The Journal of Symbolic Logic
Volume 79, Number 3, September 2014

SHEAF RECURSION AND A SEPARATION THEOREM

NATHANAEL LEEDOM ACKERMAN

Abstract. Define a second order tree to be a map between trees (with fixed codomain). We show that many properties of ordinary trees have analogs for second order trees. In particular, we show that there is a notion of "definition by recursion on a well-founded second order tree" which generalizes "definition by transfinite recursion". We then use this new notion of definition by recursion to prove an analog of Lusin's Separation theorem for closure spaces of global sections of a second order tree.

§1. Introduction. The concept of a tree is ubiquitous in mathematics and has several different formulations. This paper is motivated by three concepts all of which give rise to categories equivalent to a category of trees. First, there is the notion of a tree as a partial order with a least element such that the collection of predecessors of any element is finite and linearly ordered. Second, there is a topological space $\tilde{\mathbb{N}}$ such that the notion of a tree is equivalent to being a nonempty separated presheaf on $\tilde{\mathbb{N}}$ with no global sections. Finally, there is the notion of sheaves on $\tilde{\mathbb{N}}$. In Section 2 we show that the categories associated with each of these concepts are all equivalent.

In Section 3 we introduce the main objects studied in this paper, which we call *second order trees* (over a tree $\mathcal{T}$). Like ordinary (or *first order*) trees, second order trees (over $\mathcal{T}$) can be described in several ways. In this paper we consider three such descriptions; as a map of first order trees $\mathcal{S} : \mathcal{T}_0 \to \mathcal{T}$, as a nonempty separated presheaf of a specific type over a topological space $\tilde{\mathcal{T}}$ obtained from $\mathcal{T}$, and as a sheaf over $\tilde{\mathcal{T}}$. There is a category associated with each of these descriptions of second order trees (over $\mathcal{T}$) and in Section 3.1 we show all of these categories are equivalent. In Sections 3.2–3.4 we show that several concepts defined for first order trees have analogs for second order trees. These include the notion of well-foundedness, the notion of pruned, as well as a closure space of global sections where each closed set is the collection of global sections of some sub (second order) tree.

In Section 4 we show one of the main results of this paper, that transfinite recursive definitions on well-founded trees can be generalized to well-founded second order trees. We call the result of this generalization a *sheaf recursive definition*. We also show, in Section 4.3, the statement that "each sheaf recursive definition yields an associated partial function" is equivalent to the axiom of choice. We end this paper in

Received February 20, 2013.
2010 *Mathematics Subject Classification.* 06A99, 18F20, 54A05, 54H05.
Key words and phrases. Trees, Sheaves, Definition by Recursion, Closure Spaces, Separation Theorems.

0022-4812/14/7903-0013/$3.40
DOI:10.1017/jsl.2013.18

Section 5 by using sheaf recursion to prove a generalization of the Lusin separation theorem.

The results of this paper show many concepts associated with first order trees have analogs for second order trees. This suggests that many other such concepts should also have analogs for second order trees. In particular, understanding precisely how and when concepts related to first order trees generalize to second order trees should be a fruitful line of research.

1.1. Background. Unless otherwise specified we will work inside a background model of Zermelo-Fraenkel set theory with the Axiom of Choice. All categories in this paper will be locally small.

For each topological space $\mathcal{A}$, $\mathcal{O}(\mathcal{A})$ is the collection of open sets of $\mathcal{A}$. In this paper $\mathcal{P}$ will always be a topological space with underlying set P. There are a few categories related to topological spaces which will play an important role in this paper and which we give names to here. We let $\mathrm{Sep}^+(\mathcal{P})$ be the category of nonempty separated presheaves on $\mathcal{P}$ and we let $\mathrm{Sh}(\mathcal{P})$ be the category of sheaves on $\mathcal{P}$. Note that for each object $\mathcal{X}$ of $\mathrm{Sep}^+(\mathcal{P})$, $\mathcal{X}(\emptyset)$ has exactly one element. We will call this element $*_{\mathcal{X}}$. We say that an object $\mathcal{X}$ of $\mathrm{Sep}^+(\mathcal{P})$ is **trivial** if $*_{\mathcal{X}}$ is its only element, i.e., if $(\forall U \in \mathcal{O}(\mathcal{P}) - \{\emptyset\})\ \mathcal{X}(U) = \emptyset$.

Recall a *lower set* of a lattice is any subset which is downward closed. We say $B \subseteq \mathcal{O}(\mathcal{P})$ is a **lower set** if it is a lower set in the lattice $(\mathcal{O}(\mathcal{P}), \subseteq)$. If $B \subseteq \mathcal{O}(\mathcal{P})$ is a nonempty lower set, then we let $\mathrm{Sep}^+(B)$ be the full subcategory of $\mathrm{Sep}^+(\mathcal{P})$ consisting of those $\mathcal{X}$, where $\mathcal{X}(U) = \emptyset$ if $U \notin B$. We will make the slightly nonstandard assumption that all presheaves $\mathcal{X}$ on $\mathcal{P}$ are such that $\mathcal{X}(U) \cap \mathcal{X}(V) = \emptyset$ whenever $U, V \in \mathcal{O}(\mathcal{P})$ with $U \neq V$. This is done purely to simplify notation and there will be no loss of generality with this assumption.

If $U, V \in \mathcal{O}(\mathcal{P})$ with $U \subseteq V$ and $\mathcal{X}$ is a presheaf over $\mathcal{P}$ with $x \in \mathcal{X}(V)$, then we denote by $x|_U$ the restriction of x to U. If further $V \subseteq W$ we let x^e_W denote the set $\{y \in \mathcal{X}(W) : y|_V = x\}$, i.e., all those elements of $\mathcal{X}(W)$ whose restriction to V is x.

There are also several functors which will play an important role in this paper and which we will now describe. Let $\imath_{\mathcal{P}} : \mathrm{Sh}(\mathcal{P}) \to \mathrm{Sep}^+(\mathcal{P})$ be the inclusion functor and let $\mathbf{a}_{\mathcal{P}} : \mathrm{Sep}^+(\mathcal{P}) \to \mathrm{Sh}(\mathcal{P})$ be the sheafification functor, i.e., the left adjoint of $\imath_{\mathcal{P}}$. For any nonempty lower set $B \subseteq \mathcal{O}(\mathcal{P})$, we let $\imath_B : \mathrm{Sep}^+(B) \to \mathrm{Sep}^+(\mathcal{P})$ be the inclusion map. We also let $\mathbf{b}_B : \mathrm{Sep}^+(\mathcal{P}) \to \mathrm{Sep}^+(B)$ be the functor where, for nonempty separated presheaves $\mathcal{X}$, $\mathbf{b}_B(\mathcal{X})(U)$ is $\mathcal{X}(U)$ if $U \in B$, and $\emptyset$ otherwise (with $\mathbf{b}_B$ doing the obvious thing on morphisms). Note $\imath_B$ is left adjoint to $\mathbf{b}_B$. We will omit subscripts on these functors when they are clear from the context.

If L is a first order language and Th is a sentence of $\mathcal{L}_{\infty,\omega}(L)$ we let $\mathrm{Mod}_L(Th)$ be the full subcategory of L-structures and homomorphisms consisting of those L-structures which satisfy Th. When I is a partial function we use the notation $I(x)\downarrow$ to mean "$I(x)$ is defined" and $I(x)\uparrow$ to mean "$I(x)$ is undefined". If X is a set we denote by $\mathfrak{P}(X)$, the powerset of X. If $f : X \to Y$ and $A \subseteq X$, then $f``[A] := \{f(x) : x \in A\}$.

For any definitions or theorems not covered here the reader is referred to such standard texts as [4] or [8] in the case of set theory, to [6] in the case of category theory, to [7] in the case of sheaf theory, and to [1] in the case of model theory.

https://doi.org/10.1017/jsl.2013.18 Published online by Cambridge University Press

§2. First order trees. In this section we give a theory Th_{Tr} of trees in a language L_{Tr} and show that there is a topological space $\tilde{\mathbb{N}}$ with a basis $\mathrm{Bas}_{\tilde{\mathbb{N}}}$ such that $\mathrm{Mod}_{L_{\mathrm{Tr}}}(Th_{\mathrm{Tr}})$ is equivalent to both $\mathrm{Sep}^+(\mathrm{Bas}_{\tilde{\mathbb{N}}})$ and to $\mathrm{Sh}(\tilde{\mathbb{N}})$.

The observation that there is a category of trees equivalent to $\mathrm{Sh}(\tilde{\mathbb{N}})$ is folklore and not new here. This observation has been used in several places, including in [3], [5], and [2] where recursion in this topos is studied. One of the main differences between that work and the work in this paper is that while we do consider the relationship between various categories (e.g., those with irreducible bases), our main focus is not in the categories of second order trees but rather in concepts of particular first order trees which have generalizations to particular second order trees.

2.1. Theory of trees.

Definition 2.1. Let $L_{\mathrm{Tr}} = \{\leq, <_1, r\}$, where both $\leq$ and $<_1$ are binary relations and r is a constant. Let Th_{Tr} be the conjunction of the following sentences of $\mathcal{L}_{\omega_1,\omega}(L_{\mathrm{Tr}})$:

Partial Order:
- $(\forall x, y)\ x \leq y \wedge y \leq x \rightarrow x = y$.
- $(\forall x, y, z)\ x \leq y \wedge y \leq z \rightarrow x \leq z$.

Root:
- $(\forall x)\ r \leq x$.

Tree:
- $(\forall x, y, z)[y \leq x \wedge z \leq x] \rightarrow [y \leq z \vee z \leq y]$.

Levels:
- $(\forall y) \bigvee_{n\in\omega}(\exists x_0, \dots, x_n)(\forall z) z \leq y \rightarrow \bigvee_{0\leq i\leq n} z = x_i$.

Predecessor:
- $(\forall x, y)\ x <_1 y \leftrightarrow [x \leq y \wedge (\forall z)\ z \leq y \wedge z \neq y \rightarrow z \leq x]$.

The following formulas will be useful (where $m, n \in \mathbb{N}$ and $m \leq n$):

- $\mathrm{Lev}_n(x) := (\exists x_0, \dots x_n)\ r = x_0 \wedge x_n = x \wedge \bigwedge_{0\leq i<n} x_i <_1 x_{i+1}$.
- $\mathrm{Pr}_{m,n}(x, y) := \mathrm{Lev}_m(x) \wedge \mathrm{Lev}_n(y) \wedge x \leq y$.

We call $\leq$ and $<_1$ the **order** and **predecessor** relations, respectively. We call r the **root** of the tree. We say that x is on **level** n if $\mathrm{Lev}_n(x)$ holds, i.e., the set of elements strictly less than x has size n. We also use familial terms for the relationship between elements, e.g., x is the **parent** of y if $x <_1 y$ holds, y is a **descendant** of x if $x \leq y$ holds, etc.

Notice that the relation $<_1$ is definable from $\leq$ by a first order formula. We include $<_1$ in our language so that homomorphisms of models of Th_{Tr} will preserve the predecessor relation. This is important, because a function preserves the predecessor relation and the root if and only if it preserves the formulas $\mathrm{Lev}_n(x)$ and $\mathrm{Pr}_{m,n}(x, y)$ (for each $m, n \in \mathbb{N}$). The following lemma is immediate.

Lemma 2.2. *If $\mathcal{M} \models Th_{Tr}$ and $\mathcal{M} \models Lev_n(y)$, then for all $m \leq n$ there is a unique $x \in \mathcal{M}$ such that $\mathcal{M} \models Pr_{m,n}(x, y)$.*

It is worth pointing out that the only axiom in Th_{Tr} which is not first order is the Levels axiom, which guarantees every element is on a finite level. An infinitary

https://doi.org/10.1017/jsl.2013.18 Published online by Cambridge University Press

axiom here is necessary as any first order axiomatization of the concept of a tree must allow trees to have *nonstandard* elements, i.e., elements which are not on any finite level.

2.2. Separated presheaves.

Definition 2.3. Let $\tilde{\mathbb{N}}$ be the topological space where:

- The underlying set is $\mathbb{N}' := \mathbb{N} - \{0\} = \{1, 2, \dots\}$.
- Open sets are $\tilde{0} := \emptyset$, $\mathbb{N}'$, and $\tilde{n} = \{1, \dots, n\}$ for $n \in \mathbb{N}'$.

Note that $\mathrm{Bas}_{\tilde{\mathbb{N}}} := \{\tilde{0}, \tilde{1}, \dots\}$ is a basis for the topology.

It is easily checked that $\tilde{\mathbb{N}}$ is a topological space. Note that $\langle \mathrm{Bas}_{\tilde{\mathbb{N}}}, \subseteq \rangle \cong \langle \mathbb{N}, \leq \rangle$. This is by design and will be generalized in Definition 3.2.

Proposition 2.4. *There is an isomorphism between the category $Sep^+(Bas_{\tilde{\mathbb{N}}})$ and the category $Mod_{L_{Tr}}(Th_{Tr})$.*

Proof. We first define functors $\mathrm{F} : \mathrm{Sep}^+(\mathrm{Bas}_{\tilde{\mathbb{N}}}) \to \mathrm{Mod}_{L_{\mathrm{Tr}}}(Th_{\mathrm{Tr}})$ and $\mathrm{G} : \mathrm{Mod}_{L_{\mathrm{Tr}}}(Th_{\mathrm{Tr}}) \to \mathrm{Sep}^+(\mathrm{Bas}_{\tilde{\mathbb{N}}})$. Using Lemma 2.2 we can associate with every model T of Th_{Tr} a presheaf $\mathrm{G}(T)$ on $\tilde{\mathbb{N}}$ where:

- For $n \in \mathbb{N}$, $\mathrm{G}(T)(\tilde{n}) = \{x : T \models \mathrm{Lev}_n(x)\}$.
- $\mathrm{G}(T)(\mathbb{N}') = \emptyset$.
- If $m \leq n$ and $x \in \mathrm{G}(T)(\tilde{n})$, then $x|_{\tilde{m}}$ is the unique element of T such that $T \models \mathrm{Pr}_{m,n}(x|_{\tilde{m}}, x)$.

Further, as such a T has a unique element on level 0, $\mathrm{G}(T)$ is separated and nonempty and hence an object of $\mathrm{Sep}^+(\mathrm{Bas}_{\tilde{\mathbb{N}}})$.

To every object E of $\mathrm{Sep}^+(\mathrm{Bas}_{\tilde{\mathbb{N}}})$ we can associate a model $\mathrm{F}(E)$ of Th_{Tr} where:

- $\mathrm{F}(E)$ has underlying set $\bigcup_{n \in \mathbb{N}} E(\tilde{n})$.
- $\mathrm{F}(E) \models x <_1 y$ if and only if there is some $n \in \mathbb{N}$, where $x \in E(\tilde{n})$, $y \in E(\widetilde{n+1})$, and $x = y|_{\tilde{n}}$.
- $\leq$ is the transitive closure of $<_1$.
- r is the (necessarily unique) element of $E(\tilde{0})$.

Note that because L_{Tr} has a constant, the construction of $\mathrm{F}(E)$ makes fundamental use of the fact that E is nonempty.

It is immediate that $F \circ G(T) = T$ for any model of Th_{Tr} and $G \circ F(E) = E$ for any object of $\mathrm{Sep}^+(\mathrm{Bas}_{\tilde{\mathbb{N}}})$. Further it is clear that a function between models of Th_{Tr} is a homomorphism if and only if it is also a natural transformation between corresponding separated presheaves. Therefore if we let $\mathrm{F}(f) = \mathrm{G}(f) = f$ for any such map, then we have F and G are isomorphisms of categories. ⊣

2.3. Irreducible bases. The basis $\mathrm{Bas}_{\tilde{\mathbb{N}}}$ has a useful property: no element of the basis can be expressed as the nonempty union of strictly smaller elements. This property allows us to show an equivalence between the categories of nonempty separated presheaves on the basis and sheaves on the topological space. We now make this notion precise.

Definition 2.5. Suppose $\mathbb{A} = (A, \preceq)$ is a bounded distributive lattice. We say $U \in A$ is ***completely join irreducible***[1] if for every nonempty $\{U_i : i \in I\} \subseteq A$,

[1]Note that this definition is slightly nonstandard as for us $\perp_{\mathbb{A}}$ is completely join irreducible even though $\perp_{\mathbb{A}} = \bigvee \emptyset$.

