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Two-phase flow equations with a dynamic capillary pressure

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We investigate the motion of two immiscible fluids in a porous medium described by a two-phase flow system. In the capillary pressure relation, we include static and dynamic hysteresis. The model is well established in the context of the Richards equation, which is obtained by assuming a constant pressure for one of the two phases. We derive an existence result for this hysteresis two-phase model for non-degenerate permeability and capillary pressure curves. A discretization scheme is introduced and numerical results for fingering experiments are obtained. The main analytical tool is a compactness result for two variables that are coupled by a hysteresis relation.

Key words: Two-Phase flow; Capillary hysteresis; Finite-element scheme

1 Introduction

The two-phase flow system describes the motion of two incompressible, immiscible phases in a porous medium. We consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, occupied by the porous material, and a time interval $[0, T)$. We denote the pressures of the two fluids by $p_1, p_2 : \Omega \times [0, T) \to \mathbb{R}$ and the saturation of the first fluid by $s : \Omega \times [0, T) \to \mathbb{R}$. The saturation $s = s_1$ is defined as the volume fraction of pore space that is filled with fluid 1, we think of the non-wetting phase. The saturation of the second fluid is $s_2 = 1 - s_1 = 1 - s$. Darcy's law for both velocities and conservation of mass can be combined into the system

$$\partial_t s = \nabla \cdot (k_1(s)[\nabla p_1 + g_1]), \tag{1.1}$$

$$-\partial_t s = \nabla \cdot (k_2(s)[\nabla p_2 + g_2]). \tag{1.2}$$

We have performed a normalization of porosity and density, the gravity vectors $g_1, g_2 \in \mathbb{R}^n$ point in direction $-e_n = (0, \ldots, 0, -1) \in \mathbb{R}^n$. The permeabilities $k_1(s) = k_1(s(x, t), x)$ and $k_2(s) = k_2(s(x, t), x)$ are described by given functions $k_1, k_2 : [0, 1] \times \Omega \to [0, \infty)$. The interesting modelling problem is regarding the relation between the capillary pressure $p_1 - p_2$ and the saturation s. The simplest possibility is to assume the functional dependence $p_1 - p_2 = p_c(s)$, where $p_c : \mathbb{R} \to \mathbb{R}$ is the capillary pressure function.

In order to take into account hysteresis and dynamic effects, the model with an algebraic relation between $p_1 - p_2$ and s is replaced by

$$p_1 - p_2 \in p_c(s) + \gamma\,\mathrm{sign}(\partial_t s) + \tau\partial_t s, \tag{1.3}$$

where τ and $\gamma \geqslant 0$ are two parameters, and sign denotes the multi-valued function defined by $\mathrm{sign}(\pm\xi) = \{\pm 1\}$ for $\xi > 0$ and $\mathrm{sign}(0) := [-1, 1]$. The model (1.3) was suggested in [8] and receives considerable attention.

For $\tau > 0$, the multi-valued function $\Phi : \xi \mapsto \tau\xi + \gamma\mathrm{sign}(\xi)$ can be inverted, the inverse $\Psi := \Phi^{-1} : \mathbb{R} \to \mathbb{R}$ is the Lipschitz continuous function. With this notation, equation (1.3) transforms into

$$\partial_t s = \Psi(p_1 - p_2 - p_c(s)). \tag{1.4}$$

Our main result is an existence theorem for systems (1.1)–(1.3) of partial differential equations in the case $\tau > 0$. The proof is based on the compactness result that was derived in [17] for the treatment of the Richards equation. Loosely speaking, the compactness result provides the following: For every family of pressure and saturation functions that satisfies the evolution law (1.4) and the natural energy estimates, the saturation functions converge strongly along a subsequence.

Numerical results. We include a numerical treatment of the two-phase flow equations with hysteresis (1.1)–(1.3). In agreement with the theoretical results, the finite-element scheme turns out to be stable; this holds true even though we include spatial and temporal adaptivity. We calculate solutions for the case that corresponds to the experimental setup which is used to observe fingering effects in porous media. The presented model and the suggested discretization scheme provide numerical solutions that show gravity fingering in porous media. A description of the scheme and the investigated parameters are presented together with numerical results in Section 3.

1.1 Further literature on two-phase flow equations

Unfortunately, the name ‘two-phase flow’ is slightly ambiguous as it is not only used for the above system but also for the Richards equation. The Richards equation is the simplification of the above model obtained by assuming that the pressure of the second phase is constant, e.g. $p_2 = 0$, and by using only (1.1) instead of the set of equations (1.1) and (1.2). Even though this simplified model describes the motion of two fluid phases, we will use the term *two-phase flow equations* only when we refer to the system (1.1) and (1.2).

1.1.1 *Results on the Richards equation*

Even in the case without hysteresis, i.e. with an algebraic relation $p = p_c(s)$ instead of (1.3), the Richards equation is an interesting mathematical object due to the possible degeneracies $k(s) = 0$ for some s and $p_c(s) \to \pm\infty$ for s tending to critical saturation values. Existence results are obtained for example in [2] and [3], uniqueness is treated for instance in [20] and [12] and the physical outflow boundary conditions are treated for instance in [1] and [24]. Regarding the numerical treatment of the Richards equation without hysteresis, we mention [4,22].

We want to highlight at this point the close connection between hysteresis effects and fingering in porous media; we refer to [27] and the references therein for experimental results on fingering. The analytical contributions of [20,28] imply, loosely speaking, that fingering is not possible in the Richards equation without hysteresis. On the other hand, it was shown theoretically in [26] and with numerical experiments in [17] that fingering can occur in the Richards equation if hysteresis is included.

The hysteresis relation (1.3), without coupling to a partial differential equation, already poses interesting questions regarding the functional analytic description, we refer to [30] for the corresponding discussion. In both cases, $\tau = 0$ (rate-independent) and $\tau > 0$ (rate-dependent), the hysteresis relation may be considered as a functional relation $s(t) = \mathscr{B}(t, p|_{[0,t]})$, where $\mathscr{B}$ maps the history of p to a value $s(t)$, given initial values s_0. We emphasize that, even in the equilibrium situation $\partial_t s = 0$, we cannot determine $p(t)$ from $s|_{[0,t]}$. In this sense, the hysteresis relation cannot be inverted.

An existence result for the Richards equation with hysteresis was provided in [23] in the case that the partial differential equation is linear, i.e. in the case that $k(.)$ does not depend on s and that $p_c(.)$ is an affine function. In this situation, it was possible to treat the case $\tau = 0$. Existence results for the non-linear Richards equation with other hysteresis relations were obtained in [5] and [6] under very general assumptions, but not covering our model.

If the hysteresis model (1.3) is considered without the static part introduced by the sign function, the model can be re-written as a pseudo-parabolic equation. Using this point of view, existence results are derived in [10] and [19], including some degeneracies of coefficients. Closest to the contribution at hand is [17], where the non-linear Richards equation with hysteresis was studied and an existence result was derived. The compactness result of Lemma 3.3 in [17] was crucial for the Richards equations and will be used again in the work at hand.

1.1.2 *Results on the two-phase flow equation*

The two-phase flow equations without hysteresis have been studied under the aspect of existence results in [13] and [16], uniqueness and regularity issues are treated in [13] and [14], outflow conditions in [18] and maximum principles appear in [16] and [18]. Physical conditions across interior interfaces are investigated in [9] and [11]. We are not aware of any contribution that derives an existence result for the two-phase flow system with hysteresis. Regarding the numerical treatment of two-phase flow without hysteresis, we refer to [7] and [21] and the references therein.

1.2 Assumptions and main result

In this section we fix the assumptions on the coefficient functions and formulate our main result. We consider non-linear but non-degenerate permeabilities k_j, $j = 1, 2$, and a strictly increasing, non-degenerate capillary pressure curve p_c. On the relaxation constant we assume positivity $\tau > 0$. We will construct solutions of the two-phase system with the specific hysteresis relation of (1.3), but the compactness result will exploit only the more general relation (1.4).

1.2.1 *Initial and boundary conditions*

The unknowns in the porous media model (1.1)–(1.3) are p_1, p_2 and s. Let $\Omega \subset \mathbb{R}^n$ be such that

$$\Omega \text{ is a Lipschitz domain and } \partial\Omega \text{ is decomposed as } \partial\Omega = \bar{\Gamma}_1 \cup \bar{\Sigma}_1 = \bar{\Gamma}_2 \cup \bar{\Sigma}_2 \tag{1.5}$$

with $\Gamma_j \cap \Sigma_j = \emptyset$ two relatively open subsets of $\partial\Omega$ for $j = 1, 2$. We impose the Dirichlet condition for p_j on Σ_j and the Neumann condition for fluid j on Γ_j, for notational convenience we impose only no-flux conditions on Γ_j. We assume positivity of the Hausdorff measures $\mathscr{H}^{n-1}(\Sigma_j) > 0$ for $j = 1, 2$. The Dirichlet conditions are given by two functions $p_{0,1}, p_{0,2} \in L^2(0, T; H^1(\Omega))$. We prescribe initial values for saturation by the function $s_0 \in L^2(\Omega)$.

1.2.2 *Coefficient functions*

Our assumptions on the coefficient functions are as follows. For six positive numbers $K_j, \kappa_j, \kappa_j^0 > 0$, $j = 1, 2$, we assume

$$p_c \in C^{0,1}(\mathbb{R} \times \Omega, \mathbb{R}), \quad \gamma \in C^{0,1}(\Omega, [0, \infty)), \tag{1.6}$$

$$k_j \in C(\mathbb{R} \times \Omega, [\kappa_j, \kappa_j^0]), \ \|k_j(x, .)\|_{\mathrm{Lip}(\mathbb{R},\mathbb{R})} \leqslant K_j, \text{ for } j \in \{1, 2\}, \ x \in \Omega. \tag{1.7}$$

We denote the Lipschitz constant of the capillary pressure by $\rho := \|p_c\|_{\mathrm{Lip}}$ and emphasize that the Lipschitz continuity of p_c is assumed in x and s. We will always assume $\tau > 0$ and use $\Phi : \mathbb{R} \times \Omega \to \mathbb{R}, (\xi, x) \mapsto \tau\xi + \gamma(x)\mathrm{sign}(\xi)$. Positivity of τ implies that $\Phi(., x)$ has the Lipschitz continuous inverse, we denote the inverse by $\Psi(., x)$. This defines $\Psi : \mathbb{R} \times \Omega \to \mathbb{R}$ with $0 \leqslant \partial_\zeta \Psi(\zeta, x) \leqslant 1/\tau$ for all $\zeta \in \mathbb{R}$. We finally assume that p_c has a positive primitive, i.e.

$$\exists P_c \in C(\mathbb{R} \times \Omega, \mathbb{R}) \text{ with } P_c(s, x) \geqslant 0, \ \partial_s P_c(s, x) = p_c(s, x) \text{ for all } s \in \mathbb{R}, x \in \Omega. \tag{1.8}$$

The gravity vectors $g_1, g_2 \in \mathbb{R}^n$ are constant vectors. The above assumptions cover physically relevant models. The main restriction is that we assume the capillary pressure curve p_c to be non-degenerate for $s \in \mathbb{R}$ such that, in particular, all saturation values $s \in \mathbb{R}$ could be observed. Physical models use degenerate capillary pressures, for an analysis see [25] and the references therein. Our model can be used in practice once the p_c-relation is truncated at pressure levels that cannot be obtained in physical setting.

1.2.3 *Weak form of equations* (1.1) *and* (1.2)

The first two evolution equations are expressed in the usual weak form. We say that $s, p_1, p_2 \in L^2(0, T; L^2(\Omega))$ with $\partial_t s \in L^2(0, T; L^2(\Omega))$ and $\nabla p_1, \nabla p_2 \in L^2(0, T; L^2(\Omega))$ solve (1.1) and (1.2) and the no-flux condition on Γ_j in the weak form if, for all test-functions

$\varphi_j \in L^2(0,T;H^1(\Omega))$ with $\varphi_j = 0$ on Σ_j, there holds

$$\int_{\Omega_T} k_1(s)[\nabla p_1 + g_1]\nabla\varphi_1 + \int_{\Omega_T} k_2(s)[\nabla p_2 + g_2]\nabla\varphi_2 = -\int_{\Omega_T} (\varphi_1 - \varphi_2)\,\partial_t s. \tag{1.9}$$

Main Theorem. Our main result concerns the existence of solutions to the non-degenerate two-phase flow equations.

