X} = \coprod_{i=1}^{r} X_i$ and

$$\tilde{X}_{[i]} = \coprod_{k_0 < \dots < k_i} (X_{k_0} \cap \dots \cap X_{k_i}).$$

The maps $\pi_i\colon \tilde{X}_{[i]} \to X_{[i]}$, $i \geqslant 0$, are the natural ones. Now, by definition, $\tau_X \subset \Omega_X$ is the sheaf of sections of Ω_X supported on the singular locus of X. Hence, it is the kernel

https://doi.org/10.1017/S0013091515000024 Published online by Cambridge University Press

of the natural map $\delta\colon \Omega_X \to \pi_* \Omega_{\tilde{X}}$. Now define the sequence of maps

$$0 \to \tau_X \to \Omega_X \xrightarrow{\delta} (\pi_0)_* \Omega_{\tilde{X}} \xrightarrow{\delta_1} (\pi_1)_* \Omega_{\tilde{X}_{[1]}} \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_i} (\pi_i)_* \Omega_{\tilde{X}_{[i]}} \xrightarrow{\delta_{i+1}} (\pi_{i+1})_* \Omega_{\tilde{X}_{[i+1]}} \to \cdots, \quad (4.4)$$

where δ_i are the Čech coboundary maps. This is clearly a complex and we proceed to show that it is in fact exact. We use induction on the number r of components of X. For $r = 1$ there is nothing to prove. Suppose now that the sequence (4.4) is exact for all simple normal crossing affine schemes with at most $r-1$ components.

Let $X' = \bigcup_{i=1}^{r-1} X_i$ and $Y = X' \cap X_r$. Then, $X = X' \cup X_r$ and $\tilde{X}_{[k]} = \tilde{X}'_{[k]} \amalg \tilde{Y}_{[k-1]}$ for all $k \geqslant 0$, where we also set $Y_{[-1]} = X_r$. By the induction hypothesis, the corresponding sequences (4.4) for X' and Y, are exact.

From the previous discussion, it follows that $\mathrm{Ker}(\delta) = \tau_X$. Next we show exactness at the next step, i.e. that $\mathrm{Ker}(\delta_1) = \mathrm{Im}(\delta)$. Now, since

$$(\pi_0)_* \Omega_{\tilde{X}} = (\pi_0)_* \Omega_{\tilde{X}'} \oplus \Omega_{X_r},$$
$$(\pi_1)_* \Omega_{\tilde{X}_{[1]}} = (\pi_1)_* \Omega_{\tilde{X}'_{[1]}} \oplus \Omega_{\tilde{Y}},$$

any element of $(\pi_0)_* \Omega_{\tilde{X}}$ is of the form (α, β), where $\alpha \in (\pi_0)_* \Omega_{\tilde{X}'}$ and $\beta \in \Omega_{X_r}$. Suppose that such an element is also in the kernel of δ_1. It is now clear from the induction hypothesis that $(\alpha, 0)$ is in the image of δ. Therefore, in order to show exactness at the level of δ_1, it suffices to show that if an element of the form $(0, \beta)$ is in the kernel of δ_1, it is also in the image of δ. Suppose that $\beta = \sum_{k \neq r} \alpha_r(f_k)\, \mathrm{d}x_k$ is such an element, where $f_k \in \mathcal{O}_X$ and $\alpha_r\colon \mathcal{O}_X \to \mathcal{O}_{X_r}$ is the natural map. Therefore, since $(\pi_0)_* \Omega_{\tilde{Y}} = \bigoplus_{i=1}^{r-1} \Omega_{X_i \cap X_r}$, it follows that the restriction of β on $X_i \cap X_r$ is zero for all $i \leqslant r-1$. Hence, $f_k \in (x_1 \cdots \hat{x}_k \cdots x_{r-1}, x_r)$ for $1 \leqslant k \leqslant r$ and $f_k \in (x_1 \cdots x_{r-1}, x_r)$ for $k > r$. Therefore, $\delta(\sum_{k \neq r} f_k\, \mathrm{d}x_k) = (0, \beta)$ and hence $(0, \beta)$ is in the image of δ.

There is an exact sequence

$$0 \to (\pi_{k-1})_* \Omega_{\tilde{Y}_{[k-1]}} \to (\pi_k)_* \Omega_{\tilde{X}_{[k]}} \to (\pi_k)_* \Omega_{\tilde{X}'_{[k]}} \to 0. \quad (4.5)$$