$\bigvee_{i\in I} U_i = U$ implies $U = U_i$ for some $i \in I$. We say $\mathrm{Bas}_{\mathbb{A}} \subseteq \mathbb{A}$ is an ***irreducible basis*** of $\mathbb{A}$ if:

- The bottom element of $\mathbb{A}$, $\bot_{\mathbb{A}}$, is contained in $\mathrm{Bas}_{\mathbb{A}}$.
- For all $U \in A$ there is a nonempty $\{U_i : i \in I\} \subseteq \mathrm{Bas}_{\mathbb{A}}$ covering U, i.e., with $U = \bigvee_{i\in I} U_i$.
- $\mathrm{Bas}_{\mathbb{A}}$ is a lower set, i.e., if $U, V \in A$ with $U \preceq V$ and $V \in \mathrm{Bas}_{\mathbb{A}}$, then $U \in \mathrm{Bas}_{\mathbb{A}}$.
- Every element of $\mathrm{Bas}_{\mathbb{A}}$ is completely join irreducible.

If $\mathcal{P}$ is a topological space we say $\mathrm{Bas}_{\mathcal{P}}$ is an irreducible basis for $\mathcal{P}$ if it is an irreducible basis for $\mathcal{O}(\mathcal{P})$.

While we will not make use of the following fact, it is worth mentioning that if $\mathbb{A}$ is a bounded distributive lattice with irreducible basis $\mathrm{Bas}_{\mathbb{A}}$, then the only element of $\mathrm{Bas}_{\mathbb{A}}$ with a nontotal cover is $\bot_{\mathbb{A}}$ (which is covered by $\emptyset$). Hence if $\mathrm{Bas}_{\mathbb{A}}^- := \mathrm{Bas}_{\mathbb{A}} - \{\bot_{\mathbb{A}}\}$ with $in_{\mathbb{A}} : \mathrm{Bas}_{\mathbb{A}}^- \to \mathrm{Bas}_{\mathbb{A}}$ the inclusion map, then the functor which takes an object $\mathcal{X}$ of $\mathrm{Sep}^+(\mathrm{Bas}_{\mathbb{A}})$ to the presheaf $\mathcal{X} \circ in_{\mathbb{A}}$ on $\mathrm{Bas}_{\mathbb{A}}^-$ (and does the obvious thing on morphisms) is an equivalence of categories between $\mathrm{Sep}^+(\mathrm{Bas}_{\mathbb{A}})$ and the category of presheaves on $\mathrm{Bas}_{\mathbb{A}}^-$.

We immediately have the following.

LEMMA 2.6. *$\mathrm{Bas}_{\tilde{\mathbb{N}}} \subseteq \mathcal{O}(\tilde{\mathbb{N}})$ is an irreducible basis of $\tilde{\mathbb{N}}$.*

Note that if a lattice has an irreducible basis, it must be unique.

LEMMA 2.7. *Suppose $\mathbb{B}_0, \mathbb{B}_1$ are irreducible bases for a bounded distributive lattice $\mathbb{A}$. Then $\mathbb{B}_0 = \mathbb{B}_1$.*

PROOF. If $U \in \mathbb{B}_0$, then there must be a nonempty $\{U_i : i \in I\} \subseteq \mathbb{B}_1$ with $U = \bigvee_{i\in I} U_i$. But then $U \in \{U_i : i \in I\}$ as U is completely join irreducible. In particular, $U \in \mathbb{B}_1$ and hence $\mathbb{B}_0 \subseteq \mathbb{B}_1$. By a similar argument $\mathbb{B}_1 \subseteq \mathbb{B}_0$ and so $\mathbb{B}_0 = \mathbb{B}_1$. ⊣

DEFINITION 2.8. Suppose $\mathrm{Bas}_{\mathbb{A}}$ is an irreducible basis of a bounded distributive lattice $\mathbb{A} = (A, \preceq)$. For $U \in A$ we let $\mathrm{Bas}_{\mathbb{A}}(U) := \{V \in \mathbb{A} : V \preceq U\} \cap \mathrm{Bas}_{\mathbb{A}}$. We let $\mathrm{Bas}_{\mathcal{P}}(U)$ denote $\mathrm{Bas}_{\mathcal{O}(\mathcal{P})}(U)$.

The following lemma lets us characterize covers in terms of irreducible bases.

LEMMA 2.9. *Suppose $\mathrm{Bas}_{\mathbb{A}}$ is an irreducible basis of a bounded distributive lattice $\mathbb{A} = (A, \preceq)$ and $U \in A$. If $\{U_i : i \in I\}$ is nonempty lower set, then $\bigvee\{U_i : i \in I\}$ is a cover of U if and only if $\mathrm{Bas}_{\mathbb{A}}(U) \subseteq \{U_i : i \in I\}$.*

PROOF. First suppose $\{U_i : i \in I\}$ is a nonempty lower set which covers U. Let $V \in \mathrm{Bas}_{\mathbb{A}}$ with $V \preceq U$. Then $V = V \wedge U = V \wedge \bigvee\{U_i : i \in I\} = \bigvee\{V \wedge U_i : i \in I\}$. But as V is completely join irreducible we have for some $i \in I$ that $V = V \wedge U_i$ and hence $V \preceq U_i$. However, as $\{U_i : i \in I\}$ is a lower set we have $V \in \{U_i : i \in I\}$. Hence as V was arbitrary we have $\mathrm{Bas}_{\mathbb{A}}(U) \subseteq \{U_i : i \in I\}$.

Next assume $\mathrm{Bas}_{\mathbb{A}}(U) \subseteq \{U_i : i \in I\}$. It is immediate from the definition of irreducible basis that there is a nonempty collection $\{W_j : j \in J\} \subseteq \mathrm{Bas}_{\mathbb{A}}(U)$ such that $\bigvee\{W_j : j \in J\} = U$. Hence $\mathrm{Bas}_{\mathbb{A}}(U)$ is a cover of U and so $\{U_i : i \in I\}$ is also a cover of U. ⊣

Principle bases determine the structure of the sheaves on the topological space in the following sense.

https://doi.org/10.1017/jsl.2013.18 Published online by Cambridge University Press

PROPOSITION 2.10. *Suppose* $Bas_{\mathcal{P}}$ *is an irreducible basis of* $\mathcal{P}$. *Then* $\mathbf{b}_{Bas_{\mathcal{P}}} \circ \imath_{\mathcal{P}} : Sh(\mathcal{P}) \to Sep^{+}(Bas_{\mathcal{P}})$ *and* $\mathbf{a}_{\mathcal{P}} \circ \imath_{Bas_{\mathcal{P}}} : Sep^{+}(Bas_{\mathcal{P}}) \to Sh(\mathcal{P})$ *form an equivalence of categories.*

PROOF. First, as $\mathrm{Bas}_{\mathcal{P}}$ is an irreducible basis we have for every $U \in \mathrm{Bas}_{\mathcal{P}}$ that there are no nonempty, nontotal covers of U. Hence for every nonempty separated presheaf $\mathcal{X}$ we have $\mathcal{X}(U) = \mathbf{a} \circ \imath(\mathcal{X})(U)$, i.e., no new elements are added to $\mathcal{X}(U)$ under sheafification. In particular, this implies $(\mathbf{b} \circ \imath) \circ (\mathbf{a} \circ \imath)(\mathcal{X}) = \mathcal{X}$ for all $\mathcal{X}$ an object of $\mathrm{Sep}^{+}(\mathrm{Bas}_{\mathcal{P}})$. It is also clear that $(\mathbf{b} \circ \imath) \circ (\mathbf{a} \circ \imath)$ is the identity on morphisms and so $(\mathbf{b} \circ \imath) \circ (\mathbf{a} \circ \imath)$ is the identity functor on $\mathrm{Sep}^{+}(\mathrm{Bas}_{\mathcal{P}})$.

Now suppose $\mathcal{Y}$ is an object of $\mathrm{Sh}(\mathcal{P})$ and let $\mathcal{Y}' = \imath \circ \mathbf{b} \circ \imath(\mathcal{Y})$. Then $\mathcal{Y}'(U) = \mathcal{Y}(U)$ if $U \in \mathrm{Bas}_{\mathcal{P}}$ and $\mathcal{Y}'(U) = \emptyset$ if $U \in \mathcal{O}(\mathcal{P}) - \mathrm{Bas}_{\mathcal{P}}$. By Lemma 2.9 we have for any $U \in \mathcal{O}(\mathcal{P})$ that $\mathrm{Bas}_{\mathcal{P}}(U)$ is a cover of U. Hence for every $U \in \mathcal{O}(\mathcal{P})$ and $y \in \mathcal{Y}(U)$, $\{(y|_V, V) : V \in \mathrm{Bas}_{\mathcal{P}}(U)\}$ is a compatible collection of elements with amalgamation y.

Now consider the map $\alpha^{\mathcal{Y}} : \mathcal{Y} \to \mathbf{a}(\mathcal{Y}')$ where:

- $\alpha^{\mathcal{Y}}_U$ is the identity if $U \in \mathrm{Bas}_{\mathcal{P}}$.
- For $y \in \mathcal{Y}(U)$ with $U \notin \mathrm{Bas}_{\mathcal{P}}$, $\alpha^{\mathcal{Y}}(y)$ is the unique element of $\mathbf{a}(\mathcal{Y}')(U)$ which is an amalgamation of the compatible collection of elements $\{(y|_V, V) : V \in \mathrm{Bas}_{\mathcal{P}}(U)\}$.

Note that then for any $U_0 \subseteq U_1$ and $y \in \mathcal{Y}(U_1)$ we have $\alpha^{\mathcal{Y}}_{U_1}(y)|_{U_0}$ is an amalgamation of the compatible collection $\{(y|_V, V) : V \in \mathrm{Bas}_{\mathcal{P}}(U_1) \cap \{V \in \mathcal{O}(\mathcal{P}) : V \subseteq U_0\}\}$.

However, we also have $\mathrm{Bas}_{\mathcal{P}}(U_0) = \{V \in \mathcal{O}(\mathcal{P}) : V \subseteq U_0\} \cap \mathrm{Bas}_{\mathcal{P}} = \{V \in \mathcal{O}(\mathcal{P}) : V \subseteq U_0\} \cap (\{V : V \subseteq U_1\} \cap \mathrm{Bas}_{\mathcal{P}}) = \{V \in \mathcal{O}(\mathcal{P}) : V \subseteq U_0\} \cap \mathrm{Bas}_{\mathcal{P}}(U_1)$. Hence $\alpha^{\mathcal{Y}}_U(y)|_{U_0}$ is an amalgamation of $\{(y|_V, V) : V \in \mathrm{Bas}_{\mathcal{P}}(U_0)\} = \{((y|_{U_0})|_V, V) : V \in \mathrm{Bas}_{\mathcal{P}}(U_0)\}$. But this implies $\alpha^{\mathcal{Y}}_{U_1}(y)|_{U_0}$ and $\alpha^{\mathcal{Y}}_{U_0}(y|_{U_0})$ are both amalgamations of the same compatible collection of elements and hence, as $\mathbf{a}(\mathcal{Y}')$ is a sheaf, are equal. But as y, U_0 and U_1 were arbitrary this implies $\alpha^{\mathcal{Y}}$ is a natural transformation.

It is immediate from the fact that $\mathcal{Y}$ is separated that $\alpha^{\mathcal{Y}}$ is injective. We now show $\alpha^{\mathcal{Y}}$ is also surjective. First observe for any $U \in \mathrm{Bas}_{\mathcal{P}}$, $\mathcal{Y}(U) = \mathbf{a}(\mathcal{Y}')(U)$, and so $U \in \mathrm{Bas}_{\mathcal{P}}$, $\alpha^{\mathcal{Y}}_U$ is surjective. Next suppose $U \in \mathcal{O}(\mathcal{P}) - \mathrm{Bas}_{\mathcal{P}}$ and $z \in \mathbf{a}(\mathcal{Y}')(U)$. There then must be some nonempty compatible collection $Z_z := \{(z_V, V) : V \in C_z\}$ of elements of $\mathcal{Y}'$, where C_z is a cover of U and z is an amalgamation of Z_z. But as $\mathcal{Y}'(V) = \emptyset$ for $V \in \mathcal{O}(\mathcal{P}) - \mathrm{Bas}_{\mathcal{P}}$, we must have $C_z \subseteq \mathrm{Bas}_{\mathcal{P}}(U)$ and hence, by Lemma 2.9, we also have $C_z = \mathrm{Bas}_{\mathcal{P}}(U)$. In particular, this implies that $z_V \in \mathcal{Y}(V)$ for all $V \in C_z$ and hence as $\mathcal{Y}$ is a sheaf there must be some $y_Z \in \mathcal{Y}(U)$ which is an amalgamation of Z_z. But then by construction we have $\alpha^{\mathcal{Y}}_U(y_z) = z$. Hence as z was arbitrary we have $\alpha^{\mathcal{Y}}$ is surjective.

In particular, as $\alpha^{\mathcal{Y}}$ is both injective and surjective it is an isomorphism of sheaves. It is then easily checked that $\langle \mathcal{Y}, \alpha^{\mathcal{Y}} \rangle$ is a natural isomorphism from the identity functor on $\mathrm{Sh}(\mathcal{P})$ to $(\mathbf{a} \circ \imath) \circ (\mathbf{b} \circ \imath)$.

In particular, we have shown that $(\mathbf{b} \circ \imath) \circ (\mathbf{a} \circ \imath)$ and $(\mathbf{a} \circ \imath) \circ (\mathbf{b} \circ \imath)$ are isomorphic to the identity functors and so $\mathbf{b} \circ \imath$ and $\mathbf{a} \circ \imath$ are equivalences of categories. ⊣

https://doi.org/10.1017/jsl.2013.18 Published online by Cambridge University Press

Proposition 2.4, Lemma 2.6, and Proposition 2.10 tell us that there is an equivalence of categories between $\mathrm{Mod}_{L_{\mathrm{Tr}}}(Th_{\mathrm{Tr}})$ and $\mathrm{Sh}(\tilde{\mathbb{N}})$. In particular, this implies that $\mathrm{Mod}_{L_{\mathrm{Tr}}}(Th_{\mathrm{Tr}})$ is a localic Grothendieck topos.

PROPOSITION 2.11. *The following three categories are equivalent*:

- $Mod_{L_{Tr}}(Th_{Tr})$,
- $Sep^+(Bas_{\tilde{\mathbb{N}}})$,
- $Sh(\tilde{\mathbb{N}})$.

Proposition 2.11 gives us three different representations for the category of trees. In general the specific representation will not be important and so by $\mathrm{Cat}_{\mathrm{Tr}}$ we will mean any one of the three categories in Proposition 2.11. Similarly, by a **tree**, or a **first order tree**, we will mean an object of $\mathrm{Cat}_{\mathrm{Tr}}$. When no confusion can arise we will abuse notation and not specify which representation of a first order tree we are using at a given time. For example, if $\mathcal{T}$ is a first order tree, then T is the underlying set, $\leq^{\mathcal{T}}$ and $<_1^{\mathcal{T}}$ are the order and predecessor relations, $\mathcal{T}(\tilde{n})$ is the collection of elements on level n, $\mathcal{T}(\mathbb{N}')$ is the collections of global sections of the tree, etc. In what follows $\mathcal{T}$ will always be a first order tree.

§3. Second order trees.

In this section we will introduce second order trees and show how several concepts associated with first order trees generalize to second order trees.

3.1. Equivalent definitions.

DEFINITION 3.1. The category of second order trees over $\mathcal{T}$ is the category $\mathrm{Tree}_{\mathcal{T}} := \mathrm{Cat}_{\mathrm{Tr}}/\mathcal{T}$. A ***second order tree*** over $\mathcal{T}$ is an object of $\mathrm{Tree}_{\mathcal{T}}$, i.e., a map of first order trees with codomain $\mathcal{T}$.

Notice we can consider $\mathbb{N}$ as a tree $\langle\{0, 1, \dots\}, \leq\rangle$. Then $\mathbb{N}$ is a terminal object in $\mathrm{Cat}_{\mathrm{Tr}}$ and so there is an isomorphism of categories between $\mathrm{Cat}_{\mathrm{Tr}}$ and $\mathrm{Tree}_{\mathbb{N}}$. In this way every first order tree can be thought of as a second order tree over $\mathbb{N}$ where the map to $\mathbb{N}$ takes an element and returns its level.

We now show that whenever a topological space $\mathcal{P}$ has an irreducible basis, $\mathrm{Bas}_{\mathcal{P}}$, we can use $\mathrm{Bas}_{\mathcal{P}}$ to define from each object $\mathcal{X}$ of $\mathrm{Sep}^+(\mathrm{Bas}_{\mathcal{P}})$ a topological space $\tilde{\mathcal{X}}$ which itself has an irreducible basis. Further the lattice of open sets of $\tilde{\mathcal{X}}$ will be isomorphic to the lattice of subobjects of $\mathcal{X}$.

