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Nilsequences and multiple correlations along subsequences

ANH NGOC LE

Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, IL 60208-2730, USA
(e-mail: anhle@math.northwestern.edu)

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Abstract. The results of Bergelson, Host and Kra, and Leibman state that a multiple polynomial correlation sequence can be decomposed into a sum of a nilsequence (a sequence defined by evaluating a continuous function along an orbit in a nilsystem) and a null sequence (a sequence that goes to zero in density). We refine their results by proving that the null sequence goes to zero in density along polynomials evaluated at primes and along the Hardy sequence ($\lfloor n^c \rfloor$). In contrast, given a rigid sequence, we construct an example of a correlation whose null sequence does not go to zero in density along that rigid sequence. As a corollary of a lemma in the proof, the formula for the pointwise ergodic average along polynomials of primes in a nilsystem is also obtained.

Key words: classical ergodic theory, number theory, multiple correlation, nilsequence, null sequence

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1. *Introduction*

1.1. *History and motivation.* Let (X, μ, T) be an invertible measure-preserving system, $f_j \in L^\infty(\mu)$ and let s_j be an *integer polynomial*, i.e. taking integer values on integers, for $0 \leq j \leq k$. Then the sequence

$$a(n) = \int_X f_0(T^{s_0(n)}x) \cdot f_1(T^{s_1(n)}x) \cdots f_k(T^{s_k(n)}x)\, d\mu(x) \tag{1.1}$$

is called a *multiple polynomial correlation sequence* or, for conciseness, a *polynomial correlation*. If $s_j(n) = c_j n$ with $c_j \in \mathbb{Z}$, we call $(a(n))$ a *linear correlation*.

Understanding multiple correlations has been a main goal of ergodic theorists since Furstenberg's celebrated proof of Szemerédi theorem. A possible approach to the problem is to find connections between correlations and the sequences that have rich algebraic structures. For example, to prove the generalized Khintchine theorem, Bergelson, Host and Kra [**6**] decomposed linear correlations into a sum of a *nilsequence* and a *null sequence* (see §2.2 for precise definitions).

This decomposition for single linear correlations ($k = 1$, $s_j(n) = c_j n$) can be proved using Herglotz's theorem. In this case, there exists a measure σ on the circle $\mathbb{T}$ such that $a(n) = \int_{\mathbb{T}} e^{2\pi i n x}\, d\sigma(x)$. Decomposing σ into discrete (atomic) and continuous (non-atomic) parts, $(a(n))$ is then a sum of an *almost periodic sequence* (one-step nilsequence) and a null sequence.

Bergelson, Host, and Kra [**6**] extend this classical result to $k \geq 2$ when $s_j(n) = jn$ and (X, μ, T) is ergodic. In their result, the almost periodic sequence is replaced by a *k-step nilsequence*. By a different method, Leibman generalizes Bergelson, Host and Kra's result to the case that $s_j(n)$ are integer polynomials [**27**]. Leibman himself later removes the ergodicity assumption in [**28**].

If a sequence $(a(n))$ can be decomposed into a sum of a nilsequence and a null sequence, we say $(a(n))$ has a *nil+null decomposition*. In this case, the decomposition is in fact unique (see §2.5 for the proof). The nilsequence and null sequence are then called the *nil component* and the *null component* of $(a(n))$, respectively.

Nilsequences in general have been studied extensively since their introduction by Bergelson, Host and Kra [**6**] in 2005. Similarly, the nil components in the nil+null decomposition of multiple correlations have also been well studied. For example, Bergelson, Host and Kra [**6**] analyzed this component to prove a generalization of Khintchine's theorem. Moreira and Richter [**29**] show that this component arises from a system whose spectrum is contained in the spectrum of the original system.

On the other hand, little is known about the null component. The goal of this paper is to partially fill that gap. We show that the null component goes to zero in density along polynomials evaluated at primes and along the Hardy sequence $(\lfloor n^c \rfloor)$. Nevertheless, for any *rigid sequence* (r_n), there is a correlation whose null component is not null along (r_n) (see §2.13 for a definition of rigid sequences).

It is worth mentioning that a related conjecture has been raised by Frantzikinakis [**16**, Problem 13]. Letting p_n denote the nth prime, Frantzikinakis conjectures that for a linear correlation in an ergodic system $(a(n))$, there exists a nilsequence $(\psi(n))$ and null sequence $(\epsilon(n))$ such that $a(p_n) = \psi(p_n) + \epsilon(n)$. The same conjecture is raised for the Hardy sequence $(\lfloor n^c \rfloor)$ instead of (p_n).

Our result not only gives an affirmative answer to Frantzikinakis' conjecture, but is stronger in several senses. First, in the case of prime sequences, we work with polynomial correlations rather than linear correlations. Also, we do not need the system to be ergodic. Moreover, instead of having different nilsequences when decomposing along (p_n) and $(\lfloor n^c \rfloor)$, we show that there is a fixed nilsequence that works for both, and in fact for many others.

Before presenting the formal statement, we have a definition.

Definition 1.

(1) Let (r_n) be a increasing sequence of integers. A bounded sequence $(a(n))$ is called a *null sequence along* (r_n) if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |a(r_n)| = 0.$$

(2) If $r_n = n$, we simply call $(a(n))$ a *null sequence*.

1.2. *Statement of results.* The main goal of this paper is to prove the following.

THEOREM 1.1.

- *The null component of a polynomial correlation is null along every sequence of the form* $(Q(n))$ *or* $(Q(p_n))$ *where* $Q \in \mathbb{Z}[n]$ *is non-constant and* p_n *is the* nth *prime.*
- *The null component of a linear correlation is also null along* $(\lfloor n^c \rfloor)$ *for* $c > 0$.

Remark. By a different method, Tao and Teräväinen [**30**] proved the null component of a linear correlation is null along the primes, and used this result to prove odd cases of the logarithmic Chowla conjecture.

In fact, we prove the null component is null along a more general category of sequences, namely *good sequences*. A good sequence is one that possesses two properties: *Good for projection on nilfactors (GPN)* and *essentially good for equidistribution on nilmanifolds (EGEN)* (see §2.11 for definitions).

To show the null component is null along good sequences, we follow an argument similar to that of Leibman [**28**]. A key proposition in Leibman's proof says that an *integral of nilsequences* has nil+null decomposition (see §2.6). In §3, we refine that result by showing the following.

PROPOSITION 1.2. *The null component of an integral of nilsequences is null along any EGEN sequence.*

The fact that $(Q(n))$ is a good sequence follows from the work of Host–Kra [**22**] and Leibman [**24**, **26**]. On the other hand, the Hardy sequence $(\lfloor n^c \rfloor)$ was proved to be good by Frantzikinakis [**13**, **14**]. In §5, we show the following.

PROPOSITION 1.3. *For any* $Q \in \mathbb{Z}[n]$ *non-constant, the sequence* $(Q(p_n))$ *is good.*

The EGEN property of polynomials of primes allows us to determine the exact formula for the pointwise ergodic average along polynomials of primes for continuous functions in a *nilsystem*. Green and Tao [**20**] proved the average converges to the integral of the function in the case of a totally ergodic nilsystem. Eisner [**9**] showed that the average converges everywhere in an arbitrary nilsystem. But the exact formula is still missing for this case. As a corollary of the EGEN property of $(Q(p_n))$, we can determine the exact average.

To be precise, for an ergodic nilsystem $(X = G/\Gamma, \mu, \tau)$, let $\pi : G \to X$ be the canonical map $\pi(g) = g\Gamma$. Assume X has d connected components, and X_0 is the component containing $1_X = \pi(1_G)$. Let $X_j = \tau^j X_0$ for $j \in \mathbb{Z}$ and μ_{X_j} be the Haar

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measure of X_j. Note that $X_i = X_j$ if $i \equiv j \pmod d$ (see §2.4.1). Let ϕ be the Euler totient function. Then we have the following.

COROLLARY 1.4. *Let (X, μ, τ) be an ergodic nilsystem with d connected components $X_0, X_1, \ldots, X_{d-1}$ with $X_i = \tau^i X_0$ and f be a continuous function on X. Suppose $x \in X_k$ for some $0 \le k \le d-1$. Then*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(\tau^{Q(p_n)}x) = \frac{1}{\phi(d)} \sum_{\substack{1\le s<d \\ (s,d)=1}} \int_{X_{Q(s)+k}} f\, d\mu_{X_{Q(s)+k}}.$$

In the same spirit of Theorem 1.1, but in the opposite direction, we are also interested in those sequences (r_n) such that there exists a correlation whose null component is not null along (r_n). It turns out there is a well-known class of sequences satisfying such a condition, namely *rigid sequences*. A sequence is called *rigid* if there is a weakly mixing system (X, μ, T) such that $\|T^{r_n} f - f\|_{L^2(\mu)} \to 0$ for all $f \in L^2(\mu)$. Examples of rigid sequences include (2^n), (3^n) and $(n!)$ (see §2.13 for more details). In §6, we prove the following.

