NEW RESULTS ON THE DISTRIBUTION OF DISCOUNTED COMPOUND POISSON SUMS

BY

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ABSTRACT

This paper focuses on the distribution of Poisson sums of discounted claims over a finite or infinite time period. It gives two new results when claim amounts follow Mittag-Leffler distributions and two new results when claim amounts follow gamma distributions. Further, as Mittag-Leffler distribution is of heavy-tailed nature and its moments only exist for order strictly smaller than one, this distribution can be used for modelling insurance whose claim amounts are extremely large, that is, catastrophe insurance.

KEYWORDS

Poisson risk process, discounted aggregate claims, Mittag-Leffler distribution, heavy tailed, catastrophe insurance.

1. INTRODUCTION

Consider a discounted sum model with continuous arrival time recorded up to time t, that is,

$$Z_t = \sum_{k=1}^{N_t} e^{-\delta T_k} X_k, \quad t \geq 0, \tag{1.1}$$

where

1. δ is the discount factor, which could be the net interest rate or just the interest rate;
2. Claim amount X_k is paid at time T_k and X_k's are positive i.i.d;
3. Claim arrival process N_t is a renewal process with total waiting time T_k and inter-arrival time τ_k, that is, N_t is the number of claims up to time t and τ_k's are i.i.d;
4. Claim amounts and the claim arrival process are independent.

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Equation (1.1) is a generalised version of the key component of the surplus process in risk theory:

$$U_t = u + pt - \sum_{k=1}^{N_t} X_k, \tag{1.2}$$

where $u \geq 0$ is the initial reserve and $p > 0$ is the premium rate per unit time. As we are investigating the surplus behaviour with respect to time t, we believe it would be more realistic to introduce a discount factor to the claim payments. However, the introduction of this discount factor increases the complexity as the moments of Z_t cannot be obtained by conditioning on N_t and then applying the independence between N_t and X. Various research has been done for moments; see Léveillé and Garrido (2001a,b) and Garrido and Léveillé (2004). For comprehensive applications of this loss distribution approach in insurance analysis, see Cruz *et al.* (2014).

Although this model setting is no stranger to actuarial and financial studies, the distributions of Z_t are still mysterious under most cases. The usual technique is applying transforms on both sides of Equation (1.1) and invert back once the transformed function, for example, the moment-generating function, is achieved. However, in most cases, the transformed function is not achievable. Further, even it can be derived, it is usually in a form we are unable to invert.

The distribution of Z_∞, which can be considered as an infinite sum of the product of random variables, has some history in the probability theory. Earlier research can be traced back to Gerber (1979, p. 106) and Harrison (1977). Vervaat (1979) establishes the conditions for the existence and uniqueness of Z_∞. Dufresne (1990) solves more cases and gives applications to risk theory and pension funding. Paulsen (1993) generalises this model to stochastic rate of return on investments as well as stochastic level of inflation. Nilsen and Paulsen (1996) solve one specific result when the return process involves a Brownian motion. Dufresne (1996) finds one more result through algebraic properties of beta and gamma variables. Gjessing and Paulsen (1997) further generalise the rate of return and the arrival process to be two independent levy process. Dufresne (1998) systematically studies the algebraic properties of beta and gamma variables and gives more results for the distribution of Z_∞. Recently, Dufresne and Zhang (2017) find the distribution of Z_∞ when τ follows a mixture of exponentials and X follows an exponential distribution.

For the distribution of Z_t, $t < \infty$, limited results have been revealed. Takács (1954) proposes a formula for the characteristic function of Z_t when N_t is a Poisson process. Léveillé *et al.* (2010) propose two integral equations for the moment-generating function of Z_t and re-derive the formula under Poisson case by renewal arguments. Wang (2010) proposes that the moment-generating function of Z_t can be solved from an nth order differential equation when the inter-arrival times follow a particular gamma distribution. Wang *et al.* (2016) propose an approximation method for inverting the moment-generating function and calculating VaR and CTE of Z_t. Dufresne and Zhang (2017) derive the explicit formula of the density function of Z_t when N_t is a Poisson process and X follows an exponential distribution.

As mentioned at the beginning, most of the papers of Z_t concentrate on the moments. However, moments and the moment-generating function may not be enough. Because of the heavy tails of financial models, moments may not exist after the first several orders and therefore the moment-generating function no longer exists. For example, the second moment of claim amounts of the catastrophe insurance may go to infinity. This is the case of this paper, where the first moment of claim amounts is infinity. If that is the case, then the premium calculation and capital requirement based on the moments are unlikely to be reliable. Fortunately, once the probability distribution function is derived theoretically, we can also find the Value at Risk (VaR), at least approximately, which can be converted into the capital requirement.

In this paper, we restrict N_t to be a Poisson process. Section 2 introduces two Mittag-Leffler distributions, which are slightly different from the literature and are used for modelling claim amounts later. Their Laplace transforms are also discussed. Section 3 applies the formula in Takács (1954) and gives the distribution of Z_t when X_k follows a Mittag-Leffler distribution. This generalises the result in Dufresne and Zhang (2017, Theorem 9). This section also reveals two new cases where X_k follows two particular gamma distributions and t tends to infinity. Finally, Section 4 discusses the application of Z_t for catastrophe insurance modelling. Throughout this paper, all numerical computations were performed with Mathematica and “MachinePrecision” is adopted unless specified.

2. Mittag-Leffler distribution

Mittag-Leffler distributions have received great attention of mathematicians and statisticians recently. Pillai (1990) first proposes this non-negative distribution based on the Mittag-Leffler function and shows that the first moment of this distribution already goes to infinity. Later, Lin (1998) and Jose *et al.* (2010) generalise this distribution and the latter discuss their generalisation in the time series modelling. Further, Cahoy (2013) shows that the first two moments of logarithmic transform of a Mittag-Leffler distribution are finite. In this section, we will introduce two generalised Mittag-Leffler distributions and give the Laplace transforms of them.

We first recall three Mittag-Leffler functions:

1. $E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k+1)}$,
2. $E_{\beta,\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k+\gamma)}$,
3. $E^{\eta}_{\beta,\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\eta)_k z^k}{k!\Gamma(\beta k+\gamma)}$,

where $z,\ \beta,\ \gamma,\ \eta \in \mathbb{C}$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Gamma(z)$ is the gamma function and $(\eta)_k = \frac{\Gamma(\eta+k)}{\Gamma(\eta)}$.

In order to be parallel to the result that X being exponentially distributed in Dufresne and Zhang (2017, Theorem 9), we introduce a more generalised definition of Mittag-Leffler distribution.