DEFINITION 3.2. Suppose $\mathcal{P}$ is a topological space with irreducible basis $\mathrm{Bas}_{\mathcal{P}}$. Further suppose $\mathcal{X}$ is an object of $\mathrm{Sep}^+(\mathrm{Bas}_{\mathcal{P}})$. For each $U \in \mathrm{Bas}_{\mathcal{P}}$ and $t \in \mathcal{X}(U)$ let $\tilde{t} = \{t|_V : V \in \mathrm{Bas}_{\mathcal{P}}, \emptyset \neq V \subseteq U\}$. Let $\mathrm{Bas}_{\tilde{\mathcal{X}}} = \{\tilde{t} : t \in \mathcal{X}(U), U \in \mathrm{Bas}_{\mathcal{P}}\}$. Define the topological space $\tilde{\mathcal{X}}$ as follows:

- The underling set of $\tilde{\mathcal{X}}$ is $\mathcal{X}' = \bigcup_{U \in \mathrm{Bas}_{\mathcal{P}} - \{\emptyset\}} \mathcal{X}(U)$.
- $\mathrm{Bas}_{\tilde{\mathcal{X}}}$ is a subbasis for $\tilde{\mathcal{X}}$.

Notice that in Definition 3.2 $\widetilde{*_{\mathcal{X}}} = \emptyset$. Also notice for any first order tree $\mathcal{T}$, $\langle T, \leq^{\mathcal{T}}\rangle \cong \langle \mathrm{Bas}_{\tilde{\mathcal{T}}}, \subseteq\rangle$ and $\mathcal{T}' = T - \{r^{\mathcal{T}}\}$. In particular, this notation is consistent with the notation in Definition 2.3 considering $\mathbb{N}$ as a tree.

PROPOSITION 3.3. *If $\mathcal{P}$ has an irreducible basis $Bas_{\mathcal{P}}$ and $\mathcal{X}$ is an object of $Sep^+(Bas_{\mathcal{P}})$, then $Bas_{\tilde{\mathcal{X}}}$ is an irreducible basis for $\tilde{\mathcal{X}}$.*

https://doi.org/10.1017/jsl.2013.18 Published online by Cambridge University Press

PROOF. First we show $\text{Bas}_{\tilde{\mathcal{X}}}$ is a basis. Suppose $U, V \in \text{Bas}_{\mathcal{P}}$, $s \in \mathcal{X}(U), t \in \mathcal{X}(V)$. If $W = \bigcup\{W' \in \text{Bas}_{\mathcal{P}} : t|_{W'} = s|_{W'}\}$, then $t|_W = s|_W$ (as $\mathcal{X}$ is separated) and $\tilde{s} \cap \tilde{t} = \{(t|_W)|_Z : Z \in \text{Bas}_{\mathcal{P}}, \emptyset \neq Z \subseteq W\}$. But if $U, V \in \text{Bas}_{\mathcal{P}}$, then $W \in \text{Bas}_{\mathcal{P}}$ and so $\tilde{s} \cap \tilde{t} = \widetilde{t|_W} \in \text{Bas}_{\tilde{\mathcal{X}}}$. Hence, as s, t were arbitrary, $\text{Bas}_{\tilde{\mathcal{X}}}$ is an actual basis and not just a subbasis.

Next we show $\text{Bas}_{\tilde{\mathcal{X}}}$ is an irreducible basis. First notice that as it is a basis every open set in $\mathcal{O}(\mathcal{X})$ is the union of a nonempty collection of sets in $\text{Bas}_{\tilde{\mathcal{X}}}$. Further notice as $\emptyset \in \text{Bas}_{\mathcal{P}}$ and $*_{\mathcal{X}} \in \mathcal{X}(\emptyset)$ we have $\emptyset = \widetilde{*_{\mathcal{X}}} \in \text{Bas}_{\tilde{\mathcal{X}}}$. To show each element of $\text{Bas}_{\tilde{\mathcal{X}}}$ is completely join irreducible suppose $U \in \text{Bas}_{\mathcal{P}}$, $t \in \mathcal{X}(U)$, and $\tilde{t} = \bigcup_{i \in I} \tilde{t}_i$ with $I \neq \emptyset$. Then for each $i \in I$, $\tilde{t}_i \subseteq \tilde{t}$ and hence $t_i = t|_{U_i}$, where $U_i \in \text{Bas}_{\mathcal{P}}$ and $t_i \in \mathcal{X}(U_i)$. But we know that $t \in \tilde{t}$ and so there must be some $i \in I$ such that $t \in \tilde{t}_i$ and hence $t = t_i|_U$. But then we must have $U = U_i$ and $t = t_i$ as $t = t_i|_U$ and $t_i = t|_{U_i}$. Hence $\tilde{t}$ is completely join irreducible and as t was arbitrary every element of $\text{Bas}_{\tilde{\mathcal{X}}}$ is completely join irreducible.

Finally, to show $\text{Bas}_{\tilde{\mathcal{X}}}$ is a lower set suppose $U \in \text{Bas}_{\mathcal{X}}$, $t \in \mathcal{X}(U)$, and $V \in \mathcal{O}(\tilde{\mathcal{X}})$ with $V \subseteq \tilde{t}$. We then have $V = \bigcup_{i \in I} \tilde{t}_i$, where $t_i \in \mathcal{X}(U_i)$, for some $\{U_i : i \in I\} \subseteq \text{Bas}_{\mathcal{P}}$. Further as $\widetilde{*_{\mathcal{X}}} = \emptyset$ we can assume without loss of generality that $U_i = \emptyset$, and $t_i = *_{\mathcal{X}}$ for some $i \in I$ (and, in particular, I is nonempty). We have for each $i \in I$ that $t_i = t|_{U_i}$. So if $U' = \bigcup_{i \in I} U_i$, then $U' \subseteq U$ and hence $U' \in \text{Bas}_{\mathcal{P}}$. But then U' is completely join irreducible and so $U' = U_j$ for some $j \in I$. Hence $t_i = t_j|_{U_i}$ for all $i \in I$ and, in particular, $\tilde{t}_i \subseteq \tilde{t}_j$ for all $i \in I$. But this implies that $V = \tilde{t}_j$ and so $V \in \text{Bas}_{\tilde{\mathcal{X}}}$. But as V and t are arbitrary this implies $\text{Bas}_{\tilde{\mathcal{X}}}$ is a lower set and hence an irreducible basis. ⊣

We next show that there is a close relationship between open sets in $\mathcal{O}(\tilde{\mathcal{X}})$ and subobjects of $\mathcal{X}$ in $\text{Sep}^+(\text{Bas}_{\mathcal{P}})$.

PROPOSITION 3.4. *If $\mathcal{P}$ has an irreducible basis $Bas_{\mathcal{P}}$, $\mathcal{X}$ is an object of $Sep^+(Bas_{\mathcal{P}})$, and $Sub(\mathcal{X})$ is the lattice of subobjects of $\mathcal{X}$ in $Sep^+(Bas_{\mathcal{P}})$, then $(\mathcal{O}(\tilde{\mathcal{X}}), \subseteq)$ is isomorphic to $(Sub(\mathcal{X}), \subseteq)$ (as lattices).*

PROOF. Define $\beta : \text{Sub}(\mathcal{X}) \to \mathcal{O}(\tilde{\mathcal{X}})$ by $\beta(Z) := \bigcup_{U \in \text{Bas}_{\mathcal{P}}} \{\tilde{t} : t \in Z(U)\}$ and $\alpha : \mathcal{O}(\tilde{\mathcal{X}}) \to \text{Sub}(\mathcal{X})$ given by $\alpha(Y)(U) := \{t \in \mathcal{X}(U) : \tilde{t} \subseteq Y\}$. (Note α is well-defined as $\tilde{t} \subseteq \tilde{t'}$ whenever t is a restriction of t').

First let us show that for any $Z \in \text{Sub}(\mathcal{X})$, $Z = \alpha \circ \beta(Z)$. For any $U \in \mathcal{O}(\mathcal{P})$, $\alpha \circ \beta(Z)(U) = \{t \in Z(U) : \tilde{t} \subseteq \beta(Z)\} = \{t \in Z(U) : \tilde{t} \subseteq \bigcup_{U \in \text{Bas}_{\mathcal{P}}} \{\tilde{t'} : t' \in Z(U)\}\}$. Hence $Z \subseteq \alpha \circ \beta(Z)$. In particular, this implies that $Z(\emptyset) = \alpha \circ \beta(Z)(\emptyset) = \mathcal{X}(\emptyset)$.

Next let $t \in \alpha \circ \beta(Z)(U)$ for some $U \in \text{Bas}_{\mathcal{P}} - \{\emptyset\}$. Then $\tilde{t} \subseteq \beta(Z)$ and hence $t \in \beta(Z)$. In particular, there must be some $t' \in Z(U')$ with $U \subseteq U'$ such that $t \in \tilde{t'}$, or equivalently, $t = t'|_U$. But then we must also have $t \in Z(U)$, as Z is a presheaf. Hence $\alpha \circ \beta(Z)(U) \subseteq Z(U)$, as t was arbitrary. Finally, as $Z(U) = \emptyset$ for all $U \in \mathcal{O}(\mathcal{P}) - \text{Bas}_{\mathcal{P}}$ we have $\alpha \circ \beta(Z) = Z$.

Next we show for any $Y \in \mathcal{O}(\tilde{\mathcal{X}})$ that $\beta \circ \alpha(Y) = Y$. First note that it is immediate that $\beta \circ \alpha(Y) = \bigcup_{U \in \text{Bas}_{\mathcal{P}}} \{\tilde{t} : t \in \alpha(Y)(U)\} = \bigcup_{U \in \text{Bas}_{\mathcal{P}}} \{\tilde{t} : t \in \{t' \in \mathcal{X}(U) : \tilde{t'} \subseteq Y\}\} \subseteq Y$.

Now as $\text{Bas}_{\tilde{\mathcal{X}}}$ is a basis for $\mathcal{O}(\tilde{\mathcal{X}})$ we must have $Y = \bigcup_{i \in I} \tilde{t}_i$ for some nonempty collection $\{\tilde{t}_i : i \in I\} \subseteq \text{Bas}_{\tilde{\mathcal{X}}}$. Choose $t \in Y$. Then $t = t_i|_W$ for some $i \in I$ and

https://doi.org/10.1017/jsl.2013.18 Published online by Cambridge University Press

$W \in \mathrm{Bas}_{\mathcal{P}}$. But then $\tilde{t} \subseteq \tilde{t}_i \subseteq Y$ and so $t \in \beta \circ \alpha(Y)$. Hence $Y \subseteq \beta \circ \alpha(Y)$ and so $Y = \beta \circ \alpha(Y)$.

Finally, it is easily checked that if $Y, Y' \in \mathcal{O}(\tilde{\mathcal{X}})$ with $Y \subseteq Y'$, then $\alpha(Y) \subseteq \alpha(Y')$ and if $Z, Z' \in \mathrm{Sub}(\mathcal{X})$ with $Z \subseteq Z'$, then $\beta(Z) \subseteq \beta(Z')$. But then because α and β are (inverse) isomorphisms of the sets $\mathcal{O}(\tilde{\mathcal{X}})$ and $\mathrm{Sub}(\mathcal{X})$ we have α and β are also (inverse) isomorphisms of lattices. ⊣

Just as in the case of first order trees we have three different characterizations of second order trees over $\mathcal{T}$.

PROPOSITION 3.5. *For any first order tree $\mathcal{T}$, the following categories are equivalent*:

(1) $\mathit{Tree}_{\mathcal{T}}$,
(2) $\mathit{Sep}^{+}(\mathit{Bas}_{\tilde{\mathcal{T}}})$,
(3) $\mathit{Sh}(\tilde{\mathcal{T}})$.

PROOF. The equivalence of (2) and (3) follows from Propositions 2.10 and 3.3.

To see (1) is equivalent to (3) recall that $\mathrm{Cat}_{\mathrm{Tr}}$ is equivalent to $\mathrm{Sh}(\tilde{\mathbb{N}})$ and hence a localic Grothendieck topos. So, $\mathrm{Tree}_{\mathcal{T}} = \mathrm{Cat}_{\mathrm{Tr}}/\mathcal{T}$ is equivalent to $\mathrm{Sh}(\mathrm{Sub}(\mathcal{T}))$, where $\mathrm{Sub}(\mathcal{T})$ is the lattice of subobjects of $\mathcal{T}$ in $\mathrm{Sh}(\tilde{\mathbb{N}})$. However, because $\mathrm{Bas}_{\tilde{\mathbb{N}}}$ is an irreducible basis of $\tilde{\mathbb{N}}$, we have by Proposition 2.10 that $\mathrm{Sh}(\tilde{\mathbb{N}})$ is equivalent to $\mathrm{Sep}^{+}(\mathrm{Bas}_{\tilde{\mathbb{N}}})$. But then by Proposition 3.4 we have $\mathrm{Sub}(\mathcal{T})$ is isomorphic to $\mathcal{O}(\tilde{\mathcal{T}})$ and so $\mathrm{Sh}(\mathrm{Sub}(\mathcal{T}))$ is equivalent to $\mathrm{Sh}(\tilde{\mathcal{T}})$. ⊣

As in the case of first order trees when no confusion will arise we will abuse notation and consider a second order tree simultaneously as an object of all three categories. For example, if $\mathcal{S}$ is a second order tree over $\mathcal{T}$ we let $\mathcal{S}(\mathcal{T}')$ be the collection of global sections, $\mathrm{dom}(\mathcal{S}) = \langle S, \leq^{\mathcal{S}}, <_1^{\mathcal{S}}, r^{\mathcal{S}} \rangle$ be the domain of the $\mathcal{S}$ as a map of first order trees, etc. However, we will also sometimes mention which of the three categories it is most helpful to think of our second order tree as belonging to. In what follows $\mathcal{S}$ (and its variants) will be second order trees over $\mathcal{T}$.

3.2. Well-foundedness. We now introduce the notion of a well-founded second order tree.

DEFINITION 3.6. We say a sheaf $\mathcal{S}$ is ***well-founded*** if $\mathcal{S}(\mathcal{T}') = \emptyset$, i.e., if $\mathcal{S}$ has no global sections.

This definition of well-founded will be important when we define sheaf recursion in Section 4.2. It is worth pointing out that this notion agrees with the definition for first order trees, i.e., a first order tree $\mathcal{T}$ is well-founded if and only if there are no infinite paths through $\mathcal{T}$, which is true if and only if $\mathcal{T}(\mathbb{N}') = \emptyset$.

While this notion of well-foundedness of a second order tree is the proper analog (for our purposes) of well-foundedness of first order trees, it need not be the case that if a second order tree $\mathcal{S} : \mathrm{dom}(\mathcal{S}) \to \mathcal{T}$ is well-founded that $\mathrm{dom}(\mathcal{S})$ is well-founded.

EXAMPLE 3.7. Let $T = 2^{<\omega}$ with $t_0 \leq^{\mathcal{T}} t_1$ if $t_1|_{\mathrm{len}(t_0)} = t_0$. In particular, $\mathcal{T}$ is a complete binary branching tree. Let $\mathcal{S} : \mathrm{dom}(\mathcal{S}) \to \mathcal{T}$ be the second order tree over $\mathcal{T}$ (considered as a map of first order trees) where:

- $\mathcal{S}^{-1}(r^{\mathcal{T}})$ has a single element.
- For any sequence $t \in T$ containing at least one 0 $\mathcal{S}^{-1}(t) = \emptyset$.

https://doi.org/10.1017/jsl.2013.18 Published online by Cambridge University Press

- For any sequence $t \in T$ containing all 1's, if $x \in \mathcal{S}^{-1}(t)$, then there are two $y_0, y_1 \in \mathcal{S}^{-1}(t^\wedge\langle 1\rangle)$ such that $x <_1^{\mathcal{S}} y_0$ and $x <_1^{\mathcal{S}} y_1$.

We then have

- $\mathcal{S}$ is a well-founded second order tree.
- $\mathrm{dom}(\mathcal{S})$ is isomorphic to $\mathcal{T}$ (as first order trees).

In particular, by moving to the domain of this second order tree we loose structure which is necessary in order to ensure there are no global sections.

Similarly, it is not the case that if $\mathcal{S}$ is a second order tree with $\mathrm{dom}(\mathcal{S})$ well-founded that $\mathcal{S}$ must also be well-founded. In particular, the following is immediate from the fact that maps of trees reflect well-foundedness.

LEMMA 3.8. *If $\mathcal{T}$ is a well-founded first order tree and $\mathcal{S}$ is a second order tree over $\mathcal{T}$, then dom$(\mathcal{S})$ is a well-founded first order tree with the height of dom$(\mathcal{S})$ no more than the height of $\mathcal{T}$.*

Finally, note that if $\mathcal{T}$ is trivial, then there is a unique second order tree over $\mathcal{T}$ and that tree is not well-founded.