PROPOSITION 1.5. *For any rigid sequence (r_n), there exists a linear correlation whose null component is not null along (r_n).*

1.3. *Application.* The goal of Bergelson, Host and Kra's paper [**6**] is not to prove the nil+null decomposition for a multiple correlation. They use the decomposition to prove a generalization of Khintchine's theorem. In a similar fashion, it follows from Theorem 1.1 and Corollary 1.4 that in an ergodic system (X, μ, T), for any measurable set $A \subseteq X$ and $\delta > 0$, the set

$$\{n \in \mathbb{N} : \mu(A \cap T^{-(p_n-1)}A \cap T^{-2(p_n-1)}A) \ge \mu(A)^3 - \delta\}$$

has positive density. The same is true for the set

$$\{n \in \mathbb{N} : \mu(A \cap T^{-(p_n-1)}A \cap T^{-2(p_n-1)}A \cap T^{-3(p_n-1)}A) \ge \mu(A)^4 - \delta\}.$$

A detailed proof will appear in a forthcoming paper [**8**].

1.4. *Open question.* It is still unknown whether a similar result to nil+null decomposition exists for a set of commuting transformations. To be precise, for a measure space (X, μ) with commuting measure-preserving transformations $T_j : X \to X$ and $f_j \in L^\infty(\mu)$ for $0 \le j \le k$, we define a correlation sequence

$$a(n) = \int_X f_0(T_0^n x) f_1(T_1^n x) \ldots f_k(T_k^n x)\, d\mu(x).$$

Frantzikinakis [**15**] showed that for any $\delta > 0$ the sequence $(a(n))$ can be decomposed as $a(n) = a_{st}(n) + a_{er}(n)$ where $(a_{st}(n))$ is a k-step nilsequence and

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} |a_{er}(n)|^2 < \delta.$$

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From Frantzikinakis' result, it is natural to ask whether we have the same decomposition, but in addition

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |a_{er}(p_n)|^2 < \delta?$$

What if we replaced (p_n) by a Hardy sequence $(\lfloor n^c \rfloor)$? Our argument in this paper does not apply since we do not have sufficient information about the factors that control the multiple ergodic averages for commuting transformations [**2**, **3**]. These factors are not simply inverse limits of nilsystems, the objects that play a crucial role in our analysis.

1.5. *Outline of the paper.* Section 2 is for background and notation. In §3, we prove the Proposition 1.2 about the integral of nilsequences. In §4, we proceed to prove that the null component of a correlation is null along good sequences. Section 5 shows that the polynomials of primes are good sequences, hence effectively proving Theorem 1.1. Also in this section, we prove the limit formula of the average along polynomials of primes (Corollary 1.4). In the last section, we construct an example of a correlation whose null component is not null along a given rigid sequence.

2. *Background and notation*

2.1. *Notation.* A sequence is a function $a : \mathbb{N} \to \mathbb{C}$. We denote this sequence by $(a(n))_{n\in\mathbb{N}}$, $(a(n))$ or sometimes only a if there is no danger of confusion.

For $N \in \mathbb{N}$, we write $[N] = \{1, 2, \ldots, N\}$. For a function f on a finite non-empty set S, let $\mathbb{E}_{s\in S} f(s)$ denote $(1/|S|)\sum_{s\in S} f(s)$. In particular, for bounded sequence $(a(n))$,

$$\mathbb{E}_{n\in[N]} a(n) := \frac{1}{N}\sum_{n=1}^{N} a(n).$$

Let (X, μ, T) be a measure-preserving system and $f \in L^\infty(X)$. Tf is defined to be $Tf(x) := f(Tx)$ for all $x \in X$. If (Y, ν, S) is a factor of (X, μ, T), we denote *conditional expectation of f on Y* by $\mathbb{E}(f|Y)$.

Let $\mathbb{P}$ denote the set of all primes and p_n the nth prime. For $d, s \in \mathbb{Z}$, let $p_{s \pmod d, n}$ be the nth prime that is congruent to $s \pmod d$.

2.2. *Nilmanifolds, nilsystems and nilsequences.* Let G be a k-step nilpotent Lie group and Γ be a *uniform* (i.e. closed and cocompact) subgroup of G. The compact homogeneous space $X := G/\Gamma$ is called a *k-step nilmanifold*. Let $\pi : G \to X$ be the standard quotient map. We write $1_X = \pi(1_G)$ where 1_G is the identity element of G. Suppose G^0 is the identity connected component of G. If X is connected, then $X = \pi(G^0) = G^0/(G^0 \cap \Gamma)$.

The space X is endowed with a unique probability measure that is invariant under translations by G. This measure is called the *Haar measure* for X, and denoted by μ_X. For every $\tau \in G$, the measure-preserving system (X, μ_X, τ) is called *k-step nilsystem.*

Let $C(X)$ denote the set of continuous functions on X. For $f \in C(X)$ and $x \in X$, the sequence $\psi(n) := f(\tau^n x)$ is called a *basic k-step nilsequence*. A *k-step nilsequence* is a uniform limit of basic k-step nilsequences.

If G is not connected, we can embed X in $X' = G'/\Gamma'$ where G' is a connected and simply-connected k-step nilpotent Lie group and Γ' is a closed, discrete cocompact subgroup of G'. Extending f to a continuous function f' on X' and supposing $\tau' \in G'$ and $x' \in X'$ are elements corresponding to $\tau \in G$ and $x \in X$, we have a different representation of basic k-step nilsequence $\psi(n) = f'(\tau'^n x')$ for all $n \in \mathbb{Z}$. Therefore, if we are discussing the basic k-step nilsequence $(f(\tau^n x))_{n\in\mathbb{Z}}$, without the loss of generality, we can assume G is connected and simply connected.

Remark. Different authors may have different concepts of nilsequences. We use the original definition by Bergelson, Host and Kra [**6**]. Leibman in his series of papers [**27**, **28**] uses the same definition. However, in Green and Tao [**19**, **20**] and Frantzikinakis [**15**], the nilsequences are in fact our basic nilsequences. Frantzikinakis [**16**] even introduces the notion of *basic generalized k-step nilsequences*. They are sequences of the form $(f(\tau^n x))_{n\in\mathbb{N}}$ when f is allowed to be Riemann integrable. We do not use this concept in current paper.

2.3. *Subnilmanifolds.* Let $X = G/\Gamma$ be a k-step nilmanifold. A *subnilmanifold* Y of X is a closed subset of X of the form $Y = Hx$ where H is a closed subgroup of G and $x \in X$. The Haar measure on Y is denoted by μ_Y. This measure is invariant under translation by any $\tau \in G$.

A *normal subnilmanifold* Z of X is a subnilmanifold which is equal to Lx for some normal closed subgroup L of G and $x \in X$. The quotient nilmanifold $X/Z := G/(L\Gamma)$ is a factor of X by standard factor map $G/\Gamma \to G/(L\Gamma)$. For a subnilmanifold Y of X, the *normal closure* of Y in X is the smallest normal subnilmanifold of X that contains Y. The normal closure of a connected subnilmanifold is connected [**28**, p. 5].

For $\tau \in G$, we say the sequence $(\tau^n Y)_{n\in\mathbb{N}}$ is *equidistributed* on X if, for any $f \in C(X)$,

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]} \int_Y \tau^n f \, d\mu_Y = \int_X f \, d\mu_X.$$

2.4. *Orbit closures of subnilmanifolds.* In this section we summarize important facts about orbit closures of subnilmanifolds under linear and polynomial translations.

2.4.1. *Linear orbits.* Let Y be a connected subnilmanifold of nilmanifold $X = G/\Gamma$ and $\tau \in G$. Then the orbit closure of Y under action of τ is a subnilmanifold of X, namely $\overline{\{\tau^n Y\}}_{n\in\mathbb{N}}$, and is denoted by $\mathcal{O}_Y$. Suppose d is the number of connected components of $\mathcal{O}_Y$ and Y^0 is the component containing Y. Then all connected components of $\mathcal{O}_Y$ are $Y^0, \tau Y^0, \tau^2 Y^0, \ldots, \tau^{d-1} Y^0$. Moreover $\tau^{dn+r} Y^0 = \tau^r Y^0$ for $n \in \mathbb{N}$, $r \in \mathbb{Z}$, and the sequence $(\tau^{dn+r} Y)_{n\in\mathbb{N}}$ is equidistributed in $\tau^r Y^0$.

In particular, suppose (X, μ, τ) is an ergodic nilsystem with d connected components. Assume X_0 is the component containing $1_X = \pi(1_G)$. Then all components of X are $X_0, \tau X_0, \ldots, \tau^{d-1} X_0$. And $(\tau^{dn+r} 1_X)_{n\in\mathbb{N}}$ is equidistributed on $\tau^r X_0$. For details and proofs, see [**26**].

