Ellipse: what else?

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1. *Introduction*

Among the curves which have always fascinated mathematicians, conics have a special position, and the ellipse may be is the best known, both for its innate beauty and its various applications in mathematics and science in general. Among the many scholars who were interested in the study of the golden ellipse we remember Kapur [1, 2], but while Kapur defined the golden ellipse as the one in which the ratio of the major to the minor axis is $\sqrt{\phi}$, we prefer its classical definition first given by Huntley in 1974 [3]. The ellipse centred at the origin O and semiaxes of lengths a and b in which the ratio $a : b$ equals the golden ratio ϕ is named the *golden ellipse*. It can be used to show how the golden right triangle and other figures can be demonstrated by some of its interesting properties.

2. *The golden ellipse and its first properties*

Since $a = b\phi$, the equation of the golden ellipse is: $x^2 + \phi^2 y^2 = b^2\phi^2$.

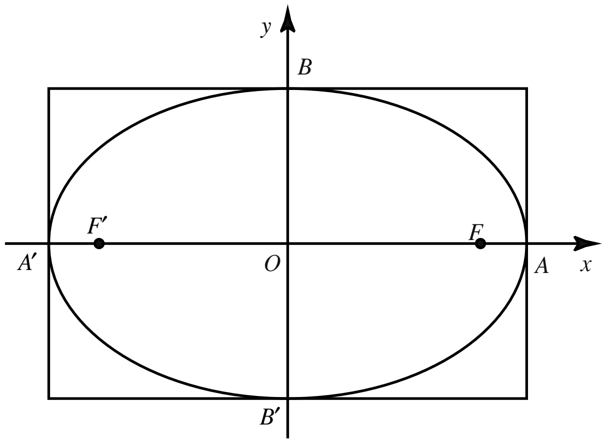

FIGURE 1: The golden ellipse and its golden rectangle

Let the focal length, OF, be c.

Since $c = \sqrt{a^2 - b^2}$ and $\phi^2 = \phi + 1$ we have

$$c = \sqrt{a^2 - b^2} = \sqrt{b^2\phi^2 - b^2} = b\sqrt{\phi^2 - 1} = b\sqrt{\phi}.$$

So, its foci are the points $F\left(b\sqrt{\phi},\ 0\right)$ and $F'\left(-b\sqrt{\phi},\ 0\right)$ and its eccentricity

$$e = \frac{c}{a} = \frac{\sqrt{\phi}}{\phi} = \sqrt{\frac{\phi}{\phi^2}} = \sqrt{\frac{1}{\phi}} = \sqrt{\phi - 1}$$

since $1/\phi = \phi - 1$. Clearly, if a rectangle with sides parallel to the axes

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be circumscribed about the ellipse, it will be what has been called the *golden rectangle*. Let us consider the straight line of equation $x = b\sqrt{\phi}$ through the focus F; it intersects the ellipse in the first quadrant at the point $P\left(b\sqrt{\phi},\ b/\phi\right)$.

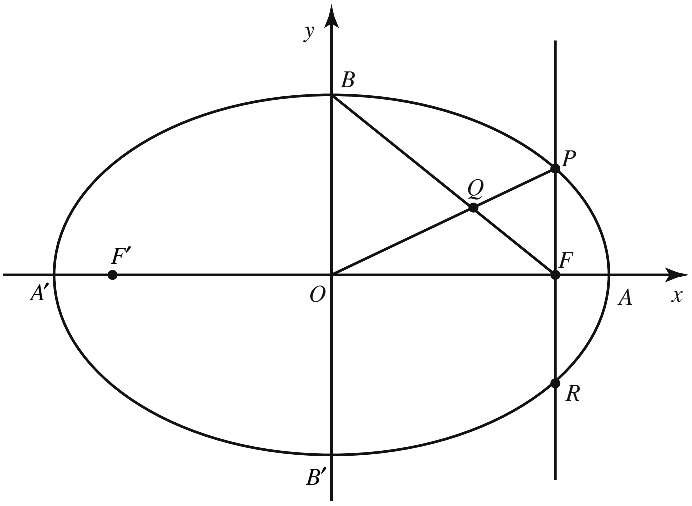

FIGURE 2: The first golden properties

The two segments $BF = b\phi$ and $OP = b\sqrt{2}$ intersect at the point $Q\left(b\sqrt{\phi}/\phi,\ b/\phi^2\right)$ and the length of BQ is

