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ZARISKI–VAN KAMPEN THEOREM FOR HIGHER-HOMOTOPY GROUPS

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Abstract This paper gives an extension of the classical Zariski–van Kampen theorem describing the fundamental groups of the complements of plane singular curves by generators and relations. It provides a procedure for computation of the first non-trivial higher-homotopy groups of the complements of singular projective hypersurfaces in terms of the homotopy variation operators introduced here.

Keywords: topology of hypersurface complements; homotopical variation operators

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1. Introduction

The classical Zariski–van Kampen theorem expresses the fundamental group of the complement of a plane algebraic curve in $\boldsymbol{CP}^2$ as a quotient of the fundamental group of the intersection of this complement and a generic element of a pencil of lines (cf. [**3**, **21**, **24**]). The latter group is always free and the quotient is taken by the normal closure of a set of elements described in terms of the monodromy action arising as a result of moving the above generic element around the special elements of the pencil. This theorem is the main tool for the study of the fundamental groups of the complements of plane algebraic curves (cf. [**14**]).

The purpose of the present paper is to describe a high-dimensional generalization of this theorem. Let V be a hypersurface of $\boldsymbol{CP}^{n+1}$ having degree d and the dimension of its singular locus equal to k. It is shown in [**15**] that, if $n-k \geqslant 2$, $\pi_1(\boldsymbol{CP}^{n+1}-V) = \boldsymbol{Z}/d\boldsymbol{Z}$ and $\pi_i(\boldsymbol{CP}^{n+1}-V) = 0$ for $2 \leqslant i \leqslant n-k-1$. Moreover, the group $\pi_{n-k}(\boldsymbol{CP}^{n+1}-V)$ depends on the local type and the position of the singularities of V. The latter homotopy group is called the *first non-trivial (higher-) homotopy group* of the complement to V in $\boldsymbol{CP}^{n+1}$. Since by the Zariski–Lefschetz hyperplane section theorem (cf. [**10**]), for a generic linear subspace H of codimension k in $\boldsymbol{CP}^{n+1}$ one has $\pi_{n-k}(\boldsymbol{CP}^{n+1} - V) = \pi_{n-k}(H - H \cap V)$, it is enough to consider only the key case when V has only isolated singularities. This remark reduces also the case $n - k = 1$ to the Zariski–van Kampen theorem mentioned above. We refer to the papers [**8**, **16**–**18**] for various applications of the homotopy groups of complements.

An analogue of the Zariski–van Kampen theorem for higher-homotopy groups of the complement to hypersurfaces with isolated singularities in $\boldsymbol{C}^{n+1}$ was given in [**15**]. There it was shown that, for a generic hyperplane L, the homotopy group $\pi_n(\boldsymbol{C}^{n+1} - V)$ is the quotient of $\pi_n(L - L \cap V)$ by a $\pi_1(L - L \cap V)$-submodule which depends not just on the monodromy around the singular members of the pencil containing the hyperplane section but also on certain 'degeneration operators' on the homotopy groups of the special members of the pencil.

The present work proposes a different approach to a high-dimensional Zariski–van Kampen theorem. It is based on the systematic use of homotopy variation operators introduced below. Homological variation operators were considered in [**5**] for a generalization of the second Lefschetz theorem (cf. [**13**, Chapter V, § 8, Théorème VI], [**22**] and [**1**]). From this point of view the main result of this paper can also be viewed as a homotopy second Lefschetz theorem.

The main result of the paper is the following (restated as Theorem 7.1 below).

Theorem 1.1. *Let V be a hypersurface in $\boldsymbol{CP}^{n+1}$ with $n \geqslant 2$ having only isolated singularities. Consider a pencil $(L_t)_{t \in \boldsymbol{CP}^1}$ of hyperplanes in $\boldsymbol{CP}^{n+1}$ with the base locus $\mathcal{M}$ transversal to V. Denote by $t_1, \dots, t_N$ the collection of those t for which $L_t \cap V$ has singularities. Let t_0 be different from either of $t_1, \dots, t_N$. Let γ_i be a good collection (cf. Definition 3.2) of paths in $\boldsymbol{CP}^1$ based in t_0. Let $e \in \mathcal{M} - \mathcal{M} \cap V$ be a base point. Let $\mathcal{V}_i$ be the variation operator (cf. § 5) corresponding to γ_i. Then the inclusion induces an isomorphism*

$$\pi_n(\boldsymbol{CP}^{n+1} - V, e) \xleftarrow{\sim} \pi_n(L_{t_0} - L_{t_0} \cap V, e) \Big/ \sum_{i=1}^{N} \operatorname{im} \mathcal{V}_i.$$

For $n = 1$, this statement reduces to the classical Zariski–van Kampen theorem as we explain it in Remark 7.2 below. But our proof does not work in the case $n = 1$. Thus we shall suppose $n \geqslant 2$ in this article.

The paper is organized as follows. We start, in § 2, by describing in detail several pencils of hyperplanes associated with a hypersurface V with isolated singularities. We also review some vanishing results and homological description of the homotopy groups of the complements of hypersurfaces from [**15**]. In §§ 3–5 we describe the monodromies, the degeneration operators and homotopy variation operators. Section 6 describes the crucial relationship between homotopy variation and degeneration operators. The last section contains the announced theorem (Theorem 7.1). Two proofs are given, one deriving it from the quoted theorem of [**5**] and another from that of [**15**].

Here is some notation we shall use throughout the paper.

Notation 1.2. The ground field in this paper is always $\boldsymbol{C}$ so we shall omit '$\boldsymbol{C}$' from our notation of complex projective space which becomes $\boldsymbol{P}^n$ for the n-dimensional space.

Notation 1.3. All inclusion maps will be denoted by 'incl' and any canonical surjection from a set to a quotient of it by 'can'. In diagrams, we shall use the same letter for a map and any other map obtained from it by restriction of the source or the target.

Notation 1.4. All homology groups will be singular homology groups with integer coefficients. Given a continuous map $f : X \to Y$ between topological spaces, we shall denote by f_* the induced homomorphism $H_n(X) \to H_n(Y)$, whatever be the integer n, and by $f_\bullet$ the induced homomorphisms $C_n(X) \to C_n(Y)$ between chain groups. If $X' \subset X$ and $f(X') \subset Y' \subset Y$, we shall write $\bar{f}_*$ for the induced homomorphisms $H_n(X, X') \to H_n(Y, Y')$. If $x \in X$ and $y = f(x) \in Y$, we shall denote by $f_\#$ the induced maps $\pi_n(X, x) \to \pi_n(Y, y)$ for $n \geqslant 0$ and, if $x \in X'$ and $y = f(x) \in Y'$, by $\bar{f}_\#$ the induced maps $\pi_n(X, X', x) \to \pi_n(Y, Y', y)$ for $n \geqslant 1$. All boundary operators in homology or homotopy will be designated by ∂. All absolute Hurewicz homomorphisms will be denoted by χ and the relative ones by $\bar{\chi}$.

Notation 1.5. Given an absolute cycle ξ, we shall write $[\xi]_X$ for its homology class in a space X containing it and, if η is a chain contained in X with boundary contained in $X' \subset X$, we shall denote by $[\eta]_{(X,X')}$ its homology class in X modulo X'. If ξ' is an(other) absolute cycle contained in X, we shall write '$\xi \sim \xi'$ in X' to mean that ξ is homologous to ξ' in X. The homotopy class of a loop γ will be denoted by $\bar{\gamma}$, the space in which this class has to be taken being made clear by the context.

Notation 1.6. The singular locus of the algebraic hypersurface V will be designated by V_{sing}.

Notation 1.7. In a blow up, the total transform of a subset E of the blown up space will be denoted by $\hat{E}$ and its strict transform by $E^\sharp$.

2. Preliminaries

2.1. General set-up

Let V be a closed algebraic hypersurface of the $(n+1)$-dimensional complex projective space $\boldsymbol{P}^{n+1}$, with only *isolated singularities*. Let d be its degree. We suppose $n \geqslant 2$ for the reasons explained in § 1 and we suppose $d \geqslant 2$, the case $d = 1$ being trivial (and appearing as a cumbersome exceptional case in what follows). Let M be a projective $(n-1)$-plane transverse to V (that is, avoiding the singular points of V and transverse to the non-singular part of V). Let L be the pencil of hyperplanes with base locus M, that is, the set of projective hyperplanes of $\boldsymbol{P}^{n+1}$ containing M. We want to compute the homotopy groups of $\boldsymbol{P}^{n+1} - V$ with the help of its sections by the elements of L.

We take homogeneous coordinates $(x_1 : \cdots : x_{n+1} : x_{n+2})$ on $\boldsymbol{P}^{n+1}$, so chosen that M is defined by the equations

$$x_{n+1} = x_{n+2} = 0.$$

We then have a one-to-one parametrization of the elements of pencil L by the elements of $\boldsymbol{P}^1$ as follows. Given also homogeneous coordinates on $\boldsymbol{P}^1$, for each $t = (\lambda : \mu) \in \boldsymbol{P}^1$, the hyperplane L_t of $\boldsymbol{P}^{n+1}$ with parameter t is defined by the equation

$$\lambda x_{n+1} + \mu x_{n+2} = 0.$$

We shall thus consider L as being the parametrized family $(L_t)_{t\in\boldsymbol{P}^1}$. The transversality of M to V entails the following claim.

Claim 2.1. *The given choice of the axis M of pencil $L = (L_t)_{t\in\boldsymbol{P}^1}$ is generic. All the members of this pencil are transverse to V except a finite number of them, say $L_{t_1}, \dots, L_{t_N}$. Each L_{t_i} is transverse to V except at a finite number of points, which may be singular points of V or tangency points of L_{t_i} to the non-singular part of V, and moreover none of them belongs to M.*

Proof. This is a consequence of [**4**, Corollaire 10.18, Corollaire 10.19 combined with Proposition 10.20 and Corollaire 10.17]. The quoted results apply when stratifying V by its singular part V_{sing} and non-singular part $V - V_{\text{sing}}$. This is indeed a Whitney stratification of V by Lemma 19.3 of [**23**], since V_{sing} is zero dimensional. □

Thus, the pencil $(L_t)_{t\in\boldsymbol{P}^1}$ looks like a stratified version of the 'Lefschetz pencils' of [**7**] but each $L_{t_i} \cap V$ may have more than one singularity and these singularities may be of any kind.

Our goal is to define variation and degeneration operators on homotopy. Each of those depends on a choice of L_{t_i} for some fixed index i and a loop γ_i running once around t_i in the parameter space $\boldsymbol{P}^1$ and surrounding none of the points $t_1, \dots, t_N$ besides t_i. The main technical tool is an interpretation of the relevant homotopy groups as the homology groups of universal coverings which was used in [**15**]. This material is discussed in the last part of this section (cf. § 2.4 below). The universal covers are obtained as restrictions of a ramified cover of $\boldsymbol{P}^{n+1}$ by a hypersurface W of $\boldsymbol{P}^{n+2}$ viewed in the next subsection. The rest describes the classical blowing up construction in our framework which we use to get rid of the base points of the pencils as will be needed for the definition of degeneration operators.

2.2. A ramified cover of $\boldsymbol{P}^{n+1}$

In the homogeneous coordinates of $\boldsymbol{P}^{n+1}$ chosen in § 2.1, let

$$f(x_1, \dots, x_{n+1}, x_{n+2}) = 0$$

be an equation of V where f is a homogeneous reduced polynomial of degree d.

Now, in $\boldsymbol{P}^{n+2}$ with homogeneous coordinates $(x_0 : x_1 : \cdots : x_{n+1} : x_{n+2})$, let A be the point of coordinates $(1 : 0 : \cdots : 0 : 0)$. Let us consider the projection with centre A

$$\text{pr} : \quad \boldsymbol{P}^{n+2} - \{A\} \to \boldsymbol{P}^{n+1}, \quad (x_0 : x_1 : \cdots : x_{n+1} : x_{n+2}) \mapsto (x_1 : \cdots : x_{n+1} : x_{n+2}).$$

Let W be the hypersurface of $\boldsymbol{P}^{n+2}$ given by the equation

$$x_0^d + f(x_1, \dots, x_{n+2}) = 0.$$

We have $A \notin W$. Thus $\pi = \text{pr}_{|W}$ is well defined. The following is a classical result.

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Claim 2.2. *The map $\pi : W \to \boldsymbol{P}^{n+1}$ is a holomorphic d-fold covering of $\boldsymbol{P}^{n+1}$ totally ramified along V.*

We consider also the embedding

$$j: \quad \boldsymbol{P}^{n+1} \to \boldsymbol{P}^{n+2}, \quad (x_1 : \cdots : x_{n+2}) \mapsto (0 : x_1 : \cdots : x_{n+2}),$$

the image of which is the projective hyperplane $j(\boldsymbol{P}^{n+1}) \subset \boldsymbol{P}^{n+2}$ given by $x_0 = 0$. We have

$$W \cap j(\boldsymbol{P}^{n+1}) = j(V) = \pi^{-1}(V), \tag{2.1}$$

each of these subsets of $\boldsymbol{P}^{n+2}$ having equations equivalent to

$$x_0 = f(x_1, \ldots, x_{n+2}) = 0.$$

The following claim is also easy to check from the equations.

Claim 2.3. *The singular points of W are the images by j of the singular points of V.*

The hypersurface W supports a natural pencil, the elements of which are the branched covers of the elements of pencil L and which can be explicitly described as follows. Let $\mathcal{M}$ be the projective n-plane of $\boldsymbol{P}^{n+2}$ defined by the same equations as M in $\boldsymbol{P}^{n+1}$, that is,

$$x_{n+1} = x_{n+2} = 0,$$

and let $\mathcal{L} = (\mathcal{L}_t)_{t \in \boldsymbol{P}^1}$ be the pencil of hyperplanes of $\boldsymbol{P}^{n+2}$ with base locus $\mathcal{M}$. Here, with the same homogeneous coordinates on $\boldsymbol{P}^1$ as in § 2.1, for each $t = (\lambda : \mu) \in \boldsymbol{P}^1$, the hyperplane $\mathcal{L}_t$ of $\boldsymbol{P}^{n+2}$ is defined by the same equations as L_t in $\boldsymbol{P}^{n+1}$, that is,

$$\lambda x_{n+1} + \mu x_{n+2} = 0.$$

As a consequence we have

$$\pi^{-1}(M) = \mathcal{M} \cap W \quad \text{and} \quad \pi^{-1}(L_t) = \mathcal{L}_t \cap W \quad \text{for any } t \in \boldsymbol{P}^1 \tag{2.2}$$

and also

$$\mathcal{M} \cap j(\boldsymbol{P}^{n+1}) = j(M) \quad \text{and} \quad \mathcal{L}_t \cap j(\boldsymbol{P}^{n+1}) = j(L_t) \quad \text{for any } t \in \boldsymbol{P}^1.$$

Unramified covers of $\boldsymbol{P}^{n+1} - V$ and of its sections by L_t and M are given by Claim 2.2. They can be specified thanks to (2.1) and (2.2) as follows.

Claim 2.4. *Map π induces the following holomorphic unramified d-fold coverings:*

(i) $W - j(V) \to \boldsymbol{P}^{n+1} - V$;

(ii) $\mathcal{L}_t \cap (W - j(V)) \to L_t - L_t \cap V$ *for any* $t \in \boldsymbol{P}^1$;

(iii) $\mathcal{M} \cap (W - j(V)) \to M - M \cap V$.