Now, define the complexes (A^*, δ_A^*), (B^*, δ_B^*) and (C^*, δ_C^*) such that $A^k = (\pi_{k-1})_* \Omega_{\tilde{Y}_{[k-1]}}$, $B^k = (\pi_k)_* \Omega_{\tilde{X}_{[k]}}$ and $C^k = (\pi_k)_* \Omega_{\tilde{X}'_{[k]}}$, $k \geqslant 0$. The coboundary maps are the Čech maps. Then (4.5) induces an exact sequence of complexes

$$0 \to A^* \to B^* \to C^* \to 0.$$

Passing to cohomology, we get an exact sequence

$$\cdots \to H^k(A^*) \to H^k(B^*) \to H^k(C^*) \to \cdots.$$

Now, by induction, $H^k(C^*) = 0$ for all $k \geqslant 1$ and $H^k(A^*) = 0$ for all $k \geqslant 2$. Hence, $H^k(B^*) = 0$ for all $k \geqslant 2$. It remains to check for $k = 1$. Then there is an exact sequence

$$H^0(C^*) \xrightarrow{\sigma} H^1(A^*) \to H^1(B^*) \to 0.$$

Moreover, $H^0(C^*) = \Omega_{X'}/\tau_{X'}$, $H^1(A^*) = \Omega_Y/\tau_Y$ and σ is the natural map

$$\Omega_{X'}/\tau_{X'} \to \Omega_Y/\tau_Y,$$

and hence it is surjective. Therefore, $H^1(B^*) = 0$ and the complex (4.4) is exact as claimed.

Step 2. The general case. So, let X be a scheme with normal crossing singularities.

Claim 4.7. *For any $x \in X$ there are pointed étale maps*

$$\begin{array}{ccc} & (U,u) & \\ {}^{f}\swarrow & & \searrow^{g} \\ (X,x) & & (W,w) \end{array} \tag{4.6}$$

where $u \in U$, $w \in W$, $f(u) = x$, $g(u) = w$, such that the following hold.

(1) *f and g induce isomorphisms of residue fields $k(u) \cong k(x) \cong k(w)$.*

(2)
$$W = \operatorname{Spec} \frac{k[x_1, \dots, x_n]}{(x_1 \cdots x_{r(x)})}$$
and $w \in W$ corresponds to the maximal ideal $(x_1, \dots, x_n)$.

(3) *All irreducible components of U pass through u.*

(4) *U is a simple normal crossing scheme and it has exactly $r = r(x)$ irreducible components, exactly as many as W. Let $U = \bigcup_{k=1}^{r} U_k$ be the decomposition of U into its irreducible components. Then there is an exact sequence*

$$0 \to \tau_U \to \Omega_U \xrightarrow{\delta} (\pi_0)_* \Omega_{\tilde{U}} \xrightarrow{\delta_1} (\pi_1)_* \Omega_{\tilde{U}_{[1]}} \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_k} (\pi_k)_* \Omega_{\tilde{U}_{[k]}} \xrightarrow{\delta_{k+1}} \cdots, \tag{4.7}$$

where, as before,
$$U_{[k]} = \bigcup_{s_0 < \cdots < s_k} (U_{s_0} \cap \cdots \cap U_{s_k}),$$
$\tilde{U}_{[k]}$ is the normalization of $U_{[k]}$ and the boundary maps are the Čech maps.

A diagram of maps such as (4.6) that satisfies (1), (2) and (3) is called an étale neighbourhood of $x \in X$. Note that the map g induces an ordering on the irreducible components $U_i = g^{-1} W_i$ of U, W_i being the irreducible component of W given by $x_i = 0$, $i = 1, \dots, r(x)$.

We proceed to show the claim. Let $x \in X$ be a point. Then, by assumption,

$$\hat{\mathcal{O}}_{X,x} \cong \frac{k[\![x_1, \dots, x_n]\!]}{(x_1 \cdots x_{r(x)})}.$$

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Let $W = \operatorname{Spec}(k[x_1, \dots, x_n]/(x_1 \cdots x_{r(x)}))$ and let w be the closed point corresponding to the maximal ideal $(x_1, \dots, x_n)$. Then, by [**2**], since $\hat{\mathcal{O}}_{X,x} \cong \hat{\mathcal{O}}_{W,w}$, there is a common étale neighbourhood of $x \in X$ and $w \in W$, i.e. there are pointed étale maps $f\colon (U,u) \to (X,x)$ and $g\colon (U,u) \to (W,w)$ as in (4.6) that satisfy the properties (1) and (2) of the claim.

Shrinking U we may assume that all its irreducible components pass through $u \in U$. Let $W = \bigcup_{i=1}^{r} W_i$, $r = r(x)$, be the decomposition of W into irreducible components, where W_i is given by $x_i = 0$, $i = 1, \dots, r$. Then, $U = \bigcup_{i=1}^{r} g^{-1}(W_i)$. Since g is étale and all irreducible components of U pass through the same point, it follows that $U_i = g^{-1}(W_i)$ is smooth and irreducible and hence $U = \bigcup_{i=1}^{r} U_i$. In particular, U_i is a simple normal crossing and has the same number of irreducible components as W.