$$BQ = \sqrt{\left(\frac{b\sqrt{\phi}}{\phi}\right)^2 + \left(b - \frac{b}{\phi^2}\right)^2} = b = OB,$$

so that the triangle OBQ is isosceles and $\dfrac{BF}{BQ} = \phi$. This means that BQ is the *golden section* of BF. On the other hand, given any ellipse of equation $b^2x^2 + a^2y^2 = a^2b^2$, if the triangle OBQ is isosceles then we have a golden ellipse. In fact, in this case one has

$$Q = \left(\frac{ac}{a + b}, \frac{b^2}{a + b}\right), \qquad BQ = \frac{a^2}{a + b}.$$

Since $BQ = OB$ one gets

$$\frac{a^2}{b^2} = 1 + \frac{a}{b}.$$

So $\frac{a}{b}$ is the positive solution of the equation $x^2 - x - 1 = 0$, i.e. $\frac{a}{b} = \phi$.

Moreover, the right-angled triangle OBF is *golden* because $\dfrac{BF}{OB} = \phi$; for its properties see [4]. But the triangle PFQ is isosceles too because

$$QF = BF - BQ = b\phi - b = b(\phi - 1) = FP.$$

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Moreover

$$\frac{OB}{PF} = \frac{b}{\frac{b}{\phi}} = \phi$$

and so PF is the golden section of OB. This means that the *latus rectum* of the ellipse, that is the segment PR is the golden section of BB'. But the converse of this fact is not sufficient to state that any ellipse must be golden.

Since $Q\left(\frac{b\sqrt{\phi}}{\phi}, \frac{b}{\phi^2}\right)$ we have:

$$OQ = \sqrt{\frac{b^2}{\phi} + \frac{b^2}{\phi^4}} = b\sqrt{\frac{1}{\phi} + \frac{1}{\phi^4}} = b\sqrt{\frac{\phi^3 + 1}{\phi^4}} = b\sqrt{\frac{2\phi^2}{\phi^4}} = b\sqrt{\frac{2}{\phi^2}} = \frac{b\sqrt{2}}{\phi}.$$

One has $\frac{OP}{OQ} = \phi$, so OQ is the golden section of OP. Therefore, we have a right-angled trapezoid $OFPB$ in which the minor base, the shorter of the two parallel sides, is the golden section of the major base, the longer of the two parallel sides, and its diagonals are divided in the extreme and mean ratio. We could define it the *golden trapezoid*. The two similar triangles OBQ and QFP are interesting too for their similarity ratio is ϕ.

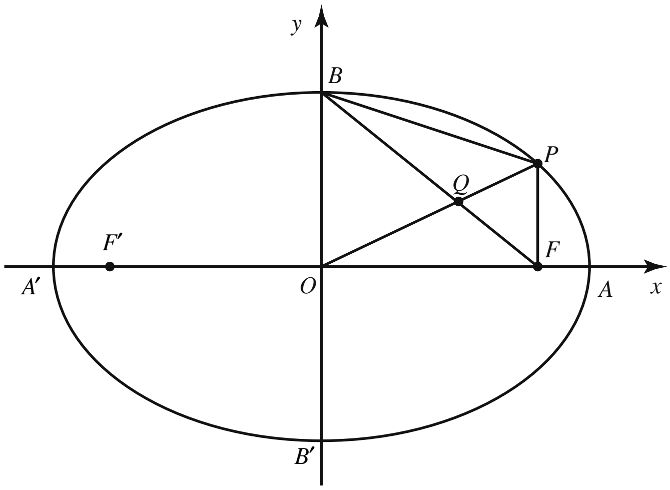

FIGURE 3: The golden trapezoid

2. *Properties related to the directrix*

Now consider the directrix r of the golden ellipse through the first and fourth quadrant whose equation is $x = b\phi\sqrt{\phi}$.

