Canad. Math. Bull. Vol. 26 (2), 1983

EXTREMAL POSITIVE SOLUTIONS OF SEMILINEAR SCHRÖDINGER EQUATIONS

BY

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ABSTRACT. Necessary and sufficient conditions are proved for the existence of maximal and minimal positive solutions of the semilinear differential equation $\Delta u = -f(x, u)$ in exterior domains of Euclidean n-space. The hypotheses are that $f(x, u)$ is nonnegative and Hölder continuous in both variables, and bounded above and below by $ug_i(|x|, u)$, $i = 1, 2$, respectively, where each $g_i(r, u)$ is monotone in u for each $r > 0$.

1. **Introduction.** The semilinear Schrödinger equation

$$Lu \equiv \Delta u + f(x, u) = 0, \qquad x \in \Omega_a \tag{1}$$

will be considered in exterior domains of R^n, $n \geq 2$, of the type

$$\Omega_a = \{x \in R^n : |x| \geq a\}, \qquad a > 0, \tag{2}$$

under the following hypotheses:

H1. For some $\delta > 0$, $f(x, u) \geq 0$ whenever $x \in \Omega_\delta$, $u \geq 0$;

H2. f belongs to the Hölder space $C^\alpha(\bar{M} \times \bar{J})$ for some α in $0 < \alpha < 1$, fixed in the sequel, for every bounded domain $M \subset \Omega_\delta$, and for every bounded positive interval J;

H3. $f(x, u) \leq ug(|x|, u)$ for all $x \in \Omega_\delta$, $u \geq 0$, where $g \in C^\alpha(\bar{I} \times \bar{J})$ for all bounded positive intervals I and J, and $g(r, u)$ is monotone in u for each $r > 0$ (either nondecreasing or nonincreasing).

H4. $f(x, u) \geq ug_0(|x|, u)$ for all $x \in \Omega_\delta$, $u \geq 0$, where $g_0(r, u)$ is continuous and nonnegative for $0 < r < \infty$, $0 < u < \infty$, and monotone in u for each r.

A *solution of* $Lu = 0$ ($Lu \leq 0$, $Lu \geq 0$, respectively) is understood throughout to be a function $u \in C^{2+\alpha}(\bar{M})$ for every bounded subdomain $M \subset \Omega_a$, with α as in H2, such that $(Lu)(x) = 0$ ($(Lu)(x) \leq 0$, $(Lu)(x) \geq 0$, respectively) for every $x \in \Omega_a$.

In this note our purpose is to prove necessary and sufficient conditions for the existence of *maximal solutions* $u^*(x)$ and *minimal solutions* $u_*(x)$ of (1),

Received by the editors September 4, 1981 and, in revised form, January 8, 1982.

* Support from NSERC (Canada) under Grant A3105 is acknowledged with gratitude.

1980 Mathematics Subject Classifications: Primary: 35B05 Secondary: 35J60.

defined below. Let ϕ_n and ψ_n be the functions in $(0, \infty)$ defined by

$$\begin{cases} \phi_2(r) = 1;\ \phi_n(r) = r^{2-n}, & n \geq 3 \\ \psi_2(r) = \log r;\ \psi_n(r) = 1, & n \geq 3. \end{cases} \tag{3}$$

A *minimal solution* satisfies $(A-\varepsilon)\phi_n(|x|) \leq u_*(x) \leq A\phi_n(|x|)$, uniformly in some exterior domain Ω_a, for some constants A and $\varepsilon, 0<\varepsilon<A$. If a minimal solution of (1) exists, the proof given in §3 shows that such a solution $u_*(x;\varepsilon)$ exists in some $\Omega_{a(\varepsilon)}$ for arbitrary ε in $(0, A)$. The definition of a *maximal solution* is the same with ψ_n replacing ϕ_n. The solution $u_0^*(x)$ are maximal in the sense that no positive solution $u(x)$ of (1) has a spherical mean