From Step 1 there is an exact sequence

$$0 \to \tau_W \to \Omega_W \xrightarrow{\delta} (\pi_{w,0})_* \Omega_{\tilde{W}} \xrightarrow{\delta_{w,1}} (\pi_{w,1})_* \Omega_{\tilde{W}_{[1]}} \xrightarrow{\delta_{w,2}} \cdots . \tag{4.8}$$

Since g is étale, by Lemma 4.6 there is a fibre square diagram

$$\begin{array}{ccc} \tilde{U}_{[k]} & \xrightarrow{\pi_{u,k}} & U_{[k]} \\ {\scriptstyle \tilde{g}}\downarrow & & \downarrow{\scriptstyle g} \\ \tilde{W}_{[k]} & \xrightarrow{\pi_{w,k}} & W_{[k]} \end{array}$$

By flat base change, it follows that $g^*(\pi_{w,k})_* = (\pi_{u,k})_*(\tilde{g}_p)^*$. Moreover, since both g and $\tilde{g}$ are étale,

$$g^* \Omega_{W_{[k]}} = \Omega_{U_{[k]}},$$
$$(\tilde{g})^* \Omega_{\tilde{W}_{[k]}} = \Omega_{\tilde{U}_{[k]}},$$

Therefore, (4.8) pulls back via g to an exact sequence in U,

$$0 \to \tau_U \to \Omega_U \xrightarrow{\delta} (\pi_{u,0})_* \Omega_{\tilde{U}} \xrightarrow{\delta_{u,1}} (\pi_{u,1})_* \Omega_{\tilde{U}_{[1]}} \xrightarrow{\delta_{u,2}} \cdots , \tag{4.9}$$

where the coboundary maps are the Čech maps corresponding to the numbering of the components of U induced from the numbering of the components of W. This concludes the proof of the claim.

Next we claim that étale neighbourhoods of X form a basis for the étale topology of X. This means that for any étale map $f\colon Y \to X$ and a point $y \in Y$, there exists an étale neighbourhood $g\colon (U,u) \to (X,x)$, $x = f(y)$, and a factorization $h\colon U \to Y$ such that $fh = g$ and $h(u) = y$. Indeed, if $h\colon (U,u) \to (Y,y)$ is an étale neighbourhood of (Y,y), then $fh\colon (U,u) \to (X,x)$ is an étale neighbourhood of (X,x).

Let $E_U^{\bullet}$ denote the exact sequence (4.7) corresponding to the étale neighbourhood $f\colon U \to X$. Then, since étale neighbourhoods form a basis for the étale topology of X, descent theory says that in order to construct an exact sequence on X that pulls back to $E_U^{\bullet}$, it suffices to construct, for any X-map $\Phi_{vu}\colon V \to U$ between étale neighbourhoods $f\colon V \to X$ and $g\colon U \to X$ of X, exact sequence isomorphisms

$$\Psi_{vu}\colon \Phi_{vu}^*(E_U^{\bullet}) \to E_V^{\bullet}$$

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such that for any commutative diagram

$$\begin{array}{ccc} & U & \\ {}^{\Phi_{uv}}\swarrow & & \searrow^{\Phi_{uw}} \\ V & \xrightarrow{\Phi_{vw}} & W \end{array}$$

of étale neighbourhoods of X, the following diagram commutes:

$$\begin{array}{ccc} & E_U^{\bullet} & \\ {}^{\Psi_{uv}}\swarrow & & \searrow^{\Psi_{uw}} \\ \Phi_{uv}^{*}E_V^{\bullet} & \xrightarrow{\Phi_{uw}^{*}(\Psi_{vw})} & \Phi_{uv}^{*}\Phi_{vw}^{*}E_W^{\bullet} \end{array}$$

In order to have a uniform numbering of the irreducible components of all étale neighbourhoods, given an étale neighbourhood $f\colon W \to X$ with irreducible components $W_1, \ldots, W_{r(w)}$, we extend the definition of W_i for all $i \in \{1, 2, \ldots, n\}$, by setting $W_i = \emptyset$ for $n \geqslant i > r(w)$, where $n = \dim X + 1$.

Let $\Phi\colon U \to V$ be a map between two étale neighbourhoods of X, $U \xrightarrow{f} X$ and $V \xrightarrow{g} X$. Let $U = \bigcup_{i=1}^{n} U_i$ and $V = \bigcup_{j=1}^{n} V_j$ be the decompositions of U and V into irreducible components, respectively, taking into account the conventions on the irreducible components stated in the previous paragraph. Then,

$$U = \bigcup_{i=1}^{n} U_i = \bigcup_{j=1}^{n} \Phi^{-1}(V_j).$$

Moreover, since all irreducible components of U pass through the same point, $\Phi^{-1}(V_j)$ is irreducible, otherwise it would be a disjoint union of smooth irreducible components of U, which is impossible since all irreducible components of U intersect. Therefore, there exists a permutation $\sigma \in S_{n+1}$ such that $U_{\sigma(i)} = \Phi^{-1}(V_i)$, $1 \leqslant i \leqslant n$.