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It is worth noticing that the pencil $\mathcal{L}$ is good with respect to W and $j(V)$ as L was good with respect to V. To make this precise, we first stratify W. The following claim is proved using again [**23**, Lemma 19.3].

Claim 2.5. *The partition*

$$\Sigma = \{j(V_{\mathrm{sing}}), j(V - V_{\mathrm{sing}}), W - j(V)\}$$

of W is an algebraic Whitney stratification.

The statement analogous to Claim 2.1 is then the following. It is a consequence of Claim 2.1 that can be checked on the equations.

Claim 2.6. *The base locus $\mathcal{M}$ of $\mathcal{L}$ is transverse to the strata of Σ and so is $\mathcal{L}_t$ for all $t \in \boldsymbol{P}^1$ distinct from $t_1, \dots, t_N$. Each $\mathcal{L}_{t_i}$ is transverse to $W - j(V)$, non-transverse to $j(V_{\mathrm{sing}})$ wherever it meets this finite set and transverse to $j(V - V_{\mathrm{sing}})$ except at the points $j(x)$ corresponding to the finite number of points x where L_{t_i} is tangent to the non-singular part of V.*

2.3. Blowing up the cover

The homotopical degeneration and variation operators we want will be obtained by considering homological counterparts on the cover dealt with in the preceding subsection. But the definition of the homological degeneration operators will in turn require to blow up this cover along the base locus of the pencil we considered. In fact we do it first for the ambient space $\boldsymbol{P}^{n+2}$. Let

$$\hat{\boldsymbol{P}}^{n+2} = \{(x, \tau) \in \boldsymbol{P}^{n+2} \times \boldsymbol{P}^1 \mid x \in \mathcal{L}_\tau\}$$

be the blow up of $\boldsymbol{P}^{n+2}$ along $\mathcal{M}$. It is given by the equation

$$\tau_1 x_{n+1} + \tau_2 x_{n+2} = 0,$$

which is separately homogeneous in the homogeneous coordinates $(x_0 : x_1 : \cdots : x_{n+1} : x_{n+2})$ of x and $(\tau_1 : \tau_2)$ of τ. This is an $(n+1)$-dimensional complex analytic compact connected submanifold of $\boldsymbol{P}^{n+2} \times \boldsymbol{P}^1$.

The restrictions to $\hat{\boldsymbol{P}}^{n+2}$ of the projections of $\boldsymbol{P}^{n+2} \times \boldsymbol{P}^1$ onto its first and second factors give, respectively, the blowing-down morphism Φ and the projection P,

$$\boldsymbol{P}^{n+2} \overset{\Phi}{\longleftarrow} \hat{\boldsymbol{P}}^{n+2} \overset{P}{\longrightarrow} \boldsymbol{P}^1.$$

These are holomorphic mappings and P is submersive.

For any subset $\mathcal{E} \subset \boldsymbol{P}^{n+2}$, we shall, following Notation 1.7, denote by $\hat{\mathcal{E}}$ its total transform, that is,

$$\hat{\mathcal{E}} = \Phi^{-1}(\mathcal{E}).$$

If $\mathcal{E} \subset \mathcal{M}$, then its total transform has a product structure,

$$\hat{\mathcal{E}} = \mathcal{E} \times \boldsymbol{P}^1 \quad \text{for any } \mathcal{E} \subset \mathcal{M}, \tag{2.3}$$

and the restrictions of Φ and P to $\hat{\mathcal{E}}$ coincide with the first and second projections. In particular, $\hat{\mathcal{M}} = \mathcal{M} \times \boldsymbol{P}^1$. The blowing-down morphism induces an analytic isomorphism from $\hat{\boldsymbol{P}}^{n+2} - \hat{\mathcal{M}}$ onto $\boldsymbol{P}^{n+2} - \mathcal{M}$. We shall be interested in the total transform $\hat{W}$ of the cover W of the preceding subsection.

For each $t \in \boldsymbol{P}^1$, we consider the strict transform of $\mathcal{L}_t$ which we denote by $\mathcal{L}_t^\sharp$ following Notation 1.7. We have

$$\mathcal{L}_t^\sharp = \mathcal{L}_t \times \{t\} = P^{-1}(t). \tag{2.4}$$

The blowing-down morphism induces an analytic isomorphism from $\mathcal{L}_t^\sharp$ onto $\mathcal{L}_t$ and hence an analytic isomorphism

$$\mathcal{L}_t^\sharp \cap \hat{\mathcal{E}} \xrightarrow{\sim} \mathcal{L}_t \cap \mathcal{E} \quad \text{for any } \mathcal{E} \subset \boldsymbol{P}^{n+2}. \tag{2.5}$$

At this point it will be convenient to introduce some abbreviations to save space in diagrams.

Notation 2.7. We shall refer by E' to the intersection of a subspace E of $\boldsymbol{P}^{n+1}$ (respectively, $\boldsymbol{P}^{n+2}$, $\hat{\boldsymbol{P}}^{n+2}$) with $\boldsymbol{P}^{n+1} - V$ (respectively, $W - j(V)$, $\hat{W} - \widehat{j(V)}$). For instance, $L_t' = L_t - L_t \cap V$, $\mathcal{M}' = \mathcal{M} \cap (W - j(V))$, $\mathcal{L}_t^{\sharp\prime} = \mathcal{L}_t^\sharp \cap (\hat{W} - \widehat{j(V)})$ and even $\hat{W}' = \hat{W} - \widehat{j(V)}$. We shall denote by P' the restriction of the projection P to $\hat{W} - \widehat{j(V)}$.

2.4. Homological description of homotopy groups

We obtain relevant homotopy groups of the base spaces of the covers of Claim 2.4 as homology groups of their total spaces except in some exceptional cases where we shall content ourselves with a morphism from a subgroup of the fundamental group of the base space onto the first homology group of the total space.

Lemma 2.8. *We use Notation 2.7. Let $e \in M'$ and $\varepsilon \in \pi^{-1}(e) \subset \mathcal{M}'$ be base points.*

(i) *If $n \geqslant 2$, there are isomorphisms η and α_t, for $t \in \boldsymbol{P}^1 - \{t_1, \dots, t_N\}$, defined by composition as follows,*

$$\eta: \quad \pi_n(\boldsymbol{P}^{n+1} - V, e) \xrightarrow[\sim]{\pi_\#^{-1}} \pi_n(W - j(V), \varepsilon) \xrightarrow[\sim]{\chi} H_n(W - j(V)),$$

$$\alpha_t: \quad \pi_n(L_t', e) \xrightarrow[\sim]{\pi_\#^{-1}} \pi_n(\mathcal{L}_t', \varepsilon) \xrightarrow[\sim]{\chi} H_n(\mathcal{L}_t'),$$

where the arrows labelled χ are Hurewicz isomorphisms and the arrows labelled $\pi_\#^{-1}$ are the inverses of isomorphisms induced by the projections of the coverings of Claim 2.4 (which all are restrictions of the map π). Furthermore, for any $t \in \boldsymbol{P}^1 - \{t_1, \dots, t_N\}$, the following diagram is commutative:

$$\begin{array}{ccc} H_n(\mathcal{L}_t') & \xrightarrow{\text{incl}_*} & H_n(W - j(V)) \\ \uparrow \wr \alpha_t & & \uparrow \wr \eta \\ \pi_n(L_t', e) & \xrightarrow{\text{incl}_\#} & \pi_n(\boldsymbol{P}^{n+1} - V, e) \end{array}$$

(*following Notation 1.3, all inclusion maps are denoted by* incl).

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(ii) *If $n \geqslant 3$, there are isomorphisms β_i, for $1 \leqslant i \leqslant N$, and γ defined by composition as follows,*

$$\beta_i: \quad \pi_{n-1}(L'_{t_i}, e) \xrightarrow[\sim]{\pi_\#^{-1}} \pi_{n-1}(\mathcal{L}'_{t_i}, \varepsilon) \overset{\chi}{\underset{\sim}{\to}} H_{n-1}(\mathcal{L}'_{t_i}),$$
$$\gamma: \quad \pi_{n-1}(M', e) \xrightarrow[\sim]{\pi_\#^{-1}} \pi_{n-1}(\mathcal{M}', \varepsilon) \overset{\chi}{\underset{\sim}{\to}} H_{n-1}(\mathcal{M}'),$$

where isomorphisms χ and $\pi_\#^{-1}$ are as in (i). Furthermore, for $1 \leqslant i \leqslant N$, the following diagram is commutative:

$$\begin{array}{ccc} H_{n-1}(\mathcal{M}') & \xrightarrow{\text{incl}_*} & H_{n-1}(\mathcal{L}'_{t_i}) \\ \uparrow\wr\gamma & & \uparrow\wr\beta_i \\ \pi_{n-1}(M', e) & \xrightarrow{\text{incl}_\#} & \pi_{n-1}(L'_{t_i}, e) \end{array} \tag{2.6}$$

(iii) *When $n = 2$, the projections of the coverings considered in (ii) induce isomorphisms*

$$\pi_1(\mathcal{L}'_{t_i}, \varepsilon) \xrightarrow[\sim]{\pi_\#} G_i \subset \pi_1(L'_{t_i}, e),$$
$$\pi_1(\mathcal{M}', \varepsilon) \xrightarrow[\sim]{\pi_\#} H \subset \pi_1(M', e),$$

where G_i is, for $1 \leqslant i \leqslant N$, a subgroup of index d of $\pi_1(L'_{t_i}, e)$ and H a subgroup of index d of $\pi_1(M', e)$ (the latter is free of rank $d - 1$). One can then define homomorphisms β_i, for $1 \leqslant i \leqslant N$, and γ by composition as follows,

$$\beta_i: \quad G_i \xrightarrow[\sim]{\pi_\#^{-1}} \pi_1(\mathcal{L}'_{t_i}, \varepsilon) \xrightarrow{\chi} H_1(\mathcal{L}'_{t_i}),$$
$$\gamma: \quad H \xrightarrow[\sim]{\pi_\#^{-1}} \pi_1(\mathcal{M}', \varepsilon) \xrightarrow{\chi} H_1(\mathcal{M}'),$$

where the Hurewicz homomorphisms χ are here abelianizations. The homomorphisms β_i, for $1 \leqslant i \leqslant N$, and γ are thus onto. Furthermore, for $1 \leqslant i \leqslant N$, the image of H by the natural map $\text{incl}_\# : \pi_1(M', e) \to \pi_1(L'_{t_i}, e)$ is included in G_i and the following diagram is commutative:

$$\begin{array}{ccc} H_1(\mathcal{M}') & \xrightarrow{\text{incl}_*} & H_1(\mathcal{L}'_{t_i}) \\ \uparrow\gamma & & \uparrow\beta_i \\ H & \xrightarrow{\text{incl}_\#} & G_i \end{array} \tag{2.7}$$

Proof. Let $E \to B$ be one of the unramified coverings of Claim 2.4. Its projection is a restriction of the map π. It induces an isomorphism from the fundamental group of E

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onto a subgroup of index d of the fundamental group of B and isomorphisms of the kth homotopy groups for $k \geqslant 2$. But these vanish for $2 \leqslant k \leqslant n-1$ (this range may be empty) if $B = \boldsymbol{P}^{n+1} - V$ because V is a hypersurface with isolated singularities (see [**15**, Lemma 1.5]). The same is true if $B = L'_t = L_t - L_t \cap V$ with $t \neq t_i$ for $1 \leqslant i \leqslant N$ because, by Claim 2.1, $L_t \cap V$ is a non-singular hypersurface of $L_t \simeq \boldsymbol{P}^n$ (see [**15**, Lemma 1.1]). On the other hand, the kth homotopy groups vanish for $2 \leqslant k \leqslant n-2$ (a range which may be empty) if $B = L'_{t_i}$ with $1 \leqslant i \leqslant N$ because then $L_{t_i} \cap V$ is, still by Claim 2.1, a hypersurface with isolated singularities of $L_{t_i} \simeq \boldsymbol{P}^n$. And the same is true if $B = M'$ because M was chosen transverse to V so that $M \cap V$ is a non-singular hypersurface of $M \simeq \boldsymbol{P}^{n-1}$.

If, moreover, E is simply connected, that is to say the cover universal, then the Hurewicz homomorphism $\chi : \pi_n(E, \varepsilon) \to H_n(E)$ in the first two cases and $\chi : \pi_{n-1}(E, \varepsilon) \to H_{n-1}(E)$ in the last two will be an isomorphism due to the Hurewicz isomorphism theorem. By the next lemma, this indeed happens for the values of n listed in the first two items of the statement. In the cases of the last item, χ is an epimorphism of abelianization. Besides when $n = 2$, M' is a projective line minus d points and its fundamental group is free of rank $d-1$. In the same case, the image of H by the natural map from $\pi_1(M', e)$ to $\pi_1(L'_{t_i}, e)$ is contained in G_i because the projections of the coverings commute with inclusions. As to the commutativity of the diagrams, it results from this and the functoriality of the Hurewicz homomorphisms. □

Lemma 2.9.

(i) *If $n \geqslant 2$, the first covering of Claim 2.4 is a universal covering and so is the second one if $t \neq t_i$ for $1 \leqslant i \leqslant N$.*

(ii) *If $n \geqslant 3$, the third covering of Claim 2.4 is a universal covering and so is the second covering with $t = t_i$ for $1 \leqslant i \leqslant N$.*

Proof. Let $E \to B$ be one of these coverings. According to the nature of the base space B as discussed in the proof of Lemma 2.8, it is pathwise connected and its fundamental group is $\boldsymbol{Z}/d\boldsymbol{Z}$ when $n \geqslant 2$ in the first two cases there considered and when $n \geqslant 3$ in the last two (notice that all involved hypersurfaces have degree d). Thus, for the appropriate range of n, this group has the same number of elements as the fibre. The lemma then follows once it is verified that the total space E is pathwise connected. This can be seen from the irreducibility of W, $\mathcal{L}_t \cap W$ and $\mathcal{M} \cap W$ or from the fact that, above a neighbourhood of a regular point of V, each of the coverings has a local model which is a product of the cover associated with $z \mapsto z^d$ and a disc of appropriate dimension. □

We shall also have to consider relative homotopy groups for the variation operators.