Definition 1. *If $X \sim ML(\alpha, \theta)$ for $\alpha > 0$ and $0 < \theta \leq 1$, then its cumulative distribution function is*

$$F_X(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(\alpha x)^{\theta k}}{\Gamma(1+\theta k)}, \quad x \geq 0. \tag{2.1}$$

Remarks

1. *When $\alpha = 1$, this definition goes back to Theorem* 2.1 *in Pillai* (1990).
2. *When $\theta = 1$, $X \sim$ Exp(α), that is, $f_X(x) = \alpha e^{-\alpha x}$.*
3. *$F_X(x) = 1 - E_\theta(-(\alpha x)^\theta)$ and $f_X(x) = \alpha(\alpha x)^{\theta-1} E_{\theta,\theta}\left(-(\alpha x)^\theta\right)$.*

Definition 2. *If $X \sim GML(\alpha, \theta, \gamma)$ for $\alpha > 0$, $0 < \theta \leq 1$ and $\gamma > 0$, then its cumulative distribution function is*

$$F_X(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (\gamma)_k (\alpha x)^{\theta(\gamma+k)}}{k!\Gamma(1+\theta(\gamma+k))}, \quad x \geq 0. \tag{2.2}$$

Remarks

1. *When $\alpha = 1$, this definition goes back to Equation* (2) *in Jose et al.* (2010).
2. *When $\theta = 1$, $X \sim G(\gamma, \alpha)$, that is, $f_X(x) = \frac{\alpha^\gamma}{\Gamma(\gamma)} x^{\gamma-1} e^{-\alpha x}$.*
3. *When $\gamma = 1$, this definition goes back to Equation* (2.1).
4. *$F_X(x) = (\alpha x)^{\theta\gamma} E^{\gamma}_{\theta,\theta\gamma+1}\left(-(\alpha x)^\theta\right)$ and $f_X(x) = \alpha(\alpha x)^{\theta\gamma-1} E^{\gamma}_{\theta,\theta\gamma}\left(-(\alpha x)^\theta\right)$.*

The Laplace transforms, $L_X(q)$ for $X \sim ML(1, \theta)$ and $X \sim GML(1, \theta, \gamma)$, appear several times in the references while the domain for q is not consistent. Thus we are going to re-derive them and give the following theorem:

Theorem 1. *For $\gamma \in \mathbb{N}$, $GML(\alpha, \theta, \gamma)$ is the γ-fold convolution of $ML(\alpha, \theta)$.*

Proof. For $X \sim GML(\alpha, \theta, \gamma)$, the Laplace transform of X is

$$\begin{aligned}
L_X(q) &= \int_0^\infty e^{-qx} \sum_{k=0}^{\infty} \frac{\Gamma(k+\gamma)}{\Gamma(\gamma)} \frac{(-1)^k}{k!} \frac{x^{-1+(k+\gamma)\theta}\alpha^{(k+\gamma)\theta}}{\Gamma((k+\gamma)\theta)} dx \\
&= \sum_{k=0}^{\infty} \frac{\Gamma(k+\gamma)}{\Gamma(\gamma)} \frac{(-1)^k}{k!} \frac{\alpha^{(k+\gamma)\theta}}{\Gamma((k+\gamma)\theta)} \int_0^\infty e^{-qx} x^{-1+(k+\gamma)\theta} dx \\
&= \sum_{k=0}^{\infty} \frac{\Gamma(k+\gamma)}{\Gamma(\gamma)} \frac{(-1)^k}{k!} \frac{\alpha^{(k+\gamma)\theta}}{\Gamma((k+\gamma)\theta)} q^{-\theta(k+\gamma)} \Gamma(\theta(k+\gamma)) \quad (*\, \Re(q) > 0 \,*)
\end{aligned}$$

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$$= \left(\frac{\alpha}{q}\right)^{\gamma\theta} \sum_{k=0}^{\infty} \frac{\Gamma(k+\gamma)}{\Gamma(\gamma)} \frac{(-1)^k}{k!} \left(\left(\frac{\alpha}{q}\right)^{\theta}\right)^k$$
$$= \left(\frac{\alpha}{q}\right)^{\gamma\theta} \left(1 + \left(\frac{\alpha}{q}\right)^{\theta}\right)^{-\gamma} \left(* \left|\left(\frac{\alpha}{q}\right)^{\theta}\right| < 1 * \right)$$
$$= \left(\frac{\alpha^{\theta}}{q^{\theta} + \alpha^{\theta}}\right)^{\gamma}.$$

By Definition 2 Remark 3, when $\gamma = 1$, $X \sim ML(\alpha, \theta)$ and this completes the proof. ■

Remarks

1. *We apply the Fubini theorem to exchange the order of integral and summation for the second equation. Because the same calculation can show that* $(-1)^m$ *under the summation does not affect the convergence for the same domain of* q.
2. *If* $q \in \mathbb{R}$ *and* $0 < \theta < 1$, *then we require* $q \in (\alpha, \infty)$ *for the convergence of its Laplace transform. Further, by analytic continuation, the domain of* q *can be extended to* $[0, \infty)$.
3. *Differentiating the Laplace transform shows that the moments of a* $GML(\alpha, \theta, \gamma)$ *only exist when* $\theta = 1$.

We will use notation $Y^{(\alpha,\beta)}$ to represent a random variable following a distribution $Y(\alpha, \beta)$. For example, a random variable following a beta distribution with parameters α and β is written as $B^{(\alpha,\beta)}$, that is, $f_{B^{(\alpha,\beta)}} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$. With this notation and the Theorem above, we have the additive and multiplicative properties of generalised Mittag-Leffler random variables.

Corollary 1. *For* $i \in \mathbb{N}$, $\sum_{i=1}^{n} GML^{(\alpha,\theta,\gamma_i)} \sim GML(\alpha, \theta, \sum_{i=1}^{n} \gamma_i)$.

Corollary 2. *For* $c > 0$, $c \cdot GML^{(\alpha,\theta,\gamma)} \sim GML(\frac{\alpha}{c}, \theta, \gamma)$.

Since the $ML(\alpha, \theta)$ only has moments when $\theta = 1$ by Remark 3 of Theorem 1, we are going to discuss the tail behaviour of this distribution. One famous subclass of heavy-tailed distributions is the class of distribution functions with regularly varying tails. A non-negative distribution F belongs to the class $\mathcal{R}$ if $\bar{F}(x) = 1 - F(x) > 0$ for all $x \geq 0$ and there exists some $\beta > 0$ such that

$$\lim_{x\to\infty} \frac{\bar{F}(yx)}{\bar{F}(x)} = y^{-\beta}, \tag{2.3}$$

for $y > 0$. We write $F \in \mathcal{R}_{-\beta}$ and if F is the distribution function of X, $X \in \mathcal{R}_{-\beta}$.