Theorem 1.1 (*Existence for the two-phase flow problem with hysteresis*) *Let the Lipschitz domain $\Omega \subset \mathbb{R}^n$ with boundary parts Γ_j and Σ_j be as in* (1.5). *Let initial and boundary conditions be given by $s_0 \in L^2(\Omega)$ and $p_{0,1}, p_{0,2} \in L^2(0,T;H^1(\Omega))$. Let the coefficients satisfy $\tau > 0$ and* (1.6)–(1.8).

Then there exists a solution $p_1, p_2 \in L^2(0,T;H^1(\Omega))$ and $s \in H^1(0,T;L^2(\Omega))$ of the two-phase hysteresis system (1.1)–(1.3). *More precisely, equations* (1.1) *and* (1.2)*, and the no-flux condition are satisfied in the weak form* (1.9)*, the hysteresis relation* (1.3) *holds pointwise almost everywhere, initial and the Dirichlet boundary conditions are satisfied in the sense of traces.*

1.3 *A priori* estimates and solution concept

We start our analysis of system (1.1)–(1.3) by presenting the formal *a priori* estimates. These indicate natural norms and function spaces. At the same time, we will observe a lack of spatial regularity for the saturation variable s. It is this lack of compactness that makes the existence proof interesting.

We multiply (1.1) with $p_1 - p_{0,1}$ and (1.2) with $p_2 - p_{0,2}$ and integrate over Ω. Adding the equations, we obtain (1.9) with $\varphi_j = p_j - p_{0,j}$. Inserting (1.3) gives

$$\begin{aligned}
&\int_\Omega k_1(s)[\nabla p_1 + g_1]\nabla[p_1 - p_{0,1}] + \int_\Omega k_2(s)[\nabla p_2 + g_2]\nabla[p_2 - p_{0,2}] \\
&\quad = -\int_\Omega [(p_1 - p_2) - p_{0,1} + p_{0,2}]\,\partial_t s \\
&\quad \in -\int_\Omega (p_c(s) + \gamma\,\mathrm{sign}(\partial_t s) + \tau\partial_t s)\,\partial_t s + \int_\Omega (p_{0,1} - p_{0,2})\,\partial_t s.
\end{aligned}$$

By assumption (1.8), the monotone function p_c has a convex, positive primitive P_c. Integration over $t \in [0,T]$ and application of the Cauchy–Schwarz and the Poincaré inequality yields in the standard fashion the estimate

$$\int_\Omega P_c(s)\bigg|_{t=T} + \int_{\Omega_T} \left\{k_1(s)|\nabla p_1|^2 + k_2(s)|\nabla p_2|^2 + \gamma|\partial_t s| + \tau|\partial_t s|^2\right\} \leqslant C_0, \tag{1.10}$$

where the constant C_0 depends on the data g_j, $p_{0,j}$, s_0, on τ and the other system constants introduced before (1.6). We exploited that p_c is Lipschitz continuous and that, as a consequence, P_c has at most quadratic growth in s. The domain of integration is $\Omega_T = \Omega \times (0,T)$.

Variational weak solutions. In the solution concept of Theorem 1.1 we demand that relation (1.3) holds for almost all $(x,t) \in \Omega_T$. In order to verify this condition, it is convenient to use, in addition, the notion of variational weak solutions.

Definition 1.2 (Variational weak solution) *Let (s, p_1, p_2) be a triple of functions with*

$$s \in L^\infty(0,T;L^2(\Omega)), \quad \partial_t s \in L^2(0,T;L^2(\Omega)), \quad p_1, p_2 \in L^2(0,T;H^1(\Omega)) \tag{1.11}$$

satisfying, in the sense of traces, the initial condition $s = s_0$ on $\Omega \times \{0\}$ and the boundary conditions $p_j = p_{0,j}$ on $\Sigma_j \times (0,T)$. The triple is called a variational weak solution *of the two-phase equation if the following three conditions are satisfied:*

(1) *The evolution equations* (1.1) *and* (1.2) *and the no-flux conditions are satisfied in the weak sense of* (1.9).

(2) *The relation $p_1(x,t) - p_2(x,t) - p_c(s(x,t),x) - \tau\partial_t s(x,t) \in [-\gamma(x), \gamma(x)]$ holds for almost every $(x,t) \in \Omega_T$.*

(3) *The variational inequality*

$$\begin{aligned} 0 \geqslant &\int_{\Omega_T} \left(p_c(s) - p_{0,1} + p_{0,2}\right) \partial_t s + \int_{\Omega_T} \left\{\tau\,|\partial_t s|^2 + \gamma\,|\partial_t s|\right\} \\ &+ \int_{\Omega_T} k_1(s)[\nabla p_1 + g_1]\nabla[p_1 - p_{0,1}] + \int_{\Omega_T} k_2(s)[\nabla p_2 + g_2]\nabla[p_2 - p_{0,2}] \end{aligned} \tag{1.12}$$

is satisfied.

Lemma 1.3 *Let (s, p_1, p_2) be a variational weak solution as in Definition 1.2. Then* (1.3) *is satisfied almost everywhere. In particular, (s, p_1, p_2) is a solution of* (1.1)–(1.3) *as described in Theorem 1.1.*

Proof We only have to show that (1.3) holds almost everywhere. For weak solutions, the two distributions $\nabla \cdot (k_j(s)[\nabla p_j + g_j]) = \pm\partial_t s$ are actually $L^2(\Omega_T)$ functions, hence we can perform an integration by parts in the last two integrals of (1.12). Then the inequality (1.12) simplifies to

$$0 \geqslant \int_{\Omega_T} (p_c(s) - p_1 + p_2)\,\partial_t s + \int_{\Omega_T} \left\{\tau\,|\partial_t s|^2 + \gamma\,|\partial_t s|\right\}.$$

We write this as

$$\int_{\Omega_T} \gamma\,|\partial_t s| \leqslant \int_{\Omega_T} [p_1 - p_2 - p_c(s) - \tau\partial_t s]\partial_t s.$$

By property 2 of variational weak solutions, the integrand on the right-hand side satisfies $[p_1 - p_2 - p_c(.,s) - \tau\partial_t s]\partial_t s \leqslant \gamma|\partial_t s|$ almost everywhere, and is therefore smaller or equal to the integrand on the left-hand side. Since the integral inequality is in the opposite direction, the integrands must coincide, $[p_1 - p_2 - p_c(.,s) - \tau\partial_t s]\partial_t s = \gamma\,|\partial_t s|$ holds almost everywhere. This, together with property 2 of variational weak solutions, implies the pointwise inclusion (1.3). □

We see that Theorem 1.1 is shown once that we prove the existence of a variational weak solution as in Definition 1.2.

2 Discrete system and proof of the main theorem

2.1 The discrete system

Our next aim is to define the Galerkin scheme such that the original equations (1.1)–(1.3) are approximated by a system of ordinary differential equations. With this aim we introduce space-discretization with parameter $h > 0$. We recall that three positive (and possibly small) physical parameters appear in the equations: the numbers $\kappa_1, \kappa_2 > 0$ are lower bounds for permeabilities and $\tau > 0$ is the time delay parameter.

In the case of a vanishing time delay, $\tau = 0$, the play-type relation (1.3) can be written with the multi-valued function $\Phi^0(\sigma) := \gamma\,\mathrm{sign}(\sigma)$ as $p_1 - p_2 \in p_c(s) + \Phi^0(\partial_t s)$. In the general case $\tau \geqslant 0$ and with $\gamma = \gamma(x)$ we use $\Phi^\tau := \Phi^0 + \tau\,\mathrm{id}$, or, more precisely,

$$\Phi^\tau(\sigma, x) := \begin{cases} [-\gamma(x), \gamma(x)] & \text{for} \quad \sigma = 0 \\ \gamma(x) + \tau\sigma & \text{for} \quad \sigma > 0. \\ -\gamma(x) + \tau\sigma & \text{for} \quad \sigma < 0 \end{cases} \tag{2.1}$$

With this choice, (1.3) can be written as $p_1 - p_2 \in p_c(s) + \Phi^\tau(\partial_t s)$; we suppress the dependence on x whenever possible. We denote the inverse by $\Psi^\tau(., x) := (\Phi^\tau(., x))^{-1}$. The inverse $\Psi^\tau : \mathbb{R} \times \Omega \to \mathbb{R}$ is multi-valued only in the case $\tau = 0$. For positive τ, the function Ψ^τ is single-valued with maximal slope τ^{-1}. In this sense, $\tau > 0$ can be regarded as a regularization of the system.

2.1.1 *Spatial discretization*

We next discretize the spatial domain Ω. In order to simplify notation, we describe the method for the case that the domain Ω is polygonal; in the case of a general Lipschitz domain, it poses no problem to use boundary elements that are not simplices.

Let $\mathscr{T}_h$ be a triangulation of Ω, decomposing Ω into finitely many simplices $A \in \mathscr{T}_h$. Let $h > 0$ be an upper bound for the diameter of all elements of $\mathscr{T}_h$. We denote by $\Omega_h = \{x_1, \dots, x_N\}$ a suitable subset of N points such that we can associate to every triangle $A \in \mathscr{T}_h$ a uniquely determined point $x \in \Omega_h \cap A$. The set of points $(x_k)_{k \leqslant N}$ defines a projection $X^h : \Omega \to \Omega_h$. The map X^h can also be used to define an invertible map that identifies $\mathbb{R}^N$ with piecewise constant functions (defined almost everywhere),

$$J : \mathbb{R}^N \equiv \{f : \Omega_h \to \mathbb{R}\} \longrightarrow \{\hat{f} : \Omega \to \mathbb{R} \textit{ piecewise constant}\} =: \mathscr{P}_0(\Omega, \mathscr{T}_h), \tag{2.2}$$

by $(Jf)(x) = f(X^h(x))$ for almost every $x \in \Omega$. We will furthermore use the $L^2(\Omega)$-orthogonal projection $P := P_h : L^2(\Omega) \to L^2(\Omega)$ to the space of piecewise constant functions $\mathscr{P}_0(\Omega, \mathscr{T}_h)$. A continuous function on Ω can be discretized with the help of $X^h : \Omega \to \Omega_h$. To give an example, given $\gamma = \gamma(x)$, we can restrict to the relevant corners and consider $\gamma|_{\Omega_h}$, and correspondingly the piecewise constant parameter function $\gamma^h(x) := J(\gamma|_{\Omega_h})(x) = \gamma(X^h(x))$. Accordingly, we define the piecewise constant (in x)

coefficient function $p_c^h(s,x) := p_c(s, X^h(x))$ and its primitive $P_c^h(s,x) = P_c(s, X^h(x))$ with $\partial_s P_c^h(s,x) = p_c^h(s,x)$. Analogously, the function $\Phi^\tau(\sigma, x)$ of (2.1) is discretized in space to $\Phi_h^\tau(\sigma,x) := \Phi_h^\tau(\sigma, X^h(x))$ and its inverse (in the variable σ) is $\Psi_h^\tau(.,x) = (\Phi_h^\tau(.,x))^{-1}$. With this notation, we can now define the Galerkin scheme.

Definition 2.1 (Galerkin scheme) *Our unknowns are piecewise constant functions $p_j^h : \Omega \times [0,T] \to \mathbb{R}, j = 1,2$ and $s^h : \Omega \times [0,T] \to \mathbb{R}$, identified with maps $p_1^h, p_2^h, s^h : [0,T] \to \mathscr{P}_0(\Omega, \mathscr{T}_h)$. We demand, for almost every $x \in \Omega$ and almost every $t \in (0,T)$,*

$$\begin{aligned} \partial_t s^h(x,t) &= \Psi_h^\tau(p_1^h(x,t) - p_2^h(x,t) - p_c^h(s^h,x)), \\ s^h(x,0) &= (P_h s_0)(x), \end{aligned} \tag{2.3}$$

where we suppressed the explicit dependence of Ψ_h^τ on x. The pressures p_1^h and p_2^h are reconstructed from s^h as follows. We solve with two functions $\tilde{p}_j^h \in H^1(\Omega, \mathbb{R})$, $j = 1,2$, in a weak sense the elliptic system

$$-\nabla \cdot \left(k_1(s^h,x)(\nabla \tilde{p}_1^h + g_1)\right) = \Psi_h^\tau \left(p_c(s^h,x) - P_h[\tilde{p}_1^h - \tilde{p}_2^h]\right) \;\text{ in } \Omega, \tag{2.4}$$

$$-\nabla \cdot \left(k_2(s^h,x)(\nabla \tilde{p}_2^h + g_2)\right) = -\Psi_h^\tau \left(p_c(s^h,x) - P_h[\tilde{p}_1^h - \tilde{p}_2^h]\right) \;\text{ in } \Omega, \tag{2.5}$$

$$\tilde{p}_j^h(\cdot,t) = p_{0,j}(\cdot,t) \text{ on } \Sigma_j \text{ for } j = 1,2, \tag{2.6}$$

for all $t \in [0,T]$, with no-flux conditions on Γ_j. The discrete pressures are recovered by a projection, $p_j^h = P_h \tilde{p}_j^h$ for $j = 1,2$.