$$U(r) = \frac{1}{\omega(S_1)} \int_{S_1} u(x)\, d\omega \tag{4}$$

growing more rapidly than a constant multiple of $\psi_n(r)$ as $r \to \infty$. Here ω denotes the measure on the unit sphere S_1 in R^n. In fact, $U(r)$ satisfies the ordinary differential inequality [6, p. 70] below because of H4:

$$-\frac{d}{dr}\left[r^{n-1}\frac{dU}{dr}\right] \geq \frac{r^{n-1}}{\omega(S_1)} \int_{S_1} u(x) g_0(r, u(x))\, d\omega, \tag{5}$$

so in particular, if $n=2$, $rU'(r)$ is nonincreasing. Since $U(r)>0$ for $r \geq a$, say, it follows easily that $U(r) \leq A \log r$ for some constant A, $r \geq a$. Similarly if $n \geq 3$, $U(r) \leq A$ for some constant A.

A positive solution $u(x)$ of (1) in Ω_a satisfies $\Delta u \leq 0$ by H1, and consequently the a priori lower bound [7, p. 917]

$$u(x) \geq \left[\frac{a}{|x|}\right]^{n-2} \inf_{|x|=a} u(x), \qquad |x| \geq a \tag{6}$$

shows that $u(x) \geq A\phi_n(|x|)$ for some constant A, $|x| \geq a$.

2. **Statement of theorems**

THEOREM 1. *Equation* (1) *has a maximal solution in some exterior domain* $\Omega_a \subset R^n$, $n \geq 2$, *if* H1, H2 *and* H3 *hold and*

$$\int^\infty r \log r\, g(r, c \log r)\, dr < \infty, \qquad n=2 \tag{7}$$

$$\int^\infty r g(r, c)\, dr < \infty, \qquad n \geq 3 \tag{8}$$

for some positive constant c.

THEOREM 2. *Equation* (1) *has a minimal positive solution in some exterior*

https://doi.org/10.4153/CMB-1983-028-3 Published online by Cambridge University Press

domain $\Omega_a \subset R^n$, $n \geq 2$, *if* H1, H2, *and* H3 *hold and*

(9) $$\int^{\infty} r \log r\, g(r, c)\, dr < \infty, \qquad n = 2$$

(10) $$\int^{\infty} rg(r, cr^{2-n})\, dr < \infty, \qquad n \geq 3$$

for some positive constant c.

THEOREM 3. *Under hypotheses* H1, H2, *and* H4, *a necessary condition for equation* (1) *to have a maximal solution in some exterior domain in* R^n *is*

(11) $$\int^{\infty} r \log r\, g_0(r, c \log r)\, dr < \infty, \qquad n = 2$$

(12) $$\int^{\infty} rg_0(r, c)\, dr < \infty, \qquad n \geq 3$$

for some positive constant c.

THEOREM 4. *Under hypotheses* H1, H2, *and* H4, *a necessary condition for* (1) *to have a minimal positive solution in some exterior domain in* R^n *is*

(13) $$\int^{\infty} r \log r\, g_0(r, c)\, dr < \infty, \qquad n = 2$$

(14) $$\int^{\infty} rg_0(r, cr^{2-n})\, dr < \infty, \qquad n \geq 3$$

for some positive constant c.

It is clear from Theorems 1–4 that conditions (7)–(10) are both necessary and sufficient for the existence of extremal positive solutions of (1) provided g and g_0 satisfy the growth conditions below:

H5. $$\limsup_{r\to\infty} \frac{g(r, c \log r)}{g_0(r, c \log r)} < \infty$$

H6. $$\limsup_{r\to\infty} \frac{g(r, c)}{g_0(r, c)} < \infty$$

H7. $$\limsup_{r\to\infty} \frac{g(r, cr^{2-n})}{g_0(r, cr^{2-n})} < \infty, \qquad n \geq 3$$

for every positive constant c.

COROLLARY 1. *If* H1-H4 *and* H5 [*respectively,* H6] *hold, then* (7) [*respectively,* (8)] *is a necessary and sufficient condition for* (1) *to have a maximal positive solution in some exterior domain* R^2 [*respectively,* R^n, $n \geq 3$].