On the one hand, when $\theta = 1$, $ML^{(\alpha,1)} \sim \text{Exp}(\alpha)$ and $\bar{F}_{ML^{(\alpha,1)}}(x) = e^{-\alpha x}$. On the other hand, when $\theta < 1$, from the well-known asymptotic behaviour of the

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Mittag-Leffler function (Bateman and Erdélyi, 1953, p. 210, Equation (21)),

$$E_{\beta,\gamma}(z) = -\sum_{n=1}^{\infty} \frac{z^{-n}}{\Gamma(\gamma-\beta n)}, \quad z\to\infty, \quad \left|\arg(-z) < \left(1-\frac{1}{2}\beta\right)\pi\right|, \tag{2.4}$$

it can be deducted that $ML^{(\alpha,\theta)}$ has the heavy power tail. This is illustrated in the following theorem:

Theorem 2. *If* $0<\theta<1$, $ML^{(\alpha,\theta)} \in \mathcal{R}_{-\theta}$.

Proof. For $y>0$, applying the third Remark of Definition 1 and Equation (2.4) gives

$$\lim_{x\to\infty}\frac{\bar{F}_{ML}(yx)}{\bar{F}_{ML}(x)} = \lim_{x\to\infty}\frac{yf_{ML}(yx)}{f_{ML}(x)} = \lim_{x\to\infty}\frac{y\frac{\alpha^{-\theta}(yx)^{-\theta-1}\theta}{\Gamma(1-\theta)}}{\frac{\alpha^{-\theta}x^{-\theta-1}\theta}{\Gamma(1-\theta)}} = y^{-\theta}. \qquad \blacksquare$$

3. Distribution of Z_t

In this section, we will give one new result for Z_t when claim amounts follow a Mittag-Leffler distribution and two new results for Z_∞ when the claims amounts follow two particular gamma distributions. We start by recalling one result from Takács (1954) and rewriting it into the Laplace transform context.

Let N_t follow a Poisson process with rate $\lambda>0$. Then for any $t>0$, the Laplace transform of Z_t is

$$L_{Z_t}(q) = e^{\lambda\int_0^t (L_X(qe^{-\delta v})-1)dv}. \tag{3.1}$$

Theorem 3. *If N_t follows a Poisson process with rate λ and $X\sim ML(\alpha,\theta)$, then for $\delta>0$ and $z>0$, Z_t has a defective density function:*

$$f_{Z_t}(z) = e^{-\lambda t}\sum_{n=1}^{\infty}\frac{\left(-\frac{\lambda}{\theta\delta}\right)_n\left(1-e^{\theta\delta t}\right)^n}{n!}\alpha(\alpha z)^{\theta n-1}E^n_{\theta,\theta n}\left(-(\alpha z)^\theta\right) \tag{3.2}$$

and a cumulative distribution function:

$$F_{Z_t}(z) = e^{-\lambda t} + e^{-\lambda t}\sum_{n=1}^{\infty}\frac{\left(-\frac{\lambda}{\theta\delta}\right)_n\left(1-e^{\theta\delta t}\right)^n}{n!}(\alpha z)^{\theta n}E^n_{\theta,\theta n+1}\left(-(\alpha z)^\theta\right), \tag{3.3}$$

with $\mathbb{P}(Z_t=0)=e^{-\lambda t}$.

Proof. By Equation (3.1), we have

$$\lambda\int_0^t\left(L_X\left(qe^{-\delta v}\right)-1\right)\,dv = \lambda\int_0^t\frac{\alpha^\theta}{\alpha^\theta+q^\theta e^{-\theta\delta v}}\,dv - \lambda t.$$

Let $u = q^\theta e^{-\theta\delta v}$. Then $v = -\frac{1}{\theta\delta}\log\left(\frac{u}{q^\theta}\right)$ and $dv = -\frac{1}{u\theta\delta}du$. We have

$$\lambda\int_0^t \left(L_X\left(qe^{-\delta v}\right) - 1\right)\, dv = \frac{\lambda}{\theta\delta}\log\left(\frac{\alpha^\theta + q^\theta e^{-\theta\delta t}}{\alpha^\theta + q^\theta}\right),$$

which gives

$$L_{Z_t}(q) = \left(\frac{1 + \left(\frac{q}{\alpha}e^{-\delta t}\right)^\theta}{1 + \left(\frac{q}{\alpha}\right)^\theta}\right)^{\frac{\lambda}{\theta\delta}} = e^{-\lambda t}\left(\frac{e^{\theta\delta t} + \left(\frac{q}{\alpha}\right)^\theta}{1 + \left(\frac{q}{\alpha}\right)^\theta}\right)^{\frac{\lambda}{\theta\delta}}. \tag{3.4}$$

In order to invert the Laplace transform, rewrite $L_{Z_t}(q)$ as

$$\begin{aligned} L_{Z_t}(q) &= e^{-\lambda t}\left(1 + \frac{e^{\theta\delta t} - 1}{1 + \left(\frac{q}{\alpha}\right)^\theta}\right)^{\frac{\lambda}{\theta\delta}} \\ &= e^{-\lambda t}\sum_{n=0}^{\infty}\frac{\Gamma\left(n - \frac{\lambda}{\theta\delta}\right)}{\Gamma\left(-\frac{\lambda}{\theta\delta}\right)}\frac{\left(-\frac{e^{\theta\delta t}-1}{1+\left(\frac{q}{\alpha}\right)^\theta}\right)^n}{n!} \\ &= e^{-\lambda t} + e^{-\lambda t}\sum_{n=1}^{\infty}\frac{\Gamma\left(n - \frac{\lambda}{\theta\delta}\right)}{\Gamma\left(-\frac{\lambda}{\theta\delta}\right)}\frac{\left(1 - e^{\theta\delta t}\right)^n}{n!}\left(\frac{1}{1 + \left(\frac{q}{\alpha}\right)^\theta}\right)^n. \end{aligned}$$

Applying Theorem 1, we have

$$f_{Z_t}(z) = e^{-\lambda t}\sum_{n=1}^{\infty}\frac{\left(-\frac{\lambda}{\theta\delta}\right)_n\left(1 - e^{\theta\delta t}\right)^n}{n!}\sum_{m=0}^{\infty}\frac{\Gamma(m+n)}{\Gamma(n)}\frac{(-1)^m}{m!}\frac{z^{-1+(m+n)\theta}\alpha^{(m+n)\theta}}{\Gamma((m+n)\theta)}. \quad \blacksquare$$

Remarks

1. *When $\theta = 1$, we have*

$$f_{Z_t}(z) = \frac{\lambda}{\delta}e^{-z\alpha - t\lambda}\left(-1 + e^{t\delta}\right)\alpha\ {}_1F_1\left(1 - \frac{\lambda}{\delta}, 2, \left(1 - e^{t\delta}\right)z\alpha\right), \tag{3.5}$$

where ${}_1F_1(z)$ is the Kummer confluent hypergeometric function, that is, ${}_1F_1(a, b; z) = \sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\frac{z^n}{n!}$. Further, if $\alpha = 1$, this restores Theorem 9 in Dufresne and Zhang (2017).

