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Sentinels with given sensitivity

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This work is devoted to the identification of parameters in a problem of pollution modeled by a semi-linear parabolic equation. We use the notion of sentinels introduced by J. L. Lions, (Lions, J. L. 1992 *Sentinelles pour les systèmes distribués à données incomplètes*. Masson, Paris.) re-visited in a more general framework. We prove the existence of such sentinels by solving a problem of null controllability with constraint on the control. The key of our results is an observability inequality of Carleman type adapted to the constraint.

1 Introduction

In the modeling of the problems of pollution governed by dissipative systems (for example, pollution in a river or a lake), the source terms as well as the initial or boundary conditions may be unknown. More precisely, let $N, M \in \mathbb{N}\backslash\{0\}$ and let Ω be a bounded open and connected subset of $\mathbb{R}^N$ with boundary Γ of class $\mathscr{C}^2$. For a time $T > 0$, we set $Q = \Omega \times (0, T)$ and $\Sigma = \Gamma \times (0, T)$. Then we consider the following system modeling a problem of pollution [9]:

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + f(y) = \xi + \displaystyle\sum_{i=1}^{M} \lambda_i \hat{\xi}_i & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 + \tau \hat{y}^0 & \text{in } \Omega, \end{cases} \tag{1}$$

where

- y represents the concentration of the pollutant.
- f is a real-valued given function of class $\mathscr{C}^1$.
- The source term is unknown and represents pollution source of the form $\xi + \sum_{i=1}^{M} \lambda_i \hat{\xi}_i$. The functions ξ and $\{\hat{\xi}_i\}_{i=1}^{M}$ are known whereas the real coefficients $\{\lambda_i\}_{i=1}^{M}$ are unknown.
- The initial condition is of the form $y^0 + \tau \hat{y}^0$ where the function y^0 is known while τ, real, is unknown.

We assume that

- y^0 and $\hat{y}^0$ belong to $L^2(\Omega)$, ξ and $\hat{\xi}_i$ belong to $L^2(Q)$,
- the functions $\hat{\xi}_i$, $1 \leqslant i \leqslant M$, are linearly independent,

- the real τ is sufficiently small,
- the function f verifies

$$f(0) = 0 \tag{2}$$

and satisfies the growth condition

$$|f(s_1) - f(s_2) - f'(0)(s_1 - s_2)| \leqslant C\left(|s_1|^{p-1} + |s_2|^{p-1}\right)|s_1 - s_2| \;\forall s_1,\quad s_2 \in I\!R, \tag{3}$$

for some $C > 0$ and $p > 1$ such that $p < \frac{N+4}{N}$.

In the model (1), we are interested in identifying the parameters λ_i without any attempt of computing the missing term $\tau\hat{y}^0$. To identify these parameters, we use the method of sentinels due to J. L. Lions [9] but in a more general framework.

The theory of sentinels introduced by J. L. Lions relies on three considerations:

- A *state equation* represented here by (1) whose solution $y = y(x, t; \lambda, \tau) = y(\lambda, \tau)$ depends on two families of parameters $\lambda = \{\lambda_1, \ldots, \lambda_M\}$ and τ.
- An *observation* y_{obs} which is a measurement of the concentration of the pollutant taken on a non-empty open subset O of Ω, called observatory.
- A *function* $S = S(\lambda, \tau)$ called "sentinel" defined for $h_0 \in L^2(O \times (0, T))$ by

$$S(\lambda, \tau) = \int_0^T \int_O (h_0 + w) y \, dx \, dt \tag{4}$$

where the control function w is to be found of minimal norm in $L^2(O \times (0, T))$ among functions S satisfying (4) which are stationary to the first order with respect to the missing term $\tau\hat{y}^0$, i.e.,

$$\frac{\partial S}{\partial \tau}(0, 0) = 0 \quad \forall \hat{y}^0. \tag{5}$$

Using the adjoint problem, one shows that the existence of these sentinels is reduced to the solution of exact controllability problem with constraints on the state. These types of controllability problems were the subject of many numerical methods which in fact reduce them to an approximate controllability problem with constraints on the state. It is in this context, for instance, that J. P. Kernevez *et al.* use these sentinels in [1, 2] to identify parameters of pollution in a river. O. Bodart apply them in [3] to identify an unknown boundary.

