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# Numerical simulation of the Vlasov–Maxwell system for the nonlinear Berk–Breizman phenomenology

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**Abstract.** A fusion plasma is modeled by a one-dimensional Vlasov–Maxwell system of equations. Particle annihilation is considered by a dedicated factor in the Vlasov equation. A term describing background damping mechanisms is considered in the equation relating the electric field to the current (Berk–Breizman model). Results are presented for simulations in a one-dimensional velocity space, the extension by a further velocity dimension is sketched. As an initial condition, a bump-on-tail distribution is assumed. The time evolution of the system is studied and characterized with respect to different points in parameter space. A fractional steps method using cubic splines interpolation is applied for time-integrating this system.

## 1. Introduction

An important method for the numerical solution of the Vlasov equation is the direct solution for the velocity distribution function as a partial differential equation in phase space. This differential equation is discretized on a fixed Eulerian grid. Interest in Eulerian grid-based Vlasov solvers arise from the very low noise level associated with these methods, which provide a powerful tool to accurately investigate the physics associated with the low-density regions of phase space. It is the purpose of this work to apply the second-order splitting scheme of Cheng and Knorr (1976) by studying the Vlasov–Maxwell system of equations (Berk and Breizman 1990; Arber and Vann 2002; Vann et al. 2003), which is used to model the self-consistent interaction between energetic particles and wave-fields or collective modes in tokamaks.

## 2. The problem equations

The numerical method presented here consists of fractional shifts which are applied to the distribution function (Pohn et al. 2005). The performance of this second-order splitting scheme is evaluated by studying a Berk–Breizman augmentation of the Vlasov–Maxwell system for one spatial and one velocity dimension. This system reads as follows. The Vlasov equation is written in the form

$$\partial_t f + v \cdot \partial_x f + E \cdot \partial_v f = -\nu_{\mathrm{a}}(f - F_0), \tag{2.1}$$

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where $f = f(x, v, t)$ is the density distribution function and $F_0(v)$ is the steady state solution of the system. The parameter $\nu_a$ models a particle annihilation rate. For $t > 0$ the electric field is determined from

$$\partial_t E = -j - \gamma_d E, \quad j = \int v f \, dv \tag{2.2}$$

and from Poisson's equation for $t = 0$. The parameter $\gamma_d$ describes the combined effect of all background damping mechanisms acting on the electric field. The steady state solution $F_0$ of (2.1) is assumed to be composed of two parts, $F_0 = \eta F_{\text{bulk}} + (1 - \eta) F_{\text{source}}$, with

$$F_{\text{bulk}}(v) = \frac{1}{v_t\sqrt{2\pi}} e^{-1/2(v/v_t)^2}, \quad F_{\text{source}}(v) = \frac{1}{v_{st}\sqrt{2\pi}} e^{-1/2(v-v_s/v_{st})^2} \tag{2.3}$$

where $\eta$ is the bulk particle share, $v_t$ the thermal velocity of the plasma, $v_s$ the average particle source velocity and $v_{st}$ the thermal broadening of the source particles. The system is initially disturbed, $f(x, v, t = 0) = (1 + \epsilon \cos(2\pi x/L)) F_0(v)$, where $L$ is the spatial dimension of the plasma, and $\epsilon$ is a small perturbation term.

Generalizations of splitting schemes to higher dimensions were presented a long time ago (Cheng 1977). However, due to the higher computational resources it took a long time for the applications of these methods to Eulerian Vlasov codes to be presented (see Pohn et al. (2005) and references therein). If the velocity space of the system is extended to two dimensions, $f = f(x, v_x, v_y, t)$, gyration of particles becomes active. The normalized Vlasov equation is then

$$\partial_t f + v_x \cdot \partial_x f + (E + \omega_{cp} v_y) \cdot \partial_{v_x} f - \omega_{cp} v_x \cdot \partial_{v_y} f = -\nu_a (f - F_0), \tag{2.4}$$

where $\omega_{cp} = \omega_c/\omega_p$ is the ratio of cyclotron frequency and plasma frequency. Equation (2.4) represents a system where the magnetic field is perpendicular to $x$ and $y$. Some results for this two-dimensional case will also be presented here.