2. *When $\theta = \frac{1}{2}$, we have*

$$f_{Z_t}(z) = e^{-\lambda t}\sum_{n=1}^{\infty}\frac{\left(-\frac{2\lambda}{\delta}\right)_n\left(1 - e^{\frac{1}{2}\delta t}\right)^n(\alpha z)^{\frac{n}{2}}}{n!}$$
$$\times\left[\frac{1}{z}\frac{1}{\Gamma\left(\frac{n}{2}\right)}\ {}_1F_1\left(\frac{1+n}{2}, \frac{1}{2}, z\alpha\right) - \left(\frac{\alpha}{z}\right)^{\frac{1}{2}}\frac{1}{\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)}\ {}_1F_1\left(1 + \frac{n}{2}, \frac{3}{2}, z\alpha\right)\right]. \tag{3.6}$$

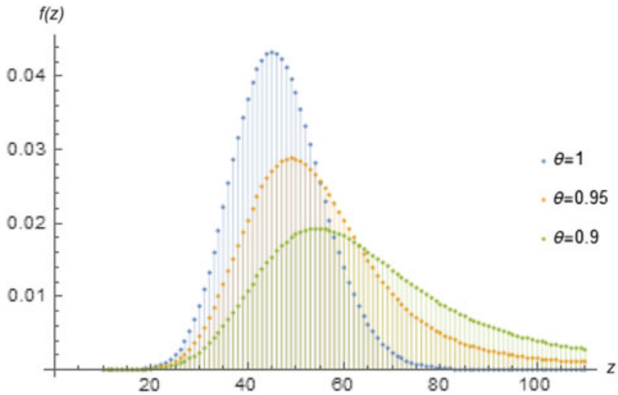

FIGURE 1: $f_{Z_{10}}$ with $\delta = 0.015$, $\alpha = 1$ and $\lambda = 5$.

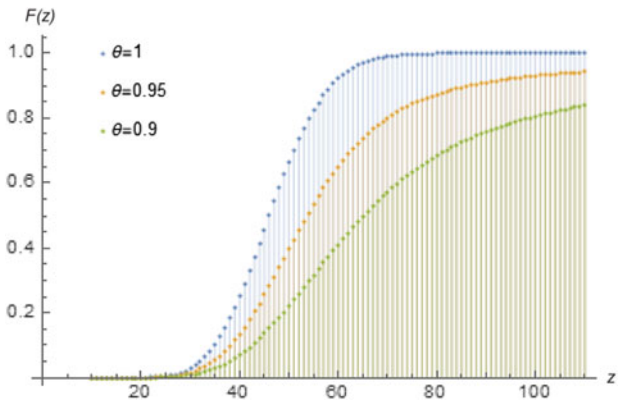

FIGURE 2: $F_{Z_{10}}$ with $\delta = 0.015$, $\alpha = 1$ and $\lambda = 5$.

Figures 1 and 2 illustrate probability density and distribution functions, respectively, of Z_{10} under different θ's, with $\delta = 0.015$, $\alpha = 1$ and $\lambda = 5$. During the computations, the infinite series in Equations (3.2) and (3.3) are truncated after 2000 terms and the "Precision" of Mathematica is set at 200. These two figures show the tail of Z_{10} increases significantly as θ decreases. The mode of $f_{Z_{10}}$ under $\theta = 0.9$ is already less than half of that under $\theta = 1$, that is, $X \sim \text{Exp}(1)$. Although the shape of the density function when $\theta = 1$ looks symmetrical, it is actually not the case, which can be verified by skewness.

When t tends to infinity, this question is simplified to some extent and more results could be found in the literature. The reason for simplification is not only that one parameter is gone but also that Markov property can be established, which leads to a well-known result (Vervaat, 1979):

$$Z_\infty \overset{\text{dist}}{=} V(X + Z_\infty), \tag{3.7}$$

where $V=e^{-\delta\tau}$ and all the random variables on the right-hand side are independent. A known instance of Equation (3.7) is

$$G^{(\alpha,1)} \overset{\text{dist}}{=} B^{(\alpha,\theta)}\left(G^{(\theta,1)}+G^{(\alpha,1)}\right). \tag{3.8}$$

Theorem 4. *If N_t follows a Poisson process with rate λ and $X\sim ML(\alpha,\theta)$, then for $\delta>0$ and $t\to\infty$, $Z_\infty\sim GML\left(\alpha,\theta,\frac{\lambda}{\theta\delta}\right)$.*

Proof. Let $t\to\infty$ in Equation (3.4) and the result follows by identifying the Laplace transform. ■

Remarks *If $\tau\sim\mathrm{Exp}(\lambda)$, then $\delta\tau\sim\mathrm{Exp}\left(\frac{\lambda}{\delta}\right)$ and therefore $e^{-\delta\tau}\sim B\left(\frac{\lambda}{\delta},1\right)$. By Equation (3.7) we have the following distribution identities:*

1. *For $\alpha>0$, $0<\theta\le 1$, $\gamma>0$, $\lambda>0$ and $\delta>0$,*

$$GML^{\left(\alpha,\theta,\frac{\lambda}{\theta\delta}\right)} \overset{dist}{=} B^{\left(\frac{\lambda}{\delta},1\right)}\left(GML^{(\alpha,\theta,1)}+GML^{\left(\alpha,\theta,\frac{\lambda}{\theta\delta}\right)}\right). \tag{3.9}$$

This can be further verified by checking whether both sides have the same Laplace transform. Starting from the right-hand side, we have

$$\begin{aligned}
&\mathbb{E}\left[e^{-qB^{\left(\frac{\lambda}{\delta},1\right)}\left(GML^{(\alpha,\theta,1)}+GML^{\left(\alpha,\theta,\frac{\lambda}{\theta\delta}\right)}\right)}\right]\\
&=\mathbb{E}\left[\mathbb{E}\left[e^{-qB^{\left(\frac{\lambda}{\delta},1\right)}\left(GML^{(\alpha,\theta,1)}+GML^{\left(\alpha,\theta,\frac{\lambda}{\theta\delta}\right)}\right)}|B^{\left(\frac{\lambda}{\delta},1\right)}\right]\right]\\
&=\frac{\lambda}{\delta}\int_0^1\left(\frac{1}{1+\left(\frac{qb}{\alpha}\right)^\theta}\right)^{\frac{\lambda}{\theta\delta}+1} b^{\frac{\lambda}{\delta}-1}db\\
&=\frac{\lambda}{\delta}\left(\frac{\alpha}{q}\right)^{\frac{\lambda}{\delta}}\frac{1}{\theta}\int_0^{\left(\frac{q}{\alpha}\right)^\theta}(1+u)^{-\frac{\lambda}{\theta\delta}-1}u^{\frac{\lambda}{\theta\delta}-1}du\\
&=\frac{\lambda}{\delta}\left(\frac{\alpha}{q}\right)^{\frac{\lambda}{\delta}}\frac{1}{\theta}\sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{\lambda}{\theta\delta}+1+k\right)}{\Gamma\left(\frac{\lambda}{\theta\delta}+1\right)}\frac{(-1)^k}{k!}\int_0^{\left(\frac{q}{\alpha}\right)^\theta}u^{k+\frac{\lambda}{\theta\delta}-1}\,du\\
&=\left(\frac{1}{1+\left(\frac{q}{\alpha}\right)^\theta}\right)^{\frac{\lambda}{\theta\delta}},
\end{aligned}$$

which is the Laplace transform of left-hand side by Theorem 1.