Lemma 2.10. *Let e and ε be base points as in Lemma 2.8. If $n \geqslant 2$ and $t \in \boldsymbol{P}^1 - \{t_1, \ldots, t_N\}$, there are homomorphisms $\bar{\alpha}_t$ defined by composition as follows (we use Notation 2.7),*

$$\bar{\alpha}_t : \quad \pi_n(L'_t, M', e) \xrightarrow[\sim]{\bar{\pi}_{\#}^{-1}} \pi_n(\mathcal{L}'_t, \mathcal{M}', \varepsilon) \xrightarrow{\bar{\chi}} H_n(\mathcal{L}'_t, \mathcal{M}'),$$

where $\bar{\chi}$ is the relative Hurewicz homomorphism and $\bar{\pi}_{\#}^{-1}$ the inverse of an isomorphism induced by the projection of the second covering of Claim 2.4. The homomorphisms $\bar{\chi}$ and $\bar{\alpha}_t$ are epimorphisms if $n = 2$ and isomorphisms if $n \geqslant 3$. Furthermore, for $n \geqslant 3$ and $t \in \boldsymbol{P}^1 - \{t_1, \dots, t_N\}$, the following diagram, where γ is the isomorphism defined in Lemma 2.8 (ii) is commutative:

$$\begin{array}{ccc} H_n(\mathcal{L}'_t, \mathcal{M}') & \xrightarrow{\partial} & H_{n-1}(\mathcal{M}') \\ \uparrow \wr\, \bar{\alpha}_t & & \uparrow \wr\, \gamma \\ \pi_n(L'_t, M', e) & \xrightarrow{\partial} & \pi_{n-1}(M', e) \end{array} \tag{2.8}$$

When $n = 2$, the image of the boundary homomorphism from $\pi_2(L'_t, M', e)$ to $\pi_1(M', e)$ is contained in the subgroup H defined in Lemma 2.8 (iii) and the following diagram, where γ is the homomorphism defined there, is commutative:

$$\begin{array}{ccc} H_2(\mathcal{L}'_t, \mathcal{M}') & \xrightarrow{\partial} & H_1(\mathcal{M}') \\ \uparrow \bar{\alpha}_t & & \uparrow \gamma \\ \pi_2(L'_t, M', e) & \xrightarrow{\partial} & H \end{array} \tag{2.9}$$

Proof. Map π induces the projection of the covering $\mathcal{L}'_t \to L'_t$ and we have $\pi^{-1}(M') = \mathcal{M}'$ by (2.1) and (2.2). That $\bar{\pi}_{\#}$ is then an isomorphism is a general fact about pairs of fibred spaces (cf. [**19**, 7.2.8]). We now come to the relative Hurewicz homomorphism $\bar{\chi}$. If $n \geqslant 2$ and $t \in \boldsymbol{P}^1 - \{t_1, \dots, t_N\}$, the spaces $\mathcal{M}'$ and $\mathcal{L}'_t$ are path-connected as seen in the proof of Lemma 2.9. Furthermore, the same lemma and the vanishing of higher-homotopy groups stated in the proof of Lemma 2.8 give then that $\pi_k(\mathcal{M}', \varepsilon) = 0$ for $0 \leqslant k \leqslant n-2$ and $\pi_k(\mathcal{L}'_t, \varepsilon) = 0$ for $1 \leqslant k \leqslant n-1$, hence $\pi_k(\mathcal{L}'_t, \mathcal{M}', \varepsilon) = 0$ for $1 \leqslant k \leqslant n-1$ by the homotopy exact sequence. The relative Hurewicz theorem (cf. [**19**, 7.5.4]) then applies and gives that the Hurewicz homomorphism $\bar{\chi}$ induces an isomorphism onto $H_n(\mathcal{L}'_t, \mathcal{M}')$ from the quotient of $\pi_n(\mathcal{L}'_t, \mathcal{M}', \varepsilon)$ obtained by identifying each element with its images by the action of $\pi_1(\mathcal{M}', \varepsilon)$. But, by Lemma 2.9, this fundamental group is trivial if $n \geqslant 3$. When $n = 2$, the image of $\partial : \pi_2(L'_t, M', e) \to \pi_1(M', e)$ is contained in H because boundary homomorphisms commute with those induced by π and $\bar{\pi}_{\#}$ is onto as we have seen. As furthermore boundary operators commute with Hurewicz homomorphisms (cf. [**19**, 7.4.3]), the last two diagrams are commutative. □

Finally, the following lemma will be useful while computing homology in the universal coverings.

Lemma 2.11. *We have the following vanishing of homology groups.*

(i) $H_k(W - j(V)) = 0$ *for* $1 \leqslant k \leqslant n-1$.

(ii) $H_k(\mathcal{L}_t \cap (W - j(V))) = 0$ *for* $1 \leqslant k \leqslant n-1$ *if* $t \neq t_i$ *for* $1 \leqslant i \leqslant N$.

(iii) $H_k(\mathcal{L}_{t_i} \cap (W - j(V))) = 0$ *for* $1 \leqslant k \leqslant n-2$ *and* $1 \leqslant i \leqslant N$.

(iv) $H_k(\mathcal{M} \cap (W - j(V))) = 0$ *for* $1 \leqslant k \leqslant n-2$.

Proof. This also results from Lemma 2.9, the vanishing of higher-homotopy groups stated in the proof of Lemma 2.8 and the Hurewicz isomorphism theorem. □

Note that the last two assertions are empty if $n = 2$.

3. Monodromies

The homological degeneration and variation operators which we must define at the universal covering level involve monodromies around the exceptional hyperplanes in the cover $W - j(V)$ and its blow up $\hat{W} - \widehat{j(V)}$. These in turn are linked with a fibration structure of $\hat{W} - \widehat{j(V)}$ outside of the exceptional hyperplanes, a structure we shall also directly use for the degeneration operator.

Claim 3.1. *The restriction of P to $(\hat{W} - \widehat{j(V)}) - \bigcup_{i=1}^{N} \mathcal{L}_{t_i}^{\sharp}$ is the projection of a fibre bundle over $\boldsymbol{P}^1 - \{t_1, \dots, t_N\}$. This bundle has*

$$\mathcal{M} \cap (W - j(V)) \times (\boldsymbol{P}^1 - \{t_1, \dots, t_N\})$$

as a trivial subbundle of it. The fibres over $t \in \boldsymbol{P}^1 - \{t_1, \dots, t_N\}$ are $\mathcal{L}_t^{\sharp} \cap (\hat{W} - \widehat{j(V)})$ and $\mathcal{M} \cap (W - j(V)) \times \{t\}$.

Proof. This results from the fact that, by Claim 2.6, $\mathcal{M}$ is transverse to the strata of a Whitney stratification of W for which $W - j(V)$ is a union of strata (see [**5**, Corollary 3.12]). The quoted corollary rests on the Thom–Mather first isotopy theorem. □

Notice, however, that there is not a similar fibration for $W - j(V)$ because its sections by the pencil $(L_t)_{t \in \boldsymbol{P}^1}$ are pinched together along the axis M. Nevertheless, we shall obtain monodromies also there, using the isomorphisms from $\mathcal{L}_t^{\sharp} \cap (\hat{W} - \widehat{j(V)})$ to $\mathcal{L}_t \cap (W - j(V))$ that, by (2.5), the blowing-down morphism Φ induces.

We shall consider the monodromies above a special set of loops in the parameter space $\boldsymbol{P}^1$. We choose a base point t_0 in $\boldsymbol{P}^1$ distinct from the points $t_1, \dots, t_N$ and consider a *good* system of generators $(\bar{\Gamma}_i)_{1 \leqslant i \leqslant N}$ of the fundamental group $\pi_1(\boldsymbol{P}^1 - \{t_1, \dots, t_N\}, t_0)$ (cf. [**14**] and [**15**, Definition 2.1]). Recall that in such a system each loop Γ_i is based at t_0 and is the boundary of a subset D_i of $\boldsymbol{P}^1$ homeomorphic to a disc with t_i as an interior point. Moreover, $D_i \cap D_j = \{t_0\}$ for $i \neq j$. If the D_i are suitably chosen, a presentation of $\pi_1(\boldsymbol{P}^1 - \{t_1, \dots, t_N\}, t_0)$ is given by these generators and the relation $\bar{\Gamma}_1 \cdots \bar{\Gamma}_N = 1$. A standard method to construct such a system is to obtain it as the homotopy classes $\bar{\gamma}_i = \bar{\Gamma}_i$ of parametrized loops $\gamma_i : [0, 1] \to \boldsymbol{P}^1 - \{t_1, \dots, t_N\}$ as described in the following definition.

Definition 3.2. Let t_0 be a base point in $\boldsymbol{P}^1 - \{t_1, \dots, t_N\}$. Let $\Delta_1, \dots, \Delta_N$ be closed discs about $t_1, \dots, t_N$ mutually disjoint and not containing t_0. For $1 \leqslant i \leqslant N$, let δ_i be a path connecting t_0 to a point d_i of the boundary $\partial\Delta_i$ of Δ_i. Each δ_i is required not to meet any of the previous discs except that the end of δ_i touches Δ_i. Moreover, paths $\delta_1, \dots, \delta_N$ are required not to meet together except at their origin. For $1 \leqslant i \leqslant N$, let ω_i be a loop based at d_i and running once counterclockwise around $\partial\Delta_i$. Finally, for

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$1 \leqslant i \leqslant N$, consider the loops $\gamma_i = \delta_i * \omega_i * \delta_i^-$, where $*$ designates a concatenation and where δ_i^- is the path opposite to δ_i. We denote by $\bar{\gamma}_i$ the homotopy class of γ_i in $\boldsymbol{P}^1 - \{t_1, \dots, t_N\}$.

We shall obtain the wanted monodromies with the help of some special isotopies above these loops.

Lemma 3.3. *For $1 \leqslant i \leqslant N$, there are isotopies*

$$\begin{aligned} G_i :&\quad \mathcal{L}_{t_0} \cap (W - j(V)) \times [0,1] \to W - j(V), \\ \hat{G}_i :&\quad \mathcal{L}^\sharp_{t_0} \cap (\hat{W} - \widehat{j(V)}) \times [0,1] \to \hat{W} - \widehat{j(V)}, \end{aligned}$$

such that

(I) *$G_i(x, 0) = x$ for any $x \in \mathcal{L}_{t_0} \cap (W - j(V))$;*

(II) *$G_i(\cdot, s)$ is a homeomorphism from $\mathcal{L}_{t_0} \cap (W - j(V))$ onto $\mathcal{L}_{\gamma_i(s)} \cap (W - j(V))$ for any $s \in [0,1]$;*

(III) *$G_i(y, s) = y$ for any $y \in \mathcal{M} \cap (W - j(V))$ and $s \in [0,1]$;*

($\hat{\mathrm{I}}$) *$\hat{G}_i(v, 0) = v$, for any $v \in \mathcal{L}^\sharp_{t_0} \cap (\hat{W} - \widehat{j(V)})$;*

($\hat{\mathrm{I}}\hat{\mathrm{I}}$) *$\hat{G}_i(\cdot, s)$ is a homeomorphism from $\mathcal{L}^\sharp_{t_0} \cap (\hat{W} - \widehat{j(V)})$ onto $\mathcal{L}^\sharp_{\gamma_i(s)} \cap (\hat{W} - \widehat{j(V)})$ for any $s \in [0,1]$;*

($\hat{\mathrm{I}}\hat{\mathrm{I}}\hat{\mathrm{I}}$) *$\hat{G}_i((y, t_0), s) = (y, \gamma_i(s))$ for any $y \in \mathcal{M} \cap (W - j(V))$ and $s \in [0,1]$.*

Moreover, G_i and $\hat{G}_i$ can be asked to fit together by making commutative the following diagram:

$$\begin{array}{ccc} \mathcal{L}^\sharp_{t_0} \cap (\hat{W} - \widehat{j(V)}) \times [0,1] & \xrightarrow{\hat{G}_i} & \hat{W} - \widehat{j(V)} \\ \big\downarrow {\scriptstyle \Phi \times \mathrm{id}_{[0,1]}} & & \big\downarrow {\scriptstyle \Phi} \\ \mathcal{L}_{t_0} \cap (W - j(V)) \times [0,1] & \xrightarrow{G_i} & W - j(V) \end{array} \tag{3.1}$$

(here, recall that the blowing-down morphism Φ induces an isomorphism between $\mathcal{L}^\sharp_{t_0} \cap (\hat{W} - \widehat{j(V)})$ and $\mathcal{L}_{t_0} \cap (W - j(V))$; following Notation 1.3, all maps induced by Φ are denoted here by the same letter).

Proof. This follows in a standard manner from Claim 3.1, starting from $\hat{G}_i$ and then going down to G_i thanks to the isomorphism between $\mathcal{L}^\sharp_t \cap (\hat{W} - \widehat{j(V)})$ and $\mathcal{L}_t \cap (W - j(V))$ (cf. [**5**, Lemmas 4.1 and 4.2]). The statements for points (II) and ($\hat{\mathrm{I}}\hat{\mathrm{I}}$) in [**5**] are weaker than here but the proof given in [**5**] is valid for the stronger form. □

It must be noticed that the isotopies G_i and $\hat{G}_i$ are not uniquely determined by the loop γ_i. But, if Diagram (3.1) is commutative, one of them determines the other.

The ending stage of these isotopies will be the geometric monodromies we want. Lemma 3.3 implies the following.

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Lemma 3.4. *For $1 \leqslant i \leqslant N$, we have the homeomorphisms*

$$H_i: \quad \mathcal{L}_{t_0} \cap (W - j(V)) \to \mathcal{L}_{t_0} \cap (W - j(V)),$$
$$\hat{H}_i: \quad \mathcal{L}^{\sharp}_{t_0} \cap (\hat{W} - \widehat{j(V)}) \to \mathcal{L}^{\sharp}_{t_0} \cap (\hat{W} - \widehat{j(V)}),$$

defined by setting

$$H_i(x) = G_i(x, 1) \quad \text{and} \quad \hat{H}_i(v) = \hat{G}_i(v, 1).$$

These homeomorphisms leave fixed the points of $\mathcal{M} \cap (W - j(V))$ and $\mathcal{M} \cap (W - j(V)) \times \{t_0\}$, respectively. Moreover, if Diagram (3.1) is commutative, so is the following:

$$\begin{array}{ccc} \mathcal{L}^{\sharp}_{t_0} \cap (\hat{W} - \widehat{j(V)}) & \xrightarrow{\hat{H}_i} & \mathcal{L}^{\sharp}_{t_0} \cap (\hat{W} - \widehat{j(V)}) \\ \downarrow \wr \Phi & & \downarrow \wr \Phi \\ \mathcal{L}_{t_0} \cap (W - j(V)) & \xrightarrow{H_i} & \mathcal{L}_{t_0} \cap (W - j(V)) \end{array} \tag{3.2}$$

Definition 3.5. We shall call H_i a *geometric monodromy of $\mathcal{L}_{t_0} \cap (W - j(V))$ relative to $\mathcal{M} \cap (W - j(V))$ above γ_i*. Similarly for $\hat{H}_i$.

Notice that, like the isotopies giving rise to them, these geometric monodromies are not uniquely determined by the choice of loop γ_i. However, we have the following invariance property.

Lemma 3.6. *Given an index i with $1 \leqslant i \leqslant N$, another choice of loop γ_i within the same homotopy class $\bar{\gamma}_i$ in $\boldsymbol{P}^1 - \{t_1, \ldots, t_N\}$ and another choice of isotopies G_i and $\hat{G}_i$ above γ_i, provided they satisfy conditions (I)–(III) and ($\hat{\mathrm{I}}$)–($\widehat{\mathrm{III}}$), respectively, would lead to geometric monodromies which are isotopic to H_i and $\hat{H}_i$ through isotopies in $\mathcal{L}_{t_0} \cap (W - j(V))$ and $\mathcal{L}^{\sharp}_{t_0} \cap (\hat{W} - \widehat{j(V)})$ leaving pointwise fixed the subsets $\mathcal{M} \cap (W - j(V))$ and $\mathcal{M} \cap (W - j(V)) \times \{t_0\}$, respectively. This is true even if the new loop has not the special form described in Definition 3.2.*

Proof. Concerning $\hat{G}_i$, this is the classical invariance property of geometric monodromies with an enhancement about fixed points given by the trivial subbundle of Claim 3.1. A similar property holds for G_i since it can be associated to an isotopy $\hat{G}_i$ making Diagram (3.1) commutative. □

Thus, though the geometric monodromy H_i is not uniquely defined, its isotopy class in $\mathcal{L}_{t_0} \cap (W - j(V))$ relative to $\mathcal{M} \cap (W - j(V))$ is unique and wholly determined by the homotopy class $\bar{\gamma}_i$. This isotopy class can be called *the* geometric monodromy of $\mathcal{L}_{t_0} \cap (W - j(V))$ relative to $\mathcal{M} \cap (W - j(V))$ *associated to $\bar{\gamma}_i$*. Similarly for $\hat{H}_i$.