## 3. Results

The parameters of the simulations are chosen as follows: periodic phase-space length $L = 21$, velocity interval is $[-V, V]$ with $V = 8$, share of bulk particles $\eta = 0.9$, thermal velocity of bulk particles $v_t = 1.0$ (i.e. velocity is normalized to thermal velocity), source particle velocity $v_s = 4.5$ with thermal broadening $v_{st} = 0.5$, perturbation $\epsilon = 0.01$, grid size $N_x = 128$ and $N_v = 512$ and a time step $\Delta t = 0.1$.

Depending on the choice of the parameter pair $(\gamma_d, \nu_a)$ the evolution of the system shows qualitatively different behavior (Vann et al. 2003). The electric field energy $W(t) = L^{-1} \int_0^L E(x,t)^2 \, dx$ is used for characterizing these differences. Figure 1 shows examples with chaotic and periodic evolution. Steady state and damped evolution is also obtained.

Figure 2(a) shows the density distribution function $f$ at $t = 1116$ for a simulation in the periodic region of the parameter space ($\gamma_d = 1.00$, $\nu_a = 0.03$). The figure is constricted to the relevant part near the phase velocity $v_{ph}$, where $f$ flattens with respect to $v$. The total particle number is conserved very well during the simulations. The relative change in density is less than $2 \times 10^{-10}$.

Previous theoretical work has been restricted to various limiting parameter regimes. For the case of small values of $\nu_a$ and $\gamma_d$ Breizman et al. (1993) showed that the system undergoes relaxation oscillation for $\nu_a < \gamma_d$ and the system saturates

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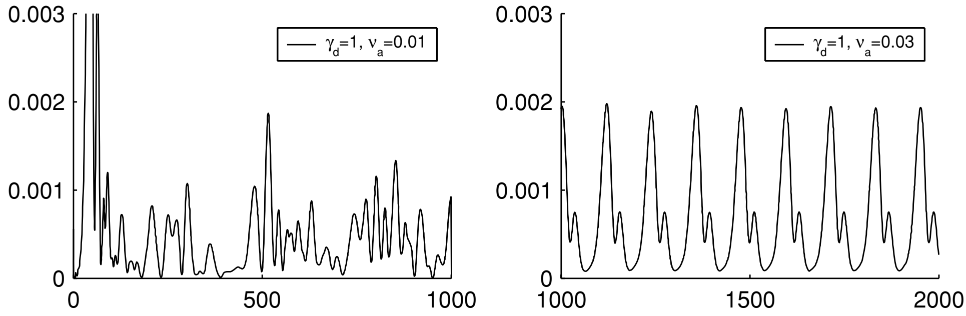
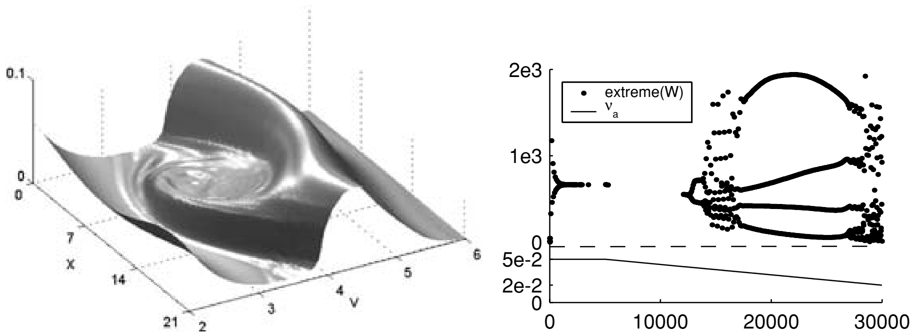



(a) (b)

**Figure 1.** Time evolution of electric field energy for different points in parameter space: (a) chaotic; (b) periodic.




(a) (b)

**Figure 2.** (a) $f(x, v, t = 1116)$ for $\gamma_\mathrm{d} = 1, \nu_\mathrm{a} = 0.03$. (b) Bifurcation.

to the steady state for $\nu_\mathrm{a} > \gamma_\mathrm{d}$. Fasoli et al. (1998) studied the case of slow linear growth. The present code, however, provides fully nonlinear behavior over the whole $(\gamma_\mathrm{d}, \nu_\mathrm{a})$ parameter space.

When varying $\nu_\mathrm{a}$ while retaining $\gamma_\mathrm{d}$ unchanged, qualitatively different regions of the parameter space are passed through. Figure 2(b) shows a bifurcation diagram coming from such a simulation with a constant value of $\gamma_\mathrm{d} = 1$ and initial value $\nu_\mathrm{a} = 0.05$. The initial value of $\nu_a$ is kept constant until $t = 5000$ and then linearly decreased until $\nu_\mathrm{a} = 0.02$ is reached at $t = 30\,000$.