Let $\pi_k\colon \tilde{U}_{[k]} \to U_{[k]}$ and $\nu_k\colon \tilde{V}_{[k]} \to V_{[k]}$ be as in Definition 4.3. Then,

$$(\pi_k)_*\mathcal{O}_{\tilde{U}_{[k]}} = \bigoplus_{i_0<i_1<\cdots<i_k} \mathcal{O}_{U_{i_0 i_1 \cdots i_k}} \quad \text{and} \quad (\nu_k)_*\mathcal{O}_{\tilde{V}_{[k]}} = \bigoplus_{i_0<i_1<\cdots<i_k} \mathcal{O}_{V_{i_0 i_1 \cdots i_k}},$$

where $U_{i_0 i_1 \cdots i_k} = U_{i_0} \cap \cdots \cap U_{i_k}$ and similarly for $V_{i_0 \cdots i_k}$. Let $U_i' = \Phi^{-1}(V_i)$ and define the map

$$\lambda_k\colon \bigoplus_{i_0<i_1<\cdots<i_k} \mathcal{O}_{U_{i_0 i_1 \cdots i_k}} \to \bigoplus_{i_0<i_1<\cdots<i_k} \mathcal{O}_{U'_{i_0 i_1 \cdots i_k}} \tag{4.10}$$

by setting, for any $\alpha \in \oplus_{i_0<i_1<\cdots<i_k} \mathcal{O}_{U_{i_0 i_1 \cdots i_k}}$,

$$\lambda_k(\alpha)_{i_0 \cdots i_k} = \operatorname{sgn}(\tau)\alpha_{\tau\sigma(i_0)\cdots\tau\sigma(i_k)},$$

where $\tau \in S_{k+1}$ is the permutation such that $\tau\sigma(i_0) < \tau\sigma(i_1) < \cdots < \tau\sigma(i_k)$.

Note that both sides of (4.10) are isomorphic to $(\pi_k)_*\mathcal{O}_{\tilde{U}_{[k]}}$. The left-hand side is written by using the ordering of the components of U coming from its structure as an

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étale neighbourhood of X, while the right-hand side is written by using the ordering of the components of U inherited from V by Φ. By this consideration, λ_k gives an automorphism of $U_{[k]}$ such that λ_k^2 is the identity. Moreover, it is straightforward to check that the diagram

$$\begin{array}{ccc} \bigoplus_{i_0<\cdots<i_k} \mathcal{O}_{U_{i_0 i_1 \cdots i_k}} & \xrightarrow{d_k} & \bigoplus_{i_0<\cdots<i_k} \mathcal{O}_{U_{i_0 i_1 \cdots i_{k+1}}} \\ \downarrow \lambda_k & & \downarrow \lambda_{k+1} \\ \bigoplus_{i_0<\cdots<i_k} \mathcal{O}_{U'_{i_0 i_1 \cdots i_k}} & \xrightarrow{\delta_k} & \bigoplus_{i_0<\cdots<i_k} \mathcal{O}_{U'_{i_0 i_1 \cdots i_{k+1}}} \end{array}$$

commutes. Therefore, λ_k gives a map between Čech complexes. Let

$$(\nu_k)_* \Omega_{V_{[k]}} \xrightarrow{\delta_{V,k}} (\nu_{k+1})_* \Omega_{V_{[k+1]}}$$

be the map at the k stage of the exact sequence $E_V^{\bullet}$. This pulls back by Φ to a map

$$(\pi_k)_* \Omega_{U_{[k]}} \xrightarrow{\delta_{V,k}} (\nu_{k+1})_* \Omega_{U_{[k+1]}}.$$

This is simply the map of Čech complexes corresponding to the ordering of the irreducible components of U induced from the ordering of V by ϕ. Moreover, λ_k induces an isomorphism

$$\Lambda_k \colon (\pi_k)_* \Omega_{U_{[k]}} \to (\pi_k)_* \Omega_{U_{[k]}}$$

such that Λ_k^2 is the identity. Moreover, a straightforward calculation shows that there is a commutative diagram

$$\begin{array}{ccc} (\pi_k)_* \Omega_{U_{[k]}} & \xrightarrow{\delta_{U,k}} & (\pi_{k+1})_* \Omega_{U_{[k+1]}} \\ \downarrow \Lambda_k & & \downarrow \Lambda_{k+1} \\ (\pi_k)_* \Omega_{U_{[k]}} & \xrightarrow{\delta_{V,k}} & (\pi_{k+1})_* \Omega_{U_{[k+1]}} \end{array} \tag{4.11}$$

Now, by descent theory, the involutions λ_k glue the structure sheaves $\mathcal{O}_{\tilde{V}_{[k]}}$ to 2-torsion sheaves L_k on $\tilde{X}_{[k]}$. Then, from (4.11), all the maps in the diagram glue as well and we get a sequence on X:

$$0 \to \tau_X \to \Omega_X \to (\pi_0)_* \Omega_{\tilde{X}} \xrightarrow{\delta_1} (\pi_1)_* (\Omega_{\tilde{X}_{[1]}} \otimes L_1) \xrightarrow{\delta_2} (\pi_2)_* (\Omega_{\tilde{X}_{[2]}} \otimes L_2) \to \cdots .$$

This sequence is exact since it pulls back locally by faithful flat maps (that correspond to local étale neighbourhoods) to exact sequences. This concludes the proof of the first part of Theorem 4.4.