The isomorphisms H_{i*} and $\bar{H}_{i*}$ of

$$H_k(\mathcal{L}_{t_0} \cap (W - j(V))) \quad \text{and} \quad H_k(\mathcal{L}_{t_0} \cap (W - j(V)), \mathcal{M} \cap (W - j(V)))$$

induced by H_i depend only on this isotopy class. Similarly for the algebraic monodromies induced by $\hat{H}_i$. In particular, to obtain them, we could use geometric monodromies arising from maps G_i and $\hat{G}_i$ satisfying the looser requirements of Lemma 3.6. For instance, monodromies above the loops Γ_i described before Definition 3.2. Or even geometric monodromies not satisfying the commutativity of Diagram (3.2); the corresponding diagrams at the homology and relative homology levels would still be commutative.

Nevertheless, for convenience in our forthcoming constructions, we shall henceforth use special geometric monodromies H_i and $\hat{H}_i$ as we have constructed above, which are associated with a special set of loops as given in Definition 3.2 and which are linked together by the commutative Diagram (3.2).

4. The degeneration operator

In [**15**, § 2], homotopical degeneration operators are introduced for generic pencils of hyperplane sections of the complement in $\boldsymbol{C}^{n+1}$ of a hypersurface with isolated singularities (including at infinity).

The purpose of this section is to define projective analogues of these for the pencil $(L_t)_{t\in \boldsymbol{P}^1}$, acting on the $(n-1)$th homotopy group of each $L_{t_i} - L_{t_i} \cap V$ when $n \geqslant 3$ and on some subgroup of the fundamental group of each when $n = 2$. According to § 2.4, these homotopy groups are, when $n \geqslant 3$, canonically identified with the homology groups of d-fold covers introduced there and hence it is enough to define a homological degeneration operator on these homology groups. This also works when $n = 2$ thanks to the morphisms of Lemma 2.8 (iii) and still the isomorphisms of (i).

4.1. Homological degeneration operator on the cover

Suppose $n \geqslant 2$. For each i, with $1 \leqslant i \leqslant N$, we define such an operator D_i so that the following diagram is commutative, where Δ_i is a disc about t_i as described in Definition 3.2, and where the arrows labelled τ_i, wn_i and $\overline{\Phi_*}$ are to be defined in the remainder of this section:

$$\begin{array}{ccc} H_n(P^{-1}(\partial\Delta_i) \cap (\hat{W} - \widehat{j(V)})) & \xrightarrow[\sim]{\mathrm{wn}_i} & H_n(\mathcal{L}^\sharp_{t_0} \cap (\hat{W} - \widehat{j(V)}))/\operatorname{im}(\hat{H}_{i\,*} - \mathrm{id}) \\ \uparrow{\scriptstyle \tau_i} & & \downarrow{\scriptstyle \wr\, \overline{\Phi_*}} \\ H_{n-1}(\mathcal{L}_{t_i} \cap (W - j(V))) & \xrightarrow{D_i} & H_n(\mathcal{L}_{t_0} \cap (W - j(V)))/\operatorname{im}(H_{i\,*} - \mathrm{id}) \end{array} \tag{4.1}$$

The arrow labelled $\overline{\Phi_*}$ is easily defined as follows. The blowing-down morphism Φ induces an isomorphism between $\mathcal{L}^\sharp_{t_0} \cap (\hat{W} - \widehat{j(V)})$ and $\mathcal{L}_{t_0} \cap (W - j(V))$ (by (2.5)) which gives an isomorphism Φ_* between the nth homology groups of these spaces. This in turn factorizes into an isomorphism $\overline{\Phi_*}$ as indicated on the diagram thanks to the commutativity of Diagram (3.2). Recall that, by the invariance property of Lemma 3.6, isomorphisms $H_{i\,*}$ and $\hat{H}_{i\,*}$ depend only on the homotopy class $\bar{\gamma}_i$ of Definition 3.2.

The arrow labelled wn_i arises from the Wang sequence of the fibration of Claim 3.1 restricted to the part above the circle $\partial\Delta_i$. This is detailed below.

The arrow labelled τ_i is essentially a tube map in the Poincaré residue sequence for the complement of $\mathcal{L}^\sharp_{t_i} \cap (\hat{W} - \widehat{j(V)})$ in $\hat{W} - \widehat{j(V)}$. Details are also given below.

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The operator D_i depends only on the homotopy class $\bar{\gamma}_i$. This will be easy to see after the comparison between the degeneration and variation operators made in § 6 (see Corollary 6.4).

4.2. Definition of the isomorphism wn_i

To define wn_i, we use the Wang sequence of the fibration indicated above. In this sequence we want to use the fibre above t_0 of the fibration of Claim 3.1, though t_0 is outside of $\partial\Delta_i$, and the monodromy $\hat{H}_i$ above the loop γ_i of Definition 3.2 instead of the monodromy above $\partial\Delta_i$.

For this purpose, let us get the monodromy $\hat{H}_i$ in three steps, following the decomposition $\gamma_i = \delta_i * \omega_i * \delta_i^-$ given in Definition 3.2. Let

$$\hat{H}'_i : \quad \mathcal{L}^\sharp_{d_i} \cap (\hat{W} - \widehat{j(V)}) \to \mathcal{L}^\sharp_{d_i} \cap (\hat{W} - \widehat{j(V)})$$

be a geometric monodromy above ω_i defined in the same way as $\hat{H}_i$, using Lemmas 3.4 and 3.3 but replacing the parameter t_0 by the base point d_i of ω_i and the loop γ_i by the loop ω_i wherever they occur. Also let

$$\hat{H}''_i : \quad \mathcal{L}^\sharp_{t_0} \cap (\hat{W} - \widehat{j(V)}) \to \mathcal{L}^\sharp_{d_i} \cap (\hat{W} - \widehat{j(V)})$$

be a homeomorphism obtained with the same definitions, this time replacing γ_i by δ_i. If $\hat{H}'_i$ is constructed using isotopy $\hat{G}'_i$ and $\hat{H}''_i$ using $\hat{G}''_i$, we can build $\hat{H}_i$ from $\hat{G}_i = \hat{G}''_i * \hat{G}'_i * \hat{G}''^-_i$, where operations on isotopies parallel those on paths, each isotopy taking the fibre up in the place it was left by the former. This is a legal choice for $\hat{G}_i$ since it can easily been verified that it satisfies the conditions of Lemma 3.3. With this setting for $\hat{G}_i$, the corresponding geometric monodromy $\hat{H}_i$ is decomposed as

$$\hat{H}_i = \hat{H}''^{-1}_i \circ \hat{H}'_i \circ \hat{H}''_i. \tag{4.2}$$

As loop ω_i runs once around Δ_i, the monodromy $\hat{H}'_i$ above ω_i fits into the Wang sequence of the fibration above $\partial\Delta_i$ we consider. We embed this sequence into the following diagram where it appears as the upper line (we use Notation 2.7):

$$\begin{array}{ccccccc}
H_n(\mathcal{L}^{\sharp\prime}_{d_i}) & \xrightarrow{\hat{H}'_{i*}-\mathrm{id}} & H_n(\mathcal{L}^{\sharp\prime}_{d_i}) & \xrightarrow{\mathrm{incl}_*} & H_n(P'^{-1}(\partial\Delta_i)) & \longrightarrow & H_{n-1}(\mathcal{L}^{\sharp\prime}_{d_i}) \\
\uparrow \wr\, \hat{H}''_{i*} & & \uparrow \wr\, \hat{H}''_{i*} & & & & \\
H_n(\mathcal{L}^{\sharp\prime}_{t_0}) & \xrightarrow{\hat{H}_{i*}-\mathrm{id}} & H_n(\mathcal{L}^{\sharp\prime}_{t_0}) & & & &
\end{array} \tag{4.3}$$

It is commutative by (4.2). But $H_{n-1}(\mathcal{L}^{\sharp\prime}_{d_i}) = 0$ because this group is isomorphic to $H_{n-1}(\mathcal{L}_{d_i} \cap (W - j(V)))$, which vanishes when $n \geqslant 2$ (cf. (2.5) and Lemma 2.11). Thus the inclusion map induces an isomorphism

$$H_n(\mathcal{L}^{\sharp\prime}_{d_i}) / \operatorname{im}(\hat{H}'_{i*} - \mathrm{id}) \xrightarrow{\sim} H_n(P'^{-1}(\partial\Delta_i)).$$

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Then, by the commutativity of Diagram (4.3), the homeomorphism $\hat{H}''_i$ followed by inclusion also induces an isomorphism

$$H_n(\mathcal{L}^{\sharp\prime}_{t_0})/\operatorname{im}(\hat{H}_{i*} - \mathrm{id}) \xrightarrow{\sim} H_n(P'^{-1}(\partial\Delta_i)). \tag{4.4}$$

The isomorphism wn_i appearing in Diagram (4.1) is the inverse of this one.

4.3. Definition of the homomorphism τ_i

The homomorphism τ_i is defined as a tube map in a Poincaré residue sequence (also named the Leray or the Thom–Gysin sequence) through the following diagram, where we still use Notation 2.7:

$$\begin{array}{ccc}
H_{n-1}(\mathcal{L}'_{t_i}) & & \\
\uparrow \wr \Phi_* & & \\
H_{n-1}(\mathcal{L}^{\sharp\prime}_{t_i}) & \xrightarrow[\sim]{T_i} & H_{n+1}(\hat{W}', \hat{W}' - \mathcal{L}^{\sharp\prime}_{t_i}) \\
& & \uparrow \wr \ \mathrm{incl}_* \\
& & H_{n+1}(P'^{-1}(\Delta_i), P'^{-1}(\Delta_i) - \mathcal{L}^{\sharp\prime}_{t_i}) \\
& & \uparrow \wr \ \mathrm{incl}_* \\
& & H_{n+1}(P'^{-1}(\Delta_i), P'^{-1}(\partial\Delta_i)) \xrightarrow{\partial} H_n(P'^{-1}(\partial\Delta_i))
\end{array} \tag{4.5}$$

The homomorphism τ_i is obtained by overall composition from the upper-left to the lower-right end of the diagram (reversing the isomorphisms when necessary). The arrow labelled Φ_* is an isomorphism induced by the blowing-down morphism Φ (see (2.5)). Following our general convention, the arrows labelled incl_* are induced by inclusion. The upper one is an excision isomorphism. The lower one is also an isomorphism because $\partial\Delta_i$ is a strong deformation retract of $\Delta_i - \{t_i\}$ and the spaces $P'^{-1}(\partial\Delta_i)$ and $P'^{-1}(\Delta_i) - \mathcal{L}^{\sharp\prime}_{t_i}$ are the parts over $\partial\Delta_i$ and $\Delta_i - \{t_i\}$ of the locally trivial fibration of Claim 3.1, so that the inclusion of the former into the latter is a homotopy equivalence (see [**4**, proof of Lemme 4.4]).

The significant arrow is the one labelled T_i, which is a Leray (or Thom–Gysin) isomorphism. Such an isomorphism arises whenever one removes a closed submanifold P from a Hausdorff paracompact complex manifold N. If P has pure complex codimension c in N, this is an isomorphism from $H_{k-2c}(P)$ onto $H_k(N, N-P)$ holding for any k, with the convention that $H_{k-2c}(P) = 0$ for $k < 2c$ (cf. [**4**, Annexe]). Here we apply it with $N = \hat{W}'$, $P = \mathcal{L}^{\sharp\prime}_{t_i}$, $c = 1$ and $k = n+1$. We must verify that these settings satisfy the above conditions on N, P and c.

First, $\hat{W}' = \hat{W} - \widehat{j(V)}$ is the total transform of $W - j(V)$ when blowing $\boldsymbol{P}^{n+2}$ up (cf. § 2.3). But $W - j(V)$ is a submanifold of $\boldsymbol{P}^{n+2}$ by Claim 2.3 and the n-plane $\mathcal{M}$ along which $\boldsymbol{P}^{n+2}$ is blown up is, by Claim 2.6, transverse to $W - j(V)$. It follows that the total transform $\hat{W}'$ of $W - j(V)$ is a submanifold of $\hat{\boldsymbol{P}}^{n+2}$ (cf. [**4**, (5.5.1)]). It is Hausdorff paracompact since $\hat{\boldsymbol{P}}^{n+2}$ is a subspace of $\boldsymbol{P}^{n+2} \times \boldsymbol{P}^1$ which is metrizable.

Second, we have $\mathcal{L}^{\sharp\prime}_{t_i} = \mathcal{L}^{\sharp}_{t_i} \cap \hat{W}'$ and $\mathcal{L}^{\sharp}_{t_i}$ is the strict transform of $\mathcal{L}_{t_i}$, a member of the pencil $\mathcal{L}$ with base locus $\mathcal{M}$ introduced in § 2.3. As $\mathcal{L}_{t_i}$ is, by Claim 2.6, also transverse to $W - j(V)$, it follows that $\mathcal{L}^{\sharp}_{t_i}$ is transverse in $\hat{\boldsymbol{P}}^{n+2}$ to the total transform $\hat{W}'$ of $W - j(V)$ (cf. [**4**, (5.5.2)]). But $\mathcal{L}^{\sharp}_{t_i}$ is a closed submanifold of $\hat{\boldsymbol{P}}^{n+2}$ of pure complex codimension 1 as it follows from (2.4) and the fact that P is a submersion. Hence $\mathcal{L}^{\sharp\prime}_{t_i}$ is a closed submanifold of $\hat{W}'$ of pure complex codimension 1. The conditions of validity of the Leray isomorphism are thus checked.

This completes the definition of the homomorphism τ_i and hence of the homological degeneration operator D_i at the d-fold cover level.

4.4. Homotopical degeneration operator

For each i, with $1 \leqslant i \leqslant N$, the isomorphism α_{t_0} and the homomorphisms β_i of Lemma 2.8 will allow us, if $n \geqslant 2$, to define a homotopical degeneration operator $\mathcal{D}_i$ from the homology operator D_i constructed at the d-fold level in the preceding subsection.

We first define monodromies on $\pi_n(L_{t_0} - L_{t_0} \cap V, e)$ as the pull-backs of the monodromies on $H_n(\mathcal{L}_{t_0} \cap (W - j(V)))$ by the isomorphism α_{t_0}.

Definition 4.1. Let e be a base point in $M - M \cap V$ as in Lemma 2.8. If $n \geqslant 2$ and for $1 \leqslant i \leqslant N$, the monodromy $h_{i\,\#}$ is defined by the commutativity of the following diagram:

$$\begin{array}{ccc} H_n(\mathcal{L}_{t_0} \cap (W - j(V))) & \xrightarrow[\sim]{H_{i\,*}} & H_n(\mathcal{L}_{t_0} \cap (W - j(V))) \\ \uparrow \wr\, \alpha_{t_0} & & \uparrow \wr\, \alpha_{t_0} \\ \pi_n(L_{t_0} - L_{t_0} \cap V, e) & \xrightarrow[\sim]{h_{i\,\#}} & \pi_n(L_{t_0} - L_{t_0} \cap V, e) \end{array} \tag{4.6}$$

As $H_{i\,*}$ depends only on the homotopy class $\bar{\gamma}_i$ of Definition 3.2, so does the monodromy $h_{i\,\#}$.

