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Inclusion Relations for New Function Spaces on Riemann Surfaces

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Abstract. We introduce and study some new function spaces on Riemann surfaces. For certain parameter values these spaces coincide with the classical Dirichlet space, BMOA, or the recently defined Q_p space. We establish inclusion relations that generalize earlier known inclusions between the above-mentioned spaces.

1 Introduction

Let R be an open Riemann surface that possesses a Green's function, *i.e.*, $R \notin O_G$, and let $g_R(z,\alpha)$ denote the Green function on R with logarithmic singularity at $\alpha \in R$. Let $A(R)$ denote the collection of all analytic functions on R. The classical Dirichlet space $AD(R)$ consists of those $F \in A(R)$ for which

$$\int_R |F'(z)|^2 \, dA(z) < \infty,$$

where $dA(z)$ is the element of the Lebesgue area measure on R. Following [7], we define $\mathrm{BMOA}(R)$ as the set of $F \in A(R)$ such that

$$\sup_{\alpha\in R} \int_R |F'(z)|^2 g_R(z,\alpha)\, dA(z) < \infty.$$

For $0 < p < \infty$, the space $Q_p(R)$, introduced in [2], consists of those $F \in A(R)$ for which

$$\sup_{\alpha\in R} \int_R |F'(z)|^2 g_R^p(z,\alpha)\, dA(z) < \infty.$$

Metzger [7] (see also [5]) showed that $\mathrm{BMOA}(R)$ contains $AD(R)$ analogously to the case of the unit disc. This result was sharpened in [2] by proving that $AD(R) \subset Q_p(R)$ for all $p > 0$; see also [1]. Notice that $Q_1(R) = \mathrm{BMOA}(R)$.

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We will generalize the above-mentioned definitions of function spaces in the following way. For $0 < p, q < \infty$, define

$$AD^q(R) = \Big\{ F \in A(R) : \sup_{\alpha \in R} \int_R |F(z) - F(\alpha)|^{q-2} |F'(z)|^2 dA(z) < \infty \Big\},$$

$$H^q_{\mathrm{BMOA}}(R) = \Big\{ F \in A(R) : \sup_{\alpha \in R} \int_R |F(z) - F(\alpha)|^{q-2} |F'(z)|^2 g_R(z,\alpha) dA(z) < \infty \Big\},$$

$$H^q_{Q_p}(R) = \Big\{ F \in A(R) : \sup_{\alpha \in R} \int_R |F(z) - F(\alpha)|^{q-2} |F'(z)|^2 g^p_R(z,\alpha) dA(z) < \infty \Big\}.$$

Then $AD^2(R) = AD(R)$, $H^q_{\mathrm{BMOA}}(R) = \mathrm{BMOA}(R)$ by [12] (see also [10]), and $H^2_{Q_p}(R) = Q_p(R)$ for all $0 < p < \infty$.

2 $AD^q(R) \subset \mathrm{BMOA}(R)$ for all $0 < q < \infty$

For $F \in A(R)$, $0 < q < \infty$ and $\alpha \in R$, let $H_{|F-F(\alpha)|^q}$ denote the least harmonic majorant of the subharmonic function $u(z) = |F(z) - F(\alpha)|^q$. We set $H_{|F-F(\alpha)|^q}(z) = \infty$ if u admits no harmonic majorant. The following result follows by [12, Corollary 2.6]; see also [10, Proposition 1].

Lemma A *Let $F \in A(R)$, $0 < q < \infty$ and $\alpha \in R$. Then*

$$H_{|F-F(\alpha)|^q}(\alpha) = \frac{q^2}{2\pi} \int_R |F(z) - F(\alpha)|^{q-2} |F'(z)|^2 g_R(z,\alpha) dA(z).$$

An application of [6, Corollary 1] gives

$$\frac{1}{\pi} \int_R |F(z) - F(\alpha)|^{q-2} |F'(z)|^2 dA(z) \geq \frac{2}{q} H_{|F-F(\alpha)|^q}(\alpha), \tag{2.1}$$

from which Lemma A yields

$$AD^q(R) \subset H^q_{\mathrm{BMOA}}(R) = \mathrm{BMOA}(R)$$

for all $0 < q < \infty$.

3 $H^q_{Q_{p_1}}(R) \subset H^q_{Q_{p_2}}(R)$ for all $0 < p_1 < p_2 < \infty$

To prove this inclusion the following lemma is needed.