2. *For $\alpha>0$, $\gamma>0$, $\lambda>0$ and $\delta>0$,*

$$G^{\left(\frac{\lambda}{\delta},\alpha\right)} \overset{\text{dist}}{=} B^{\left(\frac{\lambda}{\delta},1\right)}\left(G^{(1,\alpha)}+G^{\left(\frac{\lambda}{\delta},\alpha\right)}\right), \tag{3.10}$$

which is the $X\sim\mathrm{Exp}(\alpha)$ case. This can also be proved by that $\frac{G^{(\gamma_1,\alpha)}}{G^{(\gamma_1,\alpha)}+G^{(\gamma_2,\alpha)}}$ is independent with $G^{(\gamma_1,\alpha)}+G^{(\gamma_2,\alpha)}$ (Lukacs, 1955) and follows $B(\gamma_1,\gamma_2)$.

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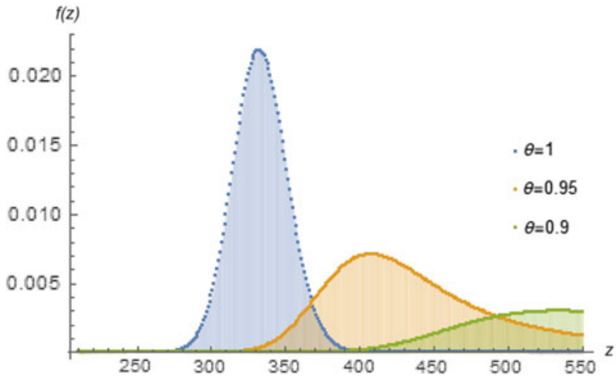

FIGURE 3: f_{Z_∞} with $\delta = 0.015$, $\alpha = 1$ and $\lambda = 5$.

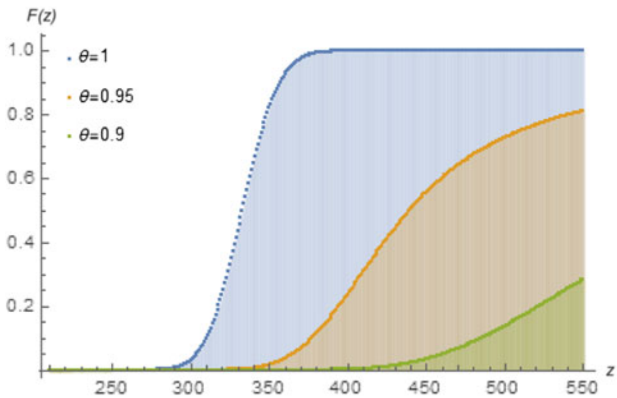

FIGURE 4: F_{Z_∞} with $\delta = 0.015$, $\alpha = 1$ and $\lambda = 5$.

Figures 3 and 4 illustrate the probability density function and probability distribution function of Z_∞ under different θ's, with $\delta = 0.015$, $\alpha = 1$ and $\lambda = 5$. During the computations, the infinite series in Definition 2 are truncated after 5000 terms and the "Precision" of Mathematica is set at 700. Figure 3 shows the tail of Z_∞ is much heavier than that of Z_{10}, as modes are much lower, only half of previous ones. Further, with smaller θ, the tails become much heavier, as the mode under $\theta = 0.9$ is only a quarter of that under $\theta = 1$. When $t = 10$, all density functions rise almost the same time, whereas when $t \to \infty$, the density function of $\theta = 0.9$ has not risen until that of $\theta = 1$ almost vanishes.

For $X \sim G(\gamma, \alpha)$, no general formula for Z_t has been found yet. Léveillé *et al.* (2010, Example 3.6) give the moment-generating function of Z_t when claim amounts have an Erlang(n) distribution. However, their expression is not friendly for inverting. The following theorems give the explicit density functions of Z_∞ under two particular cases:

Theorem 5. *If N_t follows a Poisson process with rate λ and $X \sim G(2, \alpha)$, then for $\delta > 0$ and $t \to \infty$, Z_∞ has a density function:*

$$f_{Z_\infty}(z) = e^{-\frac{\lambda}{\delta}} \sum_{n=0}^{\infty} \frac{\left(\frac{\lambda}{\delta}\right)^n}{n!} \frac{\alpha^{n+\frac{\lambda}{\delta}}}{\Gamma\left(n+\frac{\lambda}{\delta}\right)} z^{\frac{\lambda}{\delta}+n-1} e^{-\alpha z}, \tag{3.11}$$

and a cumulative distribution function:

$$F_{Z_\infty}(z) = 1 - e^{-\frac{\lambda}{\delta}} \sum_{n=0}^{\infty} \frac{\left(\frac{\lambda}{\delta}\right)^n}{n!} \frac{\Gamma\left(n+\frac{\lambda}{\delta}, \alpha z\right)}{\Gamma\left(n+\frac{\lambda}{\delta}\right)}, \tag{3.12}$$

where $\Gamma(a, z)$ is the incomplete gamma function.