Remark 4.2. In fact $h_{i\,\#}$ is indeed induced on the nth homotopy group by a geometric monodromy h_i of $L_{t_0} - L_{t_0} \cap V$ as the notation suggests. Such a monodromy is obtained in the same way as H_i was defined from the isotopy G_i, by using an isotopy g_i satisfying conditions similar to those given for G_i in Lemma 3.3. This exists by [**5**, Lemma 4.1]. Then h_i satisfies an invariance property similar to that of Lemma 3.6 and induces an automorphism of $\pi_n(L_{t_0} - L_{t_0} \cap V, e)$ depending only on $\bar{\gamma}_i$. But h_i and H_i can be chosen so that they commute with the covering projection π. To do this, one starts from g_i and defines G_i as a lift of g_i by π satisfying the initial condition (I) of Lemma 3.3. Then one can see that G_i satisfies also automatically conditions (II) and (III) thanks to the similar conditions satisfied by g_i. The corresponding geometric monodromies h_i and H_i commute with π as desired. This fact together with the functoriality of the Hurewicz homomorphisms entail that the induced automorphism $h_{i\,\#}$ of $\pi_n(L_{t_0} - L_{t_0} \cap V, e)$ makes Diagram (4.6) commutative (recall the definition of α_{t_0} in Lemma 2.8). Hence this $h_{i\,\#}$ coincides with the one of Definition 4.1.

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If $n \geqslant 2$ and for $1 \leqslant i \leqslant N$, the commutative Diagram (4.6) allows to define in turn an isomorphism $\overline{\alpha_{t_0}}$ making commutative the following diagram:

$$\begin{array}{ccc} H_n(\mathcal{L}_{t_0} \cap (W - j(V))) & \xrightarrow{\text{can}} & H_n(\mathcal{L}_{t_0} \cap (W - j(V)))/\operatorname{im}(H_{i\,*} - \mathrm{id}) \\ \uparrow \wr\, \alpha_{t_0} & & \uparrow \wr\, \overline{\alpha_{t_0}} \\ \pi_n(L_{t_0} - L_{t_0} \cap V, e) & \xrightarrow{\text{can}} & \pi_n(L_{t_0} - L_{t_0} \cap V, e)/\operatorname{im}(h_{i\,\#} - \mathrm{id}) \end{array} \tag{4.7}$$

Then, if $n \geqslant 3$ and for $1 \leqslant i \leqslant N$, the isomorphism β_i of Lemma 2.8 (ii) together with this isomorphism $\overline{\alpha_{t_0}}$ lead from the homological operator D_i to the homotopical degeneration operator $\mathcal{D}_i$ we are looking for. This is done by asking the following diagram to be commutative:

$$\begin{array}{ccc} H_{n-1}(\mathcal{L}_{t_i} \cap (W - j(V))) & \xrightarrow{D_i} & H_n(\mathcal{L}_{t_0} \cap (W - j(V)))/\operatorname{im}(H_{i\,*} - \mathrm{id}) \\ \uparrow \wr\, \beta_i & & \uparrow \wr\, \overline{\alpha_{t_0}} \\ \pi_{n-1}(L_{t_i} - L_{t_i} \cap V, e) & \xrightarrow{\mathcal{D}_i} & \pi_n(L_{t_0} - L_{t_0} \cap V, e)/\operatorname{im}(h_{i\,\#} - \mathrm{id}) \end{array} \tag{4.8}$$

When $n = 2$, the isomorphism $\overline{\alpha_{t_0}}$ and this time the homomorphism β_i of Lemma 2.8 (iii) lead to an operator $\mathcal{D}_i$ defined on the subgroup G_i introduced there, by asking the following diagram to be commutative:

$$\begin{array}{ccc} H_1(\mathcal{L}_{t_i} \cap (W - j(V))) & \xrightarrow{D_i} & H_2(\mathcal{L}_{t_0} \cap (W - j(V)))/\operatorname{im}(H_{i\,*} - \mathrm{id}) \\ \uparrow \beta_i & & \uparrow \wr\, \overline{\alpha_{t_0}} \\ G_i & \xrightarrow{\mathcal{D}_i} & \pi_2(L_{t_0} - L_{t_0} \cap V, e)/\operatorname{im}(h_{i\,\#} - \mathrm{id}) \end{array} \tag{4.9}$$

Like D_i, operator $\mathcal{D}_i$ depends only on the homotopy class $\bar{\gamma}_i$ of Definition 3.2.

5. The variation operator

In [**5**, § 4] homological variation operators are defined for generic pencils of hyperplane sections of a quasi-projective variety. They are analogous to the classical variation operator associated with the Milnor fibration of an isolated singularity (see [**2**, Chapter 2]). In our situation they give homological variation operators for the pencil $(L_t)_{t \in \boldsymbol{P}^1}$, defined on the nth relative homology group of $L_{t_0} - L_{t_0} \cap V$ modulo $M - M \cap V$ and associated with each special member L_{t_i} of the pencil, more precisely with the homotopy class $\bar{\gamma}_i$ in $\boldsymbol{P}^1 - \{t_1, \ldots, t_N\}$ of a loop γ_i surrounding t_i in the parameter space as in Definition 3.2.

In this section we want to define, when $n \geqslant 2$, homotopical analogues of these,

$$\mathcal{V}_i : \quad \pi_n(L_{t_0} - L_{t_0} \cap V, M - M \cap V, e) \to \pi_n(L_{t_0} - L_{t_0} \cap V, e)$$

associated with $\bar{\gamma}_i$ for $1 \leqslant i \leqslant N$, where e is a base point in $M - M \cap V$.

As for the degeneration operators, we shall go to the d-fold cover level and use homological variation operators defined there,

$$V_i : \quad H_n(\mathcal{L}_{t_0} \cap (W - j(V)), \mathcal{M} \cap (W - j(V))) \to H_n(\mathcal{L}_{t_0} \cap (W - j(V)))$$

associated, for $1 \leqslant i \leqslant N$, with the homotopy classes $\bar{\gamma}_i$ of Definition 3.2.

We recall the definition and properties of the operator V_i as given in [**5**, § 4], which in fact hold with $n \geqslant 1$.

For any relative n-cycle $\varXi$ on $\mathcal{L}_{t_0} \cap (W - j(V))$ modulo $\mathcal{M} \cap (W - j(V))$, one defines

$$V_i([\varXi]_{(\mathcal{L}_{t_0} \cap (W-j(V)), \mathcal{M} \cap (W-j(V)))}) = [H_{i\bullet}(\varXi) - \varXi]_{\mathcal{L}_{t_0} \cap (W-j(V))} \tag{5.1}$$

using Notations 1.4 and 1.5. Due to the fact that H_i leaves the points of $\mathcal{M} \cap (W - j(V))$ fixed (Lemma 3.4), the chain $H_{i\bullet}(\varXi) - \varXi$ is actually an absolute cycle and the correspondence $\varXi \mapsto H_{i\bullet}(\varXi) - \varXi$ induces a homomorphism V_i at the homology level (see [**5**, Lemmas 4.6 and 4.8]). Thanks to the invariance property expressed by Lemma 3.6, this homomorphism depends only on the homotopy class $\bar{\gamma}_i$ (see [**5**, Lemma 4.8]).

Now, if $n \geqslant 2$ and for $1 \leqslant i \leqslant N$, the isomorphism α_{t_0} of Lemma 2.8 and the homomorphism $\bar{\alpha}_{t_0}$ of Lemma 2.10 lead from V_i to the wanted homotopical variation operator $\mathcal{V}_i$ by asking the following diagram to be commutative:

$$\begin{array}{ccc} H_n(\mathcal{L}_{t_0} \cap (W - j(V)), \mathcal{M} \cap (W - j(V))) & \xrightarrow{V_i} & H_n(\mathcal{L}_{t_0} \cap (W - j(V))) \\ \uparrow{\scriptstyle \bar{\alpha}_{t_0}} & & \uparrow{\scriptstyle \wr\, \alpha_{t_0}} \\ \pi_n(L_{t_0} - L_{t_0} \cap V, M - M \cap V, e) & \xrightarrow{\mathcal{V}_i} & \pi_n(L_{t_0} - L_{t_0} \cap V, e) \end{array} \tag{5.2}$$

As V_i depends only on the homotopy class $\bar{\gamma}_i$ of Definition 3.2, so does the operator $\mathcal{V}_i$.

Remark 5.1. The homological variation operators

$$v_i : \quad H_n(L_{t_0} - L_{t_0} \cap V, M - M \cap V) \to H_n(L_{t_0} - L_{t_0} \cap V)$$

we talked about at the beginning of the section are given by a formula similar to 5.1 using the monodromies h_i considered in Remark 4.2. The homotopical variation operators $\mathcal{V}_i$ we have defined here are linked to those by Hurewicz homomorphisms as is shown in the following diagram:

$$\begin{array}{ccc} \pi_n(L_{t_0} - L_{t_0} \cap V, M - M \cap V, e) & \xrightarrow{\mathcal{V}_i} & \pi_n(L_{t_0} - L_{t_0} \cap V, e) \\ \downarrow{\scriptstyle \bar{\chi}} & & \downarrow{\scriptstyle \chi} \\ H_n(L_{t_0} - L_{t_0} \cap V, M - M \cap V) & \xrightarrow{v_i} & H_n(L_{t_0} - L_{t_0} \cap V) \end{array}$$

The commutativity of this diagram is a consequence of the commutativity of the Diagram (5.2) defining $\mathcal{V}_i$, of the definitions of the homomorphisms α_{t_0} and $\bar{\alpha}_{t_0}$ occurring there (see Lemmas 2.8 and 2.10), of the functoriality of the Hurewicz homomorphisms and of the commutation of the monodromies h_i and H_i with the covering projection π as stated in Remark 4.2.

We end this section by noticing that the homotopical variation operator $\mathcal{V}_i$ when restricted to absolute cycles acts like the variation $h_{i\,\#} - \mathrm{id}$ of the homotopical monodromy associated with $\bar{\gamma}_i$. This is specified by the following lemma.

Lemma 5.2. *If $n \geqslant 2$ then, for $1 \leqslant i \leqslant N$, the following diagram is commutative:*

$$\begin{array}{ccc} \pi_n(L_{t_0} - L_{t_0} \cap V, e) & \xrightarrow{\mathrm{incl}_\#} & \pi_n(L_{t_0} - L_{t_0} \cap V, M - M \cap V, e) \\ & \searrow^{h_{i\,\#} - \mathrm{id}} & \downarrow \mathcal{V}_i \\ & & \pi_n(L_{t_0} - L_{t_0} \cap V, e) \end{array} \tag{5.3}$$

Proof. This will follow from the commutativity of the corresponding homology diagram at the d-fold covering level:

$$\begin{array}{ccc} H_n(\mathcal{L}_{t_0} \cap (W - j(V))) & \xrightarrow{\mathrm{incl}_*} & H_n(\mathcal{L}_{t_0} \cap (W - j(V)), \mathcal{M} \cap (W - j(V))) \\ & \searrow^{H_{i\,*} - \mathrm{id}} & \downarrow V_i \\ & & H_n(\mathcal{L}_{t_0} \cap (W - j(V))) \end{array} \tag{5.4}$$

Indeed, Diagrams (5.3) and (5.4) are linked together by the homomorphisms α_{t_0} and $\bar{\alpha}_{t_0}$ and one can use the commutativity of Diagrams (2.8), (5.2) and (4.6) and the injectivity of α_{t_0}. The commutativity of Diagram (5.4) can be checked in turn by a straightforward computation at the chain level using formula (5.1). □

6. The link between degeneration and variation operators

In this section we make the link between the degeneration operators $\mathcal{D}_i$ defined in § 4 and the variation operators $\mathcal{V}_i$ defined in § 5. As a side result, we shall obtain the invariance property for the degeneration operators D_i and hence $\mathcal{D}_i$, stated in § 4.

The main result is the following.

Proposition 6.1. *Using Notation 2.7, we have, for $1 \leqslant i \leqslant N$, the following commutative diagram if $n \geqslant 3$:*

$$\begin{array}{ccccc} \pi_{n-1}(M', e) & \xrightarrow{\mathrm{incl}_\#} & \pi_{n-1}(L'_{t_i}, e) & \xrightarrow{\mathcal{D}_i} & \pi_n(L'_{t_0}, e)/\operatorname{im}(h_{i\,\#} - \mathrm{id}) \\ \uparrow \partial & & & & \uparrow \mathrm{can} \\ \pi_n(L'_{t_0}, M', e) & & \xrightarrow{\qquad \mathcal{V}_i \qquad} & & \pi_n(L'_{t_0}, e) \end{array}$$

and the following one if $n = 2$:

$$\begin{array}{ccccc} H & \xrightarrow{\mathrm{incl}_\#} & G_i & \xrightarrow{\mathcal{D}_i} & \pi_2(L'_{t_0}, e)/\operatorname{im}(h_{i\,\#} - \mathrm{id}) \\ \uparrow \partial & & & & \uparrow \mathrm{can} \\ \pi_2(L'_{t_0}, M', e) & & \xrightarrow{\qquad \mathcal{V}_i \qquad} & & \pi_2(L'_{t_0}, e) \end{array}$$

where the groups H and G_i are defined in Lemma 2.8 (iii) and the homomorphisms $\mathrm{incl}_\#$ and ∂ are well defined by Lemma 2.8 (iii) and Lemma 2.10.

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Before proving this result, we state a corollary relating the images of the considered operators.

Corollary 6.2. *If $n \geqslant 2$, then, for $1 \leqslant i \leqslant N$,*

$$\operatorname{im} \mathcal{D}_i = \operatorname{im} \mathcal{V}_i / \operatorname{im}(h_{i\,\#} - \mathrm{id}).$$

This makes sense since $\mathcal{D}_i$ takes its values in $\pi_n(L_{t_0} - L_{t_0} \cap V, e)/\operatorname{im}(h_{i\,\#} - \mathrm{id})$ while $\mathcal{V}_i$ takes its values in $\pi_n(L_{t_0} - L_{t_0} \cap V, e)$ with an image containing $\operatorname{im}(h_{i\,\#} - \mathrm{id})$ by Lemma 5.2.