Next we sketch the proof of the other parts of the theorem. First we will construct the exact sequences (4.2) and (4.3) locally and then glue them to get the global sequence, exactly as in the proof of (4.1). Locally in the étale topology,

$$X = \operatorname{Spec} \frac{k[x_0, \dots, x_n]}{(x_0 \cdots x_r)}.$$

In this case $X = \bigcup_{i=1}^{r} X_r$, where X_i is given by $x_i = 0$. Then,

$$(\pi_i)_* \mathcal{O}_{\tilde{X}_i} = \bigoplus_{j_0 < \cdots < j_i} \mathcal{O}_{X_{j_0} \cap \cdots \cap X_{j_i}}.$$

We then define the sequence of maps

$$0 \to \tau_X \to \Omega_X \xrightarrow{\lambda_0} \Omega_X(\log) \xrightarrow{\lambda_1} (\pi_1)_* \mathcal{O}_{\tilde{X}_{[1]}} \xrightarrow{\lambda_2} \cdots \xrightarrow{\lambda_i} (\pi_i)_* \mathcal{O}_{\tilde{X}_{[i]}} \xrightarrow{\lambda_{i+1}} (\pi_{i+1})_* \mathcal{O}_{\tilde{X}_{[i+1]}} \to \cdots, \quad (4.12)$$

$$0 \to \mathcal{O}_X \to (\pi_0)_* \mathcal{O}_{\tilde{X}} \to (\pi_1)_* \mathcal{O}_{\tilde{X}_{[1]}} \to \cdots \to (\pi_i)_* \mathcal{O}_{\tilde{X}_{[i]}} \to (\pi_{i+1})_* \mathcal{O}_{\tilde{X}_{[i+1]}} \to \cdots \quad (4.13)$$

as follows. The second sequence is simply the Čech coboundary maps. The maps λ_i are the Čech coboundary maps for $i \geqslant 2$ and λ_0 is the natural map between Ω_X and $\Omega_X(\log)$. $\Omega_X(\log)$ is a free $\mathcal{O}_X$-module generated by $\mathrm{d}x_1/x_1, \ldots, \mathrm{d}x_r/x_r, \mathrm{d}x_{r+1}, \ldots, \mathrm{d}x_n$ with the relation $\mathrm{d}x_0/x_0 + \cdots \mathrm{d}x_r/x_r = 0$. We then define $\lambda_1(\mathrm{d}x_i) = 0$ if $i > r$, and if $i \leqslant r$, $\lambda_1(\mathrm{d}x_i/x_i) = (\alpha_{j_0,j_1})$, $j_0 < j_1$, such that

$$\alpha_{j_0,j_1} = \begin{cases} 1 & \text{if } j_0 = i, \\ -1 & \text{if } j_1 = i, \\ 0 & \text{otherwise.} \end{cases}$$

Now, by either using the same method as in the first part for the sequence (4.1) or by [**4**, Corollary 3.6], we obtain that (4.11) and (4.12) are exact. Now, by using exactly the same argument as in the first part by using étale covers we get the existence of (4.2) and (4.3). □

Theorem 4.8. *Let X be a Fano variety with normal crossing singularities defined over a field k of characteristic zero. Then,*

$$H^2(T_X) = H^2(X, \mathcal{O}_X) = 0.$$

Moreover, if X has a semi-stable logarithmic structure, then

$$H^2(\mathcal{H}\mathrm{om}(\Omega_X(\log), \mathcal{O}_X)) = 0$$

as well.

Corollary 4.9. *Let X be a Fano variety defined over an algebraically closed field of characteristic zero with normal crossing singularities. Then any formal deformation of X is effective.*

Proof. This follows since $H^2(X, \mathcal{O}_X) = 0$ and from Grothendieck's criterion of effectivity [**13**, Theorem 2.5.13]. □

Theorems 3.4, 3.5 and 4.8 imply the following corollary.

Corollary 4.10. *Let X be a Fano variety with normal crossing singularities defined over a field k of characteristic zero. Assume that X has a semi-stable logarithmic structure $\mathcal{M}$. Then,* $\mathrm{LD}(X, \mathcal{M})$ *is smooth.*

Proof of Theorem 4.8. We only prove the vanishing of $H^2(T_X)$. The remaining parts of the theorem are proved in exactly the same way by using the exact sequences (4.2) and (4.3) and, in addition, that by Serre duality, $H^2(X, \mathcal{O}_X) = H^{n-2}(X, \omega_X)$, where $n = \dim X$.

By Corollary 4.2,

$$H^2(T_X) = H^{n-2}((\Omega_X/\tau_X) \otimes \omega_X), \tag{4.14}$$

where $n = \dim X$. Then, by Theorem 4.4, there is an exact sequence

$$0 \to \Omega_X/\tau_X \xrightarrow{\delta_0} (\pi_0)_* \Omega_{\tilde{X}} \xrightarrow{\delta_1} (\pi_1)_*(\Omega_{\tilde{X}_{[1]}} \otimes L_1) \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_N} (\pi_N)_*(\Omega_{\tilde{X}_{[N]}} \otimes L_N) \to 0, \tag{4.15}$$

where $N \leqslant \dim X$, τ_X is the torsion part of Ω_X and L_i is an invertible sheaf on $\tilde{X}_{[i]}$ such that $L_i^{\otimes 2} \cong \mathcal{O}_{\tilde{X}_{[i]}}$.