Lemma 3.1 *Let R be an open Riemann surface that possesses a Green's function,* i.e., *$R \notin O_G$. Let $F \in A(R)$, and let $\alpha \in R$, $0 < p_1 < p_2 < \infty$ and $0 < q < \infty$. Then*

$$\int_R |F(z) - F(\alpha)|^{q-2} |F'(z)|^2 g^{p_2}_R(z,\alpha) dA(z) \leq$$

$$C \int_R |F(z) - F(\alpha)|^{q-2} |F'(z)|^2 g^{p_1}_R(z,\alpha) dA(z),$$

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where

$$C=\begin{cases} p_2(p_2-1)e^q q^{1-p_2}\Gamma(p_2-1)+p_2+1, & \text{if } 1\le p_1<p_2<\infty,\\ \big(p_1((p_1-1)e^q q^{1-p_1}\Gamma(p_1-1,q)+1)\big)^{-1}, & \text{if } 0<p_1<p_2\le 1.\end{cases}$$

Proof By considering a regular exhaustion of R, it is sufficient to prove the assertion in the case where R is the interior of a compact bordered Riemann surface $\overline{R}$ and F is analytic on $\overline{R}$.

Let $\alpha\in R$ and $R_{1,\alpha}=\{z\in R: g_R(z,\alpha)>1\}$. Then clearly

$$\int_{R\setminus R_{1,\alpha}}|F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_R^{p_2}(z,\alpha)\,dA(z)\le \int_{R\setminus R_{1,\alpha}}|F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_R^{p_1}(z,\alpha)\,dA(z). \tag{3.1}$$

Let α,α_j, $j=1,\dots,m$, and β_k, $k=1,\dots,n$, be the distinct zeros of $F(z)-F(\alpha)$ in $R_{1,\alpha}$ and on $\partial R_{1,\alpha}$, respectively. For α,α_j,β_k, $j=1,\dots,m$ and $k=1,\dots,n$, we take the parameter discs $U(\alpha,\varepsilon)$ and $U(\alpha_j,\varepsilon)$ and the half discs $B(\beta_k,\varepsilon)$ such that they are mutually disjoint. Denote

$$R_{1,\alpha,\{\alpha_j\},\{\beta_k\}}=R_{1,\alpha}\setminus\left\{U(\alpha,\varepsilon)\bigcup\cup_{j=1}^m U(\alpha_j,\varepsilon)\bigcup\cup_{k=1}^n B(\beta_k,\varepsilon)\right\}.$$

Green's formula yields

$$\int_{R_{1,\alpha,\{\alpha_j\},\{\beta_k\}}}\left(g_R^{p_2}(z,\alpha)\triangle|F(z)-F(\alpha)|^q-|F(z)-F(\alpha)|^q\triangle g_R^{p_2}(z,\alpha)\right)dA(z)= \int_{\partial R_{1,\alpha,\{\alpha_j\},\{\beta_k\}}}\left(|F(z)-F(\alpha)|^q\frac{\partial g_R^{p_2}(z,\alpha)}{\partial n}-g_R^{p_2}(z,\alpha)\frac{\partial|F(z)-F(\alpha)|^q}{\partial n}\right)ds, \tag{3.2}$$

where $\triangle$ denotes the Laplacian, $\frac{\partial}{\partial n}$ denotes the differentiation in the inward normal direction, and ds is the arc length element on $\partial R_{1,\alpha,\{\alpha_j\},\{\beta_k\}}$. Lengthy but routine calculations show that

$$\triangle|F(z)-F(\alpha)|^q=q^2|F(z)-F(\alpha)|^{q-2}|F'(z)|^2$$

and

$$\triangle g_R^{p_2}(z,\alpha)=p_2(p_2-1)g_R^{p_2-2}(z,\alpha)|P'_\alpha(z)|^2,$$

where

$$P_\alpha(z)=g_R(z,\alpha)+ig_R^*(z,\alpha)$$

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and $g_R^*(z,\alpha)$ is a harmonic conjugate of $g_R(z,\alpha)$. It is known that $g_R^*(z,\alpha)$ is locally defined up to an additive constant, and

$$\frac{\partial g_R^{p_2}(z,\alpha)}{\partial n} = p_2 \frac{\partial g_R(z,\alpha)}{\partial n}$$

for $z \in \partial R_{1,\alpha}$.