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COROLLARY 2. *If* H1–H4 *and* H6 [*respectively,* H7] *are satisfied, then* (9) [*respectively,* (10)] *is necessary and sufficient for* (1) *to have a minimal positive solution in some exterior domain in* R^2 [*respectively* R^n, $n \geq 3$].

For example, if (1) is the Emden-Fowler equation

$$\Delta u + p(x)\,|u|^\gamma \operatorname{sgn} u = 0, \qquad x \in \Omega_a \tag{1'}$$

where γ is a positive constant, then appropriate functions g and g_0 in H3 and H4 are given by

$$g(r, u) = \left[\max_{|x|=r} p(x)\right] u^{\gamma-1} = P(r)u^{\gamma-1}$$

$$g_0(r, u) = \left[\min_{|x|=r} p(x)\right] u^{\gamma-1} = P_0(r)u^{\gamma-1}$$

and each of H5, H6, and H7 reduces to

$$\limsup_{r\to\infty} P(r)/P_0(r) < \infty.$$

In this case, the necessary and sufficient conditions (7)–(10) reduce to, respectively,

$$\int^\infty r(\log r)^\gamma P(r)\, dr < \infty, \qquad n = 2, \gamma > 0 \tag{7'}$$

$$\int^\infty rP(r)\, dr < \infty, \qquad n \geq 3, \gamma > 0 \tag{8'}$$

$$\int^\infty r \log r\, P(r)\, dr < \infty, \qquad n = 2, \gamma > 0 \tag{9'}$$

$$\int^\infty r^\sigma P(r)\, dr < \infty, \qquad n \geq 3, \gamma > 0 \tag{10'}$$

where

$$\sigma = n - 1 - \gamma(n-2).$$

One-dimensional versions of Theorems 1–4 are contained in works by Belohorec [1, Theorem 3], Coffman and Wong [2, Theorems 1 and 2], Izyumova [3, Theorem 1.1], Nehari [5, Theorems I and II] and others, concerning the ordinary differential equation

$$\frac{d^2y}{dt^2} + yg(t, y) = 0, \qquad 0 < t < \infty. \tag{15}$$

THEOREM 5 [1, 2, 3, 5]. *Let* $f(t, y) = yg(t, y)$ *be continuous and nonnegative for*

$0<t<\infty$, $0<y<\infty$, and suppose that $g(t, y)$ is either nondecreasing or nonincreasing in y for each t. Then equation (15) *has a bounded positive solution in some interval (t_0, ∞), $t_0>0$, if and only if*

$$\int^{\infty} tg(t, c)\, dt<\infty \tag{16}$$

for some positive constant c; and moreover, if (16) *holds, this solution is asymptotic to a positive constant as $t\to\infty$. Furthermore,* (15) *has a solution y such that $y(t)\sim At$ as $t\to\infty$, for some $A>0$, if and only if*

$$\int^{\infty} tg(t, ct)\, dt<\infty \tag{17}$$

for some positive constant c.

The following theorem [8, p. 125] will be needed in the proofs of Theorems 1 and 2.

THEOREM 6. *If* H1 *and* H2 *hold, and if there exist positive solutions v, w of $Lv\le 0$, $Lw\ge 0$, respectively, in Ω_a, $a\ge\delta$, such that $w(x)\le v(x)$ for all $|x|\ge a$, then equation* (1) *has at least one solution $u(x)$ satisfying $w(x)\le u(x)\le v(x)$ for all $|x|\ge a$.*

Subsolutions $w(x)$ for Theorem 6 are readily available in the form

$$\begin{cases} w(x)=A\log r+B, & n=2\\ w(x)=Ar^{2-n}+B, & n\ge 3\end{cases} \tag{18}$$

where A, B are constants and $r=|x|$, since

$$Lw\ge\Delta w=r^{1-n}\frac{d}{dr}\left[r^{n-1}\frac{dw}{dr}\right]=0.$$

Supersolutions $v(x)$ of (1) will be constructed in §3 in the form $v(x)=\zeta(r)$, $r=|x|\ge a$, where ζ is a positive solution, in the space $C^{2+\alpha}[a, b]$ for all $b>a$, of the ordinary differential equation

$$\frac{d}{dr}\left(r^{n-1}\frac{d\zeta}{dr}\right)+r^{n-1}\zeta(r)g(r, \zeta(r))=0. \tag{19}$$