Tensoring equation (4.15) with ω_X and taking into consideration that

$$(\pi_i)_*(\Omega_{\tilde{X}_{[i]}} \otimes (\pi_i)^* \omega_X) = (\pi_i)_* \Omega_{\tilde{X}_{[i]}} \otimes \omega_X,$$

we get the exact sequence

$$\begin{aligned} 0 \to (\Omega_X/\tau_X) \otimes \omega_X \xrightarrow{\delta_0} (\pi_0)_*(\Omega_{\tilde{X}} \otimes \pi_0^* \omega_X) \xrightarrow{\delta_1} (\pi_1)_*(\Omega_{\tilde{X}_{[1]}} \otimes L_1 \otimes \pi_1^* \omega_X) \xrightarrow{\delta_2} \cdots \\ \xrightarrow{\delta_N} (\pi_N)_*(\Omega_{\tilde{X}_{[N]}} \otimes L_N \otimes \pi_N^* \omega_X) \to 0. \end{aligned}$$

Let $M_k = \mathrm{Im}(\delta_k)$, $1 \leqslant k \leqslant N$. The above sequence splits into

$$\begin{gathered} 0 \to (\Omega_X/\tau_X) \otimes \omega_X \xrightarrow{\delta_1} (\pi_0)_*(\Omega_{\tilde{X}} \otimes \pi_0^* \omega_X) \xrightarrow{\delta_2} M_1 \to 0, \\ 0 \to M_k \to (\pi_k)_*(\Omega_{\tilde{X}_{[k]}} \otimes L_k \otimes \pi_k^* \omega_X) \to M_{k+1} \to 0, \end{gathered}$$

where $1 \leqslant k \leqslant N-1$, $N \leqslant n = \dim X$ and $M_N = (\pi_N)_*(\Omega_{\tilde{X}_{[N]}} \otimes L_N \otimes \pi_N^* \omega_X)$. Therefore, we get exact sequences in cohomology:

$$\begin{aligned} \cdots H^{n-3}(M_1) \to H^{n-2}((\Omega_X/\tau_X) \otimes \omega_X) \to H^{n-2}((\pi_0)_*(\Omega_{\tilde{X}} \otimes \pi_0^* \omega_X)) \to \cdots \\ \cdots H^{n-k-3}(M_{k+1}) \to H^{n-k-2}(M_k) \to H^{n-k-2}((\pi_k)_*(\Omega_{\tilde{X}_{[k]}} \otimes L_k \otimes \pi_k^* \omega_X)) \to \cdots. \end{aligned} \tag{4.16}$$

Now, since the π_k are finite, it follows that $(\pi_k^* \omega_X)^{-1}$ are ample for all $0 \leqslant k \leqslant N$, and hence $(L_k^{-1} \otimes \pi_k^* \omega_X)^{-1}$ is ample too, since L_k is 2-torsion and invertible. Moreover, $\tilde{X}_{[k]}$ is smooth of dimension $n-k$. Therefore, and by using the Kodaira–Nakano vanishing theorem [**3**, Corollary 6.4],

$$H^{n-k-2}((\pi_k)_*(\Omega_{\tilde{X}_{[k]}} \otimes L_k \otimes \pi_k^* \omega_X)) = 0$$

for all $1 \leqslant k \leqslant N$. Hence, from (4.16) and by induction, it follows that $H^{n-k-2}(M_k) = 0$ for all $0 \leqslant k \leqslant N$, and hence, again by (4.16), it follows that there is an exact sequence

$$H^{n-3}(M_1) \to H^{n-2}((\Omega_X/\tau_X) \otimes \omega_X) \to H^{n-2}((\pi_0)_*(\Omega_{\tilde{X}} \otimes \pi_0^*\omega_X)),$$

and therefore

$$H^2(T_X) = H^{n-2}((\Omega_X/\tau_X) \otimes \omega_X) = 0$$

as claimed. □

Unfortunately, in general the author cannot say much about the other obstruction space, namely $H^1(T_X^1)$. However, since T_X^1 is a line bundle on the singular locus $X_{[1]}$ of X, it is much more easily handled than $H^2(T_X)$ and it will vanish if we impose certain positivity requirements on T_X^1.

The case in which X has only double points exhibits much better behaviour and it deserves special consideration. The difference between this and the general case is that the singular locus $X_{[1]}$ of X is smooth.

Theorem 4.11. *Let X be a Fano variety with only double point normal crossing singularities such that T_X^1 is finitely generated by its global sections. Then,*

$$H^2(T_X) = H^1(T_X^1) = 0.$$

Corollary 4.12. *Let X be a Fano variety with only double point normal crossing singularities and such that T_X^1 is finitely generated by its global sections. Then* $\mathrm{Def}(X)$ *is smooth.*

Question 4.13. Is $\mathrm{Def}(X)$ smooth for any Fano variety with normal crossing singularities? If this is true, then X is smoothable if and only if T_X^1 is finitely generated by global sections, and hence this is a very natural condition to impose.

Remark 4.14. In general, $H^1(T_X^1)$ will not vanish. However, if X is smoothable, then T_X^1 must have some positivity properties and the one stated is the most natural one.