For later use we note that the evolution equation in (2.3) can also be written as

$$\Phi_h^\tau(\partial_t s^h) = \Phi_h^0(\partial_t s^h(x,t)) + \tau \partial_t s^h \ni p_1^h - p_2^h - p_c^h(s^h). \tag{2.7}$$

We note that Φ_h^τ and Φ_h^0 depend via $\gamma^h(x)$ also in a direct way on $x \in \Omega$.

2.2 Well-posedness of the Galerkin scheme

Our aim is to prove that (2.3) is an ordinary differential equation for $s^h : [0,T] \to \mathscr{P}_0(\Omega, \mathscr{T}_h)$. With this perspective, we want to show that the system (2.4)–(2.6) defines the Lipschitz-continuous map $s^h \mapsto (p_1^h, p_2^h) = (P_h \tilde{p}_1^h, P_h \tilde{p}_2^h)$. Once this is shown, we have verified that the Galerkin scheme consists of the ordinary differential equation (2.3) (the image space $\mathscr{P}_0(\Omega, \mathscr{T}_h)$ is finite dimensional) with an intricate, but Lipschitz continuous right-hand side. We exploit here the Lipschitz property of the parameter functions.

The aim of the next lemma is precisely this analysis of the stationary system (2.4)–(2.6). We write $\tilde{p}_j^h = p_{0,j} + u_j$ for $j = 1,2$ such that (2.4) and (2.5) read, omitting the h-dependence of function s,

$$\begin{aligned} -\nabla \cdot (k_1(s)\nabla u_1) &= \Psi_h^\tau \left(p_c(s) - P_h(u_1 - u_2) - P_h(p_{0,1} - p_{0,2})\right) \\ &\quad + \nabla \cdot \left(k_1(s)(\nabla p_{0,1} + g_1)\right), \\ -\nabla \cdot (k_2(s)\nabla u_2) &= -\Psi_h^\tau \left(p_c(s) - P_h(u_1 - u_2) - P_h(p_{0,1} - p_{0,2})\right) \\ &\quad + \nabla \cdot \left(k_2(s)(\nabla p_{0,2} + g_2)\right). \end{aligned}$$

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We introduce some abbreviations. Let $f : \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$ be the function

$$f(s,z,x) := -\Psi_h^\tau(p_c(s,x) - z - P_h(p_{0,1} - p_{0,2})(x), x). \tag{2.8}$$

Then $s \mapsto f(s,z,x)$ is Lipschitz continuous with the Lipschitz constant $\rho\tau^{-1}$. The map $z \mapsto f(s,z,x)$ is monotonically non-decreasing and Lipschitz continuous with the Lipschitz constant τ^{-1}. In the following, we suppress the explicit x-dependece of Ψ_h^τ. We use $F : L^2(\Omega) \times L^2(\Omega) \to L^2(\Omega)$,

$$F(s,z)(x) := f(s(x), z(x), x). \tag{2.9}$$

Later on, we will insert $P_h(u_1 - u_2)$ for variable z. With this choice, the expression $f(s,z,x)$ coincides, up to signs, with the first part on the right-hand sides of (2.4) and (2.5). To abbreviate also the other lower order terms, we use, for $j = 1, 2$, the map $\bar{G}_j : L^2(\Omega) \to L^2(\Omega)$,

$$\bar{G}_j(s)(x) := k_j(s(x), x)(\nabla p_{0,j}(x) + g_j). \tag{2.10}$$

We intend to use piecewise constant functions $s, z \in \mathscr{P}_0(\Omega, \mathscr{T}_h)$ and note that functions, such as $p_c(s,x)$, $F(s,z)$ or $\bar{G}_j(s)$, are not piecewise constant functions in general.

Lemma 2.2 (Local existence result for the discrete stationary system) *Let the data* $\Omega, p_c, k_j, g_j, p_{0,j}$ *and* $\tau > 0$ *satisfy* (1.5)–(1.8). *Let the hysteresis function* Ψ_h^τ *and the projection* P_h *be as described before Definition 2.1. Let* F *and* $\bar{G}_j$ *be as in* (2.8)–(2.10). *Then there exists a positive number* $h_0 > 0$ *such that the following statements hold.*

Existence. *Let* $h \in (0, h_0)$ *and* $S \in L^\infty(\Omega)$ *be arbitrary. We consider the spaces* $H_{0,j}(\Omega) := \{u_j \in H^1(\Omega) : u_j = 0 \text{ on } \Sigma_j\}$ *and search for solutions* $u = (u_1, u_2)$ *in the product space* $u \in H_{0,1}(\Omega) \times H_{0,2}(\Omega)$. *For arbitrary right-hand sides* $G_j \in H_{0,j}(\Omega)'$, $j = 1, 2$, *there exists a unique weak solution* $u = (u_1, u_2)$ *of*

$$\begin{aligned} -\nabla \cdot (k_1(S)\nabla u_1) &= -F(S, P_h(u_1 - u_2)) + G_1 \\ -\nabla \cdot (k_2(S)\nabla u_2) &= F(S, (P_h(u_1 - u_2)) + G_2 \end{aligned} \tag{2.11}$$

in Ω, *with a weak no-flux condition* $n \cdot [\nabla(p_{0,j} + u_j) + g_j] = 0$ *on* Γ_j.

Lipschitz continuity. *For every* $R > 0$, *there exists a positive constant* $C = C(R)$ *such that the following holds. Let* $s, \tilde{s} \in L^\infty(\Omega)$ *with* $\|s\|_\infty, \|\tilde{s}\|_\infty \leqslant R$. *Let* $u = (u_1, u_2)$ *be a solution of* (2.11) *for* $S = s$ *and* $G_j : \varphi \mapsto -\int_\Omega \bar{G}_j(s)\nabla\varphi$. *Let* $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)$ *be a solution of* (2.11) *for* $S = \tilde{s}$ *and* $G_j : \varphi \mapsto -\int_\Omega \bar{G}_j(\tilde{s})\nabla\varphi$. *Then*

$$\|u - \tilde{u}\|_{H^1(\Omega,\mathbb{R}^2)} \leqslant C\|s - \tilde{s}\|_{L^\infty(\Omega)}. \tag{2.12}$$

Proof We search for solutions in the product space $H := H_{0,1}(\Omega) \times H_{0,2}(\Omega)$. The space H is a Hilbert space with the norm of $H^1(\Omega) \times H^1(\Omega)$ and the dual space is $H' = H_{0,1}(\Omega)' \times H_{0,2}(\Omega)'$.

Step 1. Re-formulation of the system: In a later step, we want to use the continuity method. We therefore generalize the system slightly and consider, for $\lambda \in [0,1]$, the following system for $u = (u_1, u_2) \in H$,

$$\begin{aligned} -\nabla \cdot (k_1(S,x)\nabla u_1) &= -\lambda F(S, P_h(u_1 - u_2)) + G_1 \\ -\nabla \cdot (k_2(S,x)\nabla u_2) &= \lambda F(S, P_h(u_1 - u_2)) + G_2 \end{aligned} \qquad (E_\lambda)$$

in the weak form and with the same no-flux condition. With this choice of (E_λ), problem (E_1) for $\lambda = 1$ coincides with the original problem (2.11). On the space H we define a bilinear form $B_S : H \times H \to \mathbb{R}$ as

$$B_S[u, \varphi] := \int_\Omega k_1(S,x)\nabla u_1(x)\nabla\varphi_1(x) + k_2(S,x)\nabla u_2(x)\nabla\varphi_2(x)\, dx$$

for $u = (u_1, u_2), \varphi = (\varphi_1, \varphi_2) \in H$. The pair $G := (G_1, G_2)$ satisfies $G \in H'$. Equation (E_λ) now reads

$$B_S[u, \varphi] = -\lambda \int_\Omega F(S, P_h(u_1 - u_2))(\varphi_1 - \varphi_2) + \langle G, \varphi\rangle \qquad (2.13)$$

for all $\varphi \in H$; here $\langle\cdot,\cdot\rangle$ denotes the duality pairing between H' and H.

Step 2. *A priori* estimates: We use $\varphi = u$ in (2.13). Poincaré's inequality and the positiviy of k_1, k_2 imply the coercivity of B_S and we obtain with $c > 0$

$$\begin{aligned} c\|u\|_H^2 &\leqslant \int_\Omega k_1(S)\,|\nabla u_1|^2 + k_2(S)\,|\nabla u_2|^2 = B_S[u,u] \\ &= -\lambda \int_\Omega F(S, P_h(u_1 - u_2))(u_1 - u_2) + \langle G, u\rangle. \end{aligned}$$

The second term on the right-hand side can be estimated directly and treated with Young's inequality as $\langle G, u\rangle \leqslant \|G\|_{H'}\|u\|_H \leqslant \delta\|u\|_H^2 + \delta^{-1}\|G\|_{H'}^2$ for arbitrary $\delta > 0$.

Concerning the integral containing F, we exploit the monotonicity of f in z, furthermore the triangle inequality and Cauchy–Schwarz's inequality.

$$\begin{aligned} &-\int_\Omega F(S, P_h(u_1 - u_2))(u_1 - u_2) \\ &= -\int_\Omega F(S, P_h(u_1 - u_2))P_h(u_1 - u_2) + F(S, P_h(u_1 - u_2))(u_1 - u_2 - P_h(u_1 - u_2)) \\ &\leqslant -\int_\Omega F(S, 0)P_h(u_1 - u_2) + |F(S, P_h(u_1 - u_2))|\,|u_1 - u_2 - P_h(u_1 - u_2)| \\ &\leqslant \|F(S,0)\|_{L^2(\Omega)}\|P_h(u_1 - u_2)\|_{L^2(\Omega)} \\ &\quad + \|F(S, P_h(u_1 - u_2))\|_{L^2(\Omega)}\|u_1 - u_2 - P_h(u_1 - u_2)\|_{L^2(\Omega)}. \end{aligned}$$

We exploit the following properties of P_h. For $w \in H^1(\Omega)$ we have $\|P_h w\|_{L^2(\Omega)} \leqslant \|w\|_{L^2(\Omega)}$ by Hölder's inequality, and $\|P_h w - w\|_{L^2(\Omega)} \leqslant C_{\mathscr{T}} h\|w\|_{H^1(\Omega)}$ by Poincaré's inequality for some $C_{\mathscr{T}} > 0$. We use the Lipschitz continuity of f in z with constant τ^{-1}, Young's and

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Poincaré's inequality to calculate

$$
\begin{aligned}
&\|F(S,0)\|_{L^2(\Omega)}\|P_h(u_1-u_2)\|_{L^2(\Omega)} \\
&\quad + \|F(S,P_h(u_1-u_2)\|_{L^2(\Omega)}\|u_1-u_2-P_h(u_1-u_2)\|_{L^2(\Omega)} \\
&\leqslant \|F(S,0)\|_{L^2(\Omega)}\|u_1-u_2\|_{L^2(\Omega)} + \frac{1}{\tau}\|u_1-u_2\|_{L^2(\Omega)} C_{\mathscr{T}} h \|u_1-u_2\|_{H^1(\Omega)} \\
&\quad + \|F(S,0)\|_{L^2(\Omega)} C_{\mathscr{T}} h \|u_1-u_2\|_{H^1(\Omega)} \\
&\leqslant \frac{C_{\mathscr{T}} h}{\tau}\|u_1-u_2\|^2_{H^1(\Omega)} + (1+C_{\mathscr{T}} h)\|F(S,0)\|_{L^2(\Omega)}\|u_1-u_2\|_{H^1(\Omega)} \\
&\leqslant 2\left(\frac{C_{\mathscr{T}} h}{\tau} + \delta(1+C_{\mathscr{T}} h)^2\right)\|u\|_H^2 + \frac{1}{\delta}\|F(S,0)\|^2_{L^2(\Omega)}
\end{aligned}
$$

for arbitrary $\delta > 0$. Choosing δ and h_0 sufficiently small, we can absorb the first term and find the *a priori* estimate

$$\|u\|_H^2 \leqslant \frac{2}{c\delta}\|F(S,0)\|^2_{L^2(\Omega)} + C_G = C(\|S\|_{L^\infty(\Omega)}, \|G\|_{H'}). \tag{2.14}$$

Estimate (2.14) holds for every $h \in (0, h_0)$ and we emphasize that the number h_0 is independent of data S and G_j.