3. **Proofs**

Proof of Theorem 1. If $n=2$, the change of variables $r=e^s$, $y(s)=\zeta(r)$ transforms (19) into

$$y''(s)+e^{2s}y(s)g(e^s, y(s))=0. \tag{20}$$

By Theorem 5, equation (20) has a solution $y(s)\sim As$ as $s\to\infty$, for some

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positive constant A, if and only if

$$\int^{\infty} se^{2s}g(e^s, cs)\, ds<\infty$$

for some positive constant c, which is equivalent to (7). Furthermore, $y\in C^{2+\alpha}[s_0, s]$ for some s_0 and for all $s>s_0$ by standard regularity theory, see e.g. [4], since $g\in C^{\alpha}$ by H3. Then, if (7) holds, equation (19) has a solution $\zeta\in C^{2+\alpha}[a, b]$ for all $b>a=\exp s_0$ such that $\zeta(r)=y(s)\sim A\log r$ as $r\to\infty$. This implies that there exist positive numbers A_1, ε, and a_1 such that $0<\varepsilon<A_1<A$ and $(A_1-\varepsilon)\log r\leq\zeta(r)\leq A_1\log r$ for all $r\geq a_1$, and clearly $a=a_1$ without loss of generality. In view of (1) and H3, the function v defined by $v(x)=\zeta(r)$, $r=|x|$, in $\Omega_a\subset R^2$ satisfies the inequality

$$rLv=\frac{d}{dr}\left(r\frac{d\zeta}{dr}\right)+rf(x, v)\leq\frac{d}{dr}\left(r\frac{d\zeta}{dr}\right)+r\zeta(r)g(r, \zeta(r))$$

and hence $Lv\leq 0$ for all $x\in\Omega_a$ by (19). As noted in (18), $w(x)=(A_1-\varepsilon)\log r$ satisfies $Lw\geq 0$ for an arbitrary positive constant ε. Since $w(x)\leq v(x)$ for $|x|\geq a$, Theorem 6 establishes the existence of a solution $u(x)$ of (1) satisfying

$$w(x)=(A_1-\varepsilon)\log r\leq u(x)\leq v(x)=\zeta(r)\leq A_1\log r$$

for $r=|x|\geq a$. This proves that equation (1) has a maximal positive solution $u(x)$ in Ω_a.

If $n\geq 3$, the change of variables

$$r=\beta(s)=(\nu s)^{\nu},\qquad y(s)=s\zeta(\beta(s)),\qquad \nu=\frac{1}{n-2}$$

transforms (19) into

$$y''(s)+s^{-4}[\beta(s)]^{2n-2}y(s)g\left(\beta(s), \frac{y(s)}{s}\right)=0. \tag{21}$$

By Theorem 5, (21) has a solution $y(s)\sim As$ as $s\to\infty$, for some positive constant A, if and only if

$$\int^{\infty} s^{-3}[\beta(s)]^{2n-2}g(\beta(s), c)\, ds<\infty$$

for some positive constant c, which is equivalent to (8). The remainder of the proof is virtually the same as the proof for $n=2$ and is deleted.

Proof of Theorem 2. We shall outline the proof for $n\geq 3$ only since the proof for $n=2$ is similar (using (20) instead of (21)). By Theorem 5, (21) has a solution $y(s)$ with $A-\varepsilon \le y(s) \le A$ in some interval $[s_0, \infty)$, where A and ε are constants, $0<\varepsilon<A$, if and only if

$$\int^{\infty} s^{-3}[\beta(s)]^{2n-2} g\left(\beta(s), \frac{c}{s}\right) ds < \infty$$

for some $c>0$, which is equivalent to condition (10). Then, if (10) holds, (19) has a solution $\zeta \in C^{2+\alpha}[a, b]$ for all $b>a=\exp s_0$ such that