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2.4.2. *Polynomial orbits.* A nilsystem (X, μ, τ) is totally ergodic if and only if X is connected [**12**, Proposition 2.1]. In this case, for any $Q(n) \in \mathbb{Z}[n]$ non-constant, and $x \in X$, the sequence $(\tau^{Q(n)}x)$ is equidistributed on X. A stronger result is obtained in [**14**, Lemma 6.7]

2.5. *Uniqueness of nil+null decomposition.* If a sequence $(a(n))$ has two nil+null decompositions $a = \psi_1 + \epsilon_1 = \psi_2 + \epsilon_2$ where ψ_1, ψ_2 are nilsequences and ϵ_1, ϵ_2 are null sequences. Then $\psi_1 - \psi_2 = \epsilon_2 - \epsilon_1$.

$\psi_1 - \psi_2$ is a nilsequence and $\epsilon_2 - \epsilon_1$ is a null sequence. A nilsequence returns to any neighborhood of its supremum in a bounded gap set (due to the minimality of ergodic nilsystems). Hence, it is a null sequence only when the supremum is 0. Thus in our case, $\psi_1 - \psi_2 = \epsilon_2 - \epsilon_1 = 0$.

2.6. *Integral of nilsequences.* Let (Ω, ρ) be a measure space. Suppose for each $\omega \in \Omega$, there is a nilsequence $(\psi_\omega(n))_{n\in\mathbb{Z}}$. We say the family of nilsequences $\{\psi_\omega : \omega \in \Omega\}$ is *integrable with respect to* ρ if, for each $n \in \mathbb{Z}$, the function $\omega \mapsto \psi_\omega(n)$ is integrable with respect to ρ. In this case, the sequence $a(n) = \int_\Omega \psi_\omega(n)\, d\rho(\omega)$ is called *an integral of nilsequences*. Leibman [**28**, Proposition 4.2] proved that an integral of nilsequences admits a nil+null decomposition.

2.7. *Nilfactors.* Let (X, μ, T) be an ergodic measure-preserving system. Suppose $(s_j(n))_{n\in\mathbb{N}}$ is an integer valued sequence for $1 \le j \le k$. A factor (Y, ν, S) of (X, μ, T) is said to be *characteristic for* $(s_1(n), \ldots, s_k(n))$ if, for any bounded functions $f_1, \ldots, f_k$ on X, we have

$$\lim_{N\to\infty} \bigg(\mathbb{E}_{n\in[N]} \prod_{j=1}^{k} T^{s_j(n)} f_j - \mathbb{E}_{n\in[N]} \prod_{j=1}^{k} T^{s_j(n)} \mathbb{E}(f_j|Y) \bigg) = 0,$$

where the limits are taken in $L^2(X, \mu)$. Host and Kra [**23**] show that there exists a characteristic factor for $(n, 2n, \ldots, kn)$ which is an inverse limit of $(k-1)$-step nilsystems. We call this factor the $(k-1)$-*step nilfactor of* X and denote it by $\mathcal{Z}_{k-1}(X)$ (or sometimes just $\mathcal{Z}_{k-1}$ if there is no chance of confusion).

Host and Kra [**22**] show that for most families of integer polynomials Q_j, there exists a nilfactor $\mathcal{Z}_m$ that is characteristic for $(Q_1(n), \ldots, Q_k(n))$. Leibman [**24**] later showed that the result is true for all families of integer polynomials.

2.8. *Characteristic factor for integer polynomials of primes.* Frantzikinakis, Host and Kra [**17**] proved that $\mathcal{Z}_1$ factor is characteristic for a 2-tuple $(p_n, 2p_n)$ where p_n is the nth prime. For $k \ge 3$, they show that $\mathcal{Z}_{k-1}$ is characteristic for a k-tuple $(p_n, 2p_n, \ldots, kp_n)$, conditional upon results on the Mobius function and the inverse conjecture for the Gowers norms, which have been established by Green and Tao [**19**] and Green, Tao, and Ziegler [**21**] respectively.

2.9. *Relative products.* Let X_1, X_2 and Y be three sets. Suppose there are surjective maps $\delta_1 : X_1 \to Y$ and $\delta_2 : X_2 \to Y$. Then *fiber product of* X_1 *and* X_2 with respect to

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Y is defined to be $\{(x_1, x_2) \in X_1 \times X_2 : \delta_1(x_1) = \delta_2(x_2)\}$. We denote this product by $X_1 \times_Y X_2$.

Suppose (X_1, μ_1, T_1) and (X_2, μ_2, T_2) are measure-preserving systems. Let (Y, ν, S) be a common factor of (X_1, μ_1, T_1) and (X_2, μ_2, T_2). Then the *relative product* of X_1 and X_2 with respect to Y is the measure-preserving system $(X_1 \times_Y X_2, \mu_1 \times_Y \mu_2, T_1 \times T_2)$ where the following holds.

(i) The space $X_1 \times_Y X_2$ is the fiber product of X_1 and X_2 with respect to Y.

(ii) The measure $\mu_1 \times_Y \mu_2$ is characterized by

$$\int_{X_1 \times_Y X_2} f_1(x_1) \otimes f_2(x_2)\, d(\mu_1 \times_Y \mu_2)(x_1, x_2) = \int_Y \mathbb{E}(f_1|Y)\mathbb{E}(f_2|Y)\, d\nu$$

for all $f_1 \in L^2(X_1)$ and $f_2 \in L^2(X_2)$.

By an abuse of notation, let $X_1 \times_Y X_2$ denote the relative product of X_1 and X_2 with respect to Y. If X_1 and X_2 are nilsystems, and Y is common nilsystem factor, then $X_1 \times_Y X_2$ is also a nilsystem.

2.10. *Hardy sequences.* Let $\mathcal{F}$ be the collection of functions $f : \mathbb{R}_{>0} \to \mathbb{R}$. Define $\mathcal{B} = \mathcal{F}/\sim$ where $f \sim g$ if there exists a constant $c > 0$ such that $f(x) = g(x)$ for all $x > c$. A *Hardy field* is a subfield of the ring $(\mathcal{B}, +, \times)$ which is closed under differentiation. Examples of Hardy fields include the set of functions that are combinations of addition, multiplication, exponential and logarithm on real variable t and real constants. Let $\mathcal{H}$ be the union of all Hardy fields.

For $a, b \in \mathcal{H}$, we write $a(t) \succ b(t)$ if $\lim_{t\to\infty} b(t)/a(t) = 0$. We say a function $a(t)$ has polynomial growth if there exists a polynomial $p \in \mathbb{R}[t]$ such that $p(t) \succ a(t)$. We call the sequence $(\lfloor a(n) \rfloor)_{n\in\mathbb{N}}$ a *Hardy sequence* where $a \in \mathcal{H}$ and $\lfloor . \rfloor$ indicates the integral part.

Definition 2. Let $a \in \mathcal{H}$ have polynomial growth and satisfy $a(t) - cp(t) \succ \log t$ for every $c \in \mathbb{R}$ and $p \in \mathbb{Z}[t]$. Then the sequence $(\lfloor a(n) \rfloor)_{n\in\mathbb{N}}$ is called *a Hardy sequence of polynomial growth and logarithmically away from every multiple of polynomial of integer coefficients*.

Examples of sequences that satisfy the previous definition are $(\lfloor n^c \rfloor)_{n\in\mathbb{N}}$ where $c > 0$, $c \notin \mathbb{Z}$, $(\lfloor n \log n \rfloor)_{n\in\mathbb{N}}$, $(n^2\sqrt{2} + n\sqrt{3})_{n\in\mathbb{N}}$ and $(n^3 + (\log n)^3)_{n\in\mathbb{N}}$. From now on, whenever we write $(\lfloor n^c \rfloor)$, it represents the entire class of Hardy sequences of polynomial growth and logarithmically away from every multiple of polynomial of integer coefficients.

2.11. *Good sequences.*

Definition 3.

(1) The sequence $(r_n)_{n\in\mathbb{N}}$ is said to be *linearly good for projection onto nilfactors* (denoted by *linear-GPN*) if for any $h_1, h_2, \ldots, h_k \in \mathbb{Z}$, there is some m such that the m-step nilfactor is characteristic for $(h_1 r_n, h_2 r_n, \ldots, h_k r_n)$.

(2) Similarly, $(r_n)_{n\in\mathbb{N}}$ is said to be *polynomially good for projection onto nilfactors* (*polynomial-GPN*) if for any $s_1, s_2, \ldots, s_k \in \mathbb{Z}[n]$, there is some m such that the m-step nilfactor is characteristic for $(s_1(r_n), s_2(r_n), \ldots, s_k(r_n))$. It is obvious that a polynomial-GPN sequence is linear-GPN.

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By the work of Host and Kra [**22**] and Leibman [**25**],the polynomial sequence $(Q(n))$ is polynomial-GPN. On the other hand, Frantzikinakis [**14**] showed that $(\lfloor n^c \rfloor)$ with $c > 0$, $c \notin \mathbb{Z}$ is linear-GPN.

Definition 4.

(1) The sequence (r_n) is said to be *good for equidistribution on nilmanifolds* (denoted by *GEN*) if, for an ergodic nilsystem (X, μ, τ), the sequence $(\tau^{r_n} 1_X)_{n \in \mathbb{N}}$ is equidistributed on X.