Remark 1 To estimate the parameter λ_i, one proceeds as follows: Assume that the solution of the state equation (1) when $\lambda = 0$ and $\tau = 0$ is known. Then one has the following information:

$$S(\lambda, \tau) - S(0, 0) \approx \sum_{i=1}^{M} \lambda_i \frac{\partial S}{\partial \lambda_i}(0, 0).$$

Therefore, fixing $i,\, j \in \{1,\ldots,M\}$ and choosing i and j such that

$$\frac{\partial S}{\partial \lambda_j}(0,0) = 0 \quad \text{for } j \neq i \text{ and } \frac{\partial S}{\partial \lambda_i}(0,\ldots,0) = 1$$

one obtains the following estimate of the parameter λ_i:

$$\lambda_i \approx S(\lambda,\tau) - S(0,0).$$

Remark 2 Notice that for the J. L. Lions's sentinel defined by (4)–(5), the observatory $O \subset \Omega$ is the support of the control function w.

In this paper, we consider the general case where the support of the control function w is different from the observatory O. More precisely, we consider, as above, the state equation (1) whose solution $y = y(\lambda,\tau)$ depends on two families of parameters λ and τ, the observatory $O \subset \Omega$ where the measurement y_{obs} is recorded and for any non-empty open subset ω of Ω with $\omega \neq O$, we look for a function $S = S(\lambda,\tau)$ which is solution of the following problem: Given $h_0 \in L^2(O \times (0,T))$, find $w \in L^2(\omega \times (0,T))$ such that

(1) the function S defined by:

$$S(\lambda,\tau) = \int_0^T \int_O h_0 y(\lambda,\tau)\,dx\,dt + \int_0^T \int_\omega w\, y(\lambda,\tau)\,dx\,dt. \tag{6}$$

satisfies the following conditions:

- S is stationary to the first order with respect to the missing term $\tau \hat{y}^0$:

$$\frac{\partial S}{\partial \tau}(0,0) = 0 \quad \forall \hat{y}^0. \tag{7}$$

- S is sensitive to the first order with respect to the pollution terms $\lambda_i \hat{\xi}_i$:

$$\frac{\partial S}{\partial \lambda_i}(0,0) = c_i \quad 1 \leqslant i \leqslant M \tag{8}$$

where c_i, $(1 \leqslant i \leqslant M)$, are given constants not all identically zero.

(2) The control w is of minimal norm in $L^2(\omega \times (0,T))$ among "the admissible controls'," i.e.,

$$|w|_{L^2(\omega\times(0,T))} = \min_{\bar{w}\in E} |\bar{w}|_{L^2(\omega\times(0,T))} \tag{9}$$

where $E = \{\bar{w} \in L^2(\omega \times (0,T))$, such that $(\bar{w}, S(\bar{w}))$ satisfies (6)–(8)$\}$

In the sequel, we assume without loss of generality that

$$\xi = 0 \text{ in } Q \text{ and } y^0 = 0 \text{ in } \Omega. \tag{10}$$

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Under the above hypotheses on f and the data, it is proved in [4, 15] that there exists $\alpha > 0$ such that when

$$\|\tau\hat{y}^0\|_{L^2(\Omega)} + \left\| \sum_{i=1}^{M} \lambda_i \hat{\xi}_i \right\|_{L^2(Q)} \leqslant \alpha$$

the problem (1) admits a unique solution $y = y(\lambda, \tau)$ in $\mathscr{C}([0,T], L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$. Moreover, the following applications

$$\tau \mapsto y(\lambda, \tau) \text{ and } \lambda_i \mapsto y(\lambda, \tau) \quad (1 \leqslant i \leqslant M) \tag{11}$$

are in $\mathscr{C}^1(I\!R; \mathscr{C}([0,T], L^2(\Omega)))$.

Remark 3 Since $y \in L^2(0,T; H_0^1(\Omega))$, $h_0 \in L^2(\mathcal{O} \times (0,T))$ and $w \in L^2(\omega \times (0,T))$, the relation (6) is well defined. Furthermore, in view of (11), the derivatives of y with respect to τ denoted by

$$y_\tau = \frac{d}{d\tau} y(\lambda, \tau)|_{\tau=0} \tag{12}$$

and with respect to λ_i denoted by

$$y_{\lambda_i} = \frac{d}{d\lambda_i} y(\lambda, \tau)|_{\lambda_i=0} \tag{13}$$

exist. Thus the conditions (7) and (8) are well defined.

Remark 4 In the sensitivity condition (8), the c_i are chosen according to the importance associated with the component $\hat{\xi}_i$ of the pollution source.

Remark 5 If the function S defined by (6)–(8) exists, then it is unique since w verifies (9). In this case, proceeding as in Remark 1, we get

$$\lambda_i \approx \frac{1}{c_i}(S(\lambda, \tau) - S(0,0)).$$

Definition 1.1 We will refer to the function S given by (6)–(8) as sentinel with given $\{c_i\}$ sensitivity.