Proof of Theorem 4.11. In view of Theorem 4.8 we only need to show the vanishing of $H^1(T_X^1)$. In order to show this we will first show that the singular locus $Z = X_{[1]}$ of X is a smooth Fano variety of dimension $\dim X - 1$. The Fano part is the only part to be shown. Let $\pi\colon \tilde{X} \to X$ be the normalization and let $\tilde{Z} = \pi^{-1}Z$. Then $\tilde{Z} \to Z$ is étale. By subadjunction we obtain that

$$\pi^*\omega_X = \omega_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(\tilde{Z}).$$

Therefore,

$$\omega_{\tilde{Z}} = \omega_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(\tilde{Z}) \otimes \mathcal{O}_{\tilde{Z}}.$$

Hence, $\omega_{\tilde{Z}}^{-1}$ is ample. But since $\tilde{Z} \to Z$ is étale, it follows that $\pi^*\omega_Z = \omega_{\tilde{Z}}$. Therefore, ω_Z^{-1} is ample too, and hence Z is Fano as claimed. Now,

$$H^1(T_X^1) = H^1(\omega_Z \otimes (T_X^1 \otimes \omega_Z^{-1})) = 0$$

by the Kawamata–Viehweg vanishing theorem since if T_X^1 is finitely generated by global sections, then $T_X^1 \otimes \omega_Z^{-1}$ is ample, too. □

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5. Smoothings of Fanos

In this section we obtain criteria for a Fano variety X with normal crossing singularities to be smoothable. First we state a criterion for a variety X with hypersurface singularities to be smoothable and, moreover, to be smoothable with a smooth total space.

Proposition 5.1. *Let X be a reduced projective scheme with hypersurface singularities and let D be its singular locus. Then the following hold.*

(1) *If X is smoothable by a flat deformation $\mathcal{X} \to \Delta$ such that $\mathcal{X}$ is smooth, then $T^1_X = \mathcal{O}_D$.*

(2) *Suppose that T^1_X is finitely generated by its global sections and that $H^2(T_X) = H^1(T^1_X) = 0$. Then X is smoothable. Moreover, if $\mathrm{Def}(X)$ is smooth, then the converse is also true.*

Proof. The proof of the second part is given in [**15**, Theorem 12.5]. We proceed to show the first part. Let $f\colon \mathcal{X} \to \Delta$ be a smoothing of X such that $\mathcal{X}$ is smooth, where $\Delta = \mathrm{Spec}(R)$ and (R, m_R) is a discrete valuation ring. Let $T^1_{\mathcal{X}/\Delta} = \mathcal{E}\mathrm{xt}^1_{\mathcal{X}}(\Omega_{\mathcal{X}/\Delta}, \mathcal{O}_{\mathcal{X}})$ be Schlessinger's relative T^1 sheaf. Then, dualizing the exact sequence

$$0 \to f^*\omega_\Delta = \mathcal{O}_{\mathcal{X}} \to \Omega_{\mathcal{X}} \to \Omega_{\mathcal{X}/\Delta} \to 0,$$

we get the exact sequence

$$\cdots \to \mathcal{O}_{\mathcal{X}} \to T^1_{\mathcal{X}/\Delta} \to \mathcal{E}\mathrm{xt}^1_{\mathcal{X}}(\Omega_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}}) = 0.$$

Now, restricting to the special fibre and taking into consideration that $\mathcal{X} \otimes_R R/m_r \cong X$ and that $T^1_{\mathcal{X}/\Delta} \otimes_R R/m_R = T^1_X$ [**15**, Lemma 7.7], we get that there is a surjection $\mathcal{O}_X \to T^1_X$. Moreover, T^1_X is a line bundle on the singular locus D of X. Hence, restricting on Z, it follows that $T^1_X \cong \mathcal{O}_D$, as claimed. □

Remark 5.2. The condition $T^1_X = \mathcal{O}_D$ is equivalent to Friedman's d-semi-stability condition in the case of reducible simple normal crossing schemes [**4**]. One of the natural questions raised by Friedman is whether this condition is sufficient for a simple normal crossing variety to be smoothable. He showed that in the case of $K3$ surfaces this is true but Persson and Pinkham have shown that this is not true in general [**11**]. However, this is true in the case of normal crossing (not necessarily reducible) Fano schemes, as shown by Theorem 5.3.

Theorem 5.3. *Let X be a Fano variety defined over an algebraically closed field of characteristic zero with normal crossing singularities. Assume that one of the following conditions holds.*

(1) *T^1_X is finitely generated by global sections and $H^1(T^1_X) = 0$.*

(2) *X has at worst double point normal crossing singularities and T^1_X is finitely generated by global sections.*

(3) *X is d-semi-stable, i.e. $T^1_X \cong \mathcal{O}_D$, where D is the singular locus of X.*

Then X is smoothable. Moreover, X is smoothable by a flat deformation $f\colon \mathcal{X} \to \Delta$ such that $\mathcal{X}$ is smooth if and only if X is d-semi-stable.