In the beginning of the evolution the system is strongly oscillating until the energy saturates at $t \approx 1500$ and reaches a steady state due to nonlinear effects. The system then remains in a steady state until the linear decrease of $\nu_\mathrm{a}$ sets in at $t = 5000$ causing a linear dropping of the electric field energy. Then, the typical behavior of a nonlinear system—period doubling and alternating regions of chaotic and non-chaotic behavior—is observed. We note that frequency splitting of the toroidal Alfvén eigenmode (TAE) has been observed in experiments (Fasoli et al. 1998).

The transition from one- to two-dimensional velocity space (see (2.4)) is carried out in a stepwise way (Pohn et al. 2001). For obtaining a plausibility check, a simulation with identical results for one and two dimensions is done (Fig. 3, dashed

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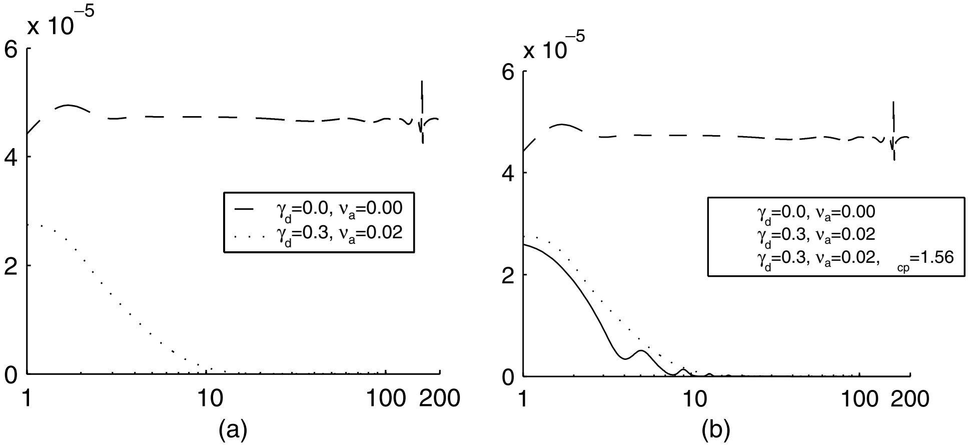


**Figure 3.** Time evolution of electric field energy: (a) one-dimensional velocity space; (b) two-dimensional velocity space.

curves). The system develops towards a steady state and a kind of numerical recurrence effect is observed (recurrence time $T_{\mathrm{R}} = 2\pi/k\Delta v \approx 160$ for $L = 5$). After switching on $\gamma_{\mathrm{d}} = 0.3$ and $\nu_{\mathrm{a}} = 0.02$ the results for the one- and two-dimensional cases still stay the same, because these parameters are only effective in space (Fig. 3, dotted curve). Based on the results from this cross-validation the investigation of the two-dimensional system can be started by switching on the magnetic field ($\omega_{\mathrm{cp}} = 1.56$). An oscillation with period $T_{\mathrm{c}} = 4.03$ due to gyration is observed (Fig. 3, solid curve).

## 4. Conclusions

We have applied a method of fractional steps (Cheng and Knorr 1976; Shoucri and Gagné 1977) associated to cubic spline interpolation to present a numerical solution of the nonlinear Berk–Breizman augmentation model of the Vlasov–Maxwell system, and to study the full range of parameters $(\gamma_{\mathrm{d}}, \nu_{\mathrm{a}})$ associated with this model. The results agree nicely with what has been presented by Vann et al. (2003). In the code we developed, the relative change in density is less than $2 \times 10^{-10}$ for all simulations. In the $(\gamma_{\mathrm{d}}, \nu_{\mathrm{a}})$ parameter space, four types of different system behavior have been identified, namely chaotic, periodic, steady state and damped. The longtime simulation presented in Fig. 2(b) by slowly varying $\nu_{\mathrm{a}}$ and the plotting of extrema observed in the electric field energy shows that the system bifurcates from steady state to periodic behavior and through a series of period doubling bifurcations to chaos. This demonstrates how the system naturally shows the phenomenology of frequency splitting as reported experimentally in Fasoli et al. (1998). It confirms that the qualitative expectation of period doubling (Breizman et al. 1997 ) may be the mechanism underlying the phenomenology observed experimentally.

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