Proof of Corollary 6.2. The inclusion $\operatorname{im} \mathcal{D}_i \supset \operatorname{im} \mathcal{V}_i / \operatorname{im}(h_{i\,\#} - \mathrm{id})$ is clear from the diagrams of Proposition 6.1. The reverse inclusion will also be clear from it, once the following lemma is proved. □

Lemma 6.3. *The homomorphisms ∂ and $\mathrm{incl}_{\#}$ in the diagrams of Proposition 6.1 are surjective.*

Proof. The case $n = 2$ forces us to go into the d-fold covers. It is then more economical to treat the general case thus. The covering projection π induces an isomorphism from $\pi_n(\mathcal{L}'_{t_0}, \mathcal{M}', \varepsilon)$ onto $\pi_n(L'_{t_0}, M', e)$ if $n \geqslant 2$ (see Lemma 2.10), an isomorphism from $\pi_{n-1}(\mathcal{M}', \varepsilon)$ onto $\pi_{n-1}(M', e)$ when $n \geqslant 3$ (respectively, H when $n = 2$) and an isomorphism from $\pi_{n-1}(\mathcal{L}'_{t_i}, \varepsilon)$ onto $\pi_{n-1}(L'_{t_i}, e)$ when $n \geqslant 3$ (respectively, G_i when $n = 2$) (see Lemma 2.8). It will then be enough to prove that the homomorphisms $\partial : \pi_n(\mathcal{L}'_{t_0}, \mathcal{M}', \varepsilon) \to \pi_{n-1}(\mathcal{M}', \varepsilon)$ and $\mathrm{incl}_{\#} : \pi_{n-1}(\mathcal{M}', \varepsilon) \to \pi_{n-1}(\mathcal{L}'_{t_i}, \varepsilon)$ are surjective. This is the case for the homomorphism ∂ due to the homotopy exact sequence of the pair $(\mathcal{L}'_{t_0}, \mathcal{M}')$ and to the fact that $\pi_{n-1}(\mathcal{L}'_{t_0}, \varepsilon)$ is trivial if $n \geqslant 3$ as noticed in the proof of Lemma 2.8 and also when $n = 2$ by Lemma 2.9. As to the homomorphism $\mathrm{incl}_{\#}$, it is surjective due to the Lefschetz hyperplane section theorem for non-singular quasi-projective varieties (cf. [**11**, Theorem 1.1.3] or [**9**, § II.5.1]) when applied to the hyperplane $\mathcal{M} \subset \mathcal{L}_{t_i}$ cutting the quasi-projective variety $\mathcal{L}'_{t_i} = \mathcal{L}_{t_i} \cap (W - j(V))$ which is non-singular and of pure dimension n by Claims 2.3 and 2.6. For the validity of the quoted theorem, the hyperplane $\mathcal{M}$ must fulfil some condition of genericity. By [**12**, Lemma of the Appendix], or [**9**, the remark ending § II.5.1], it is enough that $\mathcal{M}$ be transverse to all the strata of a Whitney stratification of $\mathcal{L}_{t_i} \cap W$ having $\mathcal{L}_{t_i} \cap j(V)$ as a union of strata. But, thanks to the transversality of $\mathcal{M}$ to the strata of the stratification Σ of W defined in Claim 2.5, the trace of Σ on $\mathcal{L}_{t_i}$ can be refined into such a stratification of $\mathcal{L}_{t_i} \cap W$ (cf. [**4**, Lemme 11.3]). □

The sequel of this section will be devoted to the proof of Proposition 6.1 and to the result on the invariance of the operator D_i mentioned above.

Proof of Proposition 6.1

We shall go up to the homology of d-fold coverings and consider for $n \geqslant 2$ the following homology diagram which corresponds at this level to the diagrams of Proposition 6.1 (it also uses Notation 2.7):

$$\begin{array}{ccccc} H_{n-1}(\mathcal{M}') & \xrightarrow{\text{incl}_*} & H_{n-1}(\mathcal{L}'_{t_i}) & \xrightarrow{D_i} & H_n(\mathcal{L}'_{t_0})/\operatorname{im}(H_{i\,*}-\mathrm{id}) \\ \uparrow{\scriptstyle\partial} & & & & \uparrow{\scriptstyle\text{can}} \\ H_n(\mathcal{L}'_{t_0},\mathcal{M}') & & \xrightarrow{\quad V_i \quad} & & H_n(\mathcal{L}'_{t_0}) \end{array} \tag{6.1}$$

It will be enough to show that this diagram is commutative since it is linked to the first or second diagram of Proposition 6.1, according to whether $n \geqslant 3$ or $n = 2$, by the commutative Diagrams (2.8) or (2.9) (right-hand parts), (2.6) or (2.7), (4.8) or (4.9), (5.2) and (4.7), and since the homomorphism $\overline{\alpha_{t_0}}$ (defined by (4.7)) linking the upper-right corners of these two diagrams is injective.

Before proving the commutativity of Diagram 6.1, we first treat the invariance property for the operators D_i mentioned in § 4.1.

Corollary 6.4. *For $n \geqslant 2$ and $1 \leqslant i \leqslant N$, the operator D_i depends only on the homotopy class $\bar{\gamma}_i$ of Definition 3.2.*

Proof. As we saw in § 5, the same invariance property holds for the operator V_i. Then, the commutativity of Diagram (6.1) implies the assertion for D_i because, as we shall see, the arrows labelled ∂ and incl_* are surjective. The surjectivity of the homomorphism ∂ results from the homology exact sequence of the pair $(\mathcal{L}'_{t_0}, \mathcal{M}')$ and the fact that

$$H_{n-1}(\mathcal{L}'_{t_0}) = 0 \quad \text{if } n \geqslant 2$$

by Lemma 2.11. The homomorphism induced by inclusion labelled incl_* is surjective due to the Lefschetz hyperplane section theorem for non-singular quasi-projective varieties with the same justification as in the proof of Lemma 6.3 but this time applied to homology. □

The commutativity of Diagram 6.1 is a consequence of the following. On one hand, the bundle of Claim 3.1 has a trivial subbundle preserved by the isotopy built in § 3 which has a trivial form on it, allowing thus the very definition of the homological variation operators. On the other hand, this subbundle extends to a product $\mathcal{M}' \times \boldsymbol{P}^1$ which is transverse to each $\mathcal{L}^{\sharp\prime}_{t_i}$, so that the tube maps entering in the definition of the homological degeneration operators have also a trivial form when restricted to $\mathcal{M}' \times \boldsymbol{P}^1$. These two facts lead to the link between the operators V_i and D_i expressed by the commutativity of Diagram 6.1. In fact similar considerations come up in the proof of Proposition 4.13 of [**5**] and our assertion will be obtained along the same lines.

More precisely, we imbed Diagram (6.1) in a larger one, putting the following diagram on its top (we still use Notation 2.7):

$$\begin{array}{ccccc} H_n(\mathcal{M}'\times\partial\Delta_i) & \xrightarrow{\mathrm{incl}_*} & H_n(P'^{-1}(\partial\Delta_i)) & \xrightarrow[\sim]{\mathrm{wn}_i} & H_n(\mathcal{L}^{\sharp\prime}_{t_0})/\operatorname{im}(\hat{H}_{i\,*}-\mathrm{id}) \\ \uparrow{\scriptstyle\kappa_i} & & \uparrow{\scriptstyle\tau_i} & & \downarrow{\scriptstyle\wr\,\overline{\varPhi_*}} \\ H_{n-1}(\mathcal{M}') & \xrightarrow{\mathrm{incl}_*} & H_{n-1}(\mathcal{L}'_{t_i}) & \xrightarrow{D_i} & H_n(\mathcal{L}'_{t_0})/\operatorname{im}(H_{i\,*}-\mathrm{id}) \end{array} \tag{6.2}$$

The right-hand square is just Diagram (4.1), which defines D_i, and the homomorphism κ_i is given by the formula

$$\kappa_i(z)=(-1)^{n-1}z\times[\omega_i]_{\partial\Delta_i}\quad\text{for } z\in H_{n-1}(\mathcal{M}'), \tag{6.3}$$

using the cross-product by the fundamental class $[\omega_i]_{\partial\Delta_i}$ of $\partial\Delta_i$, the loop ω_i of Definition 3.2 being this time considered as a 1-cycle.

Now, to prove the commutativity of Diagram 6.1, it will be enough to show that Diagram (6.2) is commutative as well as the following diagram which is the outer square of the big diagram obtained by putting Diagrams (6.1) and (6.2) on top of each other:

$$\begin{array}{ccccc} H_n(\mathcal{M}'\times\partial\Delta_i) & \xrightarrow{\mathrm{incl}_*} & H_n(P'^{-1}(\partial\Delta_i)) & \xrightarrow[\sim]{\mathrm{wn}_i} & H_n(\mathcal{L}^{\sharp\prime}_{t_0})/\operatorname{im}(\hat{H}_{i\,*}-\mathrm{id}) \\ \uparrow{\scriptstyle\kappa_i} & & & & \downarrow{\scriptstyle\wr\,\overline{\varPhi_*}} \\ H_{n-1}(\mathcal{M}') & & & & H_n(\mathcal{L}'_{t_0})/\operatorname{im}(H_{i\,*}-\mathrm{id}) \\ \uparrow{\scriptstyle\partial} & & & & \uparrow{\scriptstyle\mathrm{can}} \\ H_n(\mathcal{L}'_{t_0},\mathcal{M}') & & \xrightarrow{\qquad\qquad V_i\qquad\qquad} & & H_n(\mathcal{L}'_{t_0}) \end{array} \tag{6.4}$$

Proof of the commutativity of Diagram (6.2)

The right-hand square of this diagram is commutative since it is the commutative Diagram (4.1) defining D_i. To see the commutativity of the left-hand square, let us consider again Diagram (4.5) through which the homomorphism τ_i was defined. There is an analogous diagram obtained by restricting to $\mathcal{M}$ or $\hat{\mathcal{M}}$ in order to take advantage of the product structure $\hat{\mathcal{M}}=\mathcal{M}\times\boldsymbol{P}^1$. Here is this diagram:

$$\begin{array}{cccc} H_{n-1}(\mathcal{M}') & & & \\ \uparrow{\scriptstyle\wr\,\varPhi_*} & & & \\ H_{n-1}(\mathcal{M}'\times\{t_i\}) & \xrightarrow[\sim]{T'_i} & H_{n+1}(\mathcal{M}'\times\boldsymbol{P}^1,\mathcal{M}'\times(\boldsymbol{P}^1-\{t_i\})) & \\ & & \uparrow{\scriptstyle\wr\,\mathrm{incl}_*} & \\ & & H_{n+1}(\mathcal{M}'\times\Delta_i,\mathcal{M}'\times(\Delta_i-\{t_i\})) & \\ & & \uparrow{\scriptstyle\wr\,\mathrm{incl}_*} & \\ & & H_{n+1}(\mathcal{M}'\times\Delta_i,\mathcal{M}'\times\partial\Delta_i) & \xrightarrow{\partial} H_n(\mathcal{M}'\times\partial\Delta_i) \end{array} \tag{6.5}$$

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As in Diagram (4.5), the top isomorphism is induced by the blowing-down morphism Φ (see (2.3)), the upper arrow labelled incl_* is an excision isomorphism and the lower one is an isomorphism since $\partial\Delta_i$ is a deformation retract of $\Delta_i - \{t_i\}$. The arrow labelled T_i' is again a Leray isomorphism. The conditions of validity would be easy to check directly but they will also follow from a naturality property we shall consider in a moment.

Each space occurring in Diagram (6.5) is contained in the corresponding space of Diagram (4.5). This is clear for $\mathcal{M}' \subset \mathcal{L}'_{t_i}$ since $\mathcal{M} \subset \mathcal{L}_{t_i}$ and it can be seen, using (2.3) and (2.4), that all other spaces of (6.5) are the intersections of the corresponding spaces of (4.5) with $\hat{\mathcal{M}} = \mathcal{M} \times \boldsymbol{P}^1$. Thus Diagram (6.5) is linked to Diagram (4.5) by homomorphisms induced by inclusions. All resulting squares are commutative. This simply follows from the commutativity of the corresponding diagrams of maps or from the functoriality of the boundary homomorphism, except for the commutativity of the following diagram which deserves to be commented on:

$$\begin{array}{ccc} H_{n-1}(\mathcal{L}^{\sharp\prime}_{t_i}) & \xrightarrow[\sim]{T_i} & H_{n+1}(\hat{W}', \hat{W}' - \mathcal{L}^{\sharp\prime}_{t_i}) \\ \uparrow{\scriptstyle \mathrm{incl}_*} & & \uparrow{\scriptstyle \mathrm{incl}_*} \\ H_{n-1}(\mathcal{M}' \times \{t_i\}) & \xrightarrow[\sim]{T_i'} & H_{n+1}(\mathcal{M}' \times \boldsymbol{P}^1, \mathcal{M}' \times (\boldsymbol{P}^1 - \{t_i\})) \end{array} \tag{6.6}$$

The commutativity of this diagram results from the following naturality property for the Leray isomorphism. With the same notation and hypotheses as in the exposition we gave of it in § 4.3, suppose that N' is a closed complex submanifold of N transverse to P and let $P' = N' \cap P$. Then the validity conditions are also satisfied for a Leray isomorphism from $H_{k-2c}(P')$ onto $H_k(N', N' - P')$ and the diagram formed by the two Leray isomorphisms and the homomorphisms induced by inclusions is commutative (cf. [**4**, Annexe]). Applying these facts with $N = \hat{W}'$, $P = \mathcal{L}^{\sharp\prime}_{t_i}$, $c = 1$, $k = n+1$ as before and $N' = \mathcal{M}' \times \boldsymbol{P}^1$, we find Diagram (6.6) since $\mathcal{M}' \times \{t_i\} = (\mathcal{M}' \times \boldsymbol{P}^1) \cap \mathcal{L}^{\sharp\prime}_{t_i}$ (still by (2.3) and (2.4)). But we must verify that this setting for N' satisfies the conditions above.

Let us come back to § 4.3 where we checked the conditions of validity of the Leray isomorphism T_i. The properties which gave us that $\hat{W}'$ is a submanifold of $\hat{\boldsymbol{P}}^{n+2}$ give also, by the same reference, that $\hat{W}'$ is transverse to $\hat{\mathcal{M}}$ in $\hat{\boldsymbol{P}}^{n+2}$. Hence $\hat{\mathcal{M}} \cap \hat{W}' = \mathcal{M}' \times \boldsymbol{P}^1$ is a submanifold of $\hat{W}'$. It is closed since $\hat{\mathcal{M}}$ is closed in $\hat{\boldsymbol{P}}^{n+2}$. Next, the properties which gave us that $\mathcal{L}^{\sharp}_{t_i}$ is transverse to $\hat{W}'$ in $\hat{\boldsymbol{P}}^{n+2}$ give in fact, by the same reference, that $\mathcal{L}^{\sharp}_{t_i}$ is transverse to $\hat{\mathcal{M}} \cap \hat{W}'$ in $\hat{\boldsymbol{P}}^{n+2}$. As $\hat{W}'$ contains $\hat{\mathcal{M}} \cap \hat{W}'$, it follows that $\hat{\mathcal{M}} \cap \hat{W}' = \mathcal{M}' \times \boldsymbol{P}^1$ is transverse to $\mathcal{L}^{\sharp}_{t_i} \cap \hat{W}' = \mathcal{L}^{\sharp\prime}_{t_i}$ in $\hat{W}'$ (cf. [**4**, proof of Lemme 9.2 (iii)]). The conditions for the natural behaviour of the Leray isomorphisms T_i and T_i' are thus checked and the commutativity of Diagram (6.6) is proved.

Thus Diagrams (6.5) and (4.5) are linked in a commutative diagram by homomorphisms induced by inclusions. Let τ_i' be the homomorphism obtained by overall composition from the upper-left to the lower-right end of Diagram (6.5). As the homomorphism τ_i is obtained in the same manner in Diagram (4.5), we get the commutativity of the left square of Diagram (6.2) but with κ_i replaced by τ'_i. The commutativity of the original diagram will then follow from the next lemma.