Step 3. The continuity method: We define the set

$$\Lambda := \{\lambda \in [0,1] |\ \forall (G_1, G_2) \in H' \text{ there exists a unique solution } u \in H \text{ of } (E_\lambda)\},$$

with the aim to show that Λ contains $\lambda = 1$. It is an immediate observation that Λ contains $\lambda = 0$. Indeed, equation (E_λ) for $\lambda = 0$ consists of two decoupled linear elliptic equations that can be solved uniquely with the Lax–Milgram theorem.

We will show the following *Claim: For every* $\lambda_0 \in \Lambda$*, there exists* $\varepsilon > 0$ *independent of* λ_0*, such that* (E_λ) *has a unique solution for all* $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$. Once the claim is verified, we can apply it for a finite number of times and obtain $1 \in \Lambda$ and thus the existence and uniqueness result.

In order to prove the claim, we use a fixed point method. We define an iteration by considering, for given $\tilde{u} = (\tilde{u}_1, \tilde{u}_2) \in H$, the following equation for $u = (u_1, u_2)$,

$$
\begin{aligned}
B_S[u,\varphi] = -\lambda_0 \int_\Omega F(S, P_h(u_1-u_2))(\varphi_1-\varphi_2) \\
-\varepsilon \int_\Omega F(S, P_h(\tilde{u}_1-\tilde{u}_2))(\varphi_1-\varphi_2) + \langle G, \varphi\rangle
\end{aligned} \tag{2.15}
$$

for all $\varphi = (\varphi_1, \varphi_2) \in H$. Since λ_0 is an element of Λ, by definition of Λ, we find a unique solution (u_1, u_2) of (2.15). We exploit here that the vector $G - \varepsilon(-F(S, P_h(\tilde{u}_1 - \tilde{u}_2)), F(S, P_h(\tilde{u}_1 - \tilde{u}_2)))$ is an element of H'. The unique solvability property defines an operator $T : H \to H$, $T(\tilde{u}) = u$. We note that a fixed point $(u_1, u_2) = (\tilde{u}_1, \tilde{u}_2)$ provides a solution of (E_λ) for $\lambda = \lambda_0 + \varepsilon$.

It therefore suffices to show that the map T as above is contractive for $\varepsilon > 0$ sufficiently small (the smallness must be independent of λ_0). Let $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)$, $\tilde{v} = (\tilde{v}_1, \tilde{v}_2) \in H$ be different data, we consider solutions $T(\tilde{u}) = u = (u_1, u_2)$ and $T(\tilde{v}) = v = (v_1, v_2)$. We investigate (2.15) for u and v, subtract both equations and set $\varphi = u - v \in H$, i.e. $\varphi_1 = u_1 - v_1$ and $\varphi_2 = u_2 - v_2$. We find, for some $c > 0$,

$$\begin{aligned} c\|u-v\|_H^2 &\leqslant B_S[u-v,u-v] \\ &= -\lambda_0 \int_\Omega [F(S,P_h(u_1-u_2)) - F(S,P_h(v_1-v_2))]((u_1-v_1)-(u_2-v_2)) \\ &\quad - \varepsilon \int_\Omega [F(S,P_h(\tilde u_1-\tilde u_2)) - F(S,P_h(\tilde v_1-\tilde v_2))]((u_1-v_1)-(u_2-v_2)). \end{aligned}$$

In the first integral, we apply the monotonicity and the Lipschitz continuity of f in z, Cauchy–Schwarz and the properties of the projection P_h. We obtain

$$\begin{aligned} &-\lambda_0 \int_\Omega [F(S,P_h(u_1-u_2)) - F(S,P_h(v_1-v_2))]((u_1-v_1)-(u_2-v_2)) \\ &\leqslant \int_\Omega |F(S,P_h(u_1-u_2)) - F(S,P_h(v_1-v_2))| \cdot \\ &\qquad\qquad \cdot |u_1-u_2-(v_1-v_2)-P_h((u_1-u_2)-(v_1-v_2))| \\ &\leqslant \frac{2}{\tau}\|P_h(u-v)\|_{L^2(\Omega,\mathbb{R}^2)}\|u-v-P_h(u-v)\|_{L^2(\Omega,\mathbb{R}^2)} \leqslant \frac{2}{\tau}C_{\mathcal{T}}h\|u-v\|_H^2. \end{aligned}$$

The second integral is treated with the Lipschitz continuity of f in z,

$$\begin{aligned} &-\varepsilon \int_\Omega [F(S,P_h(\tilde u_1-\tilde u_2)) - F(S,P_h(\tilde v_1-\tilde v_2))]((u_1-v_1)-(u_2-v_2)) \\ &\leqslant \varepsilon\frac{2}{\tau}\|\tilde u-\tilde v\|_H\|u-v\|_H \leqslant \left(\frac{\varepsilon}{\tau}\right)^2\frac{1}{\delta}\|\tilde u-\tilde v\|_H^2 + \delta\|u-v\|_H^2 \end{aligned}$$

for arbitrary $\delta > 0$. Summarizing, we have now obtained

$$c\|u-v\|_H^2 \leqslant \frac{2}{\tau}C_{\mathcal{T}}h\|u-v\|_H^2 + \left(\frac{\varepsilon}{\tau}\right)^2\frac{1}{\delta}\|\tilde u-\tilde v\|_H^2 + \delta\|u-v\|_H^2\,.$$

We choose $h_0 > 0$ and $\delta > 0$ small to absorb the first and the third terms. These choices depend on τ and the lower bounds for permeabilities, but they are independent of S and λ_0. For sufficiently small ε, we obtain the contraction property of T. Using the special argument $\tilde u = 0$ and the solution $T(0)$, we find that T maps a sufficiently large ball into itself. The Banach fixed point theorem yields the existence of a solution in this ball. The fixed point is globally unique, since T is contractive on any ball. This provides that for such $\varepsilon > 0$ the equation (E_λ) has a unique solution for $\lambda < \lambda_0 + \varepsilon$ and hence the claim.

Step 4. Proof of the Lipschitz estimate (2.12)**:** In contrast to the previous steps, we now investigate how variations of the parameter function S affect solutions. With this aim, let u be a solution of (2.11) for $S = s \in L^\infty(\Omega)$ with $G_j = \nabla \cdot \bar G_j(s)$ in the sense of $G_j(s) : \varphi \mapsto -\int_\Omega \bar G_j(s)\nabla\varphi$ for $\varphi \in H_{0,j}(\Omega)$, where $\bar G_j$ is as in (2.10). For the other saturation data $\tilde s \in L^\infty(\Omega)$, let $\tilde u$ be the solution of (2.11) for $S = \tilde s$ with $G_j = \nabla \cdot \bar G_j(\tilde s)$.

In order to compare u with $\tilde{u}$, we choose $\varphi = u - \tilde{u}$ as a test-function in both variants of equation (2.11). We calculate

$$\begin{aligned} c\|u-\tilde{u}\|_H^2 \leqslant B_s[u-\tilde{u},u-\tilde{u}] &= \int_\Omega k_1(s)\,|\nabla(u_1-\tilde{u}_1)|^2 + k_2(s)\,|\nabla(u_2-\tilde{u}_2)|^2 \\ &= \sum_{j=1}^2 \int_\Omega (k_j(\tilde{s})-k_j(s))\nabla\tilde{u}_j\cdot\nabla(u_j-\tilde{u}_j) \\ &\quad - \int_\Omega [F(s,P_h(u_1-u_2)) - F(\tilde{s},P_h(\tilde{u}_1-\tilde{u}_2))]\,((u_1-\tilde{u}_1)-(u_2-\tilde{u}_2)) \\ &\quad + \sum_{j=1}^2 \int_\Omega (\bar{G}_j(s)-\bar{G}_j(\tilde{s}))\nabla(u_j-\tilde{u}_j). \end{aligned} \tag{2.16}$$

We next use the *a priori* estimate (2.14), providing $\|\tilde{u}\|_H \leqslant C_{ap}(R)$ for some constant $C_{ap}(R) > 0$. Hence, the first sum on the right-hand side can be estimated exploiting the Lipschitz continuity of k_1 and k_2, Poincaré's and Young's inequality,

$$\begin{aligned} &\sum_{j=1}^2 \int_\Omega (k_j(\tilde{s})-k_j(s))\nabla\tilde{u}_j\cdot\nabla(u_j-\tilde{u}_j) \\ &\quad \leqslant L_k\|s-\tilde{s}\|_{L^\infty(\Omega)}\|\tilde{u}\|_H\|u-\tilde{u}\|_H \leqslant L_k^2\frac{C_{ap}(R)^2}{\delta}\|s-\tilde{s}\|_{L^\infty(\Omega)}^2 + \delta\,\|u-\tilde{u}\|_H^2 \end{aligned}$$

for $L_k > 0$, which depends on the Lipschitz constants of k_1 and k_2. We can choose $\delta > 0$ sufficiently small to absorb the last term into the left-hand side.

The second integral on the right-hand side of (2.16) is treated with the monotonicity of f in z. We furthermore use the Lipschitz continuity of f in s and z and the properties of the L^2-orthogonal projection P_h,

$$\begin{aligned} &-\int_\Omega [F(s,P_h(u_1-u_2)) - F(\tilde{s},P_h(\tilde{u}_1-\tilde{u}_2))]\,((u_1-\tilde{u}_1)-(u_2-\tilde{u}_2)) \\ &\quad = -\int_\Omega [F(s,P_h(u_1-u_2)) - F(\tilde{s},P_h(u_1-u_2))]\,((u_1-\tilde{u}_1)-(u_2-\tilde{u}_2)) \\ &\qquad - \int_\Omega [F(\tilde{s},P_h(u_1-u_2)) - F(\tilde{s},P_h(\tilde{u}_1-\tilde{u}_2))]\,((u_1-\tilde{u}_1)-(u_2-\tilde{u}_2)) \\ &\quad \leqslant L_f\|s-\tilde{s}\|_{L^\infty(\Omega)}\,C_{ap}(R)\|u-\tilde{u}\|_H \\ &\qquad + \int_\Omega |F(\tilde{s},P_h(u_1-u_2)) - F(\tilde{s},P_h(\tilde{u}_1-\tilde{u}_2))| \\ &\qquad\qquad \cdot|(u_1-\tilde{u}_1)-(u_2-\tilde{u}_2) - P_h((u_1-\tilde{u}_1)-(u_2-\tilde{u}_2))| \\ &\quad \leqslant \frac{L_f^2}{\delta}C_{ap}(R)^2\|s-\tilde{s}\|_{L^\infty(\Omega)}^2 + \delta\,\|u-\tilde{u}\|_H^2 + \frac{1}{\tau}C_{\mathscr{T}}h\|u-\tilde{u}\|_H^2 \end{aligned}$$

for arbitrary $\delta > 0$, where $L_f > 0$ depends on the Lipschitz constant of f in s. Choosing $h_0, \delta > 0$ sufficiently small, we can absorb the last two terms into the left-hand side. Once more, the choice of h_0 and δ is independent of s, $\tilde{s}$ and R.

The estimate of the last integral on the right-hand side of (2.16) exploits the Lipschitz continuity of k_j,

$$\sum_{j=1}^{2}\int_{\Omega}(\bar{G}_j(s)-\bar{G}_j(\tilde{s}))\nabla(u_j-\tilde{u}_j)$$
$$\leqslant \sum_{j=1}^{2}\int_{\Omega}|k_j(s)-k_j(\tilde{s})|\,|\nabla p_{0,j}+g_j|\,|\nabla(u_j-\tilde{u}_j)|$$
$$\leqslant C\,L_k\|s-\tilde{s}\|_{L^\infty(\Omega)}\|u-\tilde{u}\|_H \leqslant \delta\|u-\tilde{u}\|_H^2 + C^2\frac{L_k^2}{\delta}\|s-\tilde{s}\|_{L^\infty(\Omega)}^2,$$

where C depends only on the data $p_{0,j}$ and g_j for $j=1,2$. Once more, we can choose $\delta>0$ sufficiently small to absorb the first term.

