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EQUILIBRIUM IN n-PERSON GAME OF SHOWCASE SHOWDOWN

VLADIMIR MAZALOV AND ANNA IVASHKO
Institute of Applied Mathematical Research
Karelian Research Center of RAS
Petrozavodsk, Russia
E-mail: vmazalov@krc.karelia.ru, afalko@krc.karelia.ru

In this article we consider a noncooperative n-person optimal stopping game of Showcase Showdown, in which each player observes the sum of independent and identically distributed random variables uniformly distributed in [0, 1]. Players can decide to stop the draw in each moment. The objective of a player is to get the maximal number of scores that does is not exceeded level 1. If the scores of all players exceed 1, then the winner is the player whose score is closest to 1. We derive the equilibrium in this game on the basis of the dynamic programming approach.

1. INTRODUCTION

We consider a noncooperative n-person optimal stopping game related with the popular TV game "The Price is Right". In this game, each of n players in turn spins the wheel once or twice, attaining some total score, and then waits for the results of the succeeding players' spins. The object of the game is to have the highest score, from one or two spins, without going over a given upper limit. This game of chance has been analyzed in Coe and Butterworth [2] and Tijms [7] and recently in Kaynar [3].

Assume that the upper limit is equal to 1 and the players observe the random numbers in [0, 1]. Each player $k \in \{1, 2, \ldots, n\}$ chooses one or two random numbers. After the first step, a player decides to choose the number or to continue the game. She makes the decision without information about the behavior of other players. The objective of a player to get the maximal number of scores that does not exceeded level 1. If the scores of all players exceed 1, then the winner is the player whose score is closest to 1.

In [3], the optimal solution for the game with two possible attempts and two and three players was constructed. The author found the payoff function in this game and after that achieved the solution from the Nash equilibrium conditions. This method needs huge calculations. For the number of players $n \geq 4$, it is difficult to derive the equation for the optimal strategy. There is alternative method [4–6] based on the dynamic programming theory, which is more convenient for analyzing this problem. We use the method to find the equilibrium in the Showcase Showdown game for any number of players and two steps (Sect. 2) and then for the case when the players observe sequentially the sums of independent and identically distributed (i.i.d.) random variables (Sect. 3).

2. SHOWCASE SHOWDOWN GAME WITH TWO STEPS

We consider a noncooperative optimal stopping game. In the game, there are n players. Each player chooses one or two random numbers $x_1^{(k)}, x_2^{(k)}, k = 1, 2, \dots, n$. We assume that these random variables are independent and uniformly distributed in [0, 1]. After the first draw, a player decides to stop or continue for a second draw. She does not know what the other players have done. The object of the game is to have the highest total score without going over 1. In case the total scores of all players exceed 1, the player whose score is closest to 1 is the winner in the game.

Assume that each player uses the following threshold strategy u: If the first number x_1 is larger than or equal to u, the player chooses this number; otherwise, she chooses the second number x_2. Thus, following this strategy u, the stopping time and the players' score are equal to

$$\tau = \begin{cases} 1 & \text{if } x_1 \geq u \\ 2 & \text{if } x_1 < u \end{cases} \quad \text{and} \quad s_\tau = \begin{cases} x_1 & \text{if } x_1 \geq u \\ x_1 + x_2 & \text{if } x_1 < u \end{cases}$$

respectively.

Let us calculate for $x \in [u, 1]$ the probability

$$\begin{aligned} P\{s_\tau \leq x\} &= P\{x_1 \in [u, x]\} + P\{x_1 < u, x_1 + x_2 \leq x\} \\ &= x - u + \int_0^u (x - y)\, dy \end{aligned} \tag{2.1}$$

and, for $x > 1$, the probability

$$\begin{aligned} P\{s_\tau > x\} &= P\{x - 1 < x_1 < u, x_1 + x_2 > x\} \\ &= \int_{x-1}^u (1 - (x - y))\, dy. \end{aligned} \tag{2.2}$$

From symmetry of the problem it follows that the optimal strategies of the players must be equal. Suppose that the players $(2, \dots, n)$ use the identical thresholds

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strategies $u^{(n-1)} = (u, \ldots, u)$ and consider the first player. Let $x_1 = x$ be the score of the first player after the first step. If she stops after the first step, then she wins the game if the scores of other players are less than x or greater than 1. Hence, the expected payoff of the first player after the first step is

$$h(x|u^{(n-1)}) = \prod_{k=2}^{n} (P\{s_\tau^{(k)} < x\} + P\{s_\tau^{(k)} > 1\}).$$