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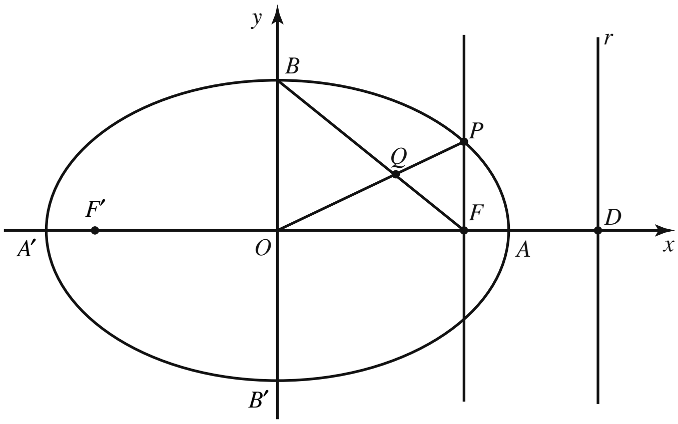

FIGURE 4: The golden ellipse and one of its directrices

We notice that

$$\frac{OD}{OF} = \frac{b\phi\sqrt{\phi}}{b\sqrt{\phi}} = \phi$$

and

$$\frac{OF}{FD} = \frac{OF}{OD - OF} = \frac{b\sqrt{\phi}}{b\phi\sqrt{\phi} - b\sqrt{\phi}} = \frac{1}{\phi - 1} = \frac{1}{\frac{1}{\phi}} = \phi;$$

hence OF is the golden section of OD and FD is the golden section of OF, so F divides OD in extreme and mean ratio. Conversely, given any ellipse, if OF is the golden section of OD, we have a golden ellipse. In fact, if $OF^2 = OD \times FD$ we have $a^2 - b^2 = ab$, and hence $\frac{a^2}{b^2} = 1 + \frac{a}{b}$ so $\frac{a}{b} = \phi$.

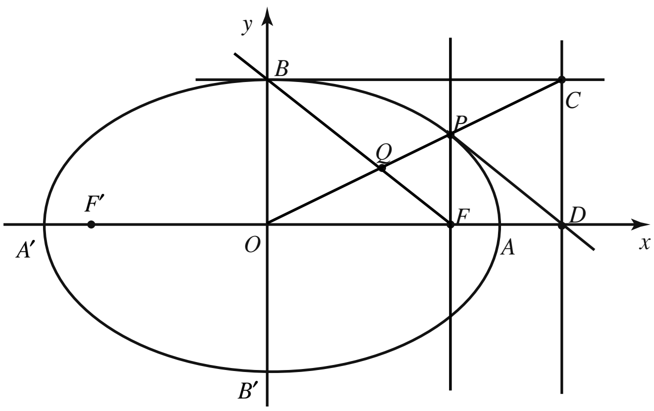

FIGURE 5: Nested properties of the golden ellipse

Moreover, the line passing through points O and P, of equation $y = \dfrac{x}{\phi\sqrt{\phi}}$, intersects the directrix at point $C\left(b\phi\sqrt{\phi}, b\right)$, hence the segment $OC = b\phi\sqrt{2}$. Therefore $\dfrac{OC}{OP} = \phi$, i.e. OP is the golden section of OC.

But $QC = b\sqrt{2}$, so we have

$$\frac{OC}{QC} = \frac{b\phi\sqrt{2}}{b\sqrt{2}} = \phi,$$

that is QC is the golden section of OC. Since $OP = OC$ one has $OQ + QP = QP + PC$, whence $OQ = PC$, as it clearly must be. Furthermore, $PB = b$, hence the triangle PDC is isosceles and congruent to the triangle OBQ. The two triangles BQC and OQF are similar too, their similarity ratio is ϕ and the right-angled triangle PFD is *golden*, since: $\dfrac{PD}{PF} = \dfrac{b}{b/\phi} = \phi$.

3. *A golden geometric puzzle*

If we consider the equation $\sqrt{\phi}x + \phi y = b\phi^2$ of the tangent to the golden ellipse at $P\left(b\sqrt{\phi}, \dfrac{b}{\phi}\right)$ it intersects the axes respectively at the points $M(0, b\phi)$ and $N\left(b\phi\sqrt{\phi}, 0\right)$. But $D\left(b\phi\sqrt{\phi}, 0\right)$ so we have $OM = a$ and $ON = OD$.