We conclude as follows. We insert the three intermediate estimates into (2.16), choose $\delta, h_0>0$ sufficiently small and absorb terms containing $\|u-\tilde{u}\|_H^2$ on the left-hand side. As a result, we obtain (2.12). □

Remark 2.3 *The Lipschitz continuity of k_j in s was not used in the existence part of the above lemma. Furthermore, the above proof is not restricted to our special choice of the non-linearity Ψ_h^τ. The essential properties are the Lipschitz continuity of Ψ_h^τ and the fact that $\Psi_h^\tau(\cdot,x)$ is monotonically nondecreasing for every $x\in\Omega$. The special choice of Ψ_h^τ is used later in the compactness Lemma 2.5 and it was used in Lemma 1.3.*

With Lemma 2.2, we have shown that (2.3) is an ordinary differential equations with Lipschitz continuous right hand side in $\mathscr{P}_0(\Omega,\mathscr{T}_h)$ for every t. Hence a local solution of the Galerkin scheme of Definition 2.1 exists. In addition, as a consequence of the general theory of ordinary differential equations, we know the following: If we can show that the norm $\|s(t)\|_\infty$ is bounded for every solution s on an arbitrary time interval $(0,T)$, with a bound that is independent of T, then the solution can be extended, and exists for all times.

In the next section we will derive such a uniform bound and thus obtain, in particular, the global existence of solutions to the Galerkin scheme.

2.3 *A priori* estimates for the time-dependent system

We intend to perform the limit $h\to 0$ for the solutions $s^h, \tilde{p}_1^h, \tilde{p}_2^h$ of the Galerkin scheme of Definition 2.1. In the first step, we derive h-independent estimates for such solutions of the time-dependent system. Lemmas 2.4 and 2.7 are very similar to the results of [17], they are essentially adaptations to the two-phase flow system. The two proofs follow the standard scheme.

Lemma 2.4 (Energy estimates) *Let the coefficient functions, initial and boundary data be given as in Lemma 2.2 and $s_0\in L^2(\Omega)$. Then there exists a number $C>0$, independent of $h>0$ and $T>0$, such that every solution $\tilde{p}_1^h, \tilde{p}_2^h, s^h$ to the Galerkin scheme of Definition 2.1*

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satisfies the uniform bound

$$\|\tilde{p}_1^h\|^2_{L^2(0,T;H^1(\Omega))} + \|\tilde{p}_2^h\|^2_{L^2(0,T;H^1(\Omega))} + \|\partial_t s^h\|^2_{L^2(0,T;L^2(\Omega))} \leqslant C. \tag{2.17}$$

Proof We abbreviate the pressure differences as $\tilde{p}^h := \tilde{p}_1^h - \tilde{p}_2^h$ and $p^h := p_1^h - p_2^h = P_h \tilde{p}^h$. Concerning the boundary data, we use $p_0 := p_{0,1} - p_{0,2}$.

We start by writing the Galerkin evolution equation in a form that is similar to the continuous formulation (1.1)–(1.2). We write (2.3) and (2.4) as

$$\begin{aligned} &\partial_t s^h - \nabla \cdot \left(k_1(s^h, x)(\nabla \tilde{p}_1^h + g_1)\right) \\ &\quad = -\Psi_h^\tau(p_c^h(s^h, x) - p_1^h + p_2^h) + \Psi_h^\tau(p_c(s^h, x) - p_1^h + p_2^h), \end{aligned} \tag{2.18}$$

and write (2.3) and (2.5) as

$$\begin{aligned} &-\partial_t s^h - \nabla \cdot \left(k_2(s^h, x)(\nabla \tilde{p}_2^h + g_2)\right) \\ &\quad = \Psi_h^\tau(p_c^h(s^h, x) - p_1^h + p_2^h) - \Psi_h^\tau(p_c(s^h, x) - p_1^h + p_2^h). \end{aligned} \tag{2.19}$$

We multiply (2.18) with $\tilde{p}_1^h - p_{0,1}$ and (2.19) with $\tilde{p}_2^h - p_{0,2}$ and integrate over Ω. Summing up the resulting equations yields

$$\begin{aligned} &\int_\Omega \partial_t s^h (\tilde{p}^h - p_0)\,dx + \sum_{j=1}^{2} \int_\Omega k_j(s^h, x)(\nabla \tilde{p}_j^h + g_j)(\nabla \tilde{p}_j^h - \nabla p_{0,j})\,dx \\ &= -\int_\Omega \left[\Psi_h^\tau(p_c^h(s^h, x) - p^h) - \Psi_h^\tau(p_c(s^h, x) - p^h)\right](\tilde{p}^h - p_0)\,dx. \end{aligned} \tag{2.20}$$

The space derivatives on the left-hand side provide a positive term,

$$\int_\Omega k_j(s^h, x)(\nabla \tilde{p}_j^h + g_j)\nabla \tilde{p}_j^h \geqslant \frac{\kappa_j}{2}\|\nabla \tilde{p}_j^h\|^2_{L^2(\Omega)} - C_j^1\,,$$

where constant C_j^1, $j = 1, 2$, depends on the bounds κ_j and κ_j^0 of permeabilities and the gravity vectors g_j.

The time derivative on the left-hand side of (2.20) is treated with the hysteresis differential equation (2.7), which reads

$$\Phi_h^0(\partial_t s^h(x,t)) + \tau \partial_t s^h(x,t) + p_c^h(s^h, x) \ni p^h(x,t).$$

Using the monotonicity $\Phi_h^0(\xi)\xi \geqslant 0$ for all $\xi \in \mathbb{R}$, we can calculate with the primitive $P_c^h(\cdot, x)$ of $p_c^h(\cdot, x)$, exploiting that P_h is an $L^2(\Omega)$-orthogonal projection,

$$\int_\Omega \partial_t s^h \tilde{p}^h = \int_\Omega \partial_t s^h p^h \geqslant \int_\Omega \tau |\partial_t s^h|^2 + p_c^h(s^h)\partial_t s^h = \tau \|\partial_t s^h\|^2_{L^2(\Omega)} + \partial_t \int_\Omega P_c^h(s^h, x)\,.$$

We have assumed that P_c and thus P_c^h can be chosen as positive functions. We have therefore recognized three relevant positive terms on the left-hand side of (2.20).

Concerning the remaining integrals on the left-hand side of (2.20) we calculate

$$\left|\int_\Omega \partial_t s^h p_0\right| + \sum_{j=1}^2 \left|\int_\Omega k_j(s^h)(\nabla \tilde{p}_j^h + g_j)\nabla p_{0,j}\right|$$
$$\leqslant \frac{\tau}{2}\|\partial_t s^h\|^2_{L^2(\Omega)} + \frac{\kappa_1}{4}\|\nabla \tilde{p}_1^h\|^2_{L^2(\Omega)} + \frac{\kappa_2}{4}\|\nabla \tilde{p}_2^h\|^2_{L^2(\Omega)} + C_1^2 + C_2^2,$$

where the constant C_j^2, $j = 1, 2$, depends on κ_j and κ_j^0 on the data $p_{0,j}$ (directly and through $p_0 = p_{0,1} - p_{0,2}$) and the gravity vectors g_j.

It remains to treat the term on the right-hand side of (2.20). Exploiting the Lipschitz continuity of p_c in x with constant ρ we find

$$\left|\left[\Psi_h^\tau(p_c^h(x, s^h) - p^h) - \Psi_h^\tau(p_c(x, s^h) - p^h)\right]\right| \leqslant \frac{\rho}{\tau} h,$$

and the corresponding product is treated with the Cauchy–Schwarz inequality.

Summarizing, we find

$$\frac{\kappa_1}{4}\|\nabla \tilde{p}_1^h\|^2_{L^2(\Omega)} + \frac{\kappa_2}{4}\|\nabla \tilde{p}_2^h\|^2_{L^2(\Omega)} + \frac{\tau}{2}\|\partial_t s^h\|^2_{L^2(\Omega)} + \partial_t \int_\Omega P_c^h(s^h, x)\, dx$$
$$\leqslant C_1^1 + C_2^1 + C_1^2 + C_2^2 + \frac{\rho h}{\tau}(\|\tilde{p}^h\|^2_{L^2(\Omega)} + \|p_0\|^2_{L^2(\Omega)}).$$

For sufficiently small h, depending only on coefficient and boundary data, we can absorb the term containing $\|\tilde{p}^h\|^2_{L^2(\Omega)}$. An integration over $(0, T)$ provides the estimate (2.17). The constant C depends on the coefficient and boundary data and, in addition, on the L^2-norm of the initial data, $\|s_0\|_{L^2(\Omega)}$, since the integral $\int_\Omega P_c^h(P_h s_0)$ enters the estimate. □

2.4 Compactness

We are now in a position to apply a compactness result that has been developed in the context of the Richards equation. Lemma 3.3 of [17] concludes from uniform estimates for the family $\tilde{p}^h$ and the hysteresis relation for s^h an $L^2(\Omega_T)$-compactness result for the sequence s^h. Here, we repeat the precise statement and recall that the Lipschitz continuity of p_c was also assumed in [17].

Lemma 2.5 ([17, Lemma 3.3], regularity and compactness from the hysteresis relation) *Let s^h and $\tilde{p}^h$ satisfy the ordinary differential equation of the hysteresis relation*

$$\partial_t s^h(x_k, t) = -\Psi_{\delta,h}^\tau(p_c(x_k, s^h) - p^h(x_k, t)) \qquad \forall x_k \in \Omega_h$$
$$s^h(x_k, 0) = P_h s_0(x_k)$$

for $p^h = P_h \tilde{p}^h$. Let $q \in [1, \infty]$ be a number and let $s_0 \in L^q(\Omega)$ define initial values. Then there holds an estimate

$$\|\partial_t s^h\|_{L^2(0,T;L^q(\Omega))} + \|s^h\|_{L^2(0,T;L^q(\Omega))} \leqslant C\|\tilde{p}^h\|_{L^2(0,T;L^q(\Omega))}, \tag{2.21}$$

where the constant C does not depend on h and δ.

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Let, in addition, the following estimate holds with C independent of h and δ,

$$\|\tilde{p}^h\|_{L^2(0,T;H^1(\Omega))} \leqslant C. \tag{2.22}$$

Then the family s^h is pre-compact in the space $L^2(\Omega \times (0,T))$.

Regarding our application of the lemma, we remark the following: (i) We use Lemma 2.5 with $\delta = 0$. The non-linear function $\Psi^\tau_{\delta,h} = \Psi^\tau_{0,h}$ of Lemma 2.5 then coincides with our function Ψ^τ_h; (ii) our non-linear function p_c satisfies the conditions that have been imposed for Lemma 2.5; (iii) the projection P_h is as in the present contribution; (iv) we apply Lemma 2.5 to $p^h := p_1^h - p_2^h$ and $\tilde{p}^h := \tilde{p}_1^h - \tilde{p}_2^h$, emphasizing that, by linearity, $P_h\tilde{p}^h = p^h$ is also satisfied. With this setting, the hysteresis equation of the lemma is identical to our equation (2.3).

Lemma 2.6 (Compactness of the family of saturations) *Let the coefficient functions, initial and boundary data be as in Lemma 2.2 and $s_0 \in L^2(\Omega)$. Let $\tilde{p}_1^h, \tilde{p}_2^h, s^h$ be a family of solutions to the Galerkin scheme of Definition 2.1 for a sequence $h \to 0$. Then*

$$\text{the sequence } s^h \text{ is pre-compact in } L^2(\Omega \times (0,T)). \tag{2.23}$$

Proof By the above remarks, we can apply Lemma 2.5 to the pressure difference $\tilde{p}^h := \tilde{p}_1^h - \tilde{p}_2^h$. We use the integrability exponent $q = 2$. The *a priori* estimates of Lemma 2.4 provide the boundedness

$$\|\tilde{p}^h\|^2_{L^2(0,T;H^1(\Omega))} \leqslant 2(\|\tilde{p}_1^h\|^2_{L^2(0,T;H^1(\Omega))} + \|\tilde{p}_2^h\|^2_{L^2(0,T;H^1(\Omega))}) \leqslant C$$

with C independent of h. This shows that (2.22) is satisfied. We can apply the second part of Lemma 2.5 and conclude (2.23). □

2.5 Limit procedure $h \to 0$

We now consider limit functions to the solution sequence $(s^h, \tilde{p}_1^h, \tilde{p}_2^h)$ for $h \to 0$. Due to the uniform estimates of Lemma 2.4, we find a subsequence $h \to 0$ and limit functions s, p_1, p_2 such that

$$\tilde{p}_j^h \rightharpoonup p_j \quad \text{in} \quad L^2(0,T;H^1(\Omega)) \text{ for } j = 1,2, \tag{2.24}$$

$$s^h \rightharpoonup s,\ \partial_t s^h \rightharpoonup \partial_t s \quad \text{in} \quad L^2(0,T;L^2(\Omega)). \tag{2.25}$$

Furthermore, by the compactness result of (2.23), we find the strong convergence

$$s^h \to s \quad \text{in} \quad L^2(0,T;L^2(\Omega)). \tag{2.26}$$

The following lemma concludes the proof of Theorem 1.1.