Let $H^1_{|F-F(\alpha)|^q}$ denote the least harmonic majorant of $|F(z)-F(\alpha)|^q$ on $R_{1,\alpha}$. It turns out that the function

$$\Phi_{1,\alpha}(z) := |(F(z)-F(\alpha))e^{P_\alpha(z)}|^q = |F(z)-F(\alpha)|^q e^{qg_R(z,\alpha)}$$

is subharmonic on $R_{1,\alpha}$ and

$$\Phi_{1,\alpha}(z) = e^q|F(z)-F(\alpha)|^q$$

for all $z \in \partial R_{1,\alpha}$. The maximum principle yields

$$|F(z)-F(\alpha)|^q \le e^q H^1_{|F-F(\alpha)|^q}(z) e^{-qg_R(z,\alpha)} \tag{3.3}$$

for all $z \in R_{1,\alpha}$.

Let $g_{R_{1,\alpha}}(z,\alpha)$ be the Green function of $R_{1,\alpha}$ with logarithmic singularity at α. Then $\triangle g_{R_{1,\alpha}}(z,\alpha) = 0$ in $R_{1,\alpha,\{\alpha_j\},\{\beta_k\}}$ and $g_{R_{1,\alpha}}(z,\alpha) = 0$ for $z \in \partial R_{1,\alpha}$. By [12, 13], we have

$$\begin{aligned} H^1_{|F-F(\alpha)|^q}(\alpha) &= \frac{1}{2\pi}\int_{\partial R_{1,\alpha}} |F(z)-F(\alpha)|^q \frac{\partial g_{R_{1,\alpha}}(z,\alpha)}{\partial n}\, ds \\ &= \frac{q^2}{2\pi}\int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_{R_{1,\alpha}}(z,\alpha)\, dA(z). \end{aligned} \tag{3.4}$$

To deal with the area integral in (3.4), denote $S_{t,\alpha} = \{z \in R : g_R(z,\alpha) = t\}$ for $t > 0$. If $z \in S_{t,\alpha}$, then $dt = \frac{\partial g_R(z,\alpha)}{\partial n} dn$. Letting $\varepsilon \to 0$ in (3.2) we see that all the integrals

$$\int_{\partial U(\alpha,\varepsilon)} |F(z)-F(\alpha)|^q \frac{\partial g_R^{p_2}(z,\alpha)}{\partial n}\, ds, \qquad \int_{\partial U(\alpha_j,\varepsilon)} |F(z)-F(\alpha)|^q \frac{\partial g_R^{p_2}(z,\alpha)}{\partial n}\, ds,$$

$$\int_{\partial B(\beta_k,\varepsilon)} |F(z)-F(\alpha)|^q \frac{\partial g_R^{p_2}(z,\alpha)}{\partial n}\, ds, \qquad \int_{\partial U(\alpha,\varepsilon)} g_R^{p_2}(z,\alpha) \frac{\partial |F(z)-F(\alpha)|^q}{\partial n}\, ds,$$

$$\int_{\partial U(\alpha_j,\varepsilon)} g_R^{p_2}(z,\alpha) \frac{\partial |F(z)-F(\alpha)|^q}{\partial n}\, ds, \qquad \int_{\partial B(\beta_k,\varepsilon)} g_R^{p_2}(z,\alpha) \frac{\partial |F(z)-F(\alpha)|^q}{\partial n}\, ds$$

tend to zero for all $j = 1,\dots,m$ and $k = 1,\dots,n$. Therefore the equality (3.2) becomes

(3.5)

$$
\begin{aligned}
I_{1,p_2,q}(\alpha) &= q^2 \int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_R^{p_2}(z,\alpha)\,dA(z)\\
&= p_2(p_2-1)\int_{R_{1,\alpha}} |F(z)-F(\alpha)|^q g_R^{p_2-2}(z,\alpha)|P'_\alpha(z)|^2\,dA(z)\\
&\qquad + p_2\int_{\partial R_{1,\alpha}} |F(z)-F(\alpha)|^q \frac{\partial g_R(z,\alpha)}{\partial n}\,ds - \int_{\partial R_{1,\alpha}} \frac{\partial |F(z)-F(\alpha)|^q}{\partial n}\,ds\\
&= p_2(p_2-1)\int_{R_{1,\alpha}} |F(z)-F(\alpha)|^q g_R^{p_2-2}(z,\alpha)|P'_\alpha(z)|^2\,dA(z)\\
&\quad + p_2\int_{\partial R_{1,\alpha}} |F(z)-F(\alpha)|^q \frac{\partial g_R(z,\alpha)}{\partial n}\,ds\\
&\quad + q^2\int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2\,dA(z),
\end{aligned}
$$

where, by Green's formula,

$$
q^2\int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2\,dA(z) = -\int_{\partial R_{1,\alpha}} \frac{\partial |F(z)-F(\alpha)|^q}{\partial n}\,ds.
$$