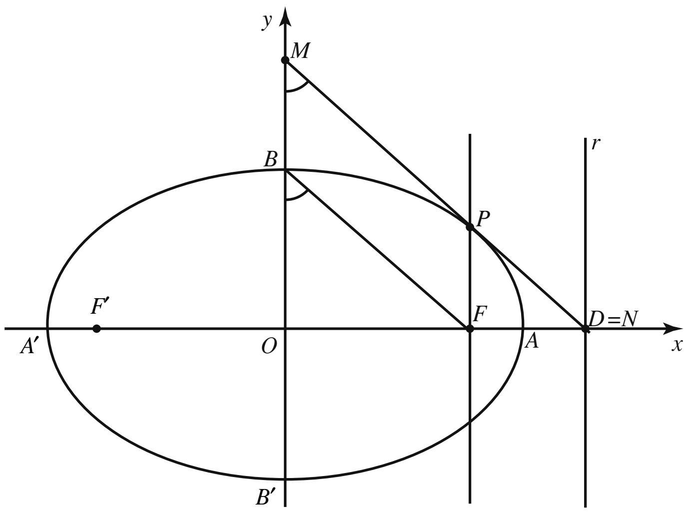

FIGURE 6: The property of the tangent at P

The angle $\angle OMD = \theta$ is given by $\tan\theta = \dfrac{OD}{OM} = \dfrac{b\phi\sqrt{\phi}}{b\phi} = \sqrt{\phi}$.

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But we also have $\tan \angle OBF = \dfrac{OF}{OB} = \dfrac{b\sqrt{\phi}}{b} = \sqrt{\phi}$, hence $\angle OBF = \theta$. Therefore the line passing through points M and D is parallel to BF and $BMPF$ is a parallelogram with $MP = a$, so the triangle OMP is isosceles and similar both to the triangle OBQ and to the triangle QFP (Figure 7).

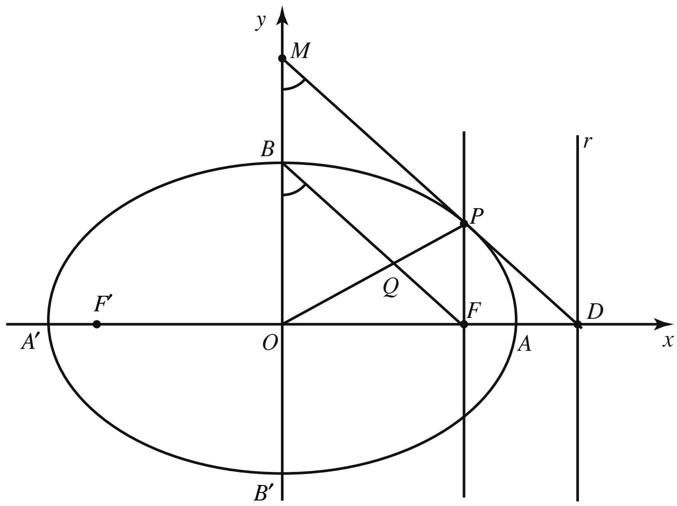

FIGURE 7: Similar triangles

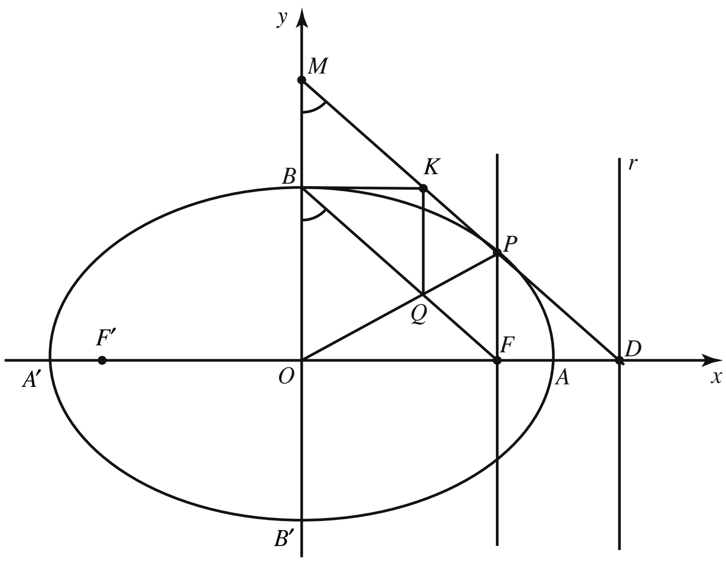

FIGURE 8: Dissection of an isosceles triangle

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This dissection of the triangle OMP can be used to create a geometrical puzzle in which two congruent right-angled triangles and two similar isosceles triangles must be combined into an isosceles triangle.