3.3. Almost flabby sheaves. There is a great deal known about the collection of global sections (i.e., infinite paths) of first order trees. In this section and Section 3.4 we show that many facts about these collections of global sections generalize to global sections of second order trees.

Recall that a sheaf is said to be *flabby* if every section can be extended to a global section, i.e., if every map from a subobject of the terminal object into a the sheaf factors through a map from the terminal object into the sheaf.

DEFINITION 3.9. If $\mathcal{X}$ is an object of $\mathrm{Sep}^+(\mathcal{P})$ we say $\mathcal{X}$ is ***almost flabby*** if either it is flabby or it is trivial. We let $\mathrm{Sub}_F(\mathcal{X})$ be the collection of almost flabby subpresheaves of $\mathcal{X}$.

The following is then immediate from the definition of almost flabby presheaf and is the reason we are dealing with almost flabby objects instead of flabby ones.

LEMMA 3.10. *For every object $\mathcal{X}$ of Sep$^+(\mathcal{P})$ the map $\alpha_{\mathcal{X}} : \mathit{Sub}_F(\mathcal{X}) \to \mathfrak{P}(\mathcal{X}(\mathcal{T}'))$ given by $\alpha_{\mathcal{X}}(Y) = Y(\mathcal{T}')$ is an isomorphism between the lattices $\langle \mathit{Sub}_F(\mathcal{X}), \subseteq\rangle$ and $\langle \mathfrak{P}(\mathcal{X}(\mathcal{T}')), \subseteq\rangle$.*

The previous lemma tells us that there is a close connection between subsets of global sections and almost flabby objects of $\mathrm{Sep}^+(\mathcal{P})$. We will see in Section 3.4 that in the case of second order trees this connection extends to a relationship between almost flabby sheaves and *closed* sets of global sections.

In the study of global sections of a first order tree, an important class of trees are those that are *pruned*. These are the trees $\mathcal{T}$ such that for every element $t \in T$ there is a path through the tree which contains t. Recall $T = \bigcup_{U \in \mathrm{Bas}_{\mathbb{N}}} \mathcal{T}(U)$ and paths through a first order tree correspond to global sections. Hence another definition of a pruned first order tree $\mathcal{T}$ is one where every completely join irreducible element in the lattice of subobjects (i.e., every finite path) has an extension to a global section (i.e., an infinite path through the tree). This suggests the following definition of a pruned second order tree.

DEFINITION 3.11. We call $\mathcal{S}$ ***pruned*** if for all $t \in T$ and $s \in \mathcal{S}(\tilde{t})$ there is an $x_s \in \mathcal{S}(\mathcal{T}')$ with $x_s|_{\tilde{t}} = s$.

In the case of first order trees, because the only set in $\mathcal{O}(\tilde{\mathbb{N}})$ which is not completely join irreducible is $\mathbb{N}'$, every pruned tree is in fact flabby as well. In the case of second order trees, however, there may be many sets $U \in \mathcal{O}(\tilde{\mathcal{T}})$ which are not completely join-irreducible. However, despite this fact we will see in Proposition 3.12 that for a second order tree to be flabby it suffices to be pruned, i.e., it suffices for the extension condition to hold only of completely join irreducible subobjects.

Proposition 3.12. *A second order tree $\mathcal{S}$ is pruned if and only if when considered as an object of $Sh(\tilde{\mathcal{T}})$ it is flabby.*

Proof. For this proof consider $\mathcal{S}$ as an object of $\mathrm{Sh}(\tilde{\mathcal{T}})$. It is immediate from Definition 3.11 that if $\mathcal{S}$ is flabby, then it is also pruned.

Next assume $\mathcal{S}$ is pruned. It suffices to show that for any $U \in \mathcal{O}(\tilde{\mathcal{T}}) - \mathrm{Bas}_{\tilde{\mathcal{T}}}$ and any $s \in \mathcal{S}(U)$ that there is some $x_s \in \mathcal{S}(\mathcal{T}')$ with $x_s|_U = s$. In particular, if there are no such U and s, then $\mathcal{S}$ is flabby and we are done. Note this will only occur if $\mathrm{Bas}_{\tilde{\mathcal{T}}} = \mathcal{O}(\mathcal{T})$, i.e., if $\mathcal{T}$ is linearly ordered and finite.

Now suppose that there is a $U_1 \in \mathcal{O}(\tilde{\mathcal{T}}) - \mathrm{Bas}_{\tilde{\mathcal{T}}}$ and $s_1 \in \mathcal{S}(U_1)$. As $\emptyset \in \mathrm{Bas}_{\tilde{\mathcal{T}}}$ we can assume $U_1 \neq \emptyset$. We can therefore choose an ordering $\mathcal{T}' := \langle t_{i+1} : 0 \leq i < \gamma \rangle$ with $t_1 \in U_1$. We next define a sequence $\langle U_i : 1 \leq i \leq \gamma \rangle$ and a sequence $\langle s_i : 1 \leq i \leq \gamma \rangle$ such that:

- for all $1 \leq i \leq \gamma$, $\widetilde{t_{i+1}} \subseteq U_{i+1}$ and $s_i \in \mathcal{S}(U_i)$, and
- for all $1 \leq i \leq j \leq \gamma$, $U_i \subseteq U_j$ and $s_j|_{U_i} = s_i$.

We define our sequences by induction.

Suppose $i = j + 1$:
Case 1: If $\tilde{t_i} \subseteq U_j$, let $U_i = U_j$ and $s_i = s_j$.

Case 2: Otherwise, let $U_i = U_j \cup \tilde{t_i}$. Define $t^* \in T$ to be such that $\tilde{t^*} = U_j \cap \tilde{t_i}$ and let $s^* = s_j|_{\tilde{t^*}}$. By assumption there is an element $x^* \in \mathcal{S}(\mathcal{T}')$ such that $x^*|_{\tilde{t^*}} = s^*$. Now if we let $x_i = x^*|_{\tilde{t_i}}$, then $\langle (s_j, U_j), (x_i, \tilde{t_i}) \rangle$ is a compatible collection of elements, so there must be an $s_i \in \mathcal{S}(U_i)$ such that $s_i|_{U_j} = s_j$.

i is a limit:
Let $U_i = \bigcup_{j<i} U_j$. By construction $\langle (s_j, U_j) : 1 \leq j < i \rangle$ is a compatible collection of elements and hence there must be a (unique) $s_i \in \mathcal{S}(U_i)$ compatible with each s_j (as $\mathcal{S}$ is a sheaf).

It is clear by construction that $\mathcal{T}' = U_\gamma$ and that $s_\gamma \in \mathcal{S}(\mathcal{T}')$ with $s_\gamma|_{U_1} = s_1$. Hence s_γ is the desired global section.

In particular, as U_1 and s_1 were arbitrary we have $\mathcal{S}$ is flabby. ⊣

3.4. Closure space of global sections. Some of the most important topological spaces in descriptive set theory, such as 2^ω or ω^ω, are those consisting of global sections of a first order tree where each closed set corresponds to all global sections of a given subtree. As we will see in this section, the global sections of a second order tree also form a closure space with each closed set being the global sections of a sub second order tree. In this section we define these closure spaces and consider some of their properties.

Lemma 3.13. *If $\mathcal{X}$ is an almost flabby object of $Sep^+(\tilde{\mathcal{T}})$, then $\mathbf{a}(\mathcal{X})$ is also almost flabby.*

https://doi.org/10.1017/jsl.2013.18 Published online by Cambridge University Press

Proof. Suppose $\mathcal{X}$ is almost flabby. First note if $\mathcal{X}$ is trivial, then $\mathbf{a}(\mathcal{X}) = \mathcal{X}$ and so $\mathbf{a}(\mathcal{X})$ is also trivial. Now assume $\mathcal{X}$ is nontrivial and hence flabby. We know $\mathcal{X}(\tilde{t}) = \mathbf{a}(\mathcal{X})(\tilde{t})$ for all $t \in T$ as each $\tilde{t}$ is completely join irreducible and $\mathcal{X}$ is nonempty. Hence for every $t \in T$ and $x \in \mathbf{a}(\mathcal{X})(\tilde{t})$ there is an $x^* \in \mathbf{a}(\mathcal{X})(\mathcal{T}')$ such that $x^*|_{\tilde{t}} = x$ (as $\mathcal{X} \subseteq \mathbf{a}(\mathcal{X})$ and $\mathcal{X}$ is flabby). In particular, $\mathbf{a}(\mathcal{X})$ is pruned and so by Proposition 3.12, $\mathbf{a}(\mathcal{X})$ is flabby as well. ⊣

Corollary 3.14. *If $\mathcal{S}$ is a sheaf over $\tilde{\mathcal{T}}$, then there is an almost flabby sheaf $\mathcal{S}^*$ over $\tilde{\mathcal{T}}$ such that $\mathcal{S}^* \subseteq \mathcal{S}$ and $\mathcal{S}(\mathcal{T}') = \mathcal{S}^*(\mathcal{T}')$.*

Proof. By Lemma 3.10 there is a unique almost flabby subpresheaf $\mathcal{S}^* \subseteq \mathcal{S}$ with $\mathcal{S}^*(\mathcal{T}') = \mathcal{S}(\mathcal{T}')$. But we have $\mathcal{S}^*(\mathcal{T}') \subseteq \mathbf{a}(\mathcal{S}^*)(\mathcal{T}') \subseteq \mathbf{a}(\mathcal{S})(\mathcal{T}') = \mathcal{S}(\mathcal{T}')$. Hence $\mathbf{a}(\mathcal{S}^*)(\mathcal{T}') = \mathcal{S}^*(\mathcal{T}')$. But, by Lemma 3.13, $\mathbf{a}(\mathcal{S}^*)$ is almost flabby and hence $\mathcal{S}^* = \mathbf{a}(\mathcal{S}^*)$ by Lemma 3.10. ⊣

Note Lemma 3.13 and Corollary 3.14 are not true for sheaves over arbitrary topological spaces.

Lemma 3.15. *If $\mathcal{S}$ is an almost flabby sheaf on $\tilde{\mathcal{T}}$, then $\mathbf{a}(\cdot)$ is a closure operator on $Sub_F(\mathcal{S})$.*

Proof. If $\mathrm{Sub}_P(\mathcal{S})$ is the collection of subpresheaves of $\mathcal{S}$, then $\mathbf{a}(\cdot)$ is a closure operator on Sub_P. But by Lemma 3.13, $\mathbf{a}(\cdot)$ takes an element of $\mathrm{Sub}_F(\mathcal{S})$ to another element of $\mathrm{Sub}_F(\mathcal{S})$ and hence is also a closure operator on $\mathrm{Sub}_F(\mathcal{S})$. ⊣

In particular, if we let $\mathrm{cl}_{\mathcal{S}}(\cdot) = \alpha_{\mathcal{S}}(\mathbf{a}(\alpha_{\mathcal{S}}^{-1}(\cdot)))$ (where $\alpha_{\mathcal{S}}$ is from Lemma 3.10), then α is an isomorphism from $(\mathrm{Sub}_F(\mathcal{S}), \subseteq, \mathbf{a})$ to $(\mathfrak{P}(\mathcal{S}(\mathcal{T}')), \subseteq, \mathrm{cl}_{\mathcal{S}})$. Hence there is a closure space of global sections of a second order tree where each closed set is the collection of global section of a sub (second order) tree. We will sometimes refer to this space by the abbreviated $(\mathcal{S}(\mathcal{T}'), \mathrm{cl}_{\mathcal{S}})$.

In general, the closure space of global sections of a second order tree will not be a topological space. However, even though $(\mathcal{S}(\mathcal{T}'), \mathrm{cl}_{\mathcal{S}})$ may not be a topological space, it still does have a basis of clopen sets.

Definition 3.16. If $x \in \mathcal{S}(\tilde{t})$ for some $t \in T$ let $\mathcal{S}_x := \{y \in \mathcal{S}(\mathcal{T}') : y|_{\tilde{t}} = x\}$ and otherwise let $\mathcal{S}_x = \emptyset$. Let $\mathbb{B}_{\mathcal{S}} = \{\mathcal{S}_x : x \in \bigcup_{t \in T} \mathcal{S}(\tilde{t})\}$. We call the elements of $\mathbb{B}_{\mathcal{S}}$ the ***basic clopen sets***.

We say a set $C \subseteq \mathcal{S}(\mathcal{T}')$ is **closed** if $\mathrm{cl}_{\mathcal{S}}(C) = C$ and a set U is **open** if $\mathcal{S}(\mathcal{T}') - U$ is closed. If U is both open and closed we say it is **clopen**.

Lemma 3.17. *If $U \in \mathbb{B}_{\mathcal{S}}$, then $cl_{\mathcal{S}}(U) = U$ and $cl_{\mathcal{S}}(\mathcal{S}(\mathcal{T}') - U) = \mathcal{S}(\mathcal{T}') - U$, i.e., U is clopen with respect to $cl_{\mathcal{S}}(\cdot)$.*

Proof. We begin with an important observation. For any $t \in T$ and $A \subseteq \mathcal{S}(\tilde{t})$ let $\mathrm{r}(A)$ is the separated presheaf, where for each $U \in \mathcal{O}(\tilde{\mathcal{T}})$ we have $\mathrm{r}(A)(U) := \{y \in \mathcal{S}(U) : (\exists x \in A) x|_{\tilde{t} \cap U} = y|_{\tilde{t} \cap U}\}$. Now suppose $\langle (y_i, U_i) : i \in I \rangle$ is a compatible collection of elements from $\mathrm{r}(A)$ with $I \neq \emptyset$. Then as $\mathcal{S}$ is a sheaf, there must be some $y^* \in \mathcal{S}(\bigcup_{i \in I} U_i)$ which is an amalgamation of $\langle (y_i, U_i) : i \in I \rangle$. There then must be some t' with $\tilde{t'} \subseteq \tilde{t}$ such that $\bigcup_{i \in I} U_i \cap \tilde{t} = \tilde{t'}$. But as $\tilde{t'}$ is completely join irreducible there must be some j such that $U_j \cap \tilde{t} = \tilde{t'}$. Then $y^*|_{\bigcup_{i \in I} U_i \cap \tilde{t}} = y^*|_{U_j \cap \tilde{t}} = y_j|_{U_j \cap \tilde{t}} = x|_{U_j \cap \tilde{t}} = x|_{\bigcup_{i \in I} U_i \cap \tilde{t}}$ for some $x \in A$. Therefore $y^* \in \mathrm{r}(A)$ and $\mathrm{r}(A)$ is a subsheaf of $\mathcal{S}$.

https://doi.org/10.1017/jsl.2013.18 Published online by Cambridge University Press

In particular, for each $t \in T$ and $x \in \mathcal{S}(\tilde{t})$, $\mathcal{S}_x = \mathrm{r}(\{x\})$ and so $\mathcal{S}_x$ is a sheaf and hence closed. Similarly, $\mathcal{S}(\mathcal{T}') - \mathcal{S}_x = \{y \in \mathcal{S}(\mathcal{T}') : y|_{\tilde{t}} \neq x\} = \mathrm{r}((\mathcal{S}(\tilde{t}) - \{x\}))$ and hence is closed. Therefore each S_x is a clopen set. ⊣

Lemma 3.18. *A set $U \subseteq \mathcal{S}(\mathcal{T}')$ is open if and only if it is the union of basic clopen sets.*

Proof. Suppose $U = \bigcup_{x \in i} \mathcal{S}_x$ is an arbitrary union of basic clopen sets. Then $\mathcal{S}(\mathcal{T}') - U = \bigcap_{x \in I} \mathcal{S}(T') - \mathcal{S}_x$. But each $\mathcal{S}(T') - \mathcal{S}_x$ is closed by Lemma 3.17. Therefore, as in any closure space the intersection of closed sets is closed, we have $\mathcal{S} - U$ is closed and U is open. Hence, as U was arbitrary, the union of any collection of basic clopen sets is open.

Next let $C \subseteq \mathcal{S}(\mathcal{T}')$ be closed to show $\mathcal{S}(\mathcal{T}') - C$ is the union of basic clopen sets. Recall $\alpha^{-1}(C)$ is the almost flabby object of $\mathrm{Sep}^{+}(\tilde{\mathcal{T}})$ with $\alpha^{-1}(C)(\mathcal{T}') = C$. If $C = \mathcal{S}(T')$, then $\mathcal{S}(\mathcal{T}') - C = \emptyset$ which is the empty union of basic clopen sets and so we are done. We can therefore there is an $x \in \mathcal{S}(\mathcal{T}') - C$. By Lemma 3.13 $\alpha^{-1}(C)$ is a sheaf and so there must be some $t \in T$ such that $x|_{\tilde{t}} \notin C^{*}(\tilde{t})$. But then $\mathcal{S}_{x|_{\tilde{t}}} \cap C = \emptyset$ and so $\mathcal{S}_{x_{\tilde{t}}} \subseteq \mathcal{S}(\mathcal{T}') - C$. Hence $\mathcal{S}(\mathcal{T}') - C$ is the union of basic clopen sets, as every element of $\mathcal{S}(\mathcal{T}') - C$ is contained in a basic clopen set contained in $\mathcal{S}(\mathcal{T}') - C$. ⊣

In particular, we have three equivalent notions of a closed set of global sections.