Lemma 2.7 *Let the data be as in Lemma 2.2 and $s_0 \in L^2(\Omega)$. Let $\tilde{p}_1^h, \tilde{p}_2^h, s^h$ be a family of solutions to the Galerkin scheme of Definition 2.1 for a sequence $h \to 0$. For a subsequence,* *let p_1, p_2, and s be limit functions as in* (2.24)–(2.26)*. Then the limit triple* $(s, p_1, p_2) \in L^2(0, T; L^2(\Omega)) \times L^2(0, T; H^1(\Omega, \mathbb{R}^2))$ *is a variational weak solution according to Definition 1.2.*

Proof The limit functions are contained in the function spaces as described in (1.11). The weak convergence allows us to take limits in the initial and boundary conditions, hence they are satisfied by the limit functions. We have to check the three items of Definition 1.2. In the calculations below we use once more $p^h = p_1^h - p_2^h$, $\tilde{p}^h = \tilde{p}_1^h - \tilde{p}_2^h$, and $p_0 = p_{0,1} - p_{0,2}$.

Item 1. We have to show that (1.1)–(1.2), i.e. $\partial_t s = \nabla \cdot (k_1(s)[\nabla p + g_1])$ and $\partial_t s = -\nabla \cdot (k_2(s)[\nabla p + g_2])$ are satisfied in the weak sense with no-flux boundary conditions. To verify these equations, it suffices to consider test functions $\varphi_j \in C_c^\infty\left((0, T) \times (\Omega \cup \Gamma_j)\right)$. We start from the ordinary differential equation (2.3) of the Galerkin scheme and the elliptic equations (2.4) to write

$$\int_0^T \int_\Omega \partial_t s^h \varphi_1 \, dx \, dt + \int_0^T \int_\Omega k_1(s^h, x)(\nabla \tilde{p}_1^h + g_1)\nabla\varphi_1 \, dx \, dt$$
$$= -\int_0^T \int_\Omega \left[\Psi_h^\tau(p_c^h(s^h, x) - p^h) - \Psi_h^\tau(p_c(s^h, x) - p^h)\right] \varphi_1 \, dx \, dt.$$

As seen already in the proof of Lemma 2.4, the Lipschitz continuity of p_c guarantees that the right-hand side tends to be zero as $h \to 0$. On the left-hand side we can pass to the limit functions, thanks to (2.24)–(2.26). This shows that the evolution equation (1.1) holds with the no-flux condition.

Equation (2.3) of the Galerkin scheme can also be combined with (2.5), tested with φ_2. The result is a relation similar to the above, but expressing $\partial_t s^h$ in terms of k_2 and $\tilde{p}_2$. The limit can be performed in the same way and provides the second evolution equation (1.2).

Item 2. We want to show the pointwise inclusion for $p^h = p_1^h - p_2^h$ as demanded in item 2 of Definition 1.2. The discrete hysteresis system (2.7) provides, pointwise in $\Omega \times (0, T)$,

$$[-\gamma^h(x), \gamma^h(x)] \ni p^h(x,t) - p_c^h(s^h(x,t), x) - \tau\partial_t s^h(x,t) \tag{2.27}$$

for almost every $x \in \Omega$ and $t \in (0, T)$. Introducing small error terms, we write this relation as

$$\begin{aligned}[-\gamma(x), \gamma(x)] \ni\ & \tilde{p}^h(x,t) - p_c(s^h(x,t), x) - \tau\partial_t s^h(x,t) \\ & + \left(p^h - \tilde{p}^h\right)(x,t) + \left(p_c(s^h(x,t), x) - p_c^h(s^h(x,t), x)\right) + r^h(x,t),\end{aligned} \tag{2.28}$$

where the error term $r^h(x,t)$ concerns the replacement of γ by γ^h and satisfies $|r^h(x,t)| \leqslant |\gamma^h(x) - \gamma(x)| \leqslant Ch$ due to the Lipschitz continuity of γ in x. Similarly, the error introduced by $p_c(s^h) - p_c^h(s^h)$ is uniformly bounded by h.

Since $[-\gamma(x), \gamma(x)] \subset \mathbb{R}$ is a convex set, the set of functions $f : \Omega_T \to \mathbb{R}$ with $f(x,t) \in [-\gamma(x), \gamma(x)]$ is convex. As a convex and closed subset of $L^2(\Omega_T)$, it is also weakly closed. The right-hand side of (2.28) converges weakly in $L^2(\Omega_T)$ to $p - p_c(s) - \tau\partial_t s$, therefore this limit again satisfies the pointwise inclusion.

Item 3. We have to prove the variational inequality (1.12) for (s, p_1, p_2). To this end we multiply equation (2.4) with $\tilde{p}_1^h - p_{0,1}$ and equation (2.5) with $\tilde{p}_2^h - p_{0,2}$ and integrate over $(0, T) \times \Omega$. We find

$$\begin{aligned} 0 = &\int_0^T \int_\Omega k_1(s^h, x)(\nabla \tilde{p}_1^h + g_1)(\nabla \tilde{p}_1^h - \nabla p_{0,1})\, dx\, dt \\ &- \int_0^T \int_\Omega \Psi_h^\tau(p_c(s^h, x) - p_1^h + p_2^h)(\tilde{p}_1^h - p_{0,1})\, dx\, dt \end{aligned}$$

and

$$\begin{aligned} 0 = &\int_0^T \int_\Omega k_2(s^h, x)(\nabla \tilde{p}_2^h + g_2)(\nabla \tilde{p}_2^h - \nabla p_{0,2})\, dx\, dt \\ &+ \int_0^T \int_\Omega \Psi_h^\tau(p_c(s^h, x) - p_1^h + p_2^h)(\tilde{p}_2^h - p_{0,2})\, dx\, dt. \end{aligned}$$

Adding these two equations yields

$$\begin{aligned} 0 = &\sum_{j=1}^{2} \left[\int_0^T \int_\Omega k_j(s^h, x)(\nabla \tilde{p}_j^h + g_j)(\nabla \tilde{p}_j^h - \nabla p_{0,j}) \right] dx\, dt \\ &- \int_0^T \int_\Omega \Psi_h^\tau(p_c(s^h, x) - p^h)(\tilde{p}^h - p_0)\, dx\, dt. \end{aligned} \tag{2.29}$$

The rest of this proof consists in performing the limit $h \to 0$ in (2.29). The limiting relation will be the variational inequality (1.12).

We start with some lower order terms. The weak convergences (2.24) and (2.25) and the strong convergence (2.26) together with the continuity of k_j allow us to take the limits

$$\begin{aligned} -\int_0^T \int_\Omega k_j(s^h, x) \left(\nabla \tilde{p}_j{}^h + g_j\right) \nabla p_{0,j}\, dx\, dt &\to -\int_0^T \int_\Omega k_j(s, x) \left(\nabla p_j + g_j\right) \nabla p_{0,j}\, dx\, dt, \\ \int_0^T \int_\Omega k_j(s^h, x) g_j \nabla \tilde{p}_j{}^h\, dx\, dt &\to \int_0^T \int_\Omega k_j(s, x) g_j \nabla p_j\, dx\, dt. \end{aligned}$$

Concerning the quadratic term, we can use lower semi-continuity of the norm. Strong convergence of s^h together with the continuity of k_j, using an argument based on Egorov's Theorem, provides

$$\liminf_{h\to 0} \int_0^T \int_\Omega k_j(s^h, x) \left|\nabla \tilde{p}_j^h\right|^2\, dx\, dt \geqslant \int_0^T \int_\Omega k_j(s, x) \left|\nabla p_j\right|^2\, dx\, dt.$$

We finally consider the terms in (2.29) containing Ψ_h^τ. We exploit the Galerkin relation (2.3) to re-write the remaining term as

$$\begin{aligned} -\int_0^T \int_\Omega \Psi_h^\tau\big(p_c(s^h) - p^h\big)(\tilde{p}^h - p_0)\, dx\, dt = &\int_0^T \int_\Omega \partial_t s^h(\tilde{p}^h - p_0)\, dx\, dt \\ + \int_0^T \int_\Omega \left[\Psi_h^\tau(p_c^h(s^h) - p^h) - \Psi_h^\tau\big(p_c(s^h) - p^h\big)\right] (\tilde{p}^h - p_0)\, dx\, dt. \end{aligned} \tag{2.30}$$

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As noted before, the last integral of (2.30) tends to zero by the Lipschitz continuity of p_c in x. In the other integral, the convergence

$$-\int_0^T \int_\Omega \partial_t s^h \, p_0 \, dx \, dt \to -\int_0^T \int_\Omega \partial_t s \, p_0 \, dx \, dt$$

is an immediate consequence of the weak convergence of $\partial_t s^h$.

In the remaining integral on the right-hand side of (2.30) we use the hysteresis relation (2.7),

$$\begin{aligned}\int_0^T \int_\Omega \partial_t s^h \, \tilde{p}^h \, dx \, dt &= \int_0^T \int_\Omega \partial_t s^h \, p^h \, dx \, dt \\ &\in \int_0^T \int_\Omega \partial_t s^h \, \left(\Phi_h^0(\partial_t s^h) + \tau \partial_t s^h + p_c^h(s^h, x) \right) \, dx \, dt \\ &= \int_0^T \int_\Omega \gamma^h(x) \, |\partial_t s^h| + \tau |\partial_t s^h|^2 + \partial_t s^h \, p_c^h(s^h) \, dx \, dt.\end{aligned}$$

In the first two terms, the limit can be estimated by the weak lower semi-continuity of the L^2-norm,

$$\liminf_{h \to 0} \int_0^T \int_\Omega \gamma^h(x) \, |\partial_t s^h| + \tau \left| \partial_t s^h \right|^2 \, dx \, dt \geqslant \int_0^T \int_\Omega \gamma(x) \, |\partial_t s| + \tau \, |\partial_t s|^2 \, dx \, dt.$$

Here we exploited the uniform convergence $\gamma^h \to \gamma$ on Ω, which is a consequence of the Lipschitz continuity of γ.

The remaining integral over $\partial_t s^h p_c^h(s^h, x)$ is a total time derivative, but this fact is not needed here. We use the weak convergence of the first factor and the strong convergence of the second factor (note that p_c^h is Lipschitz continuous in s) to conclude

$$\int_0^T \int_\Omega \partial_t s^h p_c^h(x, s^h) \, dx \, dt \to \int_0^T \int_\Omega \partial_t s \, p_c(x, s) \, dx \, dt$$

as h tends to 0. With this, we have analyzed all limits of integrals on the right-hand side of (2.30), and thus of all integrals in (2.29). The variational inequality (1.12) is derived. □

Lemma 2.7 provides the existence of a variational weak solution. By Lemma 1.3, this variational weak solution is a solution of the original problem as described in Theorem 1.1. Therefore, the existence of a solution to the two-phase problem with hysteresis is shown.