Proof. Applying Example 3.6 (Léveillé *et al.*, 2010) and letting $t \to \infty$ or applying differential equation technique on Equation (3.7) (Dufresne and Zhang, 2017, Section 2), we have

$$L_{Z_\infty}(q) = e^{-\frac{\lambda}{\delta}} e^{\frac{\lambda}{\delta}\frac{\alpha}{\alpha+q}} \left(\frac{\alpha}{\alpha+q}\right)^{\frac{\lambda}{\delta}}.$$

Expand $e^{\frac{\lambda}{\delta}\frac{\alpha}{\alpha+q}}$ into the Taylor series and the result is obtained after identifying that $\left(\frac{\alpha}{\alpha+q}\right)^{\frac{\lambda}{\delta}}$ is the Laplace transform of gamma distributions. ∎

Theorem 6. *If N_t follows a Poisson process with rate λ and $X \sim G(\frac{1}{2}, \alpha)$, then for $\delta > 0$ and $t \to \infty$, Z_∞ has a density function:*

$$f_{Z_\infty}(z) = 2^{2\frac{\lambda}{\delta}} \sum_{n=0}^{\infty} \frac{\Gamma\left(2\frac{\lambda}{\delta}+n\right)}{\Gamma\left(2\frac{\lambda}{\delta}\right)} \frac{(-1)^n}{n!} \frac{\alpha^{\frac{1}{2}\left(n+\frac{2\lambda}{\delta}\right)}}{\Gamma\left(\frac{1}{2}\left(n+\frac{2\lambda}{\delta}\right)\right)} z^{\frac{1}{2}\left(n+\frac{2\lambda}{\delta}\right)-1} e^{-\alpha z}, \tag{3.13}$$

and a cumulative distribution function:

$$F_{Z_\infty}(z) = 1 - 2^{2\frac{\lambda}{\delta}} \sum_{n=0}^{\infty} \frac{\Gamma\left(2\frac{\lambda}{\delta}+n\right)}{\Gamma\left(2\frac{\lambda}{\delta}\right)} \frac{(-1)^n}{n!} \frac{\Gamma\left(\frac{n}{2}+\frac{\lambda}{\delta}, \alpha z\right)}{\Gamma\left(\frac{n}{2}+\frac{\lambda}{\delta}\right)}. \tag{3.14}$$

Proof. Applying Equation (3.1), we have

$$L_{Z_\infty}(q) = e^{\lambda \int_0^t \left(\left(\frac{1}{1+\frac{q}{\alpha}e^{-\delta v}}\right)^{\frac{1}{2}} - 1\right) dv}.$$

For the integral part, make a change of variable $u = \frac{q}{\alpha} e^{-\delta v}$ and realise

$$\int_a^b \left(\frac{1}{1+u}\right)^{\frac{1}{2}} \frac{1}{u} du = \log\left[\frac{\left(1-\sqrt{1+b}\right)\left(1+\sqrt{1+a}\right)}{\left(1-\sqrt{1+a}\right)\left(1+\sqrt{1+b}\right)}\right].$$

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After that, doing some simplifications and letting $t \to \infty$, we end up with

$$
\begin{aligned}
L_{Z_\infty}(q) &= \left(\frac{2}{1+\sqrt{1+\frac{q}{\alpha}}} \right)^{2\frac{\lambda}{\delta}} \\
&= \left(\frac{2}{\sqrt{1+\frac{q}{\alpha}}} \right)^{2\frac{\lambda}{\delta}} \left(1 + \left(1+\frac{q}{\alpha}\right)^{-\frac{1}{2}} \right)^{-2\frac{\lambda}{\delta}} \\
&= 2^{2\frac{\lambda}{\delta}} \sum_{n=0}^{\infty} \frac{\Gamma\left(2\frac{\lambda}{\delta}+n\right)}{\Gamma\left(2\frac{\lambda}{\delta}\right)} \frac{(-1)^n \left(1+\frac{q}{\alpha}\right)^{-\frac{1}{2}\left(n+2\frac{\lambda}{\delta}\right)}}{n!}.
\end{aligned}
$$

Finally, the result follows after inverting L_{Z_∞} back. ■

The density functions in the above two theorems can be written into the hypergeometric functions, which are much easier to deal with for numerical purpose.

1. If $X \sim G(2, \alpha)$, Z_∞ has a density function:

$$
f_{Z_\infty}(z) = \frac{e^{-\frac{\lambda}{\delta}} e^{-\alpha z} \alpha^{\frac{\lambda}{\delta}} z^{-1+\frac{\lambda}{\delta}}}{\Gamma\left(\frac{\lambda}{\delta}\right)} \, {}_0F_1\left(\frac{\lambda}{\delta}, \frac{\lambda}{\delta}\alpha z\right), \tag{3.15}
$$

   where ${}_0F_1(a, z) = \sum_{n=0}^{\infty} \frac{z^n}{(a)_n n!}$.
2. If $X \sim G(\frac{1}{2}, \alpha)$, Z_∞ has a density function:

$$
\begin{aligned}
f_{Z_\infty}(z) = 2^{2\frac{\lambda}{\delta}} \alpha^{\frac{\lambda}{\delta}} e^{-\alpha z} z^{\frac{\lambda}{\delta}-1} \Bigg\{ & \frac{1}{\Gamma\left(\frac{\lambda}{\delta}\right)} \, {}_2F_1\left(\frac{1}{2}+\frac{\lambda}{\delta}, \frac{1}{2}, \alpha z\right) \\
& - \frac{\Gamma\left(\frac{\lambda}{\delta}+1\right) 2\sqrt{\alpha}\sqrt{z}}{\Gamma\left(\frac{\lambda}{\delta}\right)\Gamma\left(\frac{1}{2}+\frac{\lambda}{\delta}\right)} \, {}_2F_1\left(\frac{\delta+\lambda}{\delta}, \frac{3}{2}, \alpha z\right) \Bigg\}.
\end{aligned} \tag{3.16}
$$

Figure 5 is the comparison of f_{Z_∞} when X follows three different distributions with the same expectation, that is, Exp(1), $G(2, 2)$ and $G(\frac{1}{2}, \frac{1}{2})$. During the computations, the infinite series in Equations (3.15) and (3.16) do not need to be truncated, since the hypergeometric functions are well built in Mathematica, and the "Precision" of the software is set at 500. As the mode of f_{Z_∞} is the smallest when $X \sim G(\frac{1}{2}, \frac{1}{2})$ and the largest when $X \sim G(2, 2)$, the graph shows the tail is the heaviest under the former case and the thinnest under the latter case. This is expected because $\mathbb{E}[(E^{(\alpha)})^k] < \mathbb{E}[(G^{(\alpha,\alpha)})^k]$ for $\alpha < 1$ and $\mathbb{E}[(E^{(\alpha)})^k] \geq \mathbb{E}[(G^{(\alpha,\alpha)})^k]$ for $\alpha \geq 1$, where $k \in \mathbb{N}$. Finally, comparing with Figure 3, the tails where claim amounts follow gamma distributions are much lighter than those where claim amounts follow Mittag-Leffler distributions whose second parameter is smaller than one. This is also expected as gamma distributions have moments to any order.