Lemma 6.5. *The homomorphism τ'_i defined above is equal to the homomorphism κ_i defined in (6.3).*

Proof. This is due to the product structure of the spaces in Diagram (6.5), especially to the behaviour of the Leray isomorphism in such a case. With the same notation as in the presentation of this isomorphism in § 4.3, suppose that $N = Q \times R$ where Q and R are complex Hausdorff paracompact manifolds with R of pure complex dimension c and suppose that $P = Q \times \{r\}$ with $r \in R$. Then the conditions of validity hold for a Leray isomorphism T from $H_{k-2c}(Q \times \{r\})$ onto $H_k(Q \times R, Q \times R - Q \times \{r\})$ and this isomorphism takes the following special form. If $z^\sharp \in H_{k-2c}(Q \times \{r\})$ corresponds to $z \in H_{k-2c}(Q)$ by the canonical identification of Q to $Q \times \{r\}$, then $T(z^\sharp) = z \times w$ where $w \in H_{2c}(R, R - \{r\})$ is the fundamental class defining the canonical orientation of R about r (cf. [**4**, Annexe]). Here we are in this special case for T'_i, with $Q = \mathcal{M}'$, $R = \boldsymbol{P}^1$, $r = t_i$, $c = 1$ and $k = n + 1$. Remember indeed that $\mathcal{M}$ is transverse to $W - j(V)$ in $\boldsymbol{P}^{n+2}$ so that $\mathcal{M}' = \mathcal{M} \cap (W - j(V))$ is a submanifold of $\boldsymbol{P}^{n+2}$. Thus the Leray isomorphism T'_i has the explicit expression

$$T'_i(z^\sharp) = z \times u_i \quad \text{for } z \in H_{n-1}(\mathcal{M}'), \tag{6.7}$$

where $z^\sharp$ corresponds to z by the canonical identification of $\mathcal{M}'$ to $\mathcal{M}' \times \{t_i\}$ and where $u_i \in H_2(\boldsymbol{P}^1, \boldsymbol{P}^1 - \{t_i\})$ is the fundamental class defining the canonical orientation of $\boldsymbol{P}^1$ about t_i.

Now $\Phi_*(z^\sharp) = z$ by the remark following (2.3). Besides, we can take a representative Ω_i of u_i which is a relative 2-cycle of Δ_i modulo $\partial\Delta_i$. Then, by functoriality of the cross-product, the composition of the two isomorphisms of Diagram (6.5) called incl_* gives an isomorphism which we still denote by incl_*, such that

$$\mathrm{incl}_*(z \times [\Omega_i]_{(\Delta_i, \partial\Delta_i)}) = z \times u_i.$$

Combining these facts with (6.7), we find that, for $z \in H_{n-1}(\mathcal{M}')$,

$$\tau'_i(z) = \partial(z \times [\Omega_i]_{(\Delta_i, \partial\Delta_i)}) = (-1)^{n-1} z \times \partial[\Omega_i]_{(\Delta_i, \partial\Delta_i)}.$$

But, by the special choice of ω_i in Definition 3.2,

$$\partial[\Omega_i]_{(\Delta_i, \partial\Delta_i)} = [\omega_i]_{\partial\Delta_i}$$

and the equality $\tau'_i = \kappa_i$ follows. □

This concludes the proof of the commutativity of the left part and hence of the whole of Diagram (6.2).

Proof of the commutativity of Diagram (6.4)

It will be convenient to concentrate our work on the bundle $P'^{-1}(\partial\Delta_i)$ which already was used to define the isomorphism wn_i in § 4.1. We hence shall express the variation V_i by means of a variation operator V_i' above the loop ω_i of Definition 3.2 which runs once counterclockwise around $\partial\Delta_i$. Let us recall that d_i is the base point of ω_i. We still use Notation 2.7.

Definition 6.6. We define a homological variation operator

$$V_i': \quad H_n(\mathcal{L}'_{d_i}, \mathcal{M}') \to H_n(\mathcal{L}'_{d_i})$$

in the same way as V_i was defined in formula (5.1) but replacing $\mathcal{L}'_{t_0}$ by $\mathcal{L}'_{d_i}$ and the monodromy H_i by a monodromy H_i' above ω_i.

Just as V_i depends only on the homotopy class of γ_i in $\boldsymbol{P}^1-\{t_1,\dots,t_N\}$, the operator V_i' depends only on the homotopy class of ω_i. Therefore, the operator V_i' is specified by the requirement that ω_i runs once counterclockwise around $\partial\Delta_i$. The monodromy $H_i': \mathcal{L}'_{d_i} \to \mathcal{L}'_{d_i}$ must of course be defined in the same way as the monodromy H_i, using Lemmas 3.4 and 3.3 but replacing the parameter t_0 by the base point d_i of ω_i and the loop γ_i by the loop ω_i wherever they occur (just as we did for the monodromy $\hat{H}_i'$ at the blow-up level in the definition of the isomorphism wn_i in § 4.1). It will be moreover convenient to have H_i' and $\hat{H}_i'$ linked together by the analogue of Diagram (3.2). This is obtained by asking the commutativity of the analogue of Diagram (3.1) in Lemma 3.3 when building the isotopies leading to H_i' and $\hat{H}_i'$. Now, to make the link with V_i, we choose the monodromy H_i defining V_i by following the same process as we did for $\hat{H}_i$ in the definition of the isomorphism wn_i, so that we obtain a formula analogous to (4.2),

$$H_i = H_i''^{-1} \circ H_i' \circ H_i'', \tag{6.8}$$

where H_i'' is a homeomorphism from $\mathcal{L}'_{t_0}$ onto $\mathcal{L}'_{d_i}$ arising from an isotopy above the path δ_i of Definition 3.2. As above, we shall ask H_i'' to be linked to the homeomorphism $\hat{H}_i''$ of formula (4.2) by a diagram similar to Diagram (3.2). We notice that H_i'' leaves fixed the points of $\mathcal{M}'$ since the isotopy above δ_i giving rise to it satisfies condition (III) of Lemma 3.3. Following Notation 1.4, we denote by $\bar{H}''_{i*}$ the isomorphism induced by H_i'' between $H_n(\mathcal{L}'_{t_0}, \mathcal{M}')$ and $H_n(\mathcal{L}'_{d_i}, \mathcal{M}')$ to distinguish it from the isomorphism H''_{i*} induced between $H_n(\mathcal{L}'_{t_0})$ and $H_n(\mathcal{L}'_{d_i})$. The link between V_i and V_i' is then given by the next lemma.

Lemma 6.7. *The following diagram is commutative:*

$$\begin{array}{ccc} H_n(\mathcal{L}'_{d_i}, \mathcal{M}') & \xrightarrow{V_i'} & H_n(\mathcal{L}'_{d_i}) \\ \uparrow \wr\, \bar{H}''_{i*} & & \uparrow \wr\, H''_{i*} \\ H_n(\mathcal{L}'_{t_0}, \mathcal{M}') & \xrightarrow{V_i} & H_n(\mathcal{L}'_{t_0}) \end{array}$$

Proof. This is a straightforward check using the definitions of V_i and V_i' and formula (6.8). □

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Next, we make, as earlier, a reduction to the bundle $P'^{-1}(\partial\Delta_i)$ for the right-hand side of Diagram (6.4), this time by going back to the definition of the isomorphism wn_i. We consider the following diagram:

$$\begin{array}{ccccc} H_n(P'^{-1}(\partial\Delta_i)) & & \xrightarrow[\sim]{\mathrm{wn}_i} & & \searrow \\ \big\uparrow \scriptstyle\mathrm{incl}_* & & & & \\ H_n(\mathcal{L}^{\sharp\prime}_{d_i}) & \xleftarrow[\sim]{\hat{H}''_{i\,*}} & H_n(\mathcal{L}^{\sharp\prime}_{t_0}) & \xrightarrow{\mathrm{can}} & H_n(\mathcal{L}^{\sharp\prime}_{t_0})/\,\mathrm{im}(\hat{H}_{i\,*}-\mathrm{id}) \\ \big\downarrow \scriptstyle\wr\,\Phi_* & & \big\downarrow \scriptstyle\wr\,\Phi_* & & \big\downarrow \scriptstyle\wr\,\overline{\Phi_*} \\ H_n(\mathcal{L}'_{d_i}) & \xleftarrow[\sim]{H''_{i\,*}} & H_n(\mathcal{L}'_{t_0}) & \xrightarrow{\mathrm{can}} & H_n(\mathcal{L}'_{t_0})/\,\mathrm{im}(H_{i\,*}-\mathrm{id}) \end{array} \tag{6.9}$$

The upper triangle is commutative by the very definition of the isomorphism wn_i (cf. (4.3) and (4.4)). The right-hand square is commutative by the definition of $\overline{\Phi_*}$ given after Diagram (4.1). Finally, the left-hand square is also commutative, since we took care of defining H''_i and $\hat{H}''_i$ coherently.

Finally, we come to the left side of Diagram (6.4). We have the following diagram:

$$\begin{array}{ccc} & \overset{\partial}{\nearrow} & H_{n-1}(\mathcal{M}') \\ & & \big\uparrow \scriptstyle\partial \\ H_n(\mathcal{L}'_{t_0},\mathcal{M}') & \xrightarrow[\sim]{\bar{H}''_{i\,*}} & H_n(\mathcal{L}'_{d_i},\mathcal{M}'). \end{array} \tag{6.10}$$

It is commutative due to the fact that H''_i leaves fixed the points of $\mathcal{M}'$ as already pointed out.

Using now the commutativity of the diagram of Lemma 6.7 and of Diagrams (6.9) and (6.10) and taking into account that some of the arrows, as indicated, are isomorphisms, we see that we only need to prove that the following diagram commutes once the isomorphism labelled Φ_* is reversed:

$$\begin{array}{ccc} H_n(\mathcal{M}'\times\partial\Delta_i) & \xrightarrow{\mathrm{incl}_*} & H_n(P'^{-1}(\partial\Delta_i)) \\ \big\uparrow \scriptstyle\kappa_i & & \big\uparrow \scriptstyle\mathrm{incl}_* \\ H_{n-1}(\mathcal{M}') & & H_n(\mathcal{L}^{\sharp\prime}_{d_i}) \\ \big\uparrow \scriptstyle\partial & & \big\downarrow \scriptstyle\wr\,\Phi_* \\ H_n(\mathcal{L}'_{d_i},\mathcal{M}') & \xrightarrow{V'_i} & H_n(\mathcal{L}'_{d_i}). \end{array} \tag{6.11}$$

To show this, we shall work at the chain level and use a cross-product defined at this level as in [**5**, Notation 4.5]. Let $c_i : \mathcal{L}'_{d_i} \to \mathcal{L}^{\sharp\prime}_{d_i}$ be the inverse isomorphism of the one induced by Φ. It will be convenient to denote $c_{i\,\bullet}(\Gamma)$ by $\Gamma^\sharp$ for any singular chain Γ of $\mathcal{L}'_{d_i}$. Then using Notations 1.5 and 1.4, the commutativity of Diagram (6.11), with the

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arrow Φ_* reversed, amounts to the following homology: for any relative n-cycle Γ on $\mathcal{L}'_{d_i}$ modulo $\mathcal{M}'$,

$$(-1)^{n-1}\partial\Gamma \times \omega_i \sim (H'_{i\bullet}(\Gamma) - \Gamma)^\sharp \quad \text{in } P'^{-1}(\partial\Delta_i). \tag{6.12}$$

To prove this homology, first observe that

$$(H'_{i\bullet}(\Gamma) - \Gamma)^\sharp = H'_{i\bullet}(\Gamma)^\sharp - \Gamma^\sharp = \hat{H}'_{i\bullet}(\Gamma^\sharp) - \Gamma^\sharp,$$

since we have ensured that H'_i and $\hat{H}'_i$ commute with the blowing-down morphism. Homology (6.12) will then be given by the isotopy $\hat{G}'_i$ giving rise to $\hat{H}'_i$. More precisely, if ι is the 1-simplex of $[0,1]$ consisting of the identity map, then $\Gamma^\sharp \times \iota$ is a chain of $\mathcal{L}^{\sharp\prime}_{d_i} \times [0,1]$ to which we can apply $\hat{G}'_i$, obtaining a chain of $\hat{W}'$, in fact of $P'^{-1}(\partial\Delta_i)$ by condition $(\hat{\mathrm{I}}\hat{\mathrm{I}})$ of Lemma 3.3 (where t_0 must be replaced by d_i and γ_i by ω_i). We shall show that

$$\partial\hat{G}'_{i\bullet}(\Gamma^\sharp \times \iota) = \partial\Gamma \times \omega_i - (-1)^{n-1}(\hat{H}'_{i\bullet}(\Gamma^\sharp) - \Gamma^\sharp). \tag{6.13}$$

Here is the computation; it can already be found in [**5**, p. 540] in the course of the proof of [**5**, Proposition 4.13] but no result is stated there which we could refer to. We have

$$\partial\hat{G}'_{i\bullet}(\Gamma^\sharp \times \iota) = \hat{G}'_{i\bullet}(\partial\Gamma^\sharp \times \iota) + (-1)^n \hat{G}'_{i\bullet}(\Gamma^\sharp \times \partial\iota).$$

Concerning the first term of this sum, observe that $\partial\Gamma^\sharp$ is a chain of $\mathcal{M}' \times \{d_i\}$ and that the restriction of $\hat{G}'_i$ to $(\mathcal{M}' \times \{d_i\}) \times [0,1]$ coincides with that of $\Phi \times \omega_i$ by condition $(\hat{\mathrm{I}}\hat{\mathrm{I}}\hat{\mathrm{I}})$ of Lemma 3.3. Then

$$\begin{aligned}\hat{G}'_{i\bullet}(\partial\Gamma^\sharp \times \iota) &= (\Phi \times \omega_i)_\bullet(\partial\Gamma^\sharp \times \iota)\\ &= \Phi_\bullet(\partial\Gamma^\sharp) \times \omega_{i\bullet}(\iota)\\ &= \partial\Gamma \times \omega_i.\end{aligned}$$

As to the second term of the sum above, let $\hat{0}$ and $\hat{1}$ be the 0-simplices of $[0,1]$ with respective values 0 and 1. Then $\Gamma^\sharp \times \partial\iota = \Gamma^\sharp \times \hat{1} - \Gamma^\sharp \times \hat{0}$, a difference of chains of $\mathcal{L}^{\sharp\prime}_{d_i} \times \{1\}$ and $\mathcal{L}^{\sharp\prime}_{d_i} \times \{0\}$. But, if ϖ is the projection of $\mathcal{L}^{\sharp\prime}_{d_i} \times [0,1]$ onto the first factor, the restriction of $\hat{G}'_i$ to $\mathcal{L}^{\sharp\prime}_{d_i} \times \{0\}$ is the same as the restriction of ϖ to the same space, by condition $(\hat{\mathrm{I}})$ of Lemma 3.3. And the restriction of $\hat{G}'_i$ to $\mathcal{L}^{\sharp\prime}_{d_i} \times \{1\}$ is the same as the restriction of $\hat{H}'_i \circ \varpi$, by the definition of $\hat{H}'_i$ in Lemma 3.4. Hence

$$\begin{aligned}\hat{G}'_{i\bullet}(\Gamma^\sharp \times \partial\iota) &= \hat{G}'_{i\bullet}(\Gamma^\sharp \times \hat{1}) - \hat{G}'_{i\bullet}(\Gamma^\sharp \times \hat{0})\\ &= \hat{H}'_{i\bullet}(\varpi_\bullet(\Gamma^\sharp \times \hat{1})) - \varpi_\bullet(\Gamma^\sharp \times \hat{0})\\ &= \hat{H}'_{i\bullet}(\Gamma^\sharp) - \Gamma^\sharp,\end{aligned}$$

since the chain cross-product we have considered has the property that, for any spaces E and F, the projection $\varpi : E \times F \to E$ acts as $\varpi_\bullet(\gamma \times \sigma) = \gamma$ for every chain γ of E and every 0-simplex σ of F (cf. [**5**, Notation 4.5]).

This shows equality (6.13), proving homology (6.12) and hence the commutativity of Diagram (6.11). The commutativity of Diagram (6.4), which was reduced to the former, follows.

The commutativity of Diagrams (6.2) and (6.4) implies that of Diagram (6.1) and hence proves Proposition 6.1.