3 Numerical treatment

We propose a straightforward generalization of the numerical scheme presented in [17]. We introduce Φ_δ^τ as a regularization of Φ^τ in (2.1) with a positive regularizing parameter δ, see (3.4), the inverse is denoted by $\Psi_\delta^\tau : \mathbb{R} \to \mathbb{R}$, see (3.5). Here we assume that $\gamma > 0$

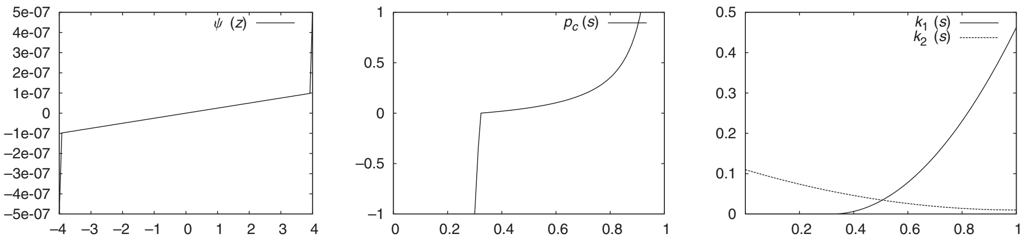

FIGURE 1. Plots of the functions Ψ_δ^τ, p_c, k_1, k_2 for parameters $\gamma = 4$, $\tau = 0.1$, $\delta = 10^{-7}$, $\kappa_1 = 10^{-4}$, $\kappa_2 = 10^{-2}$, $k_{0,1} = 1$, $k_{0,2} = 0.1$, $a_1 = 0.32$, $a_2 = 0$, $\alpha_+ = 0.1$, $\alpha_- = 50$, $\varepsilon_p = 10^{-10}$.

and $\tau > 0$ are constant. The two-phase system that we consider then reads

$$\partial_t s = \nabla \cdot (k_1(s)(\nabla p_1 + \varrho_1 e_n)), \quad (x,t) \in \Omega \times (t_0, T), \tag{3.1}$$

$$-\partial_t s = \nabla \cdot (k_2(s)(\nabla p_2 + \varrho_2 e_n)), \quad (x,t) \in \Omega \times (t_0, T), \tag{3.2}$$

$$\partial_t s = \Psi_\delta^\tau(p_1 - p_2 - p_c(s)), \quad (x,t) \in \Omega \times (t_0, T). \tag{3.3}$$

The functions Φ_δ^τ and Ψ_δ^τ read

$$\Phi_\delta^\tau = \Phi_\delta^\tau(\sigma) = \begin{cases} \gamma + \tau\sigma & \text{for} \quad \sigma > \delta, \\ (\frac{\gamma}{\delta} + \tau)\sigma & \text{for} \quad \sigma \in [-\delta, \delta], \\ -\gamma + \tau\sigma & \text{for} \quad \sigma < -\delta, \end{cases} \tag{3.4}$$

and

$$\Psi_\delta^\tau = \Psi_\delta^\tau(z) = \begin{cases} \dfrac{z - \gamma}{\tau} & \text{for} \quad z > \gamma + \tau\delta, \\ \left(\dfrac{\gamma}{\delta} + \tau\right)^{-1} z & \text{for} \quad z \in [-(\gamma + \tau\delta), \gamma + \tau\delta], \\ \frac{z+\gamma}{\tau} & \text{for} \quad z < -(\gamma + \tau\delta). \end{cases} \tag{3.5}$$

In all numerical experiments, we use the permeabilities

$$k_1(s) = \begin{cases} \kappa_1 & \text{for} \quad s < a_1, \\ \kappa_1 + k_{0,1}(s - a_1)^2 & \text{for} \quad s \geqslant a_1, \end{cases}$$

and

$$k_2(s) = \begin{cases} \kappa_2 & \text{for} \quad s > 1 - a_2, \\ \kappa_2 + k_{0,2}((1 - a_2) - s)^2 & \text{for} \quad s \leqslant 1 - a_2, \end{cases}$$

with $a_1, a_2 \in [0, 1]$ and $\kappa_1, \kappa_2, k_{0,1}, k_{0,2} > 0$. For p_c we assume the van Genuchten-type relation

$$p_c(s) = \begin{cases} \alpha_+ \left(\frac{1}{1 - a_2 - s + \varepsilon_p} - \frac{1}{1 - a_2 - a_1 + \varepsilon_p}\right) & \text{for} \quad s > a_1, \\ \alpha_-(s - a_1) \quad \text{for} \quad s \leqslant a_1 \end{cases}$$

with $\alpha_\pm \in \mathbb{R}$ and a small regularizing parameter $\varepsilon_p > 0$. For plots of the previously defined functions, see Figure 1.

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For the numerical results presented here, we consider domains $\Omega := (-L, L)^n \subset \mathbb{R}^n$ with $L > 0$. With the definition $\Gamma_\pm := \{x \in \overline{\Omega} : x_n = \pm L\} \subset \partial\Omega$ and given functions $p_{1,-}, j_{2,-} : \Gamma_- \times (t_0, T] \to \mathbb{R}$ and $j_{1,+}, p_{2,+} : \Gamma_+ \times (t_0, T] \to \mathbb{R}$, we assume the Dirichlet boundary conditions $p_1 = p_{1,-}$ for $x \in \Gamma_-$ and $p_2 = p_{2,+}$ for $x \in \Gamma_+$ and the Neumann boundary conditions

$$j_1 := -k_1(s)(\nabla p_1 + \varrho_1 e_n) \cdot \nu_+ = j_{1,+} \quad \text{for} \quad x \in \Gamma_+, \tag{3.6}$$

$$j_2 := -k_2(s)(\nabla p_2 + \varrho_2 e_n) \cdot \nu_- = j_{2,-} \quad \text{for} \quad x \in \Gamma_-, \tag{3.7}$$

$\nu_\pm = \pm e_n$ denoting the outer normals to $\Gamma_\pm$. In the lateral directions, i.e. for $x_i \in \{-L, L\}$, $i \in \{1, \dots, n-1\}$, we assume periodic pressures p_1 and p_2.

3.1 Discretization

The numerical approach is to discretize the Galerkin scheme of Definition 2.1 and to regard the corresponding set of equations (for every time-step) as one large system for saturation and pressures. We split the time interval $[t_0, T]$ by discrete time instants $t_0 < t_1 < \dots$, which leads to time-steps $\Delta t_m := t_{m+1} - t_m$, $m = 0, 1, \dots$. Moreover, for $m = 0, 1, \dots$ we introduce time-discrete solutions $s_i^{(m)}$ and $p_i^{(m)}$, $i = 1, 2$. Then all non-linear terms are linearized in a similar way as proposed in [17]: We approximate

$$\begin{aligned}\Psi(p_1^{(m+1)} - p_2^{(m+1)} - p_c(s^{(m+1)}))\\ \approx \Psi_{\mathrm{i}}(p_1^{(m+1)}, p_2^{(m+1)}, s^{(m+1)}, p_1^{(m)}, p_2^{(m)}, s^{(m)}) + \Psi_{\mathrm{e}}(p_1^{(m)}, p_2^{(m)}, s^{(m)}),\end{aligned}$$

where we drop δ and τ in the Ψ-notation and define

$$\Psi_{\mathrm{i}}(p_1^{(m+1)}, p_2^{(m+1)}, s^{(m+1)}, p_1^{(m)}, p_2^{(m)}, s^{(m)})$$
$$:= \begin{cases} \dfrac{p_1^{(m+1)} - p_2^{(m+1)} - p_c'(s^{(m)})s^{(m+1)}}{\tau} & \text{for} \quad p_1^{(m)} - p_2^{(m)} - p_c(s^{(m)}) > \gamma + \tau\delta, \\ \dfrac{p_1^{(m+1)} - p_2^{(m+1)} - p_c'(s^{(m)})s^{(m+1)}}{\gamma/\delta + \tau} & \text{for} \quad p_1^{(m)} - p_2^{(m)} - p_c(s^{(m)}) \in [-(\gamma + \tau\delta), \gamma + \tau\delta], \\ \dfrac{p_1^{(m+1)} - p_2^{(m+1)} - p_c'(s^{(m)})s^{(m+1)}}{\tau} & \text{for} \quad p_1^{(m)} - p_2^{(m+1)} - p_c(s^{(m)}) < -(\gamma + \tau\delta) \end{cases}$$

and

$$\Psi_{\mathrm{e}}(p_1^{(m)}, p_2^{(m)}, s^{(m)})$$
$$:= \begin{cases} \dfrac{-\gamma - p_c(s^{(m)}) + p_c'(s^{(m)})s^{(m)}}{\tau} & \text{for} \quad p_1^{(m)} - p_2^{(m)} - p_c(s^{(m)}) > \gamma + \tau\delta, \\ \dfrac{-p_c(s^{(m)}) + p_c'(s^{(m)})s^{(m)}}{\gamma/\delta + \tau} & \text{for} \quad p_1^{(m)} - p_2^{(m)} - p_c(s^{(m)}) \in [-(\gamma + \tau\delta), \gamma + \tau\delta], \\ \dfrac{\gamma - p_c(s^{(m)}) + p_c'(s^{(m)})s^{(m)}}{\tau} & \text{for} \quad p_1^{(m)} - p_2^{(m+1)} - p_c(s^{(m)}) < -(\gamma + \tau\delta). \end{cases}$$

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Furthermore, for $i = 1, 2$ we linearize

$$k_i(s^{(m+1)}) \approx k_i'(s^{(m)})s^{(m+1)} + k_i(s^{(m)}) - k_i'(s^{(m)})s^{(m)}.$$

Then from system (3.1)–(3.3) we obtain the semi-implicit Euler scheme

$$\begin{aligned}
\frac{s^{(m+1)} - s^{(m)}}{\Delta t_m} &= \nabla \cdot (k_1(s^{(m)})\nabla p_1^{(m+1)}) + \nabla \cdot (k_1'(s^{(m)})s^{(m+1)}\varrho_1 e_n) \\
&\quad + \nabla \cdot ((k_1(s^{(m)}) - k_1'(s^{(m)})s^{(m)})\varrho_1 e_n) \quad \text{in} \quad \Omega, \\
-\frac{s^{(m+1)} - s^{(m)}}{\Delta t_m} &= \nabla \cdot (k_2(s^{(m)})\nabla p_2^{(m+1)}) + \nabla \cdot (k_2'(s^{(m)})s^{(m+1)}\varrho_2 e_n) \\
&\quad + \nabla \cdot ((k_2(s^{(m)}) - k_2'(s^{(m)})s^{(m)})\varrho_2 e_n) \quad \text{in} \quad \Omega, \\
\frac{s^{(m+1)} - s^{(m)}}{\Delta t_m} &= \Psi_{\mathrm{i}}(p_1^{(m+1)}, p_2^{(m+1)}, s^{(m+1)}, p_1^{(m)}, p_2^{(m)}, s^{(m)}) \\
&\quad + \Psi_{\mathrm{e}}(p_1^{(m)}, p_2^{(m)}, s^{(m)}) \quad \text{in} \quad \Omega
\end{aligned}$$

for $m = 0, 1, \ldots$.

In space, we apply linear finite elements to discretize the above system. We apply adaptivity in space with an L^2-like error indicator for the discrete saturation based on the jump residual as discussed in [17]. Furthermore, we use a simple adaptive strategy in time, where the time-step Δt_m is inversely proportional to the maximum of the discrete time derivative of saturation [17]. We present numerical examples which show the validity of the algorithm implemented in the FEM toolbox AMDiS [29]. The resulting linear system of equations is solved by a direct solver for sparse linear systems (UMFPACK; [15]).

3.2 Numerical results

For the numerical results presented in the following, we have used a time-dependent boundary flux

$$j_{1,+} = \begin{cases} j_{1,+}^0 & \text{for} \quad t < t_s, \\ j_{1,+}^s & \text{for} \quad t \geqslant t_s, \end{cases}$$

with $t_s > t_0$ and $j_{1,+}^0, j_{1,+}^s \in \mathbb{R}$. The change in the upper boundary condition at a switching time t_s is important in the modelling of fingering experiments, see [26]. On the lower boundary, we assume the constant flux,

$$j_{2,-} \equiv j_{2,-}^0$$

for a given value $j_{2,-}^0 \in \mathbb{R}$. The Dirichlet boundary conditions are

$$p_{1,-} \equiv \gamma - \alpha_- a_1, \quad p_{2,+} \equiv p_{2,+}^0$$

with a constant pressure $p_{2,+}^0$. We study perturbations of the initial condition $s = 0$ of the form

$$s_0(x) = \sum_{i=1}^{10} A_i(1 - \tanh(3(|x - x_{0,i}| - 1/2))), \tag{3.8}$$

Table 1. *Parameters used for numerical results*

Parameter	γ	κ_1	κ_2	$k_{0,1}$	$k_{0,2}$	δ	τ	a_1	a_2	
Value	4	10^{-4}	10^{-2}	1	0.1	10^{-7}	0.1; 0.5	0.32	0	
Parameter	ϱ_1	ϱ_2	ε_p	α_+	α_-	$j^0_{1,+}$	$j^s_{1,+}$	$p^0_{2,+}$	$j^0_{2,-}$	L
Value	1	0	10^{-10}	0.1	50	0.524	0.01	0	0	24

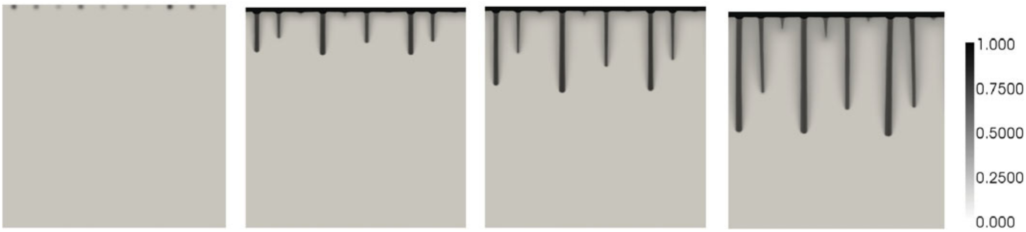

FIGURE 2. (Colour online) Discrete saturation for $\tau = 0.5$ at times $t = -2$, $t \approx 106$, $t \approx 205$ and $t \approx 421$.

where $x_{0,i} = (-L + \frac{(2i-1)}{10}L, 23.5)$, $i = 1, \ldots, 10$ and amplitudes $A_1 = 0.6$, $A_2 = 0.4$, $A_3 = 0.2$, $A_4 = 0.5$, $A_5 = 0.2$, $A_6 = 0.3$, $A_7 = 0.1$, $A_8 = 0.7$, $A_9 = 0.5$, $A_{10} = 0.1$. We provide a list of all other parameters in Table 1.