We first concentrate on the case $1 \le p_1 < p_2 < \infty$. By the formulae (3.3), (3.5), and (2.1), and by using the inequality $g_{R_{1,\alpha}}(z,\alpha) \le g_R(z,\alpha)$, $z \in R_{1,\alpha}$, we obtain

(3.6)

$$
\begin{aligned}
I_{1,p_2,q}(\alpha) &\le p_2(p_2-1)e^q \int_{R_{1,\alpha}} H^1_{|F-F(\alpha)|^q}(z) g_R^{p_2-2}(z,\alpha)|P'_\alpha(z)|^2 e^{-q g_R(z,\alpha)}\,dA(z)\\
&\qquad + 2\pi p_2 H^1_{|F-F(\alpha)|^q}(\alpha) + q^2\int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2\,dA(z)\\
&\le p_2(p_2-1)e^q \int_1^\infty \Big(\int_{S_{t,\alpha}} H^1_{|F-F(\alpha)|^q}(z)\frac{\partial g_R(z,\alpha)}{\partial n}\,ds\Big) g_R^{p_2-2}(z,\alpha)e^{-q g_R(z,\alpha)}\,dt\\
&\qquad + p_2 q^2 \int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_{R_{1,\alpha}}(z,\alpha)\,dA(z)\\
&\qquad + q^2 \int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_R(z,\alpha)\,dA(z)\\
&\le 2\pi p_2(p_2-1)e^q H^1_{|F-F(\alpha)|^q}(\alpha)\int_1^\infty t^{p_2-2}e^{-qt}\,dt\\
&\qquad + p_2 q^2 \int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_R(z,\alpha)\,dA(z)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + q^2 \int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_R(z,\alpha)\,dA(z)\\
&\le p_2(p_2-1)q^2e^q \int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_{R_{1,\alpha}}(z,\alpha)\,dA(z)\\
&\quad \cdot \frac{1}{q^{p_2-1}} \int_q^\infty u^{p_2-2}e^{-u}\,du\\
&\quad + q^2(p_2+1) \int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_R^{p_1}(z,\alpha)\,dA(z)\\
&\le q^2\big(p_2(p_2-1)e^q q^{1-p_2}\Gamma(p_2-1)+p_2+1\big)\\
&\quad \cdot \int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_R^{p_1}(z,\alpha)\,dA(z),
\end{aligned}
$$

where $\Gamma(p_2-1)=\int_0^\infty u^{p_2-2}e^{-u}\,du$ is the gamma function. By combining (3.1) and (3.6) we obtain the desired inequality for $1 \le p_1 < p_2 < \infty$.

Let now $0 < p_1 < p_2 \le 1$. Then the estimate (3.3) gives

(3.7)
$$
\begin{aligned}
I_{1,p_1,q}(\alpha) &\ge p_1(p_1-1)e^q \int_{R_{1,\alpha}} H^1_{|F-F(\alpha)|^q}(z)e^{-qg_R(z,\alpha)} g_R^{p_1-2}(z,\alpha)|P'_\alpha(z)|^2\,dA(z)\\
&\quad + 2\pi p_1 H^1_{|F-F(\alpha)|^q}(\alpha) + q^2\int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2\,dA(z)\\
&= 2\pi p_1(p_1-1)e^q H^1_{|F-F(\alpha)|^q}(\alpha)\int_1^\infty t^{p_1-2}e^{-qt}\,dt\\
&\quad + 2\pi p_1 H^1_{|F-F(\alpha)|^q}(\alpha) + q^2\int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2\,dA(z)\\
&= 2\pi p_1 H^1_{|F-F(\alpha)|^q}(\alpha)\big((p_1-1)e^q q^{1-p_1}\Gamma(p_1-1,q)+1\big)\\
&\quad + q^2\int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2\,dA(z),
\end{aligned}
$$

where $\Gamma(p_1-1,q)=\int_q^\infty u^{p_1-2}e^{-u}\,du$ is the incomplete gamma function. We note that

$$A(p_1,q)=(p_1-1)e^q q^{1-p_1}\Gamma(p_1-1,q)+1>0,$$

and hence by dividing by q^2 in (3.7) we obtain

(3.8)
$$
\begin{aligned}
&\int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_R^{p_1}(z,\alpha)\,dA(z)\\
&\quad \ge p_1A(p_1,q)\int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_{R_{1,\alpha}}(z,\alpha)\,dA(z)\\
&\qquad + \int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2\,dA(z).
\end{aligned}
$$