Equation (2.1) yields

$$h(x|u^{(n-1)}) = (x - u + xu)^{n-1}, \quad x \in [u, 1].$$

Denote by $Ph(x|u^{(n-1)})$ the return of the first player if she continues the process with the current score after the first step being x.

$$\begin{aligned} Ph(x|u^{(n-1)}) &= \int_x^1 h(y|u^{(n-1)})\, dy + \int_1^{x+1} \prod_{k=2}^{n} P\{s_\tau^{(k)} > y\}\, dy \\ &= \int_x^1 h(y|u^{(n-1)})\, dy + \int_1^{x+1} \left(\int_{y-1}^{u} (1 - (y - z)) dz \right)^{n-1} dy \\ &= \frac{1 - (x - u + xu)^n}{n(u+1)} + \frac{u^{2n-1} - (u - x)^{2n-1}}{2^{n-1}(2n-1)}. \end{aligned}$$

The value of the optimal threshold can be achieved using the equation $h(x|u^{(n-1)}) = Ph(x|u^{(n-1)})$.

Letting $x = u$, we obtain

$$u^{2(n-1)} = \frac{1 - u^{2n}}{n(u+1)} + \frac{u^{2n-1}}{2^{n-1}(2n-1)}. \tag{2.3}$$

From (2.3) we can find the optimal strategies for different n (see Table 1).

For $n = 2,\ 3$, the optimal thresholds coincide with the values in [3]. We see also from Table 1 that for large number of players, the optimal strategy is closed to 1.

TABLE 1. Optimal Strategies u^*

n	2	3	4	5	6	7	8	9	10	100	500
u^*	0.563	0.661	0.718	0.757	0.785	0.806	0.823	0.837	0.849	0.974	0.993

3. SHOWCASE SHOWDOWN GAME WITH INFINITE NUMBER OF STEPS

Assume now that the players observe the sequence of sums of i.i.d. random variables

$$s_t^{(k)} = \sum_{i=1}^{t} x_i^{(k)}, \quad t = 1, 2, \ldots,$$

where $k = 1, \ldots, n$. Without loss of generality, suppose that for any k, the random variables $\{x_i^{(k)}\}, i = 1, 2, \ldots$ are uniformly distributed in $[0, 1]$. The critical value of the threshold is 1.

Let a player uses the strategy $u, 0 < u < 1$ and stops her sum of scores at the first moment that the sum s_t exceeds u. Denote this random stopping time as τ. Thus,

$$\tau = \min\{t \geq 1 : s_t \geq u\}.$$

To construct the payoff in this game, let us find the distribution for the stopping sum s_τ. For $u \leq x \leq 1$, we present

$$P\{s_\tau \leq x\} = \sum_{t=1}^{\infty} P\{s_t \leq x, \tau = t\} = \sum_{t=1}^{\infty} P\{s_1 < u, \ldots, s_{t-1} < u, s_t \in [u, x]\}.$$

It yields

$$P\{s_\tau \leq x\} = \sum_{t=1}^{\infty} \frac{u^{t-1}}{(t-1)!}(x - u) = \exp(u)(x - u). \qquad \textbf{(3.1)}$$

From (3.1), we obtain that the probability of ruin is

$$P\{s_\tau > 1\} = 1 - P\{s_\tau \leq 1\} = 1 - \exp(u)(1 - u). \qquad \textbf{(3.2)}$$

Let us calculate also the probability $P\{s_\tau > x\}$ for $x > 1$. First, find the probability $P\{s_t > x, \tau = t\}, t = 2, 3, \ldots$:

$$\begin{aligned} P\{s_t > x, \tau = t\} &= P\{s_1 < u, s_2 < u, \ldots, s_{t-2} < u, s_{t-1} \in (x-1, u), s_t > x\} \\ &= \int_x^{u+1} dz_t \int_{z_t - 1}^{u} dz_{t-1} \int_0^{z_{t-1}} dz_{t-2} \ldots \int_0^{z_2} dz_1 \\ &= \frac{u^{t-1}(u - x + 1)}{(t-1)!} - \frac{u^t - (x-1)^t}{t!}. \end{aligned}$$

Hence,

$$P\{s_\tau > x\} = \sum_{t=2}^{\infty} P\{s_t > x, \tau = t\} = \sum_{t=2}^{\infty} \left(\frac{u^{t-1}(u - x + 1)}{(t-1)!} - \frac{u^t - (x-1)^t}{t!} \right).$$

Simplifying, we arrive at

$$P\{s_\tau > x\} = \exp(x - 1) - (x - u)\exp(u), \qquad x > 1. \qquad \textbf{(3.3)}$$

Now, we can find the equilibrium in the game. Consider the behavior of the first player. Assume that the other $n-1$ players use the identical thresholds' strategies u and find the best reply of the first player.