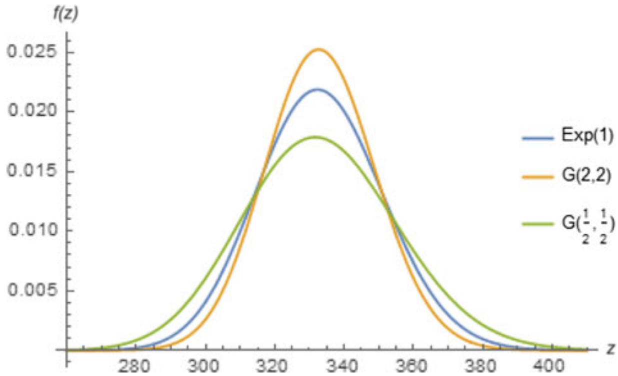

FIGURE 5: f_{Z_∞} with $\mathbb{E}[X]=1$, $\delta=0.015$ and $\lambda=5$.

4. CATASTROPHE INSURANCE MODELLING

While the probabilistic catastrophe risk modelling appeared in the late 1980s, the use of such technique was not widely accepted until Hurricane Andrew made landfall in Southern Florida in 1992. The Florida case also shows the catastrophe risk is concentrated, because it alone accounts for 80% of extreme hurricane risk in the USA. The 1990 January windstorm (Cat 90A) exposes that the problem facing the UK insurance industry is the likelihood of large losses exceeding the largest value in the data set. Both cases imply that the claim amounts of catastrophe insurances are likely to be much higher than expected and this is also pointed out in Sanders (2005). Underestimation issues also happen in Northridge Earthquake (1995) and Florida and the southern states hurricanes (2004), due to the calibration error and the unawareness of the potential for patterns of losses.

The pattern of catastrophe insurance data suggests that large claim amounts can occur in a sample with non-negligible probability, which proposes that we need distributions whose right tails decrease more slowly than exponential decay for modelling purpose. In that case, distributions with regularly varying tails, or more generally, subexponential distributions, are natural candidates. Therefore, the Mittag-Leffler distribution is one of the candidates by Theorem 2. Meanwhile, Embrechts *et al.* (2013, p. 35, Table 1.2.6) suggest that Pareto distributions, that is, if $X \sim P(\beta, \alpha)$,

$$\bar{F}_X(x) = \left(\frac{\alpha}{\alpha + x}\right)^{\beta}, \quad \beta, \alpha, x > 0,$$

and loggamma distributions, that is, if $X \sim LG(\beta, \alpha)$,

$$f_X(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}(1+x)^{-(\beta+1)} \left(\log(1+x)\right)^{\alpha-1}, \quad \beta, \alpha, x > 0,$$

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TABLE 1

α_p AND α_{lg} WITH $\alpha = 15$.

θ	0.9	0.6	0.3
Median	0.04446	0.04038	0.03812
α_p	0.03832	0.01857	0.00420
α_{lg}	0.24128	0.20765	0.17106

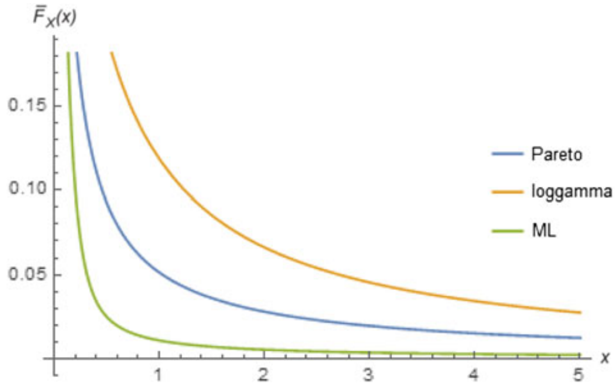

FIGURE 6: Tails with $\theta = 0.9$.

are suitable candidates as well. It should be noticed that these three types of distributions can have finite or infinite moments by feeding different values to their parameters. By Theorem 2 and Embrechts *et al.* (2013, p. 134, Example 3.3.11),

$$P^{(\theta,\alpha_p)}, LG^{(\theta,\alpha_{lg})}, ML^{(\alpha,\theta)} \in \mathcal{R}_{-\theta}, \quad 0 < \theta < 1,$$

and then none of them has a finite first moment. In order to compare these three distributions, we first fix α, $\theta \in (0, 1)$, that is, the tail behaviour and the median of them, and then use the median to find α_p and α_{lg}.

Figures 6–8 give the tail behaviour of $P^{(\theta,\alpha_p)}$, $LG^{(\theta,\alpha_{lg})}$ and $ML^{(\alpha,\theta)}$ for different θ's. For illustration purpose, we fix $\alpha = 15$ and the corresponding values of α_p and α_{lg} are given in Table 1. $ML^{(\alpha,\theta)}$ has the lightest tail when $\theta = 0.9$, dropping to zero much quicker than the other two. However, as θ decreases, all the tails become heavier. In particular, the tail of $ML^{(\alpha,\theta)}$ almost coincides with $P^{(\theta,\alpha_p)}$'s when $\theta = 0.6$ and becomes the heaviest when $\theta = 0.3$. These figures suggest that the Mittag-Leffler distribution has greater flexibility in tail behaviour and therefore is suitable for modelling claim amounts of catastrophe insurance.

The main analysis tool of a probabilistic catastrophe insurance model is the Exceedance Probability (EP) curve, which illustrates the annual probability of exceeding a certain level of loss. Once the formula of Z_t is derived, we can have the annual exceedance probability curve by Z_1. Now, for illustration purpose, assuming the claim arrival intensity is five per year, interest rate is 0.015 and

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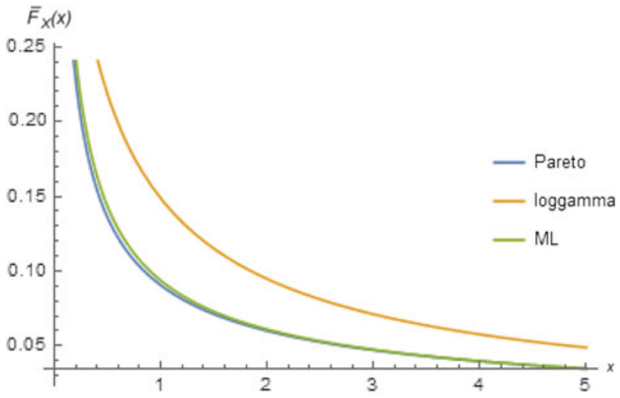

FIGURE 7: Tails with $\theta = 0.6$.

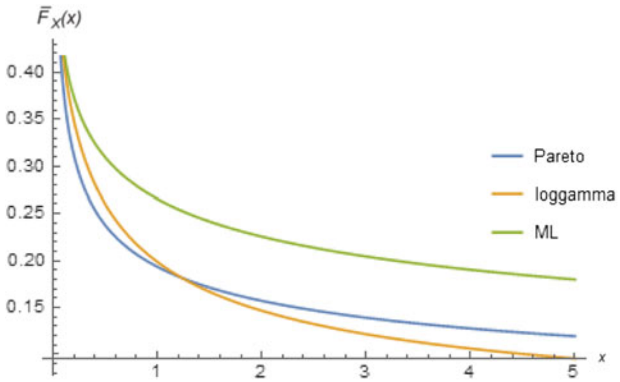

FIGURE 8: Tails with $\theta = 0.3$.

claim amount is counted in millions and follows a Mittag-Leffler distribution with parameters $(1, \theta)$, we can compare EP curves under different θ's. As for how to apply Mittag-Leffler distributions to real data sets see Cahoy (2013) and Jose and Abraham (2011).