$$(A-\varepsilon)\nu r^{2-n} = \frac{A-\varepsilon}{s} \le \zeta(r) = \frac{y(s)}{s} \le \frac{A}{s} = A\nu r^{2-n}$$

for $r \ge a$. Exactly as in Theorem 1, the function v defined in Ω_a by $v(x)=\zeta(r)$, $r=|x|$ satisfies $Lv \le 0$ in Ω_a, and by (18), $w(x)=(A-\varepsilon)\nu r^{2-n}$ satisfies $Lw \ge 0$ for arbitrary $\varepsilon>0$. Theorem 6 then shows that (1) has a solution $u(x)$ satisfying

$$(A-\varepsilon)\nu r^{2-n} \le u(x) \le v(x) = \zeta(r) \le A\nu r^{2-n}$$

for all $|x| \ge a$, from which $u(x)$ is the required minimal solution of (1).

Proof of Theorem 3. If (1) has a positive solution $u(x) \sim A \log|x|$ as $|x| \to \infty$ uniformly in $\Omega_a \subset R^2$, where A is a positive constant, there exist positive constants k_1 and k_2 such that

$$k_1 \log r \le u(x) \le k_2 \log r \quad \text{for} \quad r=|x| \ge a. \tag{22}$$

Then (4), (5), (22) and H4 yield the inequality

$$-\frac{d}{dr}\left[r\frac{dU}{dr}\right] \ge k_1 r \log r\, g_0(r, c \log r), \qquad r \ge a, \tag{23}$$

where $c=k_1$ in the case that $g_0(r, u)$ is nondecreasing in u, and $c=k_2$ if $g_0(r, u)$ is nonincreasing in u. Integration of (23) over (a, r) gives

$$-rU'(r)+aU'(a) \ge k_1 \int_a^r t \log t\, g_0(t, c \log t)\, dt. \tag{24}$$

Since $rU'(r)$ is nonincreasing by (23) and $U(r)$ is positive, it is easily seen that $U'(r)>0$ for all $r>a$. Then (24) implies the conclusion (11) of Theorem 3.

If (1) has a solution $u(x) \sim A$ as $|x| \to \infty$ uniformly in $\Omega_a \subset R^n$, $n \ge 3$, where A is a positive constant, then (23) is replaced by

$$-\frac{d}{dr}\left[r^{n-1}\frac{dU}{dr}\right] \ge k_1 r^{n-1} g_0(r, c)$$

for some positive constant c. The substitution

$$r=\beta(s)=(\nu s)^{\nu}, \qquad h(s)=sU(\beta(s)), \qquad \nu=\frac{1}{n-2}$$

transforms this into

$$-h''(s) \geq k_1 s^{-3}[\beta(s)]^{2n-2} g_0(\beta(s), c), \tag{25}$$

and integration over (b, s) gives

$$\begin{aligned} -h'(s)+h'(b) &\geq k_1 \int_b^s s^{-3}[\beta(s)]^{2n-2} g_0(\beta(s), c)\, ds \\ &= k_1 \nu \int_a^r r g_0(r, c)\, dr \end{aligned} \tag{26}$$

where $a = \beta(b)$ and $r = \beta(s) \geq a$. Since $h'(s)$ is nonincreasing by (25) and $h(s) > 0$ for $s > b$, it follows routinely that $h'(s) > 0$ for all $s > b$. The conclusion (12) of Theorem 3 is then a consequence of (26).

Proof of Theorem 4. If (1) has a minimal positive solution in $\Omega_a \subset R^2$, (23) is replaced by

$$-\frac{d}{dr}\left[r\frac{dU}{dr}\right] \geq k_1 r g_0(r, c), \qquad r \geq a$$

for some $c > 0$, and (13) follows by the same proof as in Theorem 3. Similarly for $n \geq 3$ (25) is replaced by

$$-h''(s) \geq k_1 s^{-3}[\beta(s)]^n g_0(\beta(s), c\beta(s)^{2-n}),$$

which leads to (14) after multiplication by s.

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