(2) The sequence (r_n) is called *essentially good for equidistribution on nilmanifolds (EGEN)* if the following holds: suppose for some $s, d \in \mathbb{N}$ such that the set $\{n \in \mathbb{N} : r_n \equiv s \pmod{d}\}$ has positive upper density. Let $r_{s \,(\mathrm{mod}\, d), n}$ denote the nth element of $\{r_m : m \in \mathbb{N}\}$ that is congruent to $s \pmod{d}$. Let (X, μ, τ) be an ergodic nilsystem with d connected components and X_0 be the component containing 1_X. Then the sequence $(\tau^{r_{s \,(\mathrm{mod}\, d), n}} 1_X)_{n \in \mathbb{N}}$ is equidistributed on $\tau^s X_0$.

Remark. Here is the difference between GEN and EGEN. In an ergodic nilsystem, the orbit of any point along a GEN sequence is equidistributed on the nilmanifold. On the other hand, the orbit along an EGEN sequence may not be. However, if we restrict the sequence to a suitable arithmetic progression, the orbit now is equidistributed on a connected component of the nilmanifold.

It is easy to see that a GEN sequence is EGEN.

Frantzikinakis [**13**] proved that $(\lfloor n^c \rfloor)$ with $c > 0$, $c \notin \mathbb{Z}$ is GEN. He also proved that polynomial sequences are EGEN [**14**, Lemma 6.7]. To demonstrate why polynomials satisfy EGEN (but not GEN) properties, take for example $Q(n) = n^2$, $X = \mathbb{T} \times \mathbb{Z}/3$ and $\tau = (\alpha, \bar{1})$ where α is irrational. Then since $n^2 \equiv 0$ or $1 \pmod{3}$, the sequence $(\tau^{n^2}(0, 0))$ never visits the connected component $\mathbb{T} \times \bar{2}$. So it is not equidistributed on the entire $\mathbb{T} \times \mathbb{Z}/3$. However, if we consider only those $n \equiv 0 \pmod{3}$, i.e. $n = 3m$ then the sequence $(\tau^{((3m)^2}(0, 0)$ is now equidistributed on the component $\mathbb{T} \times \bar{0}$. Similarly the sequences $(\tau^{(3m+1)^2}(0, 0))$ and $(\tau^{(3m+2)^2}(0, 0))$ are equidistributed on $\mathbb{T} \times \bar{1}$.

Definition 5. A sequence that is both linear-GPN and EGEN is called a *linear-good sequence*. Analogously, a sequence that is both polynomial-GPN and EGEN is called a *polynomial-good sequence*.

From above discussion, we see that the polynomial sequence $(Q(n))$ is polynomial-good while the Hardy sequence $(\lfloor n^c \rfloor)$ is linear-good.

2.12. *Gaussian system.* For a positive measure σ on $\mathbb{T}$, there exists a Gaussian system (X, μ, T) and function $g \in L^2(\mu)$ such that $\hat{\sigma}(n) = \int_X g T^n \bar{g}\, d\mu$ for all $n \in \mathbb{N}$. If σ is a probability measure, then $\|g\|_{L^2(\mu)} = 1$. It is worth mentioning that g is a Gaussian variable, and hence unbounded. See [**7**, pp. 369–371] for details.

2.13. *Rigid sequences.* We recall the definition of rigid sequences from the introduction. An increasing sequence of integers (r_n) is called rigid if there is a weakly mixing system (X, μ, T) such that $\|T^{r_n} f - f\|_{L^2(\mu)} \to 0$ for all $f \in L^2(\mu)$. Using

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Gaussian systems, we can show that a sequence (r_n) is rigid if and only if there is a continuous measure σ on $\mathbb{T}$ such that $\hat{\sigma}(r_n) \to 1$ as $n \to \infty$.

Examples of rigid sequences include $(q^n)_{n\in\mathbb{N}}$ for $q \in \mathbb{N}$, $q \geq 2$. Generally, an increasing sequence (r_n), such that $r_n | r_{n+1}$, is rigid [**5**, **10**]. Furthermore, there is a rigid sequence with very slow growth. Let (d_n) be an increasing sequence of integers of density zero. Then there is a rigid sequence (r_n) such that $r_n \leq d_n$ for all $n \in \mathbb{N}$ [**1**]. See [**5**, **10**, **4**] and [**11**] for more exhaustive lists of rigid sequences.

3. *Integral of nilsequences*

To prove that a correlation sequence in a non-ergodic system has a nil+null decomposition, Leibman [**28**] showed that an integral of nilsequences has such a decomposition. For this purpose, by a series of reductions, Leibman proved that it suffices to show the following.

PROPOSITION 3.1. (Leibman [**28**, Proposition 4.3]) *Let $X = G/\Gamma$ be a nilmanifold, ρ be a finite Borel measure on G and $f \in C(X)$, then the sequence $\varphi(n) = \int_G f(g^n 1_X)\, d\rho(g)$ has a nil+null decomposition.*

By the same reduction, for the purpose of showing Proposition 1.2, i.e. the null component of an integral of nilsequences is null along EGEN sequences, it suffices to show the following.

PROPOSITION 3.2. *With the set-up as in Proposition 3.1, in the nil+null decomposition of $(\varphi(n))$, the null component is null along any EGEN sequence.*

The rest of this section is devoted to proving Proposition 3.2. We start with a lemma.

LEMMA 3.3. *Let $X = G/\Gamma$ be a nilmanifold and Z be a normal subnilmanifold that contains 1_X. Suppose $\tau \in G^0$ such that $(\tau^n Z)_{n\in\mathbb{N}}$ is dense in X. Let ρ be a finite Borel measure on G such that for $\tilde{\rho} = \pi_*(\rho)$ we have $\operatorname{supp}(\tilde{\rho}) \subseteq \tau Z$ and $\tilde{\rho}(\tau W) = 0$ for any proper normal subnilmanifold W of Z. Let $\varphi(n) = \int_G f(g^n 1_X)\, d\rho(g)$ for $n \in \mathbb{N}$, $\hat{X} = X/Z$ and $\hat{f} = \mathbb{E}(f|\hat{X})$. Then $(\varphi(n) - \hat{f}(\pi(\tau^n)))_{n\in\mathbb{N}}$ is null along any EGEN sequence. In particular, the null component of φ is null along any EGEN sequence.*

Proof. Let (r_n) be an arbitrary EGEN sequence. Replacing f by $f - \hat{f}$, we can assume $\mathbb{E}(f|\hat{X}) = 0$. We are left with the need to show that $(\varphi(n))_{n\in\mathbb{N}}$ is null along (r_n).

Shift ρ to the origin by replacing it by $\tau_*^{-1}\rho$. Let L be a connected subgroup of G such that $\pi(L) = Z$. So now, $\operatorname{supp}(\rho) \subseteq L$ and $\operatorname{supp}(\tilde{\rho}) \subseteq Z$.

Let $d \in \mathbb{N}$ be the number of connected components of $X \times_{\hat{X}} X$. Note that to show φ is null along (r_n), it suffices to show that φ is null along $(r_{s\ (\mathrm{mod}\ d),n})$ for any $0 \leq s \leq d-1$ such that the set $\{n \in \mathbb{N} : r_n \equiv s \pmod{d}\}$ has positive upper density. Let s be such a number. Define

$$H(a, b) = \lim_{N\to\infty} \mathbb{E}_{n\in[N]} f \otimes \bar{f}(\pi^{\times 2}((\tau a, \tau b)^{r_{s\ (\mathrm{mod}\ d),n}}))$$

and

$$F(a, b) = \lim_{N\to\infty} \mathbb{E}_{n\in[N]} f \otimes \bar{f}(\pi^{\times 2}((\tau a, \tau b)^{dn+s}))$$

for $(a, b) \in L \times L$.

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According to the proof of Lemma 4.6 in Leibman [**28**], for $\rho^{\times 2}$-almost every $(a, b) \in L \times L$, the sequence $u_n = (\tau a, \tau b)^n(1_X, 1_X) = \pi^{\times 2}((\tau a, \tau b)^n)$ is equidistributed on $X \times_{\hat{X}} X$. Therefore, for those (a, b), by §2.4.1, the sequence $(\tau a, \tau b)^{dn+s}(1_X, 1_X)$ is equidistributed on $(\tau a, \tau b)^s(X \times_{\hat{X}} X)_o$, where $X \times_{\hat{X}} X)_o$ is the connected component of $X \times_{\hat{X}} X$ containing $(1_X, 1_X)$.