Since $g_{R_{1,\alpha}}(z,\alpha) = g_R(z,\alpha) - 1$ for $z \in R_{1,\alpha}$, (3.8) yields

$$\begin{aligned}(3.9)\qquad &\int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_R^{p_1}(z,\alpha)\,dA(z)\\ &\geq p_1A(p_1,q)\int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_R(z,\alpha)\,dA(z)\\ &\quad + \big(1-p_1A(p_1,q)\big)\int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2\,dA(z)\\ &\geq p_1A(p_1,q)\int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_R^{p_2}(z,\alpha)\,dA(z).\end{aligned}$$

The last inequality follows from the fact that $1 - p_1A(p_1,q) > 0$. The desired inequality for $0 < p_1 < p_2 \leq 1$ follows by combining (3.1) and (3.9). ∎

Theorem 3.2 *Let R be a Riemann surface such that $R \notin Q_G$, and let $0 < p_1 < p_2 < \infty$ and $0 < q < \infty$. Then the following inclusion holds:*

$$H^q_{Q_{p_1}}(R) \subset H^q_{Q_{p_2}}(R).$$

Proof If either $0 < p_1 < p_2 \leq 1$ or $1 \leq p_1 < p_2 < \infty$, then the assertion follows directly from Lemma 3.1. If $0 < p_1 \leq 1 < p_2 < \infty$, then Lemma 3.1 gives

$$H^q_{Q_{p_1}}(R) \subset H^q_{\mathrm{BMOA}}(R) \subset H^q_{Q_{p_2}}(R)$$

for all $0 < q < \infty$. ∎

4 $AD^q(R) \subset H^q_{Q_p}(R)$ for all $0 < p, q < \infty$

In Section 2, we noted that the inclusion $AD^q(R) \subset H^q_{\mathrm{BMOA}}(R) = \mathrm{BMOA}(R)$ holds for all $0 < q < \infty$. This fact is sharpened in this section by showing the following result.

Theorem 4.1 *$AD^q(R) \subset H^q_{Q_p}(R)$ for all $0 < p, q < \infty$.*

Proof Theorem 3.2 implies that $\mathrm{BMOA}(R) \subset H^q_{Q_p}(R)$ for all $1 \leq p < \infty$ and $0 < q < \infty$. Combining this with the inclusion $AD^q(R) \subset \mathrm{BMOA}(R)$, $0 < q < \infty$, we deduce

$$(4.1)\qquad AD^q(R) \subset H^q_{Q_p}(R)$$

for all $1 \leq p < \infty$ and $0 < q < \infty$.

Now let $0 < p < 1$. Recall that $R_{1,\alpha} = \{z \in R : g_R(z,\alpha) > 1\}$. By (3.5),

$$\begin{aligned}(4.2)\qquad q^2\int_{R_{1,\alpha}} &|F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_R^{p}(z,\alpha)\,dA(z) \leq\\ &2\pi p H^1_{|F-F(\alpha)|^q}(\alpha) + q^2\int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2\,dA(z),\end{aligned}$$

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because $p-1<0$. Suppose now that $F \in AD^q(R)$. Then there exists $M_1>0$ such that

$$\int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2\,dA(z) \le \int_R |F(z)-F(\alpha)|^{q-2}|F'(z)|^2\,dA(z) \le M_1 < \infty$$

for all $\alpha \in R$. By Section 2 we know that $F \in \mathrm{BMOA}(R)$. Hence, by Lemma A, there exists $M_2>0$ such that

$$H^1_{|F-F(\alpha)|^q}(\alpha) \le H_{|F-F(\alpha)|^q}(\alpha) \le M_2 < \infty \tag{4.3}$$

for all $\alpha \in R$. By (4.2) and (4.3), we deduce

$$\begin{aligned}\int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_R^p(z,\alpha)\,dA(z) &\le \frac{1}{q^2}\big(2\pi p M_2 + q^2 M_1\big)\\ &= M_1 + \frac{2\pi p}{q^2}M_2\end{aligned} \tag{4.4}$$