Proposition 3.19. *If $C \subseteq \mathcal{S}(\mathcal{T}')$, then the following are equivalent.*

(1) *$\mathcal{S}(\mathcal{T}') - C$ is the union of basic clopen sets.*
(2) *$C = cl_{\mathcal{S}}(C)$, i.e., there is a separated presheaf $\mathcal{S}^{*} \subseteq \mathcal{S}$ with $C = \mathbf{a}(\mathcal{S}^{*})(\mathcal{T}')$.*
(3) *There is an almost flabby sheaf $\mathcal{S}^{*} \subseteq \mathcal{S}$ with $C = \mathcal{S}^{*}(\mathcal{T}')$.*

Proof. That (1) is equivalent to (2) follows from Lemma 3.18. That (2) is equivalent to (3) follows from Lemma 3.10 and Corollary 3.14. ⊣

3.5. Splitting numbers and universal objects. One pictorial way to think of a first order tree is as collection of points moving forward in a series of steps where each time a point moves forward (i.e., each time we go from one level to the next larger level) a point may split into several distinct points. Building on this picture we can think of a second order tree as a collection of points where, instead of just being able to move forward in one direction, the points can move along the underlying tree $\mathcal{T}$. In this picture, just like in the picture of a first order tree, each time a point moves, it might split into several points. However, the number of distinct points a point will split into might depend on the direction the point has moved on the underlying tree T. In this picture the splitting number of a second order tree measures the smallest number larger than the maximum number of distinct points a single point can split into after moving one step.

Definition 3.20. We define the ***splitting number*** of $\mathcal{S}$ to be

$$\mathrm{Split}(\mathcal{S}) = \sup\{|x^{e}_{\tilde{V}}|^{+} : U, V \in T, U <^{T}_{1} V \text{ and } x \in \mathcal{S}(U)\}.$$

Notice that a first order tree has splitting number κ^{+}, for $\kappa > \omega$, if and only if it has size κ. However, it is possible for the splitting number of a countably infinite first order tree to be finite. For example, the splitting number of the perfect binary branching tree is 3.

Definition 3.21. Let κ be a cardinal. For $U \in \mathcal{O}(\mathcal{P})$ let $\kappa^{[\mathcal{P}]}(U) = \{f$ s.t. $f : U \to \kappa\}$ and for $U \subseteq V$ with $f \in \kappa^{[\mathcal{P}]}(V)$ let $f|_U$ be the unique function with domain U such that $(\forall u \in U) f|_U(u) = f(u)$.

It is then immediate that $\kappa^{[\mathcal{P}]}$ is a flabby sheaf over $\mathcal{P}$ and further that $\mathrm{Split}(\kappa^{[\tilde{\mathcal{P}}]}) = \kappa^+$. For any first order tree $\mathcal{T}$ and any κ, $\kappa^{[\tilde{\mathcal{T}}]}$ is universal for second order trees over $\mathcal{T}$ with splitting number at most κ^+.

Lemma 3.22. *If $\mathcal{S}$ is a sheaf on $\tilde{\mathcal{T}}$ and has splitting number $\leq \kappa^+$, then there is a monic $m : \mathcal{S} \to \kappa^{[\tilde{\mathcal{T}}]}$ in the category $Sh(\tilde{\mathcal{T}})$.*

Proof. For $t \in T$ we will define maps $m_{\tilde{t}} : \mathcal{S}(\tilde{t}) \to \kappa^{[\tilde{\mathcal{T}}]}(\tilde{t})$ which will cohere to give our monic m. We will define the maps $m_{\tilde{t}}$ by induction on the level of t.

First, $|\mathcal{S}(\widetilde{r^{\mathcal{T}}})| = |\kappa^{[\tilde{\mathcal{T}}]}(\widetilde{r^{\mathcal{T}}})| = 1$ so we can let $m_{\widetilde{r^{\mathcal{T}}}}$ be the unique isomorphism between $\mathcal{S}(\widetilde{r^{\mathcal{T}}})$ and $\kappa^{[\tilde{\mathcal{T}}]}(\widetilde{r^{\mathcal{T}}})$. In particular, as $r^{\mathcal{T}}$ is the only element of $\mathcal{T}(\tilde{0})$ we have defined $m_{\tilde{t}}$ for all $t \in \mathcal{T}(\tilde{0})$. This completes the base case of our definition.

Now assume $m_{\tilde{s}}$ has been defined for all $s \in \mathcal{T}(\tilde{n})$ and let $t \in \mathcal{T}(\widetilde{n+1})$. We know for each $x \in \mathcal{S}(t|_{\tilde{n}})$ that there is an injection $i_{x,t} : x^e_{\tilde{t}} \to \kappa$. For each $y \in x^e_{\tilde{t}}$ let $m_{\tilde{t}}(y) = m_{t|_{\tilde{n}}}(x) \cup \langle t, i_{x,t}(y)\rangle$, i.e., for all $\alpha \in \widetilde{t|_{\tilde{n}}}$, $m_{\tilde{t}}(y)(\alpha) = m_{\widetilde{t|_{\tilde{n}}}}(x)(\alpha)$ and $m_{\tilde{t}}(y)(t) = i_{x,t}(y)$.

Let m' be the function where for any $t \in T$ and $x \in \mathcal{S}(\tilde{t})$ we have $m(x) = m_{\tilde{t}}(x)$. It is clear that each $m_{\tilde{t}}$ is injective and that for all $s, t \in T$ with $\tilde{s} \subseteq \tilde{t}$ and $x \in \mathcal{S}(\tilde{t})$ that $m_{\tilde{s}}(x|_{\tilde{s}}) = m_{\tilde{t}}(x)|_{\tilde{s}}$. Hence m' is a monic map of separated presheaves and hence extends uniquely to a monic map $m : \mathcal{S} \to \kappa^{[\tilde{\mathcal{T}}]}$ in the category of sheaves on $\tilde{\mathcal{T}}$. ⊣

We now show that there is a closure space of global sections which is universal for all second order trees with splitting number at most κ^+ over a tree with splitting number at most κ^+.

Lemma 3.23. *Let $\mathcal{T}$ be a first order tree with splitting number at most κ^+ (e.g., a tree of size at most κ) and let $\mathcal{S}$ be any second order tree over $\mathcal{T}$ with splitting number at most κ^+. Then there is a monic $m_{\mathcal{S}} : \mathcal{S}(\mathcal{T}') \to \kappa^{[\widetilde{\kappa^{[\tilde{\mathbb{N}}]}}]}(\kappa^{[\tilde{\mathbb{N}}]'})$ such that for any set $C \subseteq \mathcal{S}(\mathcal{T}')$, C is a closed subset of $\mathcal{S}(\mathcal{T}')$ if and only if $m_{\mathcal{S}}``[C]$ is a closed subset of $\kappa^{[\widetilde{\kappa^{[\tilde{\mathbb{N}}]}}]}(\kappa^{[\tilde{\mathbb{N}}]'})$.*

Proof. First observe by Lemma 3.22 that it suffices to prove this lemma for $\mathcal{S} = \kappa^{[\tilde{\mathcal{T}}]}$. Next let $\kappa^{<\mathbb{N}} = \bigcup_{n \in \mathbb{N} - \{0\}} \kappa^{[\tilde{\mathbb{N}}]}(\tilde{n})$, which is the underlying set of the first order tree $\kappa^{[\tilde{\mathbb{N}}]}$ (considered as a L_{Tr}-structure).

Now by assumption and Lemma 3.22 there is a monomorphism $i_{\mathcal{T}} : \mathcal{T} \to \kappa^{[\tilde{\mathbb{N}}]}$ in the category of first order trees. Let $I_{\mathcal{T}} \subseteq \kappa^{<\mathbb{N}}$ be the image of $i_{\mathcal{T}}$. For $f \in \kappa^{[\tilde{\mathcal{T}}]}(\mathcal{T}')$ let $m_{\mathcal{S}}(f) \in \kappa^{[\widetilde{\kappa^{[\tilde{\mathbb{N}}]}}]}(\kappa^{[\tilde{\mathbb{N}}]'})$ be such that $m_{\mathcal{S}}(f)(x) := f(i^{-1}_{\mathcal{T}}(x))$ if $x \in I_{\mathcal{T}}$ and 0 if $x \in \kappa^{<\mathbb{N}} - I_{\mathcal{T}}$. Note $m_{\mathcal{S}}$ is well defined and injective as $i_{\mathcal{T}}$ is injective. Let I_m be the image of $m_{\mathcal{S}}$ on $\kappa^{[\widetilde{\kappa^{[\tilde{\mathbb{N}}]}}]}(\kappa^{[\tilde{\mathbb{N}}]'})$.

For $n \in \mathbb{N}$ and $f \in \kappa^{<\mathbb{N}}$ let $N_f = \{g \in \kappa^{[\widetilde{\kappa^{[\tilde{\mathbb{N}}]}}]}(\tilde{f})$: where $(\exists y \notin \mathrm{im}(i_{\mathcal{T}}))g(y) \neq 0\}$. Then $h \in \kappa^{[\widetilde{\kappa^{[\tilde{\mathbb{N}}]}}]}(\kappa^{[\tilde{\mathbb{N}}]'})$ is in the image of $m_{\mathcal{S}}$ if and only if $(\forall f \in \kappa^{<\mathbb{N}} h|_{\tilde{f}} \notin N_f$.

https://doi.org/10.1017/jsl.2013.18 Published online by Cambridge University Press

Now for notational convenience, for $n \in \mathbb{N}$, $f \in \kappa^{[\tilde{\mathbb{N}}]}(\tilde{n})$ and $h \in \kappa^{\widetilde{[\kappa^{[\tilde{\mathbb{N}}]}]}}(\tilde{f})$ let $K_h := \kappa^{\widetilde{[\kappa^{[\tilde{\mathbb{N}}]}]}}{}_h$ from Definition 3.16, i.e., $K_h = \{h^* \in \kappa^{\widetilde{[\kappa^{[\tilde{\mathbb{N}}]}]}}(\kappa^{[\tilde{\mathbb{N}}]'}) : h^*|_{\mathrm{dom}(h)} = h\}$. We then have $I_m = \bigcap_{f \in \kappa^{<\mathbb{N}}} \bigcap_{h \in N_f} \kappa^{\widetilde{[\kappa^{[\tilde{\mathbb{N}}]}]}}(\kappa^{[\tilde{\mathbb{N}}]'}) - K_h$ hence by Lemma 3.18, I_m is closed.

We will now prove if $C \subseteq \kappa^{[\tilde{\mathcal{T}}]}(\mathcal{T}')$ we have C is closed in $\kappa^{[\tilde{\mathcal{T}}]}(\mathcal{T}')$ if and only if $m_{\mathcal{S}}$"$[C]$ is closed in $\kappa^{\widetilde{[\kappa^{[\tilde{\mathbb{N}}]}]}}(\kappa^{[\tilde{\mathbb{N}}]'})$. First suppose C is closed. Then by Lemma 3.18 there is a set $N_C \subseteq \bigcup_{n\in\mathbb{N}} \bigcup_{t \in \mathcal{T}(\tilde{n})} \kappa^{[\tilde{\mathcal{T}}]}(\tilde{t})$ such that $C = \bigcap_{t \in N_C} \kappa^{[\tilde{\mathcal{T}}]}(\mathcal{T}') - \kappa^{[\tilde{\mathcal{T}}]}{}_t$. But then $m_{\mathcal{S}}$"$[C] = I_m \cap \bigcap_{t \in N_C} \kappa^{\widetilde{[\kappa^{[\tilde{\mathbb{N}}]}]}}(\kappa^{[\tilde{\mathbb{N}}]'}) - K_{m_{\mathcal{S}}(t)}$ which is closed as I_m is closed and by Lemma 3.18 each $\kappa^{\widetilde{[\kappa^{[\tilde{\mathbb{N}}]}]}}(\kappa^{[\tilde{\mathbb{N}}]'}) - K_{m_{\mathcal{S}}(t)}$ is closed as well.

Now suppose $m_{\mathcal{S}}$"$[C]$ is closed. Then once again by Lemma 3.18 we know that there is a collection $N_C^* \subseteq \bigcup_{t\in\kappa^{<\mathbb{N}}} \kappa^{\widetilde{[\kappa^{[\tilde{\mathbb{N}}]}]}}(\tilde{t})$ such that $m_{\mathcal{S}}$"$[C] = \bigcap_{t \in N_C^*} \kappa^{\widetilde{[\kappa^{[\tilde{\mathbb{N}}]}]}}(\kappa^{[\tilde{\mathbb{N}}]'}) - K_t$. Further as $m_{\mathcal{S}}$"$[C] \subseteq I_m$ we can assume that $N_f \subseteq N_C$ for all $f \in \kappa^{<\mathbb{N}}$. Let $N^* = N_{C^*} - \bigcup_{f \in \kappa^{<\mathbb{N}}} N_f$.

For each $g \in N_{C^*}$ there is a maximal $\imath(g) \subseteq g$ such that $\mathrm{dom}(\imath(g)) \in \mathrm{im}(i_{\mathcal{T}})$. Let $D = \bigcap_{g \in N^*} \kappa^{[\tilde{\mathcal{T}}]}(\mathcal{T}') - \kappa^{[\tilde{\mathcal{T}}]}{}_{i_{\mathcal{T}}^{-1}(\imath(g))}$. By Lemma 3.18 we have that D is closed. But by construction we have that $m_{\mathcal{S}}$"$[D] = m_{\mathcal{S}}$"$[C]$ and hence $D = C$ as $m_{\mathcal{S}}$ is injective. But this implies C is closed completing our proof. ⊣

It is worth mentioning that even though every closure space of the form $(\mathcal{S}(\mathcal{T}'), \mathrm{cl}_{\mathcal{S}})$ is a subclosure space of $(\kappa^{\widetilde{[\kappa^{[\tilde{\mathbb{N}}]}]}}(\kappa^{[\tilde{\mathbb{N}}]'}), \mathrm{cl}_{\kappa^{\widetilde{[\kappa^{[\tilde{\mathbb{N}}]}]}}})$ whenever $\mathcal{T}$ and $\mathcal{S}$ have splitting number at most κ^+, this does not mean that the there is any specific relationship between the sheaves $\mathcal{S}$ and $\kappa^{\widetilde{[\kappa^{[\tilde{\mathbb{N}}]}]}}$, as these sheaves are, in general, over different topological spaces.

Proposition 3.24. *For any $\kappa > 1$ the following are equivalent*:

(1) *$\mathcal{T}$ is linearly ordered (e.g., isomorphic to $\mathbb{N}$).*
(2) *$\langle \kappa^{[\tilde{\mathcal{T}}]}(\mathcal{T}'), cl_{\kappa^{[\tilde{\mathcal{T}}]}} \rangle$ is a topological space.*
(3) *Every two point set in $\kappa^{[\tilde{\mathcal{T}}]}(\mathcal{T}')$ is closed.*

Proof. It is immediate that (1) implies (2) and that (2) implies (3).

We now show (3) implies (1) by showing $\neg$ (1) implies $\neg$ (3). Assume $\mathcal{T}$ is not linearly ordered. So there are $t_0, t_1 \in T$ which are incomparable with common parent t^*. We can assume, without loss of generality, that t^* has minimal level among those elements with multiple children. Consider the elements $f_0, g_0 : \tilde{t_0} \to \kappa$, $f_1, g_1 : \tilde{t_1} \to \kappa$ such that

- $f_0(t) = g_0(t) = f_1(t) = g_1(t) = 0$ if $t \in \tilde{t^*}$.
- $f_0(t_0) = f_1(t_1) = 0$.
- $g_0(t_0) = g(t_1) = 1$.