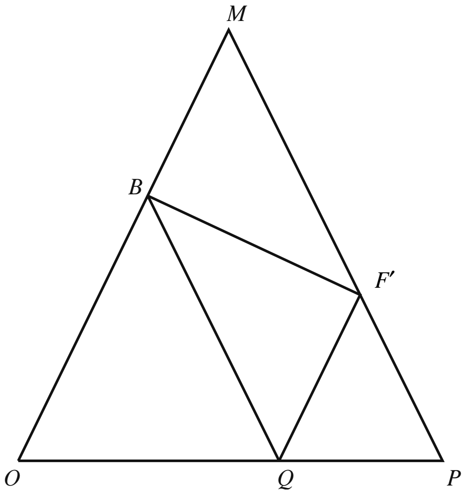

FIGURE 9: A geometrical puzzle

But, as we know, QP is the golden section of OQ, so we can get the same dissection into the isosceles triangle OBQ and iterating it we obtain the configuration of Figure 10.

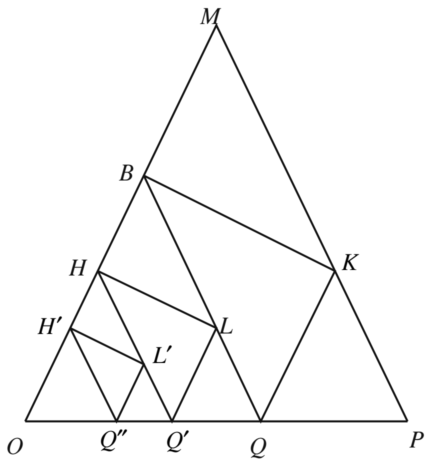

FIGURE 10: The iterated dissection

This dissection can be continued indefinitely and the points $K, L, L', \ldots, O$ are aligned, as one can verify. But the nice fact is that if we consider, for example, the line $y = \sqrt{\phi}x$ passing through the points O and $K\left(b\sqrt{\phi}/\phi, b\right)$ and the line $y = -\left(\sqrt{\phi}/\phi\right)x + b$ passing through B and F

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(Figure 8), their point of intersection is $L\left(b\sqrt{\phi}/\phi,\ b/\phi\right)$. We then have $OL = b\sqrt{\phi}/\phi$ and since $OK = b\sqrt{\phi}$ we get $OK/OL = \phi$, i.e. OL is the golden section of OK. Moreover, since $LK = b\sqrt{\phi}/\phi$ we have $OL/LK = \phi$, that is, LK is the golden section of OL, LL' is the golden section of OL' and so on, indefinitely. In the same way, QP is the golden section of OQ, QQ' is the golden section of OQ', and so on indefinitely.

4. *A theorem and the golden ellipsoid*

Let us now consider the two circles respectively with radii $OB = b$ and $OA = b\phi$ generating the golden ellipse. The following theorem holds:

> A necessary and sufficient condition for the area of the annulus between the two circles generating an ellipse to equal the area of the ellipse is that the ellipse be golden.

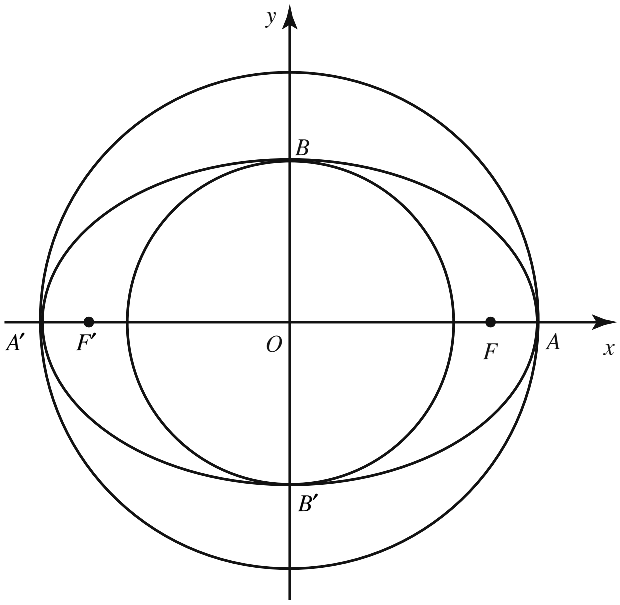

FIGURE 11: The circles generating the golden ellipse

The condition is sufficient. In fact, if we have a golden ellipse with semiaxes a and b, then $a = b\phi$ and its area is $A = \pi ab = \pi b^2\phi$. Since the area of the annulus is:

$$A_C = \pi\left(b^2\phi^2 - b^2\right) = \pi b^2\left(\phi^2 - 1\right) = \pi b^2\phi$$

we get $A = A_C$. The condition is necessary. In fact, if $A = A_C$ one must have $a^2 - b^2 = ab$ or $a^2 - b^2 - ab = 0$, from which, dividing by ab, one has $\frac{a}{b} - \frac{b}{a} - 1 = 0$. Writing $\frac{a}{b} = k$ we get the equation $k^2 - k - 1 = 0$ whose positive root is $k = \phi$; hence the ellipse must be golden.

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Now, if we pass from the golden ellipse to the golden ellipsoid with semiaxes a, b, with $a = b\phi$, whose equation is $x^2 + \phi^2 y^2 + \phi^2 z^2 = b^2\phi^2$, its volume will be $V = \frac{4}{3}\pi ab^2 = \frac{4}{3}\pi b^3\phi$ and it equals the half of the volume V' of the region between the two spheres with radii $a = b\phi$ and b. In fact, we have:

$$V' = \frac{4}{3}\pi b^3\left(\phi^3 - 1\right) = \frac{4}{3}\pi b^3 \times 2\phi = 2\left(\frac{4}{3}\pi b^3\phi\right) = 2V.$$

5. *The dual golden ellipse*

Now consider the dual ellipse of the given one, with the foci on the y-axis, whose equation is:

$$\phi^2 x^2 + y^2 = b^2\phi^2.$$

The equation of the common tangent to the two ellipses in the first quadrant is $y = -x + b\sqrt{1 + \phi^2}$.

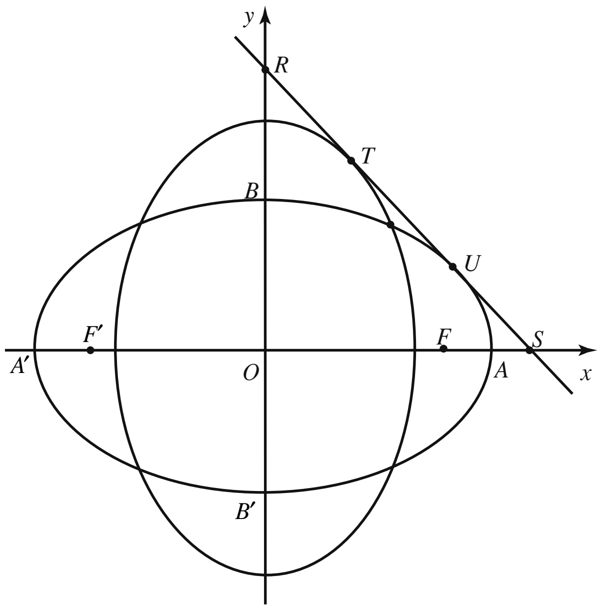

FIGURE 12: The common tangent to the two ellipses

It passes through the points T and U:

$$U = \left(\frac{b\phi^2}{\sqrt{\phi^2 + 1}}, \frac{b}{\sqrt{\phi^2 + 1}}\right), \qquad T = \left(\frac{b}{\sqrt{\phi^2 + 1}}, \frac{b\phi^2}{\sqrt{\phi^2 + 1}}\right)$$

and intersects the axes at R and S: $R\left(0, b\sqrt{1 + \phi^2}\right)$, $S\left(b\sqrt{1 + \phi^2}, 0\right)$. One

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has: $US = \dfrac{b\sqrt{2}}{\sqrt{1 + \phi^2}}$ and $TU = \dfrac{\phi b\sqrt{2}}{\sqrt{1 + \phi^2}}$ so we get $\dfrac{TU}{US} = \phi$, i.e. US is the *golden section* of TU.

6. *Conclusions*

Other properties can be found about the golden ellipse by considering, for example, its confocal hyperbola or the golden tiling of the plane obtained developing the geometrical puzzle.

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