Let y_0 be the solution of (1) when $\lambda = 0$ and $\tau = 0$. Then, in view of (10), we have

$$y_0 = 0 \text{ in } Q. \tag{14}$$

Therefore, according to (12) and (13), y_τ and y_{λ_i} are respectively solutions of

$$\begin{cases} \dfrac{\partial y_\tau}{\partial t} - \Delta y_\tau + f'(0) y_\tau = 0 & \text{in } Q, \\ y_\tau = 0 & \text{on } \Sigma, \\ y_\tau(0) = \hat{y}^0 & \text{in } \Omega \end{cases} \tag{15}$$

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and

$$\begin{cases} \dfrac{\partial y_{\lambda_i}}{\partial t} - \Delta y_{\lambda_i} + f'(0) y_{\lambda_i} = \hat{\xi}_i & \text{in } Q, \\ y_{\lambda_i} = 0 & \text{on } \Sigma, \\ y_{\lambda_i}(0) = 0 & \text{in } \Omega \end{cases} \tag{16}$$

where $f'(0)$ denotes the derivative of f at $y_0 = 0$. Thanks to (3), the problems (15) and (16) admit respectively unique solutions $y_\tau \in C([0,T], L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$ and $y_{\lambda_i} \in H^{2,1}(Q) = L^2(0,T; H^2(\Omega)) \cap H^1(0,T; L^2(\Omega))$.

Let χ_ω be the characteristic function of the set ω. We set

$$Y_\lambda = \operatorname{Span}(y_{\lambda_1}\chi_\omega, \dots, y_{\lambda_M}\chi_\omega) \tag{17}$$

the vector subspace of $L^2(\omega \times (0,T))$ generated by the M independent functions $y_{\lambda_i}\chi_\omega$, $1 \leqslant i \leqslant M$ and we denote by $Y_\lambda^\perp$ the orthogonal of Y_λ in $L^2(\omega \times (0,T))$. We also set

$$a_0 = f'(0) \tag{18}$$

and assume that

$$\begin{cases} \text{any function } k \in Y_\lambda \text{ such that} \\ \dfrac{\partial k}{\partial t} - \Delta k + a_0 k = 0 \text{ in } \omega \times (0,T) \text{ is identically zero in } \omega \times (0,T). \end{cases} \tag{19}$$

Next, we consider the following controllability problem: *Given $h \in L^2(Q)$ and $a_0 \in L^\infty(Q)$, find $v \in L^2(\omega \times (0,T))$ such that*

$$v \in Y_\lambda^\perp \tag{20}$$

and such that $q = q(x,t,v) \in H^{2,1}(Q)$ which is the solution of

$$\begin{cases} -\dfrac{\partial q}{\partial t} - \Delta q + a_0 q = h + v\chi_\omega & \text{in } Q, \\ q = 0 & \text{on } \Sigma, \\ q(T) = 0 & \text{in } \Omega, \end{cases} \tag{21}$$

satisfies

$$q(x,0,v) = 0 \quad \text{in } \Omega \tag{22}$$

with v of minimal norm in $L^2(\omega \times (0,T))$, that is

$$|v|_{L^2(\omega\times(0,T))} = \min_{\bar{v}\in\mathscr{E}} |\bar{v}|_{L^2(\omega\times(0,T))} \tag{23}$$

where

$$\mathscr{E} = \left\{ \bar{v} \in Y_\lambda^\perp \text{ such that } (\bar{v}, \bar{q} = q(x,t,\bar{v})) \text{ is subject to (21)–(22)} \right\}. \tag{24}$$

The problem (20)–(23) is a linear problem of null controllability without constraint on the control v. Few results are known for such problems. The case $Y_\lambda^\perp = L^2(\omega \times (0,T))$ has been widely studied. For instance, in [8] G. Lebeau and L. Robbiano solved this problem for the heat equation. In [12] D. Russell proved exact controllability for the heat equation as a consequence of exact controllability for the wave equation. In [14], D. Tataru showed

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that for linear equations, local and global controllability are equivalent and hold for any time $T > 0$. Considering a linear Fourier boundary condition, E. Fernandez-Cara, M. González-Burgos, S. Guerrero and J. P. Puel in [5] use the Carleman estimate for the weak solution of heat equation with non-homogeneous Neumann boundary conditions to prove the null controllability of the linear heat equation.