for all $\alpha \in R$. On the other hand, we immediately see that

$$\begin{aligned}&\int_{R\setminus R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_R^p(z,\alpha)\,dA(z)\\ &\quad\le \int_{R\setminus R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2\,dA(z)\\ &\quad\le \int_R |F(z)-F(\alpha)|^{q-2}|F'(z)|^2\,dA(z)\\ &\quad\le M_1\end{aligned} \tag{4.5}$$

for all $\alpha \in R$. Combining (4.4) and (4.5) we obtain

$$\sup_{\alpha\in R}\int_R |F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_R^p(z,\alpha)\,dA(z) \le 2M_1 + \frac{2\pi p}{q^2}M_2$$

for all $0<p<1$ and $0<q<\infty$. Thus $F \in H^q_{Q_p}(R)$ for all $0<p<1$ and $0<q<\infty$. This together with (4.1) completes the proof. ■

5 $H^q_{Q_p}(R) \subset \mathcal{B}(R)$ for all $0<p,q<\infty$

Let $\lambda_R(\alpha)$ be the density of the hyperbolic distance (Poincaré metric) on a hyperbolic Riemann surface R. The Bloch space is defined as

$$\mathcal{B}(R) := \Big\{F \in A(R) : \sup_{\alpha\in R}\frac{|F'(\alpha)|}{\lambda_R(\alpha)} < \infty\Big\}.$$

The purpose of this section is to show the maximal property of $\mathcal{B}(R)$ with respect to the spaces $H^q_{Q_p}(R)$. In the case of the unit disc, an analogous result follows by a work of Rubel and Timoney [9].

Theorem 5.1 *$H^q_{Q_p}(R) \subset \mathcal{B}(R)$ for all $0 < p, q < \infty$.*

Proof Let $\pi\colon \mathbb{D} \to R$ be a universal covering map of the unit disc $\mathbb{D}$ to the Riemann surface R. Let Ω denote the fundamental polygon of the Fuchsian group Γ. If $\alpha \in R$ and $a \in \Omega$ satisfy $\pi(a) = \alpha$, then we may take the Green function of the Riemann surface $\mathbb{D}/\Gamma$ by setting $g_\Gamma(z, a) = g_R(\pi(z), \alpha)$. By a result of Myrberg [11, p. 522], we know that

$$g_\Gamma(z, a) = \sum_{\gamma\in\Gamma} g_{\mathbb{D}}\big(z, \gamma(a)\big),$$

where $g_{\mathbb{D}}(z, a)$ is the Green function of $\mathbb{D}$ with logarithmic singularity at a. Therefore we may define the space $H^q_{Q_p}(\mathbb{D}/\Gamma) = H^q_{Q_p}(R)$ in the sense that $f \in H^q_{Q_p}(\mathbb{D}/\Gamma)$ if f is analytic in $\mathbb{D}$ and $f = F \circ \pi$, where $F \in H^q_{Q_p}(R)$. With a similar understanding, $\mathcal{B}(\mathbb{D}/\Gamma) = \mathcal{B}(R)$.

First let $1 \le p < \infty$. Suppose now that $f \in H^q_{Q_p}(\mathbb{D}/\Gamma)$, but $f \notin \mathcal{B}(\mathbb{D}/\Gamma)$. Then [3, Lemma] or [8] implies that there exist a sequence of points $\{a_n\}$ in $\mathbb{D}$ and a sequence of positive numbers $\{\rho_n\}$ such that $\rho_n/(1 - |a_n|) \to 0$, as $n \to \infty$, and $\{f(a_n+\rho_n\xi)-f(a_n)\}$ converges uniformly on compact subsets of $\mathbb{C}$ to a non-constant analytic function $f_0(\xi)$. Here, without loss of generality, we may assume that $a_n \in \Omega$ for each $n \in \mathbb{N}$. Note that in general this is not possible, but the reasoning in (5.1) below shows that we may do so. Now, for $\delta > 0$, set $K = K(\delta) = \{\xi \in \mathbb{C} : |\xi| \le \delta\}$. Denote $\varphi_n(\xi) = a_n + \rho_n\xi$ and $g_n(\xi) = f(\varphi_n(\xi)) - f(\varphi_n(0)) = f(a_n + \rho_n\xi) - f(a_n)$. Then