Denote the current value of her sum by $s_t^{(1)} = x$. If $x \geq u$, then the expected gain of the first player in case of stopping will be positive if all other $n-1$ players stop at the values that are less than x or larger than 1. Hence, the gain of the first player is

$$h(x|u) = \prod_{k=1}^{n} P\{s_\tau^{(k)} < x\} + P\{s_\tau^{(k)} > 1\} = (P\{s_\tau^{(k)} < x\} + P\{s_\tau^{(k)} > 1\})^{n-1}.$$

Equations (3.1) and (3.2) yield

$$h(x|u) = (\exp(u)(x-u) + 1 - \exp(u)(1-u))^{n-1} = (\exp(u)(x-1)+1)^{n-1}. \tag{3.4}$$

However, if the first player continues to choose and makes the stop on the next step with score y, then her payoff will be equal to $h(x+y|u)$ if $x+y \leq 1$ and $P\{s_\tau > x + y\}^{n-1}$ if $x+y > 1$. Consequently, the expected payoff under continuation is equal to

$$Ph(x|u) = \int_u^1 h(z|u)\,dz + \int_1^{u+1} P\{s_\tau > z\}^{n-1}\,dz.$$

From (3.3) and (3.4) after simplification, we obtain

$$\begin{aligned} Ph(x|u) &= \int_u^1 (\exp(u)(z-1)+1)^{n-1}\,dz + \int_1^{u+1} (\exp(z-1) - (z-u)\exp(u))^{n-1}\,dz \\ &= \frac{\exp(-u)\,(1-(\exp(u)(u-1)+1)^n)}{n} \\ &\quad + \int_1^{u+1} (\exp(z-1) - (z-u)\exp(u))^{n-1}\,dz. \end{aligned}$$

Function $h(x|u)$ for $x \geq u$ is increasing in x, but $Ph(x|u)$ is constant. Consequently, if the first player knows the strategy u of other players, then she can find her best reply comparing the gains for case 3 of stopping and continuing. The optimal threshold u_n is determined by

$$h(x|u) = Ph(x|u)$$

or

$$\frac{e^{-u}\,(1-(e^u(u-1)+1)^n)}{n} + \int_1^{u+1} (e^{z-1} - (z-u)e^u)^{n-1}\,dz = (e^u(x-1)+1)^{n-1}. \tag{3.5}$$

From monotonicity of the functions $h(x|u)$ and $Ph(x|u)$, it follows that if this threshold exists, it is unique.

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TABLE 2. Optimal Strategies u^*

n	2	3	4	5	6	7	8	9	10
u^*	0.633	0.718	0.767	0.780	0.823	0.841	0.856	0.867	0.877

Symmetry of the problem gives that the equilibrium consists of the identical strategies. Let $x = u$. From (3.5), we obtain that this strategy is satisfied by the equation

$$\frac{e^{-u}\left(1 - (e^{u}(u-1)+1)^{n}\right)}{n} + \int_{1}^{u+1} (e^{z-1} - (z-u)e^{u})^{n-1}\, dz = (e^{u}(u-1)+1)^{n-1}. \tag{3.6}$$

The solution u^* of (3.6) exists. It follows from the fact that for $u = 0$, the left-hand side of (3.6) is greater than the right-hand side $(1/n > 0)$, and for $u = 1$, the left-hand side is less than the right-hand side $\left[\int_1^2 (e^{z-1} - (z-1))^{n-1}\, dz < 1\right.$, since $e^{z-1} - (z-1) < 1$ for $\left.1 \leq z \leq 2\right]$. For instance, for $n = 2$, we find $u^* \approx 0.634$. The probability of ruin here is equal to

$$P\{s_\tau > 1\} = 1 - \exp(u^*)(1 - u^*) \approx 0.310.$$

In Table 2 the optimal strategies for various n are presented. We see that the optimal thresholds in case of infinite number of random variables are greater than in case of two random numbers. In the case of large n, the optimal thresholds also tends to 1.

Finally, describe the optimal behavior of the players. They must continue to choose until the first moment when the sum of scores exceeds the threshold u^*. After that, they make their choice.

Remark: This problem belongs to the class of optimal stopping problems for which the one-stage look-ahead (OLA) stopping rule, which compares the current gain with the expected return of continuing one stage and then stopping, is optimal. That is the so-called monotone stopping rule case [1].

In this optimal stopping problem, the current gain (3.4) is increasing function. At the same time, the expected gain of continuing for one an stage is not an increasing function. Hence, in this case, the OLA stopping rule is optimal.

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