7. A generalization of the Zariski–van Kampen theorem to higher homotopy

We give here a projective version of the van Kampen type theorem of [**15**, Theorem 2.4] using the above defined homotopy variation operators (see § 5) instead of the degeneration operators of [**15**]. In fact we shall give two proofs of this result. One depends on Theorem 5.1 from [**5**] and the other is based on an affine version of the Zariski–van Kampen type theorem from [**15**]. The first proof is simpler. The purpose of the second is to show (using § 6) the relationship between the Zariski–van Kampen type theorem of [**15**] and the one presented below in Theorem 7.1.

Theorem 7.1. *Let V be a hypersurface in $\boldsymbol{P}^{n+1}$ with $n \geqslant 2$ having only isolated singularities. Consider a pencil $(L_t)_{t\in \boldsymbol{P}^1}$ of hyperplanes in $\boldsymbol{P}^{n+1}$ with the base locus $\mathcal{M}$ transversal to V. Denote by $t_1, \dots, t_N$ the collection of those t for which $L_t \cap V$ has singularities. Let t_0 be different from $t_1, \dots, t_N$. Let γ_i be a good collection, in the sense of Definition 3.2, of paths in $\boldsymbol{P}^1$ based in t_0. Let $e \in \mathcal{M} - \mathcal{M} \cap V$ be a base point. Let $\mathcal{V}_i$ be the variation operator corresponding to γ_i (cf. section 5). Then the inclusion induces an isomorphism,*

$$\pi_n(\boldsymbol{P}^{n+1} - V, e) \xleftarrow{\sim} \pi_n(L_{t_0} - L_{t_0} \cap V, e) \Big/ \sum_{i=1}^{N} \operatorname{im} \mathcal{V}_i.$$

First proof. We apply Theorem 5.1 of [**5**] to the non-singular quasi-projective variety $W - j(V)$ in $\boldsymbol{P}^{n+2}$ (cf. § 2.2). The base locus $\mathcal{M}$ of the pencil $(\mathcal{L}_t)_{t\in \boldsymbol{P}^1}$ is transversal to the Whitney stratification Σ of W adapted to $j(V)$ (cf. Claim 2.6). Hence [**5**] gives the following isomorphism induced by inclusion,

$$H_n(W - j(V)) \xleftarrow{\sim} H_n(\mathcal{L}_{t_0} \cap (W - j(V))) \Big/ \sum_{i=1}^{N} \operatorname{im} V_i,$$

where the V_i are the homological variation operators defined in § 5.

Recall (Lemma 2.8) that we have an isomorphism η

$$H_n(W - j(V)) \xleftarrow{\sim} \pi_n(\boldsymbol{P}^{n+1} - V, e).$$

Now the result follows using the isomorphism α_{t_0} and the commutative diagrams of Lemma 2.8, and the definition of $\mathcal{V}_i$ by means of V_i from § 5. □

Second proof. The idea now is to derive the description of the homotopy groups of the complement to the hypersurface in $\boldsymbol{P}^{n+1}$ using the description of the homotopy groups of its complement in $\boldsymbol{C}^{n+1}$ given in [**15**, Theorem 2.4] via affine degeneration operators. From the latter description we shall deduce a projective analogue of it, namely that there is an isomorphism induced by inclusion

$$\left(\pi_n(L_{t_0} - L_{t_0} \cap V, e) \Big/ \sum_{i=1}^{N} \operatorname{im}(h_{i\,\#} - \mathrm{id})\right) \Big/ \sum_{i=1}^{N} \operatorname{im} \mathcal{D}_i \xrightarrow{\sim} \pi_n(\boldsymbol{P}^{n+1} - V, e), \qquad (7.1)$$

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where, by abuse, we have still written $\operatorname{im}\mathcal{D}_i$ for the canonical image of each $\operatorname{im}\mathcal{D}_i$ in the first quotient. Theorem 7.1 will then follow from Lemma 5.2 and Corollary 6.2.

We only give a sketch of proof of (7.1) when $n \geqslant 3$. According to the homological description of the homotopy groups given in § 2.4 above, it will be enough to show the exactness of the last row of the following diagram, where Notation 2.7 has been used (the other parts of the diagram will be explained just below):

$$\begin{array}{ccccccc}
\bigoplus_{i=1}^{N} H_{n-1}(\widetilde{(L^{\mathrm{a}\prime}_{t_i})_\infty}) & \xrightarrow{\sum_{i=1}^{N} D_i^\infty} & H_n(\widetilde{(L^{\mathrm{a}\prime}_{t_0})_\infty})\Big/\sum_{i=1}^{N}\operatorname{im}(H^\infty_{i*}-\mathrm{id}) & \xrightarrow{\overline{\mathrm{incl}_*}} & H_n(\widetilde{(\boldsymbol{C}^{n+1\prime})_\infty}) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \delta_*} & & \downarrow{\scriptstyle \overline{\delta_*}} & & \downarrow{\scriptstyle \delta_*} & & \\
\bigoplus_{i=1}^{N} H_{n-1}(\mathcal{L}^{\mathrm{a}\prime}_{t_i}) & \xrightarrow{\sum_{i=1}^{N} D_i^{\mathrm{a}}} & H_n(\mathcal{L}^{\mathrm{a}\prime}_{t_0})\Big/\sum_{i=1}^{N}\operatorname{im}(H^{\mathrm{a}}_{i*}-\mathrm{id}) & \xrightarrow{\overline{\mathrm{incl}_*}} & H_n(W^{\mathrm{a}\prime}) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \mathrm{incl}_*} & & \downarrow{\scriptstyle \overline{\mathrm{incl}_*}} & & \downarrow{\scriptstyle \mathrm{incl}_*} & & \\
\bigoplus_{i=1}^{N} H_{n-1}(\mathcal{L}'_{t_i}) & \xrightarrow{\sum_{i=1}^{N} D_i} & H_n(\mathcal{L}'_{t_0})\Big/\sum_{i=1}^{N}\operatorname{im}(H_{i*}-\mathrm{id}) & \xrightarrow{\overline{\mathrm{incl}_*}} & H_n(W') & \longrightarrow & 0
\end{array}$$

The starting point is the exactness of the first row of the diagram which expresses Theorem 2.4 from [**15**]. This theorem is applied to the affine situation obtained by considering the complement $\boldsymbol{C}^{n+1}$ in $\boldsymbol{P}^{n+1}$ of a generic hyperplane L_∞ belonging to the pencil $(L_t)_{t\in\boldsymbol{P}^1}$ considered in § 2.1. Then $L^{\mathrm{a}}_{t_0}$ denotes the affine part of L_{t_0} (that is, $L_{t_0} - M$ where M is the base locus of the pencil) and, for $1 \leqslant i \leqslant N$, $L^{\mathrm{a}}_{t_i}$ denotes the affine part of L_{t_i}. The primes are used, following Notation 2.7, to indicate the removal of the points of the affine part V^{a} of V. The proof of Theorem 2.4 from [**15**] uses the description of homotopy groups as homology groups of infinite cyclic covers (here denoted, as usual, by $\widetilde{(\cdot)_\infty}$). This description works as in § 2.4 above, taking into account that, this time, the covered spaces all have fundamental groups isomorphic to $\boldsymbol{Z}$ (cf. [**15**, Lemma 1.5]). There are then geometric monodromies H_i^∞ and affine homological degeneration operators D_i^∞ at the infinite cyclic cover level and Theorem 2.4 from [**15**] corresponds to the exactness of the first row of the diagram above.

The second row of the diagram is made from d-fold cyclic covers corresponding to the infinite cyclic covers of the first row. These are obtained from the coverings of Claim 2.4 by restricting to the complement in $\boldsymbol{P}^{n+2}$ of the hyperplane $\mathcal{L}_\infty$ of pencil $(\mathcal{L}_t)_{t\in\boldsymbol{P}^1}$ having the same parameter as L_∞. We have denoted by $\mathcal{L}^{\mathrm{a}}_{t_i}$, $\mathcal{L}^{\mathrm{a}}_{t_0}$ and W^{a} the affine parts of $\mathcal{L}_{t_i}$, $\mathcal{L}_{t_0}$ and W, respectively, and the primes indicate that we take the intersections of these spaces with $W^{\mathrm{a}} - j(V^{\mathrm{a}})$. We have denoted by δ all suitable restrictions of the covering $\widetilde{(\boldsymbol{C}^{n+1} - V^{\mathrm{a}})_\infty} \to W^{\mathrm{a}} - j(V^{\mathrm{a}})$ between infinite cyclic and d-fold cyclic covers of $\boldsymbol{C}^{n+1} - V^{\mathrm{a}}$. The projective geometric monodromies H_i of Definition 3.5 leave fixed the points of $\mathcal{M} \cap (W - j(V))$ and therefore induce monodromies H^{a}_i of $\mathcal{L}^{\mathrm{a}}_{t_0} \cap (W^{\mathrm{a}} - j(V^{\mathrm{a}}))$ in $W^{\mathrm{a}} - j(V^{\mathrm{a}})$. This ensures that the lower middle vertical arrow of the diagram is well defined. On the other hand, the monodromies H_i^∞ above could be constructed as lifts of the monodromies H^{a}_i by δ, hence giving rise to the upper middle arrow labelled $\overline{\delta_*}$. Now, the definition of D_i^∞ in [**15**], using a product by $\boldsymbol{S}^1$ in a trivial part of the fibration above Δ_i, implies that D_i^∞ factorizes through δ_* into an affine homological degeneration operator D_i^{a} at the d-fold cover level, defined in a way similar to D_i^∞ (observe that $H_{n-1}(\mathcal{L}_{t_0}^{\mathrm{a}} \cap (W^{\mathrm{a}} - j(V^{\mathrm{a}}))) = 0$ by [**15**, Eqn (1.6) and Lemma 1.5]). From this and the definition of D_i in § 4.1 it follows that the lower-left square in the diagram is (up to sign) commutative. This concludes the description of the whole diagram and the justification of its commutativity.

We turn to the exactness of the second row of the diagram. The arguments in the proof of Lemma 1.13 in [**15**] show that the upper-right vertical arrow labelled δ_* is surjective and that its kernel is

$$(s^d - 1)H_n(\widetilde{(\boldsymbol{C}^{n+1} - V^{\mathrm{a}})_\infty}),$$

where s is a generator of the infinite cyclic group of the deck transformations of $\widetilde{(\boldsymbol{C}^{n+1} - V^{\mathrm{a}})_\infty}$. The upper-left vertical arrow of the diagram is similarly surjective. The arguments of [**15**] hold one dimension further for $L_{t_0}^{\mathrm{a}} - L_{t_0}^{\mathrm{a}} \cap V^{\mathrm{a}}$, because $L_{t_0}^{\mathrm{a}} \cap V^{\mathrm{a}}$ is non-singular, and give that the homomorphism

$$\delta_* : \quad H_n(\widetilde{(L_{t_0}^{\mathrm{a}} - L_{t_0}^{\mathrm{a}} \cap V^{\mathrm{a}})_\infty}) \to H_n(\mathcal{L}_{t_0}^{\mathrm{a}} \cap (W^{\mathrm{a}} - j(V^{\mathrm{a}})))$$

is surjective and its kernel is equal to

$$(s^d - 1)H_n(\widetilde{(L_{t_0}^{\mathrm{a}} - L_{t_0}^{\mathrm{a}} \cap V^{\mathrm{a}})_\infty}),$$

with s considered as a generator of the group of deck transformations of

$$\widetilde{(L_{t_0}^{\mathrm{a}} - L_{t_0}^{\mathrm{a}} \cap V^{\mathrm{a}})_\infty},$$

which is isomorphic to that of $\widetilde{(\boldsymbol{C}^{n+1} - V^{\mathrm{a}})_\infty}$. A purely algebraic reasoning then leads from the exactness of the first row of the diagram to the exactness of its second row.

Finally, let us show that the last row of the diagram is exact. The lower-right vertical arrow of the diagram is an isomorphism according to the Poincaré residue sequence

$$H_{n-1}(\mathcal{L}_\infty \cap (W - j(V))) \to H_n(W^{\mathrm{a}} - j(V^{\mathrm{a}})) \to H_n(W - j(V)) \to H_{n-2}(\mathcal{L}_\infty \cap (W - j(V))),$$

with the vanishing of its first and last terms by Lemma 2.11. Similarly, the middle-lower vertical arrow is surjective by the vanishing of $H_{n-2}(\mathcal{M} \cap (W - j(V)))$ and the left-lower vertical arrow is surjective by the vanishing of $H_{n-3}(\mathcal{M} \cap (W - j(V)))$ when $n \neq 3$. In fact this arrow is surjective when $n = 3$ too because the homomorphism which follows it in the Poincaré residue sequence is equal to zero. This can be seen on the following diagram where it appears as the left vertical arrow (we use Notation 2.7):

$$\begin{array}{ccccc}
H_2(\mathcal{L}'_{t_i}) & \xrightarrow{\tau_i} & H_3(P'^{-1}(\partial\Delta_i)) & \xleftarrow{j_{i\,*}} & H_3(\mathcal{L}'_{t_0}) \\
\Big\downarrow{\scriptstyle \eta_i} & & \Big\downarrow & & \Big\downarrow \\
H_0(\mathcal{M}') & \xrightarrow{\tau'_i} & H_1(\mathcal{M}' \times \partial\Delta_i) & \xleftarrow{j'_{i\,*}} & H_1(\mathcal{M}')
\end{array}$$

The other vertical arrows come also from Poincaré residue sequences. Homomorphism τ_i is essentially a tube map; it is described in detail in § 4.3. Homomorphism τ'_i is of the same kind but reduces trivially to the product by $\boldsymbol{S}^1$. Homomorphism $j_{i\,*}$ is induced by parallel transport followed by the inclusion as a fibre of $P'^{-1}(\partial\Delta_i)$; this is detailed in § 4.2, where it is also shown that $j_{i\,*}$ is surjective. Finally, homomorphism $j'_{i\,*}$ is also induced by the inclusion as a fibre. It follows from the triviality of P' in a neighbourhood of $\mathcal{M}' \times \Delta_i$ that this diagram is up to sign commutative. Now, by the Künneth formula, $H_1(\mathcal{M}' \times \partial\Delta_i)$ is the direct sum of the images of τ'_i and $j'_{i\,*}$ which are injective. Chasing in the diagram then shows that η_i is equal to zero. Hence the left-lower vertical arrow of the original big diagram is surjective for $n \geqslant 3$. Recalling that the middle-lower vertical arrow is surjective too and that the right-lower one is an isomorphism, the exactness of the last row of the diagram can be deduced from the exactness of its second row.

This concludes our sketch of proof of (7.1) when $n \geqslant 3$. We leave the case $n = 2$ to the reader. □

Remark 7.2. We presented Theorem 7.1 as a generalization of the classical Zariski–van Kampen theorem. However, the latter concerns the case $n = 1$ and our definition of the homotopy variation operators does not work in this situation. But, as M is then reduced to a point, Lemma 5.2 suggests that taking the image of $\mathcal{V}_i$ should be replaced by the identification of each x of $\pi_1(L_{t_0} - L_{t_0} \cap V, e)$ with each $h_{i\#}(x)$. Our theorem then actually reduces to the classical Zariski–van Kampen theorem. Nevertheless, our proof does not work in the case $n = 1$. Only the statements generalize, but not the proofs.

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Note added in proof

Since the completion of the present paper, two more publications appeared dealing with the material related to the present work. [**20**] concerns the homological variation operators in non-generic pencils. In [**6**], the homotopical variation operators are generalized to quasi-projective varieties via a definition which does not go through homology as in the present article; this opens the way to further generalizations of the Zariski–van Kampen theorem.

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