$$|g_n(\xi)|^{q-2} \to |f_0(\xi)|^{q-2} \ge \delta_1 > 0 \quad \text{and} \quad |g_n'(\xi)|^2 \to |f_0'(\xi)|^2 \ge \delta_2 > 0$$

uniformly in

$$K_1 = K \setminus \big(\cup_{j=1}^n D(\xi_j, \varepsilon) \cup \cup_{i=1}^m D(\eta_i, \varepsilon)\big),$$

where $D(\xi_j, \varepsilon) = \{\xi : |\xi - \xi_j| < \varepsilon\} \subset K$ and $D(\eta_i, \varepsilon) = \{\xi : |\xi - \eta_i| < \varepsilon\}$, $\eta_i \in \partial K$, for all $j = 1, \dots, n$ and $i = 1, \dots, m$. Here, for $0 < q < \infty$, the points ξ_j, $j = 1, \dots, n$, are the zeros and poles of f_0 in $K = \{\zeta \in \mathbb{C} : |\zeta| < \delta\}$, and the points η_i, $i = 1, \dots, m$, are the zeros and poles of f_0 in ∂K. We take $\varepsilon > 0$ so small that all the discs $D(\xi_j, \varepsilon)$ and $D(\eta_i, \varepsilon)$ are pairwise disjoint. Now

$$\begin{aligned}\log\left|\frac{1 - \overline{\varphi_n(0)}\varphi_n(\xi)}{\varphi_n(\xi) - \varphi_n(0)}\right| &= \log\left|\frac{1 - \overline{a_n}(a_n + \rho_n\xi)}{a_n + \rho_n\xi - a_n}\right| \\ &= \log\left|\frac{1 - |a_n|}{\rho_n}\frac{1 + |a_n|}{\xi} - \overline{a_n}\right| \to \infty,\end{aligned}$$

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as $n \to \infty$, for all $\xi \in K_1$. On the other hand, by the assumption,

$$(5.1) \qquad \int_{K_1} |g_n(\xi)|^{q-2}|g_n'(\xi)|^2 g_{\mathbb{D}}^p(\varphi_n(\xi), \varphi_n(0))\, dA(\xi)$$
$$= \int_{\varphi_n(K_1)} |f(z) - f(a_n)|^{q-2}|f'(z)|^2 g_{\mathbb{D}}^p(z, a_n)\, dA(z)$$
$$\leq \int_{\mathbb{D}} |f(z) - f(a_n)|^{q-2}|f'(z)|^2 g_{\mathbb{D}}^p(z, a_n)\, dA(z)$$
$$= \sum_{\gamma\in\Gamma} \int_{\Omega} |f(z) - f(a_n)|^{q-2}|f'(z)|^2 g_{\mathbb{D}}^p(\gamma(z), a_n)\, dA(z)$$
$$= \int_{\Omega} |f(z) - f(a_n)|^{q-2}|f'(z)|^2 \Big(\sum_{\gamma\in\Gamma} g_{\mathbb{D}}^p(\gamma(z), a_n)\Big)\, dA(z)$$
$$\leq \int_{\Omega} |f(z) - f(a_n)|^{q-2}|f'(z)|^2 \Big(\sum_{\gamma\in\Gamma} g_{\mathbb{D}}(\gamma(z), a_n)\Big)^p\, dA(z)$$
$$= \int_{\Omega} |f(z) - f(a_n)|^{q-2}|f'(z)|^2 g_{\Gamma}^p(z, a_n)\, dA(z) \leq C < \infty$$

for all $n \in \mathbb{N}$. But this is a contradiction, since the left-hand side of (5.1) tends to infinity as $n \to \infty$. Thus $H^q_{Q_p}(\mathbb{D}/\Gamma) \subset \mathcal{B}(\mathbb{D}/\Gamma)$ for all $1 \leq p < \infty$ and $0 < q < \infty$. The assertion follows from the nesting property in Theorem 3.2. ■

6 $H^q_{Q_p}(R) \neq \mathcal{B}(R)$

Using the same idea as in the proof of [4, Theorem 4.2] we can prove that there exists a Riemann surface R such that $H^q_{Q_p}(R) \neq \mathcal{B}(R)$. Since the proof is almost identical to the original one, we omit the details.

Theorem 6.1 *For every $0 < p, q < \infty$ there exists a Riemann surface R such that $H^q_{Q_p}(R) \neq \mathcal{B}(R)$.*

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