On the other hand, by the definition of EGEN, the sequence $(\tau a, \tau b)^{r_{s \pmod d),n}}(1_X, 1_X)$ is also equidistributed on $(\tau a, \tau b)^s(X \times_{\hat{X}} X)_o$. This implies that $H(a, b) = F(a, b) = \int_{(\tau a,\tau b)^s(X\times_{\hat{X}}X)_o} f \otimes \bar{f}\, d\mu_{(\tau a,\tau b)^s(X\times_{\hat{X}}X)_o}$. This equality holds for $\rho^{\times 2}$-almost every $(a, b) \in L \times L$. So by taking the integral on $L \times L$, with respect to $\rho^{\times 2}$, we obtain

$$\lim_{N\to\infty} \mathbb{E}_{n\in\mathbb{N}}|\varphi(dn + s)|^2 = \lim_{N\to\infty} \mathbb{E}_{n\in\mathbb{N}}|\varphi(r_{s \pmod d),n})|^2.$$

The sequence $(\varphi(n))_{n\in\mathbb{N}}$ is a null sequence along $(n)_{n\in\mathbb{N}}$. The subsequence $(dn + s)_{n\in\mathbb{N}}$ has density $1/d$ in $\mathbb{N}$. It follows that $(\varphi(n))_{n\in\mathbb{N}}$ is also a null sequence along $(dn + s)_{n\in\mathbb{N}}$. Thus it follows that φ is null along $(r_{s \pmod d),n})_{n\in\mathbb{N}}$. This fact holds true for any $0 \le s \le d - 1$ such that $\{n \in \mathbb{N} : r_n \equiv s \pmod d\}$ has positive upper density. Hence φ is null along (r_n). Since (r_n) is an arbitrary EGEN sequence, we have that φ is null along any EGEN sequence.

We have just shown that $(\varphi(n) - \hat{f}(\pi(\tau^n)))$ is null along any EGEN sequence. By definition, $\hat{f}(\pi(\tau^n))$ is a nilsequence. Thus $(\varphi(n) - \hat{f}(\pi(\tau^n))$ is the null component of $\varphi(n)$, and it is null along any EGEN sequence. This finishes our proof. □

We need a lemma from Leibman [**28**].

LEMMA 3.4. (Leibman [**28**, Lemma 4.4]) *Let $X = G/\Gamma$ be a nilmanifold with standard quotient map $\pi : G \to X$. Suppose ρ is a finite Borel measure on G. Then there exists an at most countable collection $\mathcal{V}$ of connected subnilmanifolds of X and finite Borel measure ρ_V for $V \in \mathcal{V}$ on G, such that $\rho = \sum_{V\in\mathcal{V}} \rho_V$ and for every $V \in \mathcal{V}$,* supp$(\tilde{\rho}_V) \subseteq V$ *and $\tilde{\rho}_V(S) = 0$ for any proper subnilmanifold S of V where $\tilde{\rho}_V = \pi_*(\rho_V)$.*

We are ready to prove Proposition 3.2.

Proof of Proposition 3.2. By Lemma 3.4, the measure ρ can be decomposed as $\rho = \sum_{V\in\mathcal{V}} \rho_V$ where supp$(\tilde{\rho}_V) \subseteq V$ and $\tilde{\rho}_V(S) = 0$ for any proper subnilmanifold S of V.

Fix $V \in \mathcal{V}$. Let V' be the normal closure of V in X. For any proper normal subnilmanifold S' of V', the intersection $S' \cap V$ is a proper subnilmanifold of V by the minimality of V'. Therefore $\tilde{\rho}_V(S' \cap V) = 0$. Since supp$(\tilde{\rho}_V) \subseteq V$, we have $\tilde{\rho}_V(S') = \tilde{\rho}_V(S' \cap V) + \tilde{\rho}_V(S' \backslash V) = 0$.

Write $V' = \tau Z$ for $\tau \in G^0$ and a normal subnilmanifold Z of X that contains 1_X. By considering only the orbit closure $(\tau^n Z)_{n\in\mathbb{N}}$ of Z, without the loss of generality, we can assume that $(\tau^n Z)_{n\in N}$ is dense in X. Applying Lemma 3.3, the null component of the sequence $\int_G f(\pi(g^n))\, d\rho_V(g)$ is null along any EGEN sequence.

A convergent countable sum of nilsequences is a nilsequence. Likewise, a convergent countable sum of null sequences along any EGEN is a null sequence along any EGEN sequence. Therefore

$$\varphi(n) = \int_G f(\pi(g^n))\, d\rho(g) = \sum_{V\in\mathcal{V}} \int_G f(\pi(g^n))\, d\rho_V(g)$$

has a nil+null decomposition, and its null component is null along any EGEN sequence. This finishes our proof of Proposition 3.2. □

4. *Null along good sequences*

In this section, we prove that the null component of a polynomial correlation is null along any polynomial-good sequence. By the same proof, the null component of a linear correlation is null along any linear-good sequence. The argument proceeds in three stages: first in a nilsystem, then in an arbitrary ergodic system, and lastly in a general measure-preserving system.

4.1. *In a nilsystem.*

PROPOSITION 4.1. *The null component of a polynomial correlation in a nilsystem is null along any EGEN sequence.*

Proof. Let $(X = G/\Gamma, \mu_X, \tau)$ be a nilsystem, $f_j \in L^\infty(\mu_X)$ and let $s_j \in \mathbb{Z}[n]$. Let

$$a(n) = \int_X T^{s_0(n)} f_0 \ldots T^{s_k(n)} f_k \, d\mu.$$

By approximation, we can assume that f_j is a continuous function on X for all j. Then

$$\begin{aligned} a(n) &= \int_X f_0(\tau^{s_0(n)x} \ldots f_k(\tau^{s_k(n)x} \, d\mu_X(x) \\ &= \int_X f_0 \otimes \cdots \otimes f_k((\tau^{s_0(n)}, \ldots, \tau^{s_k(n)})(x, x, \ldots, x)) \, d\mu_X(x). \end{aligned}$$

The function $F = f_0 \otimes \cdots \otimes f_k$ is continuous on X^{k+1}. On the other hand, the sequence $(\tau^{s_0(n)}, \ldots, \tau^{s_k(n)}) = g_0^{s_0(n)} \ldots g_k^{s_k(n)}$ is a polynomial sequence on G^{k+1} where $g_j = (1_G, \ldots, 1_G, \tau_j, 1_G, \ldots, 1_G)$. Hence $f_0 \otimes \cdots \otimes f_k((\tau^{s_0(n)}, \ldots, \tau^{s_k(n)})$ $(x, x, \ldots, x))$ is a polynomial nilsequence for all $x \in X$. By [**25**], a polynomial nilsequence is also a nilsequence (of higher degree of nilpotency). Therefore, $(a(n))$ is an integral of nilsequences. By Proposition 1.2, the null component of $(a(n))$ is null along any EGEN sequence. Our proof finishes. □

4.2. *In an ergodic system.*

LEMMA 4.2. *Let $(a(n))$ be a polynomial correlation in an ergodic system and let (r_n) be a polynomial-GPN sequence. Then there exists an $m \in \mathbb{N}$ such that if $(\tilde{a}(n))$ is the projection of $(a(n))$ onto an m-step nilfactor $\mathcal{Z}_m$, then $a - \tilde{a}$ is a null sequence along (n) and (r_n).*

Proof. Since both (n) and (r_n) are polynomial-GPN sequences, then m_1 and m_2 exist such that $\mathcal{Z}_{m_1}$ is characteristic for $(s_0(n), \ldots, s_k(n))$, and $\mathcal{Z}_{m_2}$ is characteristic for $(s_0(r_n), \ldots, s_k(r_n))$. Let $m = \max\{m_1, m_2\} + 1$ and let $\tilde{a}$ be the projection of a onto $\mathcal{Z}_m$. Then with the same proof as in [**6**, Corollary 4.5], we obtain the conclusion. □

PROPOSITION 4.3. *The null component of a polynomial correlation in an ergodic system is null along any polynomial-good sequence.*

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Proof. Let $(a(n))$ be a polynomial correlation in an ergodic system and let (r_n) be a polynomial-good sequence. The goal is to show the null component of $(a(n))$ is null along (r_n). Let $(\tilde{a}(n))$ be as described in Lemma 4.2. Then by this lemma, $a - \tilde{a}$ is null along (n) and (r_n).

On the other hand, $\tilde{a}$ is a polynomial correlation arising from a nilfactor Y. Y is an inverse limit of nilsystems, say $Y = \varprojlim Y_l$. Let $\tilde{a}_l$ be the projection of $\tilde{a}$ on to Y_l for each $l \in \mathbb{N}$. By Proposition 4.1, $\tilde{a}_l$ can be written as $\tilde{a}_l = \psi_l + \epsilon_l$ where ψ_l is a nilsequence and ϵ_l is null along any EGEN sequence, in particular along (n) and (r_n). Note that $\tilde{a}_l$ converges to $\tilde{a}$ uniformly as $l \to \infty$, since $Y = \varprojlim Y_l$. Hence it is easy to see that ψ_l converges to a nilsequence ψ uniformly (see [**6**, §7.4]). Likewise, ϵ_l converges uniformly to a null sequence ϵ, which is also null along (r_n). Now $\tilde{a}$ can be written as $\tilde{a} = \psi + \epsilon$. Hence it has nil+null decomposition, and its null component is null along (r_n).