In the following, we will compare numerical results for the two-phase system with results for the hysteresis Richards system which is solved by the numerical method described in [17]. In order to compare solutions at similar times t, we have chosen a time-dependent parameter,

$$\kappa_2 = \kappa_2(t) = \begin{cases} 10^5 & \text{for} \quad t < t_s, \\ 10^{-2} & \text{for} \quad t \geqslant t_s. \end{cases}$$

Formally, in the limit $\kappa_2 \to \infty$, the two-phase system reduces to the hysteresis Richards model treated in [17]. The above choice of κ_2 has the effect that the discrete saturation fields for the two models are almost identical at time $t = t_s$. We emphasize that, also after the switching time t_s, in regions of low saturation, the permeability of the second fluid is much larger than the permeability of the first fluid, see Figure 1.

3.2.1 *First results for two-phase flow evolution*

Numerical results for the evolution of fingers in the two-phase flow model are presented in Figure 2. The grey-scale picture indicates the saturation s_h for the parameter $\tau = 0.5$ and the initial condition (3.8). From the perturbations, fingers start to grow and evolve basically into the direction of gravity.

We observe that fingers can develop in the two-phase flow system with hysteresis. Some fingers cease to grow after some time, the surviving fingers have a comparable length, but there are differences in width and length. We also observe that fingers can repel each other, we refer to the last two long fingers in Figure 2.

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Table 2. *Description of the data set generating four different solutions*

Two-phase flow, $\tau = 0.5$ Figure 3, left	Richards equation, $\tau = 0.5$ Figure 3, middle
Two-phase flow, $\tau = 0.1$ Figure 4, left	Richards equation, $\tau = 0.1$ Figure 4, middle

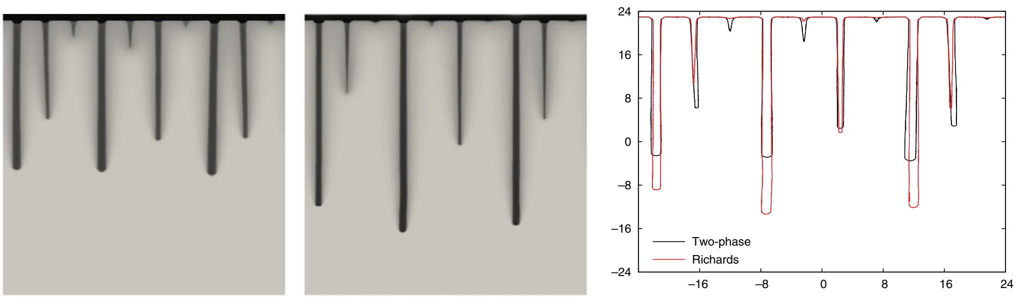

FIGURE 3. (Colour online) Discrete saturation for two-phase flow (left) and Richards equation (middle) at time, $t \approx 421$, plot of level sets $\{s_h = \frac{1}{2}\}$ (right, black: two-phase; red/grey: Richards equation). In both cases $\tau = 0.5$ has been used.

We recall that the hysteresis terms contribute only in an indirect way to the evolution of fingers. Below the highly wetted top layer of the domain, the system is in an imbibition process, hence the static hysteresis is not of relevance in that region. In contrast, in the top layer of the domain, the effect of the hysteresis terms is that the variations of the saturation distribution cannot be removed during the evolution.

3.2.2 *Comparison with the Richards equation*

Our aim is to compare numerically the two-phase flow equations with the Richards equation. More precisely, we investigate if the presence of the second fluid has an effect that can be compared with the effect of variations in the time-delay parameter τ. For this study, we compare four different solutions as indicated in Table 2.

Three qualitative observations can be made. First, the deviations of growth directions from the direction of gravity are present for several fingers in the case of two-phase flow, but not for the Richards equation. Second, for the two-phase flow system, fingers tend to be thicker. Third, long fingers are shorter and short fingers are longer if compared with the results for the Richards equation. These differences are further illustrated in the right part of Figure 3, where level sets $\{s_h = \frac{1}{2}\}$ are displayed in one plot for both cases.

A similar comparison is shown in Figure 4 for $\tau = 0.1$. Basically, the same qualitative differences between the two-phase flow and the Richards equation are visible. In addition, in both cases, reduction of τ corresponds to reduction of the thickness of fingers.

We conclude with the observation that in the parameter setup investigated here, the influence of including the second phase in the model is not comparable with the influence of reducing parameter τ.

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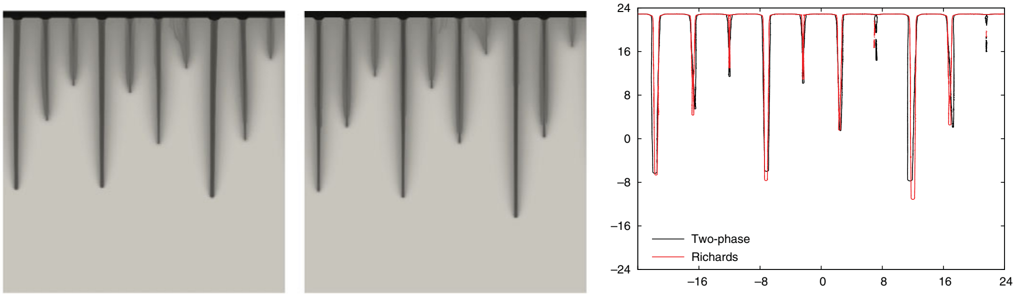

Figure 4. (Colour online) Discrete saturation for two-phase flow (left) and Richards equation (middle) at times $t \approx 529$ and $t \approx 528$, respectively, plot of level sets $\{s_h = \frac{1}{2}\}$ (right, black: two-phase; red/grey: Richards equation). In both cases, $\tau = 0.1$ has been used.

4 Conclusion

We have analyzed a hysteresis model for two-phase flow in porous media. The model extends a standard physical model from the Richards equation to the two-phase flow system. The existence of a solution and the convergence of a finite element scheme are shown. Numerical results show the appearance of gravity fingers in a perfectly homogeneous medium. The fingering effect is comparable to the corresponding situation in the Richards equation, but we obtain qualitative differences, e.g. that fingers can be weakly repelling in the two-phase flow evolution.

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References

[1] Alt, H. W. & DiBenedetto, E. (1985) Nonsteady flow of water and oil through inhomogeneous porous media. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4th series) **12**(3), 335–392.
[2] Alt, H. W. & Luckhaus, S. (1983) Quasilinear elliptic-parabolic differential equations. *Math. Z.* **183**(3), 311–341.
[3] Alt, H. W., Luckhaus, S. & Visintin, A. (1984) On nonstationary flow through porous media. *Ann. Mat. Pura Appl.* **136**(4), 303–316.
[4] Arbogast, T., Wheeler, M. F. & Zhang, N.-Y. (1996) A nonlinear mixed finite element method for a degenerate parabolic equation arising in flow in porous media. *SIAM J. Numer. Anal.* **33**(4), 1669–1687.
[5] Bagagiolo, F. & Visintin, A. (2000) Hysteresis in filtration through porous media. *Z. Anal. Anwendungen* **19**(4), 977–997.
[6] Bagagiolo, F. & Visintin, A. (2004) Porous media filtration with hysteresis. *Adv. Math. Sci. Appl.* **14**(2), 379–403.
[7] Bastian, P. & Helmig, R. (1999) Efficient fully coupled solution techniques for two-phase flow in porous media. *Adv. Water Resour.* **23**(3), 199–216.
[8] Beliaev, A. Y. & Hassanizadeh, S. M. (2001) A theoretical model of hysteresis and dynamic effects in the capillary relation for two-phase flow in porous media. *Transp. Porous Media* **43**(3), 487–510.

[9] Buzzi, F., Lenzinger, M. & Schweizer, B. (2009) Interface conditions for degenerate two-phase flow equations in one space dimension. *Analysis* (Munich) **29**(3), 299–316.

[10] Cancès, C., Choquet, C., Fan, Y. & Pop, I. (2010) *Existence of Weak Solutions to a Degenerate Pseudo-Parabolic Equation Modeling Two-Phase Flow in Porous Media*. Technical Report, Eindhoven University of Technology, CASA.

[11] Cancès, C. & Pierre, M. (2011) An existence result for multidimensional immiscible two-phase flows with discontinuous capillary pressure field. HAL: hal-00518219.

[12] Carrillo, J. (1994) On the uniqueness of the solution of the evolution dam problem. *Nonlinear Anal.* **22**(5), 573–607.

[13] Chen, Z. (2001) Degenerate two-phase incompressible flow. I. Existence, uniqueness and regularity of a weak solution. *J. Differ. Equ.* **171**(2), 203–232.

[14] Chen, Z. (2002) Degenerate two-phase incompressible flow. II. Regularity, stability and stabilization. *J. Differ. Equ.* **186**(2), 345–376.

[15] Davis, T. A. (2004) Algorithm 832: UMFPACK V4.3—an unsymmetric-pattern multifrontal method. *ACM Trans. Math. Softw.* **30**(2), 196–199.

[16] Kröner, D. & Luckhaus, S. (1984) Flow of oil and water in a porous medium. *J. Differ. Equ.* **55**(2), 276–288.

[17] Lamacz, A., Rätz, A. & Schweizer, B. (2011) A well-posed hysteresis model for flows in porous media and applications to fingering effects. *Adv. Math. Sci. Appl.* **21**(1), 33–64.

[18] Lenzinger, M. & Schweizer, B. (2010) Two-phase flow equations with outflow boundary conditions in the hydrophobic-hydrophilic case. *Nonlinear Anal.* **73**(4), 840–853.

[19] Mikelić, A. (2010) A global existence result for the equations describing unsaturated flow in porous media with dynamic capillary pressure. *J. Differ. Equ.* **248**(6), 1561–1577.

[20] Otto, F. (1997) L^1-contraction and uniqueness for unstationary saturated-unsaturated porous media flow. *Adv. Math. Sci. Appl.* **7**(2), 537–553.

[21] Radu, F. & Pop, I. S. (2011) Mixed finite element discretization and Newton iteration for a reactive contaminant transport model with non-equilibrium sorption: Convergence analysis and error estimates. *Comput. Geosci.* **15**(3), 431–450.

[22] Radu, F., Pop, I. S. & Knabner, P. (2004) Order of convergence estimates for an Euler implicit, mixed finite element discretization of Richards' equation. *SIAM J. Numer. Anal.* **42**(4), 1452–1478 (electronic).

[23] Schweizer, B. (2007) Averaging of flows with capillary hysteresis in stochastic porous media. *European J. Appl. Math.* **18**(3), 389–415.

[24] Schweizer, B. (2007) Regularization of outflow problems in unsaturated porous media with dry regions. *J. Differ. Equ.* **237**(2), 278–306.

[25] Schweizer, B. (2012a) The Richards equation with hysteresis and degenerate capillary pressure. *J. Differ. Equ.* **252**(10), 5594–5612.

[26] Schweizer, B. (2012b) Instability of gravity wetting fronts for Richards equations with hysteresis. *Interfaces Free Boundaries* **14**, 37–64.

[27] Selker, J. S., Parlange, J.-Y. & Steenhuis, T. S. (1992) Fingered flow in two dimensions. Part 2. Predicting finger moisture profile. *Wat. Resour. Res.* **28**(9), 2523–2528.

[28] van Duijn, C. J., Pieters, G. J. M. & Raats, P. A. C. (2004) Steady flows in unsaturated soils are stable. *Transp. Porous Media* **57**(2), 215–244.

[29] Vey, S. & Voigt, A. (2007) AMDiS – adaptive multidimensional simulations. *Comput. Visual. Sci.* **10**, 57–67.

[30] Visintin, A. (1994) *Differential Models of Hysteresis.* Applied Mathematical Sciences, Vol. 111, Springer-Verlag, Berlin, Germany.

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