As F_{Z_t} is expressed in terms of infinite summations, it is difficult to ascertain the value of z for a given EP(z). However, once the exact formula is given, we can always approximate it using some well-behaved functions. Thus, based on the analytical formula of F_{Z_t} in Section 3, we plot EP(z) from $z = 0$ with step 0.5 and then use linear interpolations to find z for a given EP(z). During the numerical calculations, it is expected that the tails of $\theta = 0.95$ and $\theta = 0.9$ are so heavy that they cannot be put in one figure with $\theta = 1$ case neatly. Therefore, we cut the tail at $z = 80$. For $z > 80$, claim amounts are all calculated by the line interpolated by (79, EP(79)) and (80, EP(80)). We place these numbers in brackets if that happens. However, the linear interpolation will underestimate claim amounts as the tails of EP(z) fall much slower as z increases.

TABLE 2

SUMMARY OF EP ANALYSIS.

		Loss amount ($m)		
EP (%)	Return period	$\theta = 1$	$\theta = 0.95$	$\theta = 0.9$
10	10	9.2	11.4	14.9
5	20	10.9	14.9	22.7
2	50	12.9	23.4	47.1
1	100	14.3	38.7	(89.7)
0.5	200	15.7	71.3	(123.9)

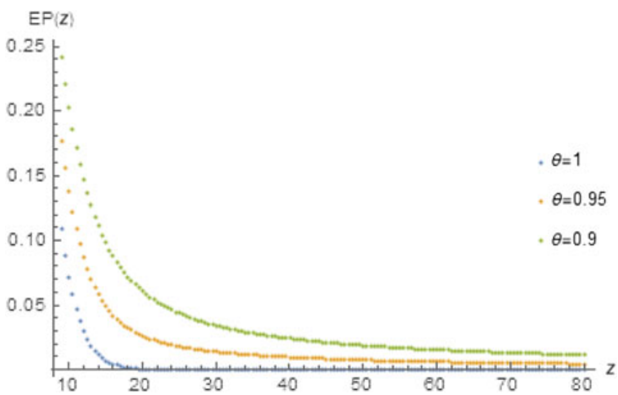

FIGURE 9: AEP with $\delta = 0.015$, $\alpha = 1$ and $\lambda = 5$.

Figure 9 and Table 2 are the main output of catastrophe modelling analysis. During the computations, the infinite series in Equation (3.3) are truncated after 2000 terms and the "Precision" of Mathematica is set at 200. Taking $\theta = 1$ as a benchmark, results show that EP(z)'s fall very slowly for $\theta < 1$. More specifically, at the return period of 50, claim amounts are more than doubled when θ decreases from 1 to 0.95 and from 0.95 to 0.9. For each return period, loss amounts increase by a larger amount when θ decreases from 0.95 to 0.9 than those when θ decreases from 1 to 0.95. Further, \$123.9m in Table 2 is underestimated as we believe the number should be more than twice of that under $\theta = 0.95$, that is, \$71.3m. For applications, we suggest to undertake interpolation across the neighbourhood of desired EP(z) for higher accuracy.

Next we are going to propose an approximation to EP. Tang (2005) gives an asymptotic formula of $\bar{F}_{Z_t}(z)$ for subexponential claim amounts. Further, if $X \in \mathcal{R}_{-\beta}$, $\bar{F}_{Z_t}(z)$ is proportional to $\bar{F}_X(z)$. More specifically, for large z,

$$\mathrm{EP}(z) \approx \frac{\lambda}{\beta\delta} \bar{F}_X(z)(1 - e^{-\beta\delta}). \tag{4.1}$$

TABLE 3

EP APPROXIMATION.

$\theta = 0.95$	$z = 80$	$z = 160$	$z = 240$	$z = 320$	$z = 400$
$X \sim P(\theta, \alpha_p)$	0.0496382	0.0257905	0.0175677	0.013375	0.0108241
$X \sim LG(\theta, \alpha_{lg})$	0.0462685	0.0234818	0.0157892	0.0119153	0.00957894
$X \sim ML(\alpha, \theta)$	0.00408678	0.00208514	0.00141168	0.00107147	0.000865504
$\theta = 0.9$	$z = 80$	$z = 160$	$z = 240$	$z = 320$	$z = 400$
$X \sim P(\theta, \alpha_p)$	0.0580848	0.0312271	0.0217029	0.0167612	0.013716
$X \sim LG(\theta, \alpha_{lg})$	0.0531649	0.0278114	0.01904	0.0145546	0.0118187
$X \sim ML(\alpha, \theta)$	0.010454	0.00551549	0.00380852	0.0029316	0.00239412

In comparison with formulas in Theorem 3, this approximation can be calculated much quicker, as those cumulative distribution functions are well built in software.

Table 3 gives the approximated values for EP(z), where α_p and α_{lg} are estimated in the same way as those in Table 1. From EP analysis, theoretically, EP(80) = 0.00441851 when $\theta = 0.95$ and EP(80) = 0.0114112 when $\theta = 0.9$. In comparison with the corresponding numbers in Table 3, it is safe to conclude that the approximation works well. This is expected as we only apply this approximation to the tail of Z_1. When θ decreases from 0.95 to 0.9, EP(z) increases the most when $X \sim ML(\alpha, \theta)$, larger than doubled. This is also expected because EP(z) is determined by $\bar{F}_X(z)$ in the approximation and $ML^{(\alpha,\theta)}$ is more flexible in tail control, that is, the tail changes from the thinnest to the heaviest among these distributions as θ decreases.

Another possible issue is the pricing of this insurance product. Since most of the premium calculation principles involve calculations of the first and second moments, we need to investigate alternatives as these indexes do not exist for $\theta < 1$. One alternative is to use the median and the interquartile range, that is, $F^{-1}(\frac{3}{4}) - F^{-1}(\frac{1}{4})$, for replacements of the first and second moments. Another alternative is to use the moments of $\theta = 1$ as benchmarks and add loadings for $\theta < 1$ cases, where the calculation of loadings could be based on the EP analysis. Similarly, we can start with the inverse of the mode of the density function, which can be derived by various approximation techniques or graphically. Under this numerical illustration, loss amounts are \$3.5m, \$3m and \$3m for θ =1, 0.95 and 0.9, respectively. Then, premiums can be calculated by adjusting different risk loadings.

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