In summary, we have just shown that both $a - \tilde{a}$ and the null component of $\tilde{a}$ is null along (r_n). Therefore the null component of $a = \tilde{a} + (a - \tilde{a})$ is null along (r_n). Our proof finishes. □

4.3. *In a general measure-preserving system.*

PROPOSITION 4.4. *The null component of a polynomial correlation is null along any polynomial-good sequence.*

Proof. Let $\mu = \int_\Omega \mu_\omega \, dP(\omega)$ be the ergodic decomposition of μ with respect to T. Then $a(n) = \int_\Omega a_\omega(n) \, dP(\omega)$ where

$$a_\omega(n) = \int_X T^{s_0(n)} f_0 \ldots T^{s_k(n)} f_k \, d\mu_\omega.$$

For almost every $\omega \in \Omega$, the system (X, μ_ω, T) is ergodic. Hence by Proposition 4.3, the null component of a_ω is null along any polynomial-good sequence. To be precise, $a_\omega = \psi_\omega + \epsilon_\omega$ where ψ_ω is a nilsequence and ϵ_ω is a null sequence along every polynomial-good sequence. Thus $a = \int_\Omega \psi_\omega \, dP + \int_\Omega \epsilon_\omega \, dP$.

On one hand, the sequence $\int_\Omega \psi_\omega \, dP$ is an integral of nilsequences. Hence its null component is null along any polynomial-good sequence, by Proposition 1.2. On the other hand, $\int_\Omega \epsilon_\omega \, dP$ is obviously null along any polynomial-good sequence (an integral of a null sequence is still a null sequence). Therefore, the null component of $(a(n))$ is null along any polynomial-good sequence. Our proof finishes. □

5. *Polynomials of primes are polynomial-good sequences*

In this section, we show that polynomials of primes are polynomial-good sequences. The proof has two parts. One is to show such sequences are polynomial-GPN. The other part is to show they are EGEN. Some notation is needed before going into the details.

The *modified von Mangoldt function* is defined to be

$$\Lambda'(n) = \begin{cases} \log n & (n \in \mathbb{P}), \\ 0 & (n \in \mathbb{N} \backslash \mathbb{P}). \end{cases}$$

The *Euler totient function* $\phi(n)$ is the number of positive integers not greater than n and relatively prime to n.

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For $r < M \in \mathbb{N}$, define

$$\Lambda'_{M,r}(n) = \frac{\phi(M)}{M}\Lambda'(Mn + r).$$

For $\omega \in \mathbb{N}$, define $W = \prod_{p\in\mathbb{P}, p<\omega} p$.

The symbol $o_{\omega\to\infty}(1)$ (or $o_{N\to\infty}(1)$) represents a function of ω that approaches zero as $\omega \to \infty (N \to \infty$ respectively). Furthermore, $o_{\omega,N\to\infty}(1)$ is a function of ω and N such that for a fixed ω, the function approaches zero as $N \to \infty$.

For two sequences a and b, by writing $a(N) \sim b(N)$ we mean $\lim_{N\to\infty} a(N)/b(N) = 1$.

5.1. *Polynomials of primes are polynomial-GPN.* Let $Q(n), s_j(n) \in \mathbb{Z}[n]$ for $1 \leq j \leq k$. The results of Host and Kra [**22**] and Leibman [**24**] show that there exists some $m \in \mathbb{N}$ such that an m-step nilfactor $\mathcal{Z}_m$ is characteristic for $(s_1(Q(n)), \ldots, s_k(Q(n))$. The optimal value for m may depend intrinsically on the polynomials and is very hard to pinpoint. However, there is always an m that only depends on the degrees of the polynomials. With that m, we prove that $\mathcal{Z}_m$ is characteristic for $(s_1(Q(p_n)), \ldots, s_k(Q(p_n))$. Denote

$$a(n) = T^{s_1(Q(n))} f_0 \ldots T^{s_k(Q(n))} f_k \in L^\infty(\mu)$$

and

$$B_{W,r}(N) = \mathbb{E}_{n\in[N]} a(Wn + r).$$

The key ingredient in our proof is a proposition from Frantzikinakis, Host and Kra [**18**] that compares ergodic averages along primes to the averages along integers.

LEMMA 5.1. (Frantzikinakis, Host and Kra [**18**, Proposition 3.6])

$$\max_{r<W,(r,W)=1} \|\mathbb{E}_{n\in[N]}\Lambda'_{W,r}(n)a(Wn + r) - B_{W,r}(N)\|_{L^2(\mu)} = o_{N\to\infty,\omega}(1) + o_{\omega\to\infty}(1).$$

We are ready for the main result of this section.

PROPOSITION 5.2. *For any $Q \in \mathbb{Z}[n]$ non-constant, the sequence $(Q(p_n))$ is polynomial-GPN.*

Proof. With the notation as before, assume $\mathbb{E}(f_j|\mathcal{Z}_m) = 0$ for some j. We need to prove that

$$\lim_{N\to\infty} \mathbb{E}_{[N]}a(p_n) = 0 \quad \text{in } L^2(\mu).$$

From Lemma 5.1, taking an average along $r < W$, $(r, W) = 1$, we obtain

$$\begin{aligned} &\|\mathbb{E}_{(r,W)=1}\mathbb{E}_{n\in[N]}\Lambda'_{\omega,r}(n)a(Wn + r) - \mathbb{E}_{(r,W)=1}B_{\omega,r}(N)\|_{L^2(\mu)} \\ &\quad = o_{\omega,N\to\infty}(1) + o_{\omega\to\infty}(1). \end{aligned} \tag{5.1}$$

Note that

$$\mathbb{E}_{(r,W)=1}\mathbb{E}_{n\in[N]}\Lambda'_{\omega,r}(n)a(Wn + r) = \mathbb{E}_{n\in[WN]}\Lambda'(n)a(n).$$

Hence (5.1) becomes

$$\|\mathbb{E}_{n\in[WN]}\Lambda'(n)a(n) - \mathbb{E}_{(r,W)=1}B_{\omega,r}(N)\|_{L^2(\mu)} = o_{\omega,N\to\infty}(1) + o_{\omega\to\infty}(1). \tag{5.2}$$

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And since $\mathcal{Z}_m$ is characteristic for $(s_1(Q(Wn+r)), \ldots, s_k(Q(Wn+r)))$ for any W and r (remember that m only depends on the degrees of the polynomials), we have $\lim_{N\to\infty} B_{\omega,r}(N) = 0$ in $L^2(\mu)$. Hence,

$$\lim_{N\to\infty} \mathbb{E}_{(r,W)=1} B_{\omega,r}(N) = 0 \quad \text{in } L^2(\mu).$$

On the other hand, by results of Wooley and Ziegler [**31**] and Frantzikinakis, Host and Kra [**18**], the limit

$$\lim_{N\to\infty} \mathbb{E}_{[N]} \Lambda'(n) a(n)$$

exists in $L^2(\mu)$ (equal to $\lim_{N\to\infty} \mathbb{E}_{[N]} a(p_n)$). Call that limit $F \in L^2(\mu)$.

Then for any $W \in \mathbb{N}$,

$$F = \lim_{N\to\infty} \mathbb{E}_{[WN]} \Lambda'(n) a(n).$$

Taking the limit as $N \to \infty$ in (5.2), we get

$$\|F - 0\|_{L^2(\mu)} = o_{\omega\to\infty}(1). \tag{5.3}$$

The left hand side of (5.3) no longer depends on ω. Let $\omega \to \infty$, we get $F = 0$ in $L^2(\mu)$. This completes our proof. □

5.2. *Polynomial of primes is EGEN.* We need the following proposition by Green and Tao.

PROPOSITION 5.3. (Green and Tao [**20**, Theorem 7.1]) *For sufficiently large $\omega \in \mathbb{N}$, define $W = W(\omega) = \prod_{p\in\mathbb{P}, p<\omega} p$. Suppose $(X = G/\Gamma, g)$ is a nilsystem, $x \in X$ and $F \in C(X)$. Then*

$$\max_{b<W,(b,W)=1} \left| \lim_{N\to\infty} \mathbb{E}_{n\in[N]} (\Lambda'_{W,b}(n) - 1) F(g^{Q(n)} x) \right| = o_{\omega\to\infty}(1).$$

Remark. The Green–Tao version of Proposition 5.3 is slightly different to the one we introduce here. In Green and Tao's version, in the place of $F(g^{Q(n)})x$ there is a Lipschitz nilsequence that arises from a connected and simply-connected nilpotent group G. However, it is immediately clear that Green–Tao's version implies our result. First, as discussed in §2.2, any basic nilsequence can be seen as arising from a nilmanifold whose Lie group is connected and simply-connected. Second, every polynomial nilsequence is a nilsequence (see [**25**, Theorem B* Proof]). Lastly, we can take F to be any continuous function since the set of Lipschitz functions is dense in $C(X)$.

We have a corollary.