Proof. Theorem 5.3 (1) follows directly from Theorem 4.8 and Proposition 5.1 (2). Theorem 5.3 (2) follows from Theorem 4.11 and Proposition 5.1 (2). Finally, suppose that $T^1_X \cong \mathcal{O}_D$, i.e. X is d-semi-stable. Then, according to Proposition 3.9, X admits a semi-stable logarithmic structure $\mathcal{M}$. Moreover, by Theorem 4.8 and Corollary 4.10, $(X, \mathcal{M})$ has unobstructed logarithmic deformations. Let s be a nowhere vanishing section of T^1_X as a sheaf on D. Then, locally in the étale topology,

$$X = \operatorname{Spec} \frac{k[x_0, \ldots, x_d]}{(x_0 \cdots x_r)}$$

and s corresponds to the first-order deformation of X,

$$X_1 = \operatorname{Spec} \frac{A_1[x_0, \ldots, x_d]}{(x_0 \cdots x_r - t)} \to \operatorname{Spec} A_1,$$

where $A_1 = k[t]/(t^2)$. This deformation is also a log deformation for the semi-stable log structure of X. This is evident from the diagram

$$\begin{array}{ccc} \mathbb{N}^d & \xrightarrow{\ \alpha\ } & \dfrac{A_1[x_0, \ldots, x_d]}{(x_0 \cdots x_r - t)} \\ {\scriptstyle \Delta}\uparrow & & \uparrow \\ \mathbb{N} & \xrightarrow{\ \beta\ } & A_1 \end{array}$$

where Δ is the diagonal, α the semi-stable log structure of X and $\beta(n) = t^n$. Since X has unobstructed log deformations, this first order deformation lifts to a formal log deformation of X over $k[\![t]\!]$. By Corollary 4.9, any formal deformations are effective. Let then $f\colon (\mathcal{X}, \mathcal{N}) \to (\Delta, \mathbb{N})$, where $\Delta = \operatorname{Spec} k[\![t]\!]$ is the lifting. Then, as in the proof of Proposition 5.1, there is an exact sequence

$$\cdots \to \mathcal{O}_{\mathcal{X}} \xrightarrow{\sigma} T^1_{\mathcal{X}/\Delta} \to \mathcal{E}\mathrm{xt}^1_{\mathcal{X}}(\Omega_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}}) \to 0.$$

Now, by the assumption that $T^1_X = \mathcal{O}_D$ and the non-triviality of the deformation, it follows that σ is surjective, and hence

$$\mathcal{E}\mathrm{xt}^1_{\mathcal{X}}(\Omega_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}}) = 0.$$

Therefore, since $\mathcal{X}$ has complete intersection singularities, $\mathcal{X}$ is smooth. □

6. Examples

In this section we construct one example of a smoothable and one of a non-smoothable normal crossing Fano threefold.

Example 6.1. Let $P \in Y \subset \mathbb{P}^4$ be a quadric surface with one ordinary double point locally analytically isomorphic to $(xy - zw = 0) \subset \mathbb{C}^4$. Let $f\colon X \to Y$ be the blow up of $P \in Y$. Then X is smooth and the f-exceptional divisor E is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Moreover, $-K_X - E$ is ample and $\mathcal{N}_{E/X} = \mathcal{O}_E(-1, -1)$.

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Next we construct an embedding $E \subset X'$ of E into a smooth scheme X' such that $\mathcal{N}_{E/X'} = \mathcal{O}_E(1,1)$ and $-K_{X'} - E$ is ample. Let $Z \subset \mathbb{P}^3$ be a smooth quadric surface. Then, $\mathcal{N}_{Z/\mathbb{P}^3} = \mathcal{O}_Z(2,2)$. Let $\pi\colon X' \to \mathbb{P}^3$ be the cyclic double cover of $\mathbb{P}^3$ ramified over Z. This is defined by the line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{P}^3}(1)$ and the section s of $\mathcal{L}^{\otimes 2}$ that corresponds to Z. Let $E = (\pi^{-1}(Z))_{\text{red}} \cong Z$. Then $\pi^*Z = 2E$ and $\omega_X = \pi^*(\omega_{\mathbb{P}^3} \otimes \mathcal{L})$. Let $l' \subset E$ be one of the rulings and $l = \pi_*(l')$. Then,

$$l' \cdot E = \tfrac{1}{2}(l' \cdot \pi^* Z) = \tfrac{1}{2}(l \cdot Z) = 1,$$

and hence $\mathcal{N}_{E/\mathcal{X}'} = \mathcal{O}_E(1,1)$. Now let Y be the scheme obtained by gluing X and X' along E. This is a normal crossing Fano threefold with only double points. Then, $T^1_Y = \mathcal{N}_{E/X} \otimes \mathcal{N}_{E/X'} = \mathcal{O}_E$. Therefore, by Theorem 5.3, Y is smoothable.

Example 6.2. Let $E \subset X$ be as in Example 1. Then let Y be obtained by gluing two copies of X along E. Then,

$$T^1_Y = \mathcal{N}_{E/X} \otimes \mathcal{N}_{E/X} = \mathcal{O}_E(-2,-2)$$

and hence $H^0(T^1(Y)) = 0$. Hence, Y is not smoothable [**14**, Theorem 12.3]. In fact, every deformation of Y is locally trivial.

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