Now let c_0, c_1 be the functions where c_0, c_1 are the constant function 0 on $\tilde{t^*}$ and which take values 0, 1, respectively, on every element of $\mathcal{T}' - \tilde{t^*}$. If $\{c_0, c_1\}$ is closed, then by Proposition 3.19 there is a flabby sheaf C with $C(\mathcal{T}') = \{c_0, c_1\}$. But then $f_0, g_0 \in C(\tilde{t_0})$ and $f_1, g_1 \in C(\tilde{t_1})$. So, as C is a flabby sheaf and $\{f_0, g_1\}$ is a compatible collection of elements, there must be some $x \in C(\mathcal{T}')$ with $x|_{\tilde{t_0}} = f_0$ and $x|_{\tilde{t_1}} = g_1$. But this is a contradiction as for any such x we would have $x \neq c_0$

and $x \neq c_1$. Hence $\{c_0, c_1\}$ is not closed and (3) does not hold. In particular, $\neg$ (1) implies $\neg$ (3) so (3) implies (1). ⊣

§4. Sheaf recursion. The method of "definition by recursion" is one of the most useful tools we have for defining a function whose domain is the natural numbers. The key feature of definition by recursion is the reduction of the definition at an element to the definition at a *simpler* element. When infinite well-founded trees began to be studied it was realized that the methods of definition by recursion worked when the domain $\mathbb{N}$ was replaced by any well-founded tree. The resulting method is known as definition by transfinite recursion. In this section we show that transfinite recursion can be further expanded to what we call definition by *sheaf recursion*, i.e., recursion on well-founded second order trees.

4.1. Transfinite recursion. In this section we give a presentation of definition by transfinite recursion. We will present this notion in such a way that it can be easily generalized to definition by sheaf recursive in Section 4.2.

For the rest of this section fix a well-founded first order tree $\mathcal{W}$ on which we wish to define a function using transfinite recursion. If $\mathrm{Split}(\mathcal{W}) \leq \kappa^+$, then by Lemma 3.22 there is a monic $m : \mathcal{W} \rightarrowtail \kappa^{[\tilde{\mathbb{N}}]}$. Hence we can assume, without loss of generality, that $\mathcal{W} \subseteq \kappa^{[\tilde{\mathbb{N}}]}$ (for some κ) with $\mathcal{W}(\mathbb{N}') = \emptyset$.

When using transfinite recursion to define a function f on a well-founded tree $\mathcal{W}$, we will assume that $f(x)$ is defined from the set $\{\langle y, f(y)\rangle : x <_1^{\mathcal{W}} y\}$. In other words there will be a function F such that if I is a function with domain $\{\langle y, f(y)\rangle : x <_1^{\mathcal{W}} y\}$ that could arise via the definition by transfinite recursion, then $f(x) = F(I)$. In this way the value of $f(x)$ will depend only on the values of f at the immediate successors of x.

In the description of transfinite recursion of the previous paragraph we have only required f to be defined on $\{y : x <_1^{\mathcal{W}} y\}$ in order for $f(x)$ to be defined. However, when dealing with transfinite recursion we will have that f is defined on $\{y : x <_1^{\mathcal{W}} y\}$ if and only if f is defined on $\{y : x \leq^{\mathcal{W}} y \text{ and } y \neq x\}$. As such we might think to instead give a definition by transfinite recursion where the value of $f(x)$ depends on the values of the entire set $\{y : x \leq^{\mathcal{W}} y \text{ and } y \neq x\}$ and not just the values $\{\langle y, f(y)\rangle : x <_1^{\mathcal{W}} y\}$.

It is worth stressing though that we loose no generality by assuming our definition of f only depends on $\{\langle y, f(y)\rangle : x <_1^{\mathcal{W}} y\}$. The reason is that if we had a definition of a function f where the value of x depended on $\{\langle y, f(y)\rangle : x \leq^{\mathcal{W}} y \text{ and } y \neq x\}$ we could consider the function f^*, where $f^*(x) = \langle f(y) : x \leq^{\mathcal{W}} y \text{ and } x \neq y\rangle$. It would then be easy to transform the definition of f into a transfinite definition of $f^*(x)$ where the value of $f^*(x)$ only depends on $\{\langle y, f^*(y)\rangle : x <_1^{\mathcal{W}} y\}$. Further we could then read off the values of f from the values of f^*.

The reason why we require our definition by transfinite recursion only to make use of values at the immediate successors of an element is that when we move to sheaf recursion it will not be the case that because our function is defined at every child of a node that it will be defined on all descendants of the node as well.

Definition 4.1. Suppose X is a function with domain $\bigcup_{n\in\mathbb{N}} \kappa^{[\tilde{\mathbb{N}}]}(\tilde{n})$ and range the universe of sets. We say a partial function H is an ***X-function*** if the domain of H is a subset of $\bigcup_{n\in\mathbb{N}} \kappa^{[\tilde{\mathbb{N}}]}(\tilde{n})$ and $H(x) \in X(x)$ whenever $H(x)\downarrow$. We also say

https://doi.org/10.1017/jsl.2013.18 Published online by Cambridge University Press

H is an X***-function in the first variable*** if H is a partial function on $\bigcup_{n\in\mathbb{N}} \kappa^{[\tilde{\mathbb{N}}]}(\tilde{n}) \times A$ (for some set A) where $H(x,a) \in X(x)$ whenever $H(x,a)\downarrow$.

It is worth emphasizing that in Definition 4.1 X is a set function and we do not assume any compatibility with the presheaf structure.

Definition 4.2. We say partial functions X, G, and $\langle F_n : n \in \mathbb{N} - \{0\}\rangle$ from a ***transfinite recursive definition*** on the pair $(\mathcal{W}, \kappa^{[\mathbb{N}]})$ if

- The domain of G is $\bigcup_{n\in\mathbb{N}} \left(\kappa^{[\tilde{\mathbb{N}}]}(\tilde{n}) - \mathcal{W}(\tilde{n})\right)$ and G is an X-function.
- For each $n \in \mathbb{N} - \{0\}$, each F_n takes two arguments, the first of which is an element of $\kappa^{[\tilde{\mathbb{N}}]}(\tilde{n})$ and the second of which is an X-function I^*. Further F_n is an X-function in its first argument.
- For each $n \in \mathbb{N}$ and all $x \in \kappa^{[\tilde{\mathbb{N}}]}(\tilde{n})$, I^* is total on $x\underset{n+1}{\overset{e}{\frown}}$ if and only if $F_{n+1}(x, I^*)$ is defined.
- For each $n \in \mathbb{N}$ and all $x \in \kappa^{[\tilde{\mathbb{N}}]}(\tilde{n})$ if I_0^*, I_1^* are defined and equal on $x\underset{n+1}{\overset{e}{\frown}}$, then $F_{n+1}(x, I_0^*) = F_{n+1}(x, I_1^*)$.

In Definition 4.2 G represents the base case of the transfinite recursion while $\langle F_n : n \in \mathbb{N} - \{0\}\rangle$ represents the induction case. Note that we have broken the inductive case into infinitely many parts, one for each level extending the first argument. This is done so that the notion of transfinite recursion will be a special case of sheaf recursion. While in the case of transfinite recursion the functions $\langle F_n : n \in \mathbb{N} - \{0\}\rangle$ can be combined into a single function, in the case of sheaf recursion we might not be able to do this as a pair (x, I^*) could be assigned many possible values, depending on the "direction" of extensions we are considering.

What makes transfinite recursion work is the following result.

Proposition 4.3 (Transfinite recursion). *Suppose X, G, and $\langle F_n : n \in \mathbb{N} - \{0\}\rangle$ are a transfinite recursive definition on $(\mathcal{W}, \kappa^{[\tilde{\mathbb{N}}]})$. Then there is an X-function I such that*:

(0) *For all* $x \in \bigcup_{n\in\mathbb{N}} \left(\kappa^{[\tilde{\mathbb{N}}]}(\tilde{n}) - \mathcal{W}(\tilde{n})\right)$, $I(x) = G(x)$.

(1) $(\forall n \in \mathbb{N})(\forall x \in \mathcal{W}(\tilde{n}))I(x)\downarrow \to [(\forall y \in x\underset{n+1}{\overset{e}{\frown}})I(y)\downarrow$ *and* $I(x) = F_{n+1}(x, I)]$.

(2) $(\forall n \in \mathbb{N})(\forall x \in \kappa^{[\tilde{\mathbb{N}}]}(\tilde{n}))[(\forall y \in x\underset{n+1}{\overset{e}{\frown}})I(y)\downarrow] \to I(x)\downarrow$.

(3) $(\forall n \in \mathbb{N})(\forall x \in \kappa^{[\tilde{\mathbb{N}}]}(\tilde{n}))I(x)\downarrow \to [(\forall y \in x\underset{n+1}{\overset{e}{\frown}})I(y)\downarrow]$.

(4) *I is defined on the unique element of* $\kappa^{[\tilde{\mathbb{N}}]}(\tilde{0})$.

Condition (0) is the base case, it tells us where to send those elements which are not in the well-founded tree. Condition (1) guarantees that if I is defined at a point x on level n in our well-founded tree, the value of I is determined by the corresponding function F_{n+1}. In particular, because of how F_{n+1} is defined, our function only depends on the values of the immediate successors of x. Condition (2) says that if I is defined on all successors of x (for any $x \in \bigcup_{n\in\mathbb{N}} \kappa^{[\tilde{\mathbb{N}}]}(\tilde{n})$), then I is defined on x. Condition (3) is the converse of Condition (2), i.e., that if I is defined at x, then I is defined on all successors of x. Finally, Condition (4) says that I is defined on the root.

Notice that Conditions (3) and (4) together imply that I is total on $\bigcup_{n\in\mathbb{N}} \kappa^{[\tilde{\mathbb{N}}]}(\tilde{n})$. However, in Proposition 4.3 we have not explicitly stated that I is total as in the analogous proposition for sheaf recursion the resulting function will not (necessarily) be total, and all we will be able to guarantee is that it is defined on the root. Similarly, it is also the case that such a function I is uniquely determined by G and $\langle F_n : n \in \mathbb{N} - \{0\}\rangle$ and Conditions (0)–(4). However, we have not mentioned this in Proposition 4.3 as this will not be the case when we consider sheaf recursion.

We now move on to the proof of Proposition 4.3.

Proof. We define partial functions I_α for $\alpha \in \mathrm{ORD}$ as follows:

(a) $I_0(x) := G(x)$ if $x \in \bigcup_{n\in\mathbb{N}} \left(\kappa^{[\tilde{\mathbb{N}}]}(\tilde{n}) - \mathcal{W}(\tilde{n})\right)$ and undefined otherwise.

(b) $I_\alpha := \bigcup_{\gamma<\alpha} I_\gamma$ if α is a limit ordinal.

(c) $I_\alpha(x)$ breaks into three cases if $\alpha = \beta + 1$, $x \in \kappa^{[\tilde{\mathbb{N}}]}(\tilde{n})$, and $n \in \mathbb{N}$:
  - (i) If $I_\beta(x)$ is defined, then $I_\alpha(x) := I_\beta(x)$.
  - (ii) If $I_\beta(x)$ is undefined and $(\forall y \in x^{e}_{\widetilde{n+1}}) I_\beta(y)\downarrow$, then $I_\alpha(x) := F_{n+1}(x, I_\beta)$.
  - (iii) Otherwise $I_\alpha(x)$ is undefined.

We let $I = \bigcup_{\alpha\in\mathrm{ORD}} I_\alpha$. Note that each I_α as well as I are X-functions.

It is immediate that Conditions (0)–Condition (2) hold for I.

To see that Condition (3) holds suppose $I(x)\downarrow$ and $x \in \kappa^{[\tilde{\mathbb{N}}]}(\tilde{n})$ with $n \in \mathbb{N}$. If $x \notin \mathcal{W}(\tilde{n})$, then we know $x^{e}_{\widetilde{n+1}} \cap \mathcal{W}(\widetilde{n+1}) = \emptyset$ (as $\mathcal{W}$ is a tree and hence closed under predecessors). In particular, this means that for all $y \in x^{e}_{\widetilde{n+1}}$ we have $I(y) = G(y)$ and hence $I(y)$ is defined. However, if $x \in \mathcal{W}(\tilde{n})$, then there must be a least α such that $I_{\alpha+1}(x)$ is defined . But then because of how I was constructed, we must have that I_α is defined for all $y \in x^{e}_{\widetilde{n+1}}$ and hence I must also be defined on $x^{e}_{\widetilde{n+1}}$.

All that is left is to show Condition (4) holds. As this is the most important condition, we will make it its own claim.

Claim 4.4. *I is defined on the unique element x_0 of $\mathcal{W}(\tilde{0}) = \kappa^{[\tilde{\mathbb{N}}]}(\tilde{0})$.*

Proof. Assume to get a contradiction that I is not defined on x_0. We will first use ordinary recursion on $\mathbb{N}$ to construct a sequence $\langle x_n : n \in \mathbb{N}\rangle$ such that

- for each $n \in \mathbb{N}$, $x_n \in \mathcal{W}(\tilde{n})$,
- for all $m, n \in \mathbb{N}$ with $\tilde{m} \subseteq \tilde{n}$, $x_n|_{\tilde{m}} = x_m$, and
- for all $n \in \mathbb{N}$, $I(x_n)$ is undefined.

Note that the base case of x_0 is done by assumption. Now suppose x_n is defined as above in order to find an x_{n+1} which also satisfies the above conditions. Note because $I(x_n)$ is undefined and $x_n \in \mathcal{W}(\tilde{n})$, there must be some $y \in \kappa^{[\tilde{\mathbb{N}}]}(\widetilde{n+1})$ such that $x_n <^{\kappa^{[\tilde{\mathbb{N}}]}}_{\tilde{1}} y$ and $I(y)$ is undefined. But as $I(y)$ is undefined we must also have $y \in \mathcal{W}(\widetilde{n+1})$. Therefore if we let $x_{n+1} = y$, we are done.

By ordinary recursion we can therefore find some sequence $\langle x_n : n \in \mathbb{N}\rangle$ as above. Then as $\mathcal{W}$ is a sheaf, and $\langle (x_n, \tilde{n}) : n \in \mathbb{N}\rangle$ is a compatible collection of elements from $\mathcal{W}$ there must be an $x^* \in \mathcal{W}(\mathbb{N}')$. But this contradicts our assumption that $\mathcal{W}$ is well-founded.

Therefore $I(x_0)$ must be defined. ⊣

⊣

https://doi.org/10.1017/jsl.2013.18 Published online by Cambridge University Press

4.2. Sheaf recursion. We now have all of the components necessary to define sheaf recursion. Fix a flabby second order tree $\mathcal{K}$ over $\mathcal{T}$, and a well-founded second order tree $\mathcal{W} \subseteq \mathcal{K}$.

DEFINITION 4.5. Suppose X is a function with domain $\bigcup_{t \in T} \mathcal{K}(\tilde{t})$ and range the universe of sets. We say a partial function H is an ***X-function*** if the domain of H is a subset of $\bigcup_{t \in T} \mathcal{K}(\tilde{t})$ and $H(x) \in X(x)$ whenever $H(x) \downarrow$. We also say H is an ***X-function in the first variable*** if H is a partial function on $\bigcup_{t \in T} \mathcal{K}(\tilde{t}) \times A$ (for some set A) where $H(x, a) \in X(x)$ whenever $H(x, a) \downarrow$.

Note that it is consistent for $X(x) = \emptyset$ and yet still to have H be an X function so long as $H(x)$ is undefined.

DEFINITION 4.6. We say partial functions X, G and $\langle F_V : V \in \mathcal{T} - \{r^{\mathcal{T}}\}\rangle$ form a ***sheaf recursive definition*** on the pair $(\mathcal{W}, \mathcal{K})$ if

- The domain of G is $\bigcup_{V \in T} \left(\mathcal{K}(\tilde{V}) - \mathcal{W}(\tilde{V})\right)$ and G is an X-function.
- For each $V \in T - \{r^{\mathcal{T}}\}$, F_V takes two arguments, the first of which is an element of $\mathcal{K}(\tilde{U})$ where $U <_1^{\mathcal{T}} V$ and the second of which is an X-function I^*. Further F_V is an X-function in its first argument.
- For each $U, V \in T$ with $U <_1^{\mathcal{T}} V$ and all $x \in \mathcal{W}(\tilde{U})$, I^* is total on $x^e_{\tilde{V}}$ if and only if $F_V(x, I^*)$ is defined.
- For each $U, V \in T$ with $U <_1^{\mathcal{T}} V$ and all $x \in \mathcal{W}(\tilde{U})$ if I_0^*, I_1^* are defined and equal on $x^e_{\tilde{V}}$, then $F_V(x, I_0^*) = F_V(x, I_1^*)$.

Notice that every transfinite recursive definition is also a sheaf recursive definition by considering first order trees $\kappa^{[\tilde{\mathbb{N}}]}$ and $\mathcal{W}$ as second order trees over $\tilde{\mathbb{N}}$. The key to making sheaf recursion work is the following result which is a direct analog of Proposition 4.3.