COROLLARY 5.4. *Fix $d \in \mathbb{N}$. For sufficiently large $\omega \in \mathbb{N}$, define $W = W(\omega) = \prod_{p\in\mathbb{P}, p<\omega} p$. Suppose $(X = G/\Gamma, g)$ is a nilsystem, $x \in X$ and $F \in C(X)$. Then*

$$\max_{b<dW,(b,dW)=1} \left| \lim_{N\to\infty} \mathbb{E}_{n\in[N]} (\Lambda'_{dW,b}(n) - 1) F(g^{Q(n)} x) \right| = o_{\omega\to\infty}(1).$$

Proof. By [**25**], the sequence $(F(g^{Q(n)} x))_{n\in\mathbb{Z}}$ is a basic nilsequence. Therefore, according to Leibman [**27**, Lemma 2.4], there exists a basic nilsequence $(F'(g'^n x'))_{n\in\mathbb{N}}$ such that

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$F'(g'^{dn+r}x') = F(g^{Q(n)}x)$ for $r = 0$ and $F'(g'^{dn+r}x') = 0$ for $1 \le r \le d-1$ and $n \in \mathbb{N}$. Note that for sufficiently large ω, all prime divisors of d divide W. Therefore for $b \in \mathbb{N}$ we have $(b, W) = 1$ if and only if $(b, dW) = 1$. It follows that $\phi(dW) = d\phi(W)$. Thus $\phi(dW)/(dW) = \phi(W)/W$. By Proposition 5.3, we have

$$\max_{b<dW,(b,dW)=1} \left| \lim_{N\to\infty} \mathbb{E}_{n\in[N]} \left(\frac{\phi(dW)}{dW} \Lambda'(Wn+b) - 1 \right) F'(g'^{n}x') \right| = o_{\omega\to\infty}(1). \quad (5.4)$$

Note that the sequence $(F'(g'^{n}x'))_{n\in\mathbb{N}}$ is supported on the set $\{n \in \mathbb{N} : n = dm \text{ for some } m \in \mathbb{N}\}$. Replacing n by dm, the left hand side of Equation 5.4 is now equal to

$$\frac{1}{d} \max_{b<dW,(b,dW)=1} \left| \lim_{N\to\infty} \mathbb{E}_{m\in[N]} \left(\frac{\phi(dW)}{dW} \Lambda'(Wdm+b) - 1 \right) F'(g'^{dm}x') \right|.$$

Our proof is finished by noting that $F'(g'^{dm}x') = F(g^{Q(m)}x)$ for all $m \in \mathbb{N}$. □

We need two more lemmas before going to the main theorem.

LEMMA 5.5. *Let $a(n)$ be a bounded sequence and let $0 \le r < d$ be positive integers. Define $Q_{r,d}(N) := \{1 \le q \le N : qd + r \in \mathbb{P}\}$. Then*

$$\mathbb{E}_{q\in Q_{r,d}(N)} a(q) - \mathbb{E}_{n\in[N]} \Lambda'_{d,r}(n) a(n) = o_{N\to\infty}(1).$$

Proof. Let $\pi_{r,d}(N)$ be the cardinality of $Q_{r,d}(N)$. By Dirichlet's theorem about primes in arithmetic progessions, the set of primes that are congruent to $r \pmod d$ has density $1/\phi(d)$ relative to the set of all primes. Therefore,

$$\pi_{r,d}(N) \sim \frac{\pi(dN+r)}{\phi(d)} \sim \frac{dN+r}{\log(dN+r)\phi(d)} \sim \frac{dN}{(\log N)\phi(d)}.$$

Note that $\Lambda'(dn+r) = 0$ if $dn+r \notin \mathbb{P}$. Therefore,

$$\begin{aligned} u(N) &:= \left| \mathbb{E}_{q\in Q_{r,d}(N)} a(q) - \mathbb{E}_{n\in[N]} \frac{\phi(d)}{d} \Lambda'(dn+r) a(n) \right| \\ &= \left| \frac{1}{N} \sum_{q\in Q_{r,d}(N)} a(q) \left(\frac{N}{\pi_{r,d}(N)} - \frac{\phi(d)}{d} \Lambda'(dq+r) \right) \right| \\ &= \left| \frac{1}{N} \sum_{q\in Q_{r,d}(N)} a(q) \left(\frac{\phi(d)}{d} \log N - \frac{\phi(d)}{d} \log(dq+r) \right) \right| + o_{N\to\infty}(1). \end{aligned}$$

Note that $\log(dq+r) = \log(q) + \log(d + r/q)$. Thus we have

$$\begin{aligned} \left| \frac{1}{N} \sum_{q\in Q_{r,d}(N)} a(q) \log\left(d + \frac{r}{q} \right) \right| &\le \|a\|_\infty \frac{1}{N} \sum_{q\in Q_{r,d}(N)} \log\left(d + \frac{r}{q} \right) \\ &= \|a\|_\infty \frac{\pi_{r,d}(N) \log(d+r/q)}{N} \\ &= \|a\|_\infty \frac{(Nd+r)\log(d+r/q)}{\phi(d)\log(Nd+r)N} + o_{N\to\infty}(1) = o_{N\to\infty}(1). \end{aligned}$$

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Therefore,

$$u(N) = \left| \frac{1}{N} \sum_{q \in Q_{r,d}(N)} a(q) \left(\frac{\phi(d)}{d} \log N - \frac{\phi(d)}{d} \log(q) \right) \right| + o_{N\to\infty}(1)$$
$$= \frac{\phi(d)}{d} \left| \frac{1}{N} \sum_{q \in Q_{r,d}(N)} a(q) \log \frac{N}{q} \right| + o_{N\to\infty}(1).$$

Since $\log(N/q) > 0$ for all $q \in Q_{r,d}(N)$, we have

$$u(N) \le \|a\|_\infty \frac{\phi(d)}{d} \frac{1}{N} \sum_{q \in Q_{r,d}(N)} \log \frac{N}{q} + o_{N\to\infty}(1).$$

For any $0 < c < 1$,

$$\frac{1}{N} \sum_{q \in Q_{r,d}(N)} \log \frac{N}{q} = \frac{1}{N} \sum_{\substack{q \in Q_{r,d}(N) \\ q < cN}} \log \frac{N}{q} + \frac{1}{N} \sum_{\substack{q \in Q_{r,d}(N) \\ q \ge cN}} \log \frac{N}{q}. \tag{5.5}$$

It is easy to see that when N goes to infinity, the first average in the right hand side of (5.5) approaches a value less than $cd/\phi(d)$, while the second average goes to 0. Since c is arbitrary, we deduce $u(N) \to 0$ as $N \to \infty$. Our proof finishes. □

PROPOSITION 5.6. *Fix $d \in \mathbb{N}$ and $r \in \{0, 1, \ldots, d-1\}$. For $n \in \mathbb{N}$, define q_n to be the nth integer such that $dq_n + r$ is a prime. Let $(X = G/\Gamma, \mu_X, \tau)$ be a totally ergodic nilsystem. Then for any $Q(n) \in \mathbb{Z}[n]$ non-constant and $x \in X$, the sequence $(\tau^{Q(q_n)}x)_{n\in\mathbb{N}}$ is equidistributed on X.*

Proof. Let $f \in C(X)$. Replacing f by $f - \int f$, we can assume $\int f = 0$. For any $x \in X$, we want to show that

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]} f(\tau^{Q(q_n)}x) = 0.$$

For sufficiently large $\omega \in \mathbb{N}$, let $B_\omega = \{0 \le b < dW : (b, dW) = 1, b \equiv r \pmod{d}\}$. By Corollary 5.4, for $b \in B$,

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]} (\Lambda'_{dW,b}(n) - 1) f(\tau^{Q(Wn+(b-r)/d)}x) = o_{\omega\to\infty}(1).$$

Note that the term $o_{\omega\to\infty}(1)$ does not depend on b. Therefore

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]} \Lambda'_{dW,b}(n) f(\tau^{Q(Wn+(b-r)/d)}x)$$
$$= \lim_{N\to\infty} \mathbb{E}_{n\in[N]} f(\tau^{Q(Wn+(b-r)/d)}x) + o_{\omega\to\infty}. \tag{5.6}$$

Since (X, μ_X, g) is totally ergodic, for any $b \in B$, the sequence $(\tau^{Q(Wn+(b-r)/d)})$ is equidistributed on X (see §2.4.2). That means

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]} f(\tau^{Q(Wn+(b-r)/d)}x) = 0.$$

Hence, (5.6) implies

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]} \Lambda'_{dW,b}(n) f(\tau^{Q(Wn+(b-r)/d)}x) = o_{\omega\to\infty}(1). \tag{5.7}$$

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Let $B'_\omega = \{0 \le b < dW : b \equiv r \pmod d\}$. In (5.7), summing over $b \in B'_\omega$ and noting that $\Lambda'(dWn + b) = 0$ if $b \in B'_\omega \backslash B_\omega$, we get

$$\sum_{b \in B'_\omega} \lim_{N\to\infty} \mathbb{E}_{n\in[N]} \Lambda'_{dW,b}(n) f(\tau^{Q(Wn+(b-r)/d)}x) = |B_\omega| o_{\omega\to\infty}(1). \tag{5.8}$$

Now dividing both sides by W (which is the cardinality of B'_ω), we get

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]} \mathbb{E}_{b\in B'_\omega} \Lambda'_{dW,b}(n) f(\tau^{Q(Wn+(b-r)/d)}x) = \frac{|B_\omega|}{W} o_{\omega\to\infty}(1). \tag{5.9}$$