THEOREM 4.7 (Sheaf recursion). *Suppose X, G, and $\langle F_V : V \in \mathcal{T} - \{r^{\mathcal{T}}\}\rangle$ is a sheaf recursive definition. Then there is a (partial) X-function I such that:*

(0) *For all $x \in \bigcup_{V \in T} \left(\mathcal{K}(\tilde{V}) - \mathcal{W}(\tilde{V})\right)$, $I(x) = G(x)$.*
(1) $(\forall U \in T)(\forall x \in \mathcal{W}(\tilde{U}))I(x) \downarrow \rightarrow [(\exists V >_1^{\mathcal{T}} U)[(\forall y \in x^e_{\tilde{V}})I(y) \downarrow$ *and* $I(x) = F_V(x, I)]]$.
(2) $(\forall U \in T)(\forall x \in \mathcal{K}(\tilde{U}))[(\exists V >_1^{\mathcal{T}} U)(\forall y \in x^e_{\tilde{V}})I(y) \downarrow] \rightarrow I(x) \downarrow$.
(3) $(\forall U \in T)(\forall x \in \mathcal{K}(\tilde{U}))I(x) \downarrow \rightarrow [(\exists V >_1^{\mathcal{T}} U)(\forall y \in x^e_{\tilde{V}})I(y) \downarrow]$.
(4) *I is defined on the unique element of $\mathcal{K}(\widetilde{r^{\mathcal{T}}})$.*

Conditions (0)–(4) are in direct analog with conditions (0)–(4) in Proposition 4.3. The main difference is that when dealing with transfinite recursion, in order to define I at an element in the well-founded first order tree we need that I is first defined on every successor of the element. However, in the case of sheaf recursion we can think of the successors of an element in $\mathcal{K}(\tilde{U})$ as being in several different directions (indexed by elements $V \in T$, $V >_1^{\mathcal{T}} U$) and in order to define I at an element, our definition only needs that there is some direction such that I is defined on all successors of the element in that direction.

We now prove Theorem 4.7.

PROOF. First, choose an arbitrary well-ordering $\preceq$ of T. We define partial functions I_α for $\alpha \in \mathrm{ORD}$ as follows:

(a) $I_0(x) := G(x)$ if $x \in \bigcup_{V \in T} \mathcal{K}(\tilde{V}) - \mathcal{W}(\tilde{V})$ and undefined otherwise.
(b) $I_\alpha := \bigcup_{\gamma<\alpha} I_\gamma$ if α is a limit ordinal.
(c) $I_\alpha(x)$ breaks into three cases if $\alpha = \beta + 1$, $x \in \mathcal{K}(\tilde{U})$, and $U \in T$:
  (i) If $I_\beta(x)$ is defined, then $I_\alpha(x) = I_\beta(x)$.
  (ii) If $I_\beta(x)$ is undefined and $(\exists V >_1^{\mathcal{T}} U)(\forall y \in x^e_{\tilde{V}}) I_\beta(y)\downarrow$.
  Then let $Q \in T$ be the $\preceq$-minimal such V and $I_\alpha(x) := F_Q(x, I_\beta)$.
  (iii) Otherwise $I_\alpha(x)$ is undefined.

We let $I = \bigcup_{\alpha \in \mathrm{ORD}} I_\alpha$.

It is immediate that Conditions (0), (2), and (3) of Theorem 4.7 hold. Further (1) follows from the fact that $F_V(x, I)$ only depends on the values of I on $x^e_{\tilde{V}}$. All that is left is to show Condition (4) holds. As with Proposition 4.3 this is the most important part of the theorem so we make it its own claim.

Claim 4.8. *I is defined on the unique element $x_{r^\mathcal{T}}$ of $\mathcal{W}(\widetilde{r^\mathcal{T}}) = \mathcal{K}(\widetilde{r^\mathcal{T}})$.*

Proof. Assume to get a contradiction that I is not defined on $x_{r^\mathcal{T}}$. We will first use ordinary recursion on $\mathbb{N}$ to construct a sequence $\langle x_U : U \in T\rangle$ such that

- for each $U \in T$, $x_U \in \mathcal{W}(\tilde{U})$,
- for all $U, V \in T$ with $\tilde{U} \subseteq \tilde{V}$, $x_V|_{\tilde{U}} = x_U$, and
- for all $U \in T$, $I(x_U)$ is undefined.

First note $\mathcal{T}(\tilde{0}) = \{r^\mathcal{T}\}$ and so for all $U \in \mathcal{T}(\tilde{0})$ we have x_U is defined. This is the base case of the ordinary recursion.

Now assume for $n \in \mathbb{N}$ that for all $U \in \mathcal{T}(\tilde{n})$ we have x_U is defined and satisfies the above. Let $V \in \mathcal{T}(\widetilde{n+1})$ be such that $U <_1^{\mathcal{T}} V$. Note that if I is defined on all $y \in (x_U)^e_{\tilde{V}} \subseteq \mathcal{K}(\tilde{V})$, then because of how I was constructed, I is defined on x_U. Therefore there must be some $y \in (x_U)^e_{\tilde{V}}$ with $I(y)$ undefined. Further, because I is defined outside of $\mathcal{W}$, we must have that such a y is in $\mathcal{W}(\tilde{V})$. Let x_U be such a y. We have then defined x_V for all $V \in \mathcal{T}(\widetilde{n+1})$.

By ordinary recursion we can therefore find some collection $\langle x_U : U \in T\rangle$ as above. Now suppose we have $U_0, U_1 \in T$ with $U^* \in T$ such that $\tilde{U}_0 \cap \tilde{U}_1 = \tilde{U}^*$. By our construction we therefore have that $x_{U_0}|_{\tilde{U}_0 \cap \tilde{U}_1} = x_{U_0}|_{\tilde{U^*}} = x_{U^*} = x_{U_1}|_{\tilde{U^*}} = x_{U_1}|_{\tilde{U}_0 \cap \tilde{U}_1}$. Hence $\langle x_U : U \in T\rangle$ is a compatible collection of elements from $\mathcal{W}$.

But as $\mathcal{W}$ is a sheaf this implies that there must be an $x^* \in \mathcal{W}(\mathcal{T}')$ which is an amalgamation of $\langle x_U : U \in T\rangle$. This however contradicts the well-foundedness of $\mathcal{W}$.

Therefore $I(x_{r^\mathcal{T}})$ must be defined. ⊣

⊣

4.3. Axiom of choice. Our proof of Theorem 4.7 made use of the axiom of choice both in the well-ordering of the tree T as well as in our choice of elements $\langle x_V : V \in T\rangle$. In this section we show that the axiom of choice is in fact needed.

Theorem 4.9. *If M is a model of Zermelo-Fraenkel set theory, then the following are equivalent*:

(1) *$M \models$ Axiom of Choice.*
(2) *For all sheaf recursive definitions in M Theorem* 4.7 *holds in M.*

https://doi.org/10.1017/jsl.2013.18 Published online by Cambridge University Press

Proof. That (1) implies (2) is Theorem 4.7. Now assume (2) to show (1). Suppose $\langle A_j : j \in J\rangle \in M$ is an arbitrary collection of (disjoint) nonempty sets. We will show there is an element of $\prod_{j\in J} A_j$ in M. For the rest of the proof we work in M.

Let $T := \{r\} \cup \langle \omega \times J\rangle$ where

- for all $m \in \omega$ and all $\alpha \in J$, we have $r^T \leq^T \langle m, \alpha\rangle$, and
- for all $m, n \in \omega$ and all $\alpha, \beta \in J$, we have $\langle m, \alpha\rangle <_1^{\mathcal{T}} \langle n, \beta\rangle$ if and only if $\alpha = \beta$ and $n = m + 1$.

In particular, $(T, \leq)$ consists of J incompatible copies of ω with a minimal element adjoined.

We now let $\mathcal{K}(r) := \{\flat\}$ and for all $n \in \omega$ and $j \in J$ we let $\mathcal{K}(\langle n, j\rangle) := \{n\} \times (A_j \cup \{\langle *, j\rangle\})$ and for all $a \in A_j \cup \{\langle *, j\rangle\}$ let $\langle n+1, a\rangle|_{\langle n,j\rangle} := \langle n, a\rangle$. Let $\mathcal{W} \subseteq \mathcal{K}$ be such that $\mathcal{W}(\langle n, j\rangle) = \{n\} \times A_j$.

Claim 4.10. *$\mathcal{K}$ is flabby.*

Proof. Suppose $\emptyset \neq U \in \mathcal{O}(\tilde{\mathcal{T}})$ and $a \in \mathcal{K}(U)$. Notice for any $j \in J$, $P_j \subseteq \omega$ with $s = \sup\{i + 1 : i \in P_j\}$, $\bigcup_{n\in P_j} \widetilde{\langle n, j\rangle} = \bigcup_{n<s} \widetilde{\langle n, j\rangle}$. So as U is open and hence of the form $\bigcup_{j\in J} \bigcup_{n\in P_j \subseteq\omega} \widetilde{\langle n, j\rangle}$ we know there is a collection of elements of T such that $U = \bigcup\{\widetilde{\langle n, j_h\rangle} : h \in H, n < n_h\}$, where $j_h \neq j_{h'}$ for $h \neq h'$ and $n_h \leq \omega$. For $h \in H$ and $n < n_h$ let $\langle n, a_h\rangle = a|_{\widetilde{\langle n, j_h\rangle}}$. The following is then a compatible collection of elements:

- $\{\langle n, a_h\rangle : n \in \omega, h \in H\}$ and
- $\{\langle n, \langle *, l\rangle\rangle : n \in \omega, l \in (J - \{j_h : h \in H\})\}$.

Hence there is an element $a^* \in \mathcal{K}(\mathcal{T}')$ which is an amalgamation of the above collection. But then we also have $a^*|_U = a$. But as a was arbitrary and $\mathcal{K}$ is nontrivial we have $\mathcal{K}$ is flabby. ⊣

Let $\kappa \in$ ORD be infinite and such that any well-founded tree with underlying set $\bigcup_{V\in T} \mathcal{K}(\tilde{V})$ has rank in κ. Let X be the constant function on $\bigcup_{V\in T} \mathcal{K}(\tilde{V})$ with value the set κ. Let G be the constant function on $\bigcup_{V\in T} \left(\mathcal{K}(\tilde{V}) - \mathcal{W}(\tilde{V})\right)$ with value 0. G is clearly an X-function. Finally, for each $U, V \in T$ with $U <_1^{\mathcal{T}} V$ every $x \in \mathcal{W}(\tilde{U})$ and every X-function I^*, let $F_V(x, I^*) = \sup\{I^*(y) + 1 : y \in x^e_{\tilde{V}}\}$. It is immediate that F_V is an X-function in its first variable.

Claim 4.11. *$\mathcal{W}$ is not a well-founded sheaf.*

Proof. Assume to get a contradiction that $\mathcal{W}$ is well-founded. We then have $X, G, \langle F_V : V \in \mathcal{T} - \{r^{\mathcal{T}}\}\rangle$ is a definition by sheaf recursion. Hence by our assumption there is a witness I to this fact, where $I(\flat)\downarrow$. But then we must have some $\langle 0, j\rangle \in \mathcal{T}(\tilde{1})$ such that I is total on $\mathcal{W}(\langle 0, j\rangle) = \{0\} \times A_j$. However, for each $n \in \omega$, $\langle n, j\rangle$ only has one successor in $\mathcal{T}$ (i.e., $\langle n+1, j\rangle$). Hence we must have that I is total on $\mathcal{W}(\widetilde{\langle n, j\rangle})$ for each n.

Now consider the first order tree $\mathcal{W}^* := \langle W^*, \leq^{\mathcal{W}^*}, <_1^{\mathcal{W}^*}\rangle$ where

- the underlying set is $W^* := \{*_{\mathcal{W}}\} \cup \bigcup_{n\in\omega} \mathcal{W}(\langle n, j\rangle)$,
- $*_{\mathcal{W}}$ is the root, and
- for each $m, n \in \omega$ and $\alpha, \beta \in J$, we have $\langle m, \alpha\rangle \leq^{\mathcal{W}^*} \langle n, \beta\rangle$ if and only if $\beta = \alpha$ and $m \leq n$, i.e., if $\langle n, \beta\rangle|_{\widetilde{\langle m, j\rangle}} = \langle m, \alpha\rangle$.

Because of how I was defined, I assigns an ordinal number to each element of $\mathcal{W}^*$ in such a way that if $a <_1^{\mathcal{W}^*} b$, then $I(a) > I(b)$. But this is a contradiction as then $\langle I(\langle n, \alpha\rangle) : n \in \omega\rangle$ is an infinite descending sequence of ordinals (for any $\alpha \in A_j$). In particular, this implies $\mathcal{W}$ must not be well-founded. ⊣

Let $x \in \mathcal{W}(\mathcal{T}')$ and let $\langle a_j : j \in J\rangle$ be such that $x|_{\widetilde{\langle 0, j\rangle}} = \langle 0, a_j\rangle$. Then for each $j \in J$ we have $a_j \in A_j$ and hence $\langle a_j : j \in J\rangle \in \prod_{j \in J} A_j$ as desired. In particular, as $\langle A_j : j \in J\rangle$ was arbitrary, the axiom of choice holds in M. ⊣

§5. Separation theorems.

5.1. Borelian and analytic sets. The following definitions are motivated by their counterparts for κ^ω.

Definition 5.1. The set of ***κ*-*Borelian*** subsets of $\mathcal{S}(\mathcal{T}')$, $\mathrm{Bor}_\kappa(\mathcal{S})$, is the smallest collection of subsets of $\mathcal{S}(\mathcal{T}')$ such that:

- All closed subsets of $\mathcal{S}(\mathcal{T}')$ are in $\mathrm{Bor}_\kappa(\mathcal{S})$.
- $\mathrm{Bor}_\kappa(\mathcal{S})$ is closed under $< \kappa$-unions and $< \kappa$-intersections.

The set of ***κ*-*Borel*** subsets of $\mathcal{S}(\mathcal{T}')$, $\mathrm{Bor}^*_\kappa(\mathcal{S})$, is the smallest collection of subsets such that:

- All closed subsets of $\mathcal{S}(\mathcal{T}')$ are in $\mathrm{Bor}^*_\kappa(\mathcal{S})$.
- $\mathrm{Bor}^*_\kappa(\mathcal{S})$ is closed under $< \kappa$-unions and complements.

Note by Lemma 3.18 the following is immediate.

Lemma 5.2. *If* $\kappa > |\bigcup_{t \in T} \mathcal{S}(\tilde{t})|$, *then* $Bor_\kappa(\mathcal{S}) = Bor^*_\kappa(\mathcal{S})$.

If $\kappa \leq |\bigcup_{t \in T} \mathcal{S}(\tilde{t})|$ though, the κ-Borelian and κ-Borel sets need not coincide.

Definition 5.3. Suppose $\mathcal{K}$ and $\mathcal{N}$ are flabby second order trees over $\mathcal{T}$. A set $X \subseteq \mathcal{N}(\mathcal{T}')$ is said to be ***K*-*Suslin*** if there is a closed $D \subseteq \mathcal{K}(\mathcal{T}') \times \mathcal{N}(\mathcal{T}')$ with $X = p_{\mathcal{K}}[D] = \{x : (\exists y \in \mathcal{K}(\mathcal{T}'))(y, x) \in D\}$.

In the case where $\mathcal{T}$ is $\tilde{\mathbb{N}}$, $\mathcal{N} = \omega^{[\tilde{\mathbb{N}}]}$, and $\mathcal{K} = \kappa^{[\tilde{\mathbb{N}}]}$, then a set $Y \subseteq \kappa^{[\tilde{\mathbb{N}}]}(\mathbb{N}')$ is κ-Borelian ($\kappa^{[\tilde{\mathbb{N}}]}$-Suslin) if and only if it is κ-Borelian (κ-Suslin) in the usual sense.

It is worth mentioning that in the case of second order trees, unlike in the case of first order trees, the $\mathcal{K}$-Souslin subsets need not be closed under all finite unions. An easy example of this is when $\mathcal{K} = 1^{[\tilde{\mathcal{T}}]}$ is a terminal second order tree $1^{[\tilde{\mathcal{T}}]}$-Souslin sets are just the closed sets. Hence by Proposition 3.24 the $1^{[\tilde{\mathcal{T}}]}$-Souslin sets are closed under finite unions if and only if $\mathcal{T}$ is a linear order.

We say a pair of sets C_A, C_B **separates** A **and** B if $A \subseteq C_A$, $B \subseteq C_B$(,) and $C_A \cap C_B = \emptyset$. We say a class of sets Γ can be **separated** by κ-Borelian sets if for every $A, B \in \Gamma$ with $A \cap B = \emptyset$, there is a pair of κ-Borelian sets which separates A from B. The following is then immediate.

Lemma 5.4. *The class of all subsets of* $\mathcal{S}(\mathcal{T}')$ *can be separated by* $|\mathcal{S}(\mathcal{T}')|^+$-*Borelian sets.*

Proof. Because every one point set in $\mathcal{S}(\mathcal{T}')$ is closed, every set is the $|\mathcal{S}(\mathcal{T}')|^+$-union of closed sets and hence $|\mathcal{S}(\mathcal{T}')|^+$-Borelian. ⊣

Given a class of sets we can think of the smallest κ needed to separate any two elements of the class by κ-Borelain sets as a measure of the complexity of the class.

https://doi.org/10.1017/jsl.2013.18 Published online by Cambridge University Press

5.2. Lusin separation theorem. One of the most important theorems of descriptive set theory is the Lusin separation theorem.[2]

THEOREM 5.5 (Lusin separation theorem). *Suppose κ is an infinite cardinal and A and B are disjoint κ-Suslin subsets of ω^ω. Then there is a pair of κ^+-Borel sets which separates A and B.*

This is the best possible, as we could not hope to have all κ-Souslin sets separated by κ-Borel subsets seeing as every κ^+-Borelian subset is also κ-Souslin. We can think of κ-Souslin sets as being "close" to κ^+-Borel sets. In fact a consequence of this theorem is that any set which is both κ-Souslin and whose complement is κ-Souslin is actually κ^+-Borel.

We will now give an analog of this theorem for second order trees. Just as the proof of the Lusin separation theorem made fundamental use of transfinite recursion, the proof of our separation theorem will make fundamental use of sheaf recursion. The interested reader is encouraged to compare the below proof to a standard proof of the Lusin separation theorem such as that given in [8] (2E.1:Constructive Proof).

We first need a standard lemma.

LEMMA 5.6. *Suppose $A = \bigcup_{i\in I} A_i$ and $B = \bigcup_{j\in J} B_j$ and suppose $C^A_{i,j}, C^B_{i,j}$ separate A_i and B_j. Then the sets $C_A = \bigcup_{i\in I}\bigcap_{j\in J} C^A_{i,j}$ and $C_B = \bigcup_{j\in J}\bigcap_{i\in I} C^B_{i,j}$ separate A and B.*

PROOF. First notice that as $A_i \subseteq \bigcap_{j\in J} C^A_{i,j}$ for each $i \in I$, we have $A = \bigcup_{i\in I} A_i \subseteq \bigcup_{i\in I}\bigcap_{j\in J} C^A_{i,j} = C_A$. Similarly, we have $B \subseteq C_B$.

Now notice

$$C_A \cap C_B = \left[\bigcup_{i\in I}\bigcap_{j\in J} C^A_{i,j}\right] \cap \left[\bigcup_{j'\in J}\bigcap_{i'\in I} C^B_{i',j'}\right]$$
$$= \bigcup_{i\in I}\bigcup_{j'\in J}\bigcap_{j\in J}\bigcap_{i'\in I} C^A_{i,j} \cap C^B_{i',j'}.$$

However, for each $i \in I, j' \in J$, $\bigcap_{j\in J}\bigcap_{i'\in I} C^A_{i,j} \cap C^B_{i',j'} = \emptyset$ as $C^A_{i,j'} \cap C^B_{i,j'} = \emptyset$ by assumption. Hence $C_A \cap C_B = \emptyset$. ⊣

THEOREM 5.7 (Separation theorem). *Suppose $\mathcal{N}$ and $\mathcal{K}$ are flabby second order trees over $\mathcal{T}$ with $Split(\mathcal{K}\times\mathcal{N}) = \kappa$, a regular cardinal. Further suppose A and B are disjoint $\mathcal{K}$-Suslin subsets of $\mathcal{N}(\mathcal{T}')$. Then there are κ^+-Borelian sets $C_A, C_B \subseteq \mathcal{N}(\mathcal{T}')$ which separates A and B.*

PROOF. In the proof we will treat $\mathcal{K}$ and $\mathcal{N}$ as sheaves. For each $U \in T$ and $\langle k_0, k_1, n\rangle \in \mathcal{K}\times\mathcal{K}\times\mathcal{N}(\tilde{U})$ we will let $\sigma(\langle k_0, k_1, n\rangle) = \langle k_0, n\rangle$ and $\tau(\langle k_0, k_1, n\rangle) = \langle k_1, n\rangle$.

As A and B are $\mathcal{K}$-Souslin, there are sheaves $A^+, B^+ \subseteq \mathcal{K}\times\mathcal{N}$ be such that $A = p_{\mathcal{K}}[A^+(\mathcal{T}')]$ and $B = p_{\mathcal{K}}[B^+(\mathcal{T}')]$. Now let $\mathcal{W} \subseteq \mathcal{K}\times\mathcal{K}\times\mathcal{N}$, where for all $U \in \mathcal{O}(\tilde{\mathcal{T}})$, $\mathcal{W}(U) = \{s \in \mathcal{K}\times\mathcal{K}\times\mathcal{N}(U) : \sigma(s) \in A^+(U) \text{ and } \tau(s) \in B^+(U)\}$.

[2]This is sometimes, such as in [8], called the *Souslin-Kleene Separation Theorem* or the *Souslin-Kleene-Addison Separation Theorem*.

https://doi.org/10.1017/jsl.2013.18 Published online by Cambridge University Press

Claim 5.8. *$\mathcal{W}$ is a well-founded sheaf over $\tilde{\mathcal{T}}$.*

Proof. That $\mathcal{W}$ is a separated presheaf is immediate from the fact that A^+ and B^+ are separated presheaves. Now suppose we have $\langle(\langle x_i, y_i, z_i\rangle, U_i) : i \in I\rangle$ is a compatible collection of elements from $\mathcal{W}$ with $U = \bigcup_{i\in I} U_i$. Then $\langle(\langle x_i, z_i\rangle, U_i) : i \in I\rangle$ and $\langle(\langle y_i, z_i\rangle, U_i) : i \in I\rangle$ are compatible collections of elements from $\mathcal{K}\times\mathcal{N}$ and hence must have amalgamations $\langle x, z\rangle \in \mathcal{K}\times\mathcal{N}(U)$ and $\langle y, z'\rangle \in \mathcal{K}\times\mathcal{N}(U)$, respectively. But then both z and z' are amalgamations of $\langle(z_i, U_i) : i \in I\rangle$, hence $z = z'$ as $\mathcal{K}$ is a sheaf. Therefore $\langle x, y, z\rangle \in \mathcal{K}\times\mathcal{K}\times\mathcal{N}(U)$ is an amalgamation of $\{(\langle x_i, y_i, z_i\rangle, U_i) : i \in I\}$. In particular, this implies that $\mathcal{W}$ is a sheaf.

Finally, because $p_{\mathcal{K}}[A^+(\mathcal{T}')] \cap p_{\mathcal{K}}[B^+(\mathcal{T}')] = \emptyset$, we have that $\mathcal{W}(\mathcal{T}') = \emptyset$ and hence $\mathcal{W}$ is well-founded. ⊣

Let X be the function with domain $\bigcup_{U\in T} \mathcal{K}\times\mathcal{K}\times\mathcal{N}(\tilde{U})$ such that $X(x)$ is the collection of pairs $C_A, C_B \subseteq \mathcal{N}(\mathcal{T}')$ such that C_A, C_B are κ-Borelian and separate $p_{\mathcal{K}}[A^+_{\sigma(x)}(\mathcal{T}')]$ from $p_{\mathcal{K}}[B^+_{\tau(x)}(\mathcal{T}')]$ (where $A^+_{\sigma(x)}$ and $B^+_{\tau(x)}$ are as in Definition 3.16). We will now use sheaf recursion on $\mathcal{W}$ to produce an element of $X(\widetilde{r^{\mathcal{T}}})$.

First we take care of the base case and define G. Suppose $s \in \mathcal{K}\times\mathcal{K}\times\mathcal{N}(\tilde{U}) - \mathcal{W}(\tilde{U})$ for some $U \in T$. We now have two cases:

Case 1: $\sigma(s) \notin A^+(\tilde{U})$. In this case let $G(s) = (\emptyset, \mathcal{N}(\mathcal{T}'))$.

Case 2: Otherwise. In this case we must have $\tau(s) \notin B^+(\tilde{U})$ so let $G(s) = (\mathcal{N}(\mathcal{T}'), \emptyset)$.

It is easy to check that in Case 1 $A^+_{\sigma(s)}(\mathcal{T}') = \emptyset$ and hence $\emptyset, \mathcal{N}(\mathcal{T}')$ separates $A^+_{\sigma(s)}(\mathcal{T}')$ and $B^+_{\tau(s)}(\mathcal{T}')$ and in Case 2 $B^+_{\tau(s)}(\mathcal{T}') = \emptyset$ and hence $\mathcal{N}(\mathcal{T}'), \emptyset$ separates $A^+_{\sigma(s)}(\mathcal{T}')$ and $B^+_{\tau(s)}(\mathcal{T}')$. Therefore in either case $G(s) \in X(s)$ and so G is an X-function.

Now, for each $U <^{\mathcal{T}}_1 V$, $x \in \mathcal{K}\times\mathcal{K}\times\mathcal{N}(\tilde{U})$ and I^* which is an X-function on $\bigcup_{Z\in T} \mathcal{K}\times\mathcal{K}\times\mathcal{N}(\tilde{Z})$ we need to define $F_V(x, I^*)$. First notice that we only need $F_V(x, I^*)$ to be defined if I^* is total on $x^e_{\tilde{V}}$ and in this case $F_V(x, I^*)$ should only depend on x and the values of I^* restricted to $x^e_{\tilde{V}}$.

Suppose, for all $a \in \sigma(x)^e_{\tilde{V}}$ and $b \in \tau(x)^e_{\tilde{V}}$ we can find, using only x and I^* restricted to $x^e_{\tilde{V}}$, κ^+-Borelain $D_{a,b}$ and $E_{a,b}$ which separate $p_{\mathcal{K}}[A^+_a(\mathcal{T}')]$ from $p_{\mathcal{K}}[B^+_b(\mathcal{T}')]$. Then, because $\mathcal{K}$ is flabby and

$$p_{\mathcal{K}}[A^+_{\sigma(x)}(\mathcal{T}')] = \bigcup_{a\in\sigma(x)^e_{\tilde{V}}} p_{\mathcal{K}}[A^+_a(\mathcal{T}')] \text{ and } p_{\mathcal{K}}[B^+_{\tau(x)}(\mathcal{T}')] = \bigcup_{b\in\tau(x)^e_{\tilde{V}}} p_{\mathcal{K}}[B^+_b(\mathcal{T}')],$$

we have by Lemma 5.6 the sets $D^+ := \bigcup_{a\in\sigma(x)^e_{\tilde{V}}} \bigcap_{b\in\tau(x)^e_{\tilde{V}}} D_{a,b}$ and $E^+ := \bigcup_{b\in\tau(x)^e_{\tilde{V}}} \bigcap_{a\in\sigma(x)^e_{\tilde{V}}} E_{a,b}$ separate $p_{\mathcal{K}}[A^+_{\sigma(x)}(\mathcal{T}')]$ and $p_{\mathcal{K}}[B^+_{\tau(x)}(\mathcal{T}')]$. Further, as κ is the splitting number of $\mathcal{K}\times\mathcal{N}$, each $D_{a,b}, E_{a,b}$ are κ-Borelian, and κ is regular, D^+, E^+ are also κ-Borelian. We can then let $F_V(x, I^*) = (D^+, E^+)$.

All that is left in the definition of F_V is to find such $D_{a,b}, E_{a,b}$ for all $a \in \sigma(x)^e_{\tilde{V}}$ and $b \in \tau(x)^e_{\tilde{V}}$. To do this we break into cases.

Case 1: There is a $y \in \mathcal{W}(\tilde{V})$, where $a = \sigma(y), b = \tau(y)$.
In this case we can choose $D_{a,b}, E_{a,b}$ so that $(D_{a,b}, E_{a,b}) = I^*(y)$. These have the desired properties as $I(y) \in X(y)$.

Case 2: There is a $y \in \mathcal{K} \times \mathcal{K} \times \mathcal{N}(\tilde{V}) - \mathcal{W}(\tilde{V})$, where $a = \sigma(y), b = \tau(y)$.

Case 2a: $a = \sigma(y) \notin A^+(\tilde{V})$. In this case $p_{\mathcal{K}}[A^+_a(\mathcal{T}')] = \emptyset$ and we can let $D_{a,b} = \emptyset$, while $E_{a,b} = \mathcal{N}(\mathcal{T}')$.

Case 2b: $b = \tau(y) \notin B^+(\tilde{V})$. In this case $p_{\mathcal{K}}[B^+_b(\mathcal{T}')] = \emptyset$ and we can let $D_{a,b} = \mathcal{N}(\mathcal{T}')$, while $E_{a,b} = \emptyset$.

Case 3: Otherwise, i.e., $a = (k_a, n_a)$ and $b = (k_b, n_b)$ with $n_a \neq n_b$.
In this case $p_{\mathcal{K}}[(\mathcal{K} \times \mathcal{N})_a(\mathcal{T}')] \subseteq \mathcal{N}_{n_a}(\mathcal{T}')$ and $p_{\mathcal{K}}[(\mathcal{K} \times \mathcal{N})_b(\mathcal{T}')] \subseteq \mathcal{N}_{n_b}(\mathcal{T}')$ with $\mathcal{N}_{n_a}(\mathcal{T}') \cap \mathcal{N}_{n_b}(\mathcal{T}') = \emptyset$. We can therefore let $D_{a,b} = \mathcal{N}_{n_a}(\mathcal{T}')$ and $E_{a,b} = \mathcal{N}_{n_b}(\mathcal{T}')$.

Note $F_V(x, I^*)$ is defined if and only if I^* is total on $x^e_{\tilde{V}}$. Hence $X, G, \langle F_V : V \in \mathcal{T}' \rangle$ form a sheaf recursive definition on the pair $(\mathcal{W}, \mathcal{K} \times \mathcal{K} \times \mathcal{N})$. In particular, there must be some X-function I which is defined on the unique element $* \in \mathcal{K} \times \mathcal{K} \times \mathcal{N}(\widetilde{r^T})$. But then $I(*) \in X(*)$ (and, in particular, $X(*) \neq \emptyset$). So if $I(*) = (C_A, C_B)$, then C_A, C_B must be κ-Borelian and must separate $p_{\mathcal{K}}[A^+_{\sigma(*)}(\mathcal{T}')] = p_{\mathcal{K}}[A^+(\mathcal{T}')] = A$ and $p_{\mathcal{K}}[B^+_{\tau(*)}(\mathcal{T}')] = p_{\mathcal{K}}[B^+(\mathcal{T}')] = B$. ⊣

§6. **Acknowledgments.** This paper is based on material from my doctorial dissertation at the Massachusetts Institute of Technology under Gerald Sacks, who I would like to thank for his unwavering encouragement and support. I would also like to thank the people at the logic seminars at MIT, University of Pennsylvania, University of Maryland-College Park, and University of California-Berkeley for letting me present this material and for giving me useful feedback. These presentations helped improved the quality of this paper. Finally, I would like to thank an anonymous referee for several helpful suggestions which greatly improved the presentation of the paper.

REFERENCES

[1] Jon Barwise, ***Admissible sets and structures***, Springer-Verlag, Berlin, 1975, An approach to definability theory, Perspectives in Mathematical Logic.

[2] Lars Birkedal, Rasmus Ejlers Møgelberg, Jan Schwinghammer, and Kristian Støvring. *First steps in synthetic guarded domain theory: Step-indexing in the topos of trees.* ***Logical Methods in Computer Science***, vol. 8 (2012), no. 4, pp. 4:1, 45.

[3] Pietro Di Gianantonio and Marino Miculan, ***Unifying recursive and co-recursive definitions in sheaf categories, Foundations of software science and computation structures***, Lecture Notes in Computer Science, vol. 2987, Springer, Berlin, 2004, pp. 136–150.

[4] Thomas Jech, ***Set theory***, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, The third millennium edition, revised and expanded.

[5] Stefano Kasangian and Sebastiano Vigna, *The topos of labelled trees: A categorical semantics for SCCS.* ***Fundamenta Informaticae***, vol. 32 (1997), no. 1, pp. 27–45.

[6] Saunders Mac Lane, ***Categories for the working mathematician***, second ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998.

[7] Saunders Mac Lane and Ieke Moerdijk, ***Sheaves in geometry and logic***, Universitext, Springer-Verlag, New York, 1994. A first introduction to topos theory, Corrected reprint of the 1992 edition.

[8] Yiannis N. Moschovakis, ***Descriptive set theory***, second ed., Mathematical Surveys and Monographs, vol. 155, American Mathematical Society, Providence, RI, 2009.

DEPARTMENT OF MATHEMATICS
HARVARD UNIVERSITY
ONE OXFORD STREET
CAMBRIDGE, MA 02138, USA
E-mail: nate@math.harvard.edu