The left hand side now is equal to

$$\lim_{N\to\infty} \mathbb{E}_{n\in[WN]} \frac{\phi(dW)}{dW} \Lambda'(dn + r) f(\tau^{Q(n)}x).$$

Multiplying both sides of (5.9) by $\phi(d)W/\phi(dW)$, we get

$$\lim_{N\to\infty} \mathbb{E}_{n\in[WN]} \Lambda'_{d,r}(n) f(\tau^{Q(n)}x) = \frac{\phi(d)|B_\omega|}{\phi(dW)} o_{\omega\to\infty}(1) = o_{\omega\to\infty}(1). \tag{5.10}$$

Note that

$$\mathbb{E}_{n\in[N]} \Lambda'_{d,r}(n) f(\tau^{Q(n)}x) = \mathbb{E}_{n\in[W\lfloor N/W\rfloor]} \Lambda'_{d,r}(n) f(\tau^{Q(n)}x) + o_{\omega,N\to\infty}(1).$$

Therefore (5.10) implies

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]} \Lambda'_{d,r}(n) f(\tau^{Q(n)}x) = o_{\omega\to\infty}(1).$$

The left hand side no longer depends on ω. By letting ω approach infinity, we get

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]} \Lambda'_{d,r}(n) f(\tau^{Q(n)}x) = 0.$$

Applying Lemma 5.5 to $a(n) = f(\tau^{Q(n)}x)$, our proposition is proved. □

We are ready to prove the main result of this section.

PROPOSITION 5.7. *For $Q \in \mathbb{Z}[n]$ non-constant, the sequence $(Q(p_n))$ is EGEN.*

Proof. Let $(X = G/\Gamma, \mu_X, \tau)$ be an ergodic nilsystem. Suppose X has d connected components and X_0 is the component containing 1_X. For $(r, d) = 1$, we need to show that the sequence $(\tau^{Q(p_{r \pmod d),n})}1_X)_{n\in\mathbb{N}}$ is equidistributed on $\tau^{Q(r)}X_0$.

Let $q_n = (p_{r \pmod d),n} - r)/d$. Applying Proposition 5.6 to the totally ergodic nilsystem $(\tau^{Q(r)}X_0, \tau^{Q(r)}\mu_{X_0}, \tau^d)$, with polynomial $P(n) = (Q(dn + r) - Q(r))/d \in \mathbb{Z}[n]$, we get that the sequence $(\tau^{P(q_n)}\tau^{Q(r)}1_X)$ is equidistributed on $\tau^{Q(r)}X_0$. Our proposition follows. □

5.3. *Proof of Proposition 1.3.* By Proposition 5.2, $(Q(p_n))$ is polynomial-GPN. By Proposition 5.7, the same sequence is EGEN. Hence by definition, $(Q(p_n))$ is polynomial-good.

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5.4. *Proof of Theorem 1.1.* By Proposition 4.4, the null component of a polynomial correlation is null along any polynomial-good sequence. The sequence $(Q(n))$ is polynomial-good, as mentioned in §2.11. On the other hand, the sequence $(Q(p_n))$ is polynomial-good by Proposition 1.3. Hence the first part of Theorem 1.1 is proved. The same argument applies to linear correlations and linear-good sequences $(\lfloor n^c \rfloor)$. That proves the second part of the theorem.

5.5. *Proof of Corollary 1.4.* Let $(X = G/\Gamma, \mu, \tau)$ be an ergodic nilsystem with d connected component and the set-up as in Corollary 1.4. Since $(Q(p_n))$ is EGEN, for each $s < d$ with $(s, d) = 1$, the sequence $(\tau^{Q(p_{s \, (\text{mod } d), n})} x)$ is equidistributed on $X_{Q(s)+k}$. Hence

$$\mathbb{E}_{p \in \mathbb{P}, p \equiv s \, (\text{mod } d)} f(\tau^{Q(p)} x) = \int_{X_{Q(s)+k}} f \, d\mu_{X_{Q(s)+k}}. \tag{5.11}$$

For each $s < d$ with $(s, d) = 1$, the set $\{p \in \mathbb{P} : p \equiv s \ (\text{mod } d)\}$ has density $1/\phi(d)$. Taking the average for all s, we have Corollary 1.4.

6. *Proof of Proposition 1.5*

In this section, we prove Proposition 1.5. Some preliminary facts are needed before going into the proof. By Herglotz's theorem, for $f \in L^2(\mu)$, there exists a complex measure σ on circle $\mathbb{T}$ such that $a(n) := \int_X f \cdot T^n \bar{f} \, d\mu = \int_{\mathbb{T}} e^{2\pi i n t} \, d\sigma(t) =: \hat{\sigma}(n)$ for all $n \in \mathbb{Z}$. The measure σ is called the spectral measure of f. Sometimes we denote it by σ_f to indicate the dependence on f. By decomposing σ to discrete and continuous parts, we get $a(n) = \hat{\sigma}_d(n) + \hat{\sigma}_c(n)$. The sequence $(\hat{\sigma}_d(n))_{n \in \mathbb{N}}$ and $(\hat{\sigma}_c(n))_{n \in \mathbb{N}}$ are the nil and null components of $(a(n))_{n \in \mathbb{N}}$, respectively. We have a proposition.

PROPOSITION 6.1. *Let (r_n) be a increasing sequence of integers such that there is a continuous measure σ on $\mathbb{T}$ such that $(\hat{\sigma}(n))$ is not null along (r_n). Then there exists a linear correlation whose null component is not null along (r_n).*

Proof. Let (r_n) be a increasing sequence of integers and σ be a continuous measure satisfying the Proposition 6.1 hypothesis. Our proposition would have been proved if there existed a bounded function f on a system (X, μ, T) whose spectral measure σ_f is equal to σ. However, we are only guaranteed the existence of an unbounded L^2-function (let us call it g) from a Gaussian system (see §2.12) that satisfies the required condition. To achieve our goal, a small modification is needed.

Since $L^\infty(\mu)$ is dense in $L^2(\mu)$, for a small ϵ_1, there exists $f \in L^\infty(\mu)$ such that $\|f - g\|_{L^2} < \epsilon_1$. That implies

$$|\hat{\sigma}_f(n) - \hat{\sigma}_g(n)| = \left| \int f T^n \bar{f} - \int g T^n \bar{g} \right| < 2\epsilon_1 \|g\|_{L^2} + \epsilon_1^2 =: \epsilon_2. \tag{6.1}$$

Let σ_{gd} and σ_{gc} denote discrete and continuous parts of σ_g respectively. By Wiener's lemma, since σ_g is continuous

$$\sigma_{gd}(\mathbb{T}) = \lim_{N \to \infty} \mathbb{E}_{n \in [N]} |\hat{\sigma}_g(n)|^2 = 0 \tag{6.2}$$

and

$$\sigma_{fd}(\mathbb{T}) = \lim_{N \to \infty} \mathbb{E}_{n \in [N]} |\hat{\sigma}_f(n)|^2. \tag{6.3}$$

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From (6.1), (6.2) and (6.3), we get $\sigma_{gd}(\mathbb{T}) < \epsilon_2^2$. That implies $|\hat{\sigma}_{fd}(n)| < \epsilon_2^2$ for all $n \in \mathbb{N}$. Hence, for all $n \in \mathbb{N}$:

$$|\hat{\sigma}_{fc}(n) - \hat{\sigma}_g(n)| = |\hat{\sigma}_f(n) - \hat{\sigma}_{fd}(n) - \hat{\sigma}_g(n)| < \epsilon_2 + \epsilon_2^2. \tag{6.4}$$

Since $(\hat{\sigma}_g(n))$ is not null along (r_n), there is $\delta > 0$ such that

$$\limsup_{N\to\infty} \mathbb{E}_{n\in[N]}|\hat{\sigma}_g(r_n)| = \delta. \tag{6.5}$$

If we choose ϵ_1 sufficiently small that $\epsilon_2 + \epsilon_2^2 < \delta$, then from (6.4) and (6.5), we have

$$\limsup_{N\to\infty} \mathbb{E}_{n\in[N]}|\hat{\sigma}_{fc}(r_n)| > \delta - \epsilon_2 - \epsilon_2^2 > 0. \tag{6.6}$$

That would imply $(\hat{\sigma}_{fc}(n))$ is not null along (r_n). Therefore, the null component of $\int f T^n \bar{f}$ is not null along (r_n). Our proof finishes. □

Proof of Proposition 1.5. If (r_n) is a rigid sequence, then there is a continuous measure σ on $\mathbb{T}$ such that $\hat{\sigma}(r_n) \to 1$ as $n \to \infty$ (see §2.13) . Hence $(\hat{\sigma}(n))$ is not null along (r_n), so (r_n) satisfies the Proposition 6.1 hypothesis. □

Remark. In a recent paper, Badea and Grivaux [**4**] show that there exists a continuous measure σ on $\mathbb{T}$ such that $\liminf_{n,m\in\mathbb{N}} \hat{\sigma}(2^n 3^m) > 0$. Hence the sequence $(2^n 3^m)$, when ordering in the increasing fashion, also satisfies the Proposition 6.1 hypothesis.

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