In the non-linear case, the problem of finite dimensional null controllability is studied by E. Zuazua in [15]. The author proved that for a rather general and natural class of non-linearities, the problem is solvable if the initial data are small enough. In [6] A. Fursikov and O. Yu. Imanuvilov showed that, when the control acts on the boundary, null controllability holds for bounded continuous and sufficiently small initial data.

When $Y_\lambda^\perp \neq L^2(\omega \times (0,T))$, J. L. Lions gives in [9] an optimality system for the optimal control assuming that the problem (20)–(23) has a unique solution. The proof uses Hilbert Uniqueness Method (HUM). The success of this method rests on the unique continuation results due to S. Mizohata [11] and J. C. Saut and B. Scheurer [13].

In this paper, we solve the null controllability problem with constraint on the control (20)–(23) assuming that (19) holds. This allows us to prove the existence of the sentinel (6)–(9). More precisely, we have the following results:

Theorem 1.1 *Assume that the above hypotheses on Ω, ω, $\mathcal{O}$, f and the data of the equation (1) are satisfied. Then the existence of the sentinel (6)–(9) holds if and only if null controllability problem with constraint on the control (20)–(23) holds.*

The proof of the null controllability problem with constraint on the control (20)–(23) lies on the existence of a function θ and a Carleman inequality adapted to the constraint (*cf.* Subsection 2.2) for which we have the following result:

Theorem 1.2 *Assume that the hypotheses of Theorem 1.1 and the condition (19) are satisfied. Then there exists a positive weight function θ such that, for any function $h \in L^2(Q)$ with $\theta h \in L^2(Q)$, null controllability problem with constraint on the control (20)–(23) holds. Moreover, the control is given by*

$$\hat{v}_\theta = -(\hat{\rho}_\theta \chi_\omega - P\hat{\rho}_\theta) \tag{25}$$

where $\hat{\rho}_\theta$ is a solution of

$$\begin{cases} \dfrac{\partial \hat{\rho}_\theta}{\partial t} - \Delta \hat{\rho}_\theta + a_0 \hat{\rho}_\theta = 0 & \text{in } Q, \\ \hat{\rho}_\theta = 0 & \text{on } \Sigma \end{cases} \tag{26}$$

and P is the orthogonal projection operator from $L^2(\omega \times (0,T))$ into Y_λ.

Remark 6 The assumption (19) has already been introduced by J. L. Lions in [9], p. 33. This assumption is satisfied for instance if we are in the following case: Let $(\omega_i)_{i=1}^M$ be a sequence of open sets such that

$$\omega_i \subset \omega, \quad \omega_i \cap \omega_j \neq \emptyset \quad \text{for } i \neq j.$$

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Assume that the functions y_{λ_i} are such that $\frac{\partial y_{\lambda_i}}{\partial t} - \Delta y_{\lambda_i} + a_0 y_{\lambda_i} \neq 0$ in $\omega_i \times (0, T)$. Then if $y_{\lambda_i} \in Y_\lambda$ and $\frac{\partial y_{\lambda_i}}{\partial t} - \Delta y_{\lambda_i} + a_0 y_{\lambda_i} = 0$ in $\omega \times (0, T)$ we have $y_{\lambda_i} = 0$ in $\omega \times (0, T)$, and the assumption (19) is satisfied.

The rest of the paper is organized as follows. Section 2 is devoted to some preliminary results. In this section, we prove Theorem 1.1 and establish the inequality adapted to the constraint (20). In Section 3, we prove the existence and the uniqueness of the solution for the controllability problem (20)–(23) of Theorem 1.1 and give the proof of Theorem 1.2. We finish with Section 4 where the expression of the sentinel S defined by (6)–(9) and the estimate of the parameters λ_i are given.

2 Preliminary results

2.1 Proof of Theorem 1.1

Since y_τ and y_{λ_i} are respectively solutions of (15) and (16), the stationary condition (7) and respectively the sensitivity condition (8) hold if and only if

$$\int_0^T \int_O h_0 y_\tau \, dx \, dt + \int_0^T \int_\omega w y_\tau \, dx \, dt = 0 \quad \forall \hat{y}^0 \in L^2(\Omega) \tag{27}$$

and

$$\int_0^T \int_O h_0 y_{\lambda_i} \, dx \, dt + \int_0^T \int_\omega w y_{\lambda_i} \, dx \, dt = c_i \quad 1 \leqslant i \leqslant M. \tag{28}$$

In order to transform equation (27), we introduce the classical adjoint state. More precisely, we consider the solution $q = q(x, t)$ of the linear problem

$$\begin{cases} -\dfrac{\partial q}{\partial t} - \Delta q + a_0 q = h_0 \chi_O + w \chi_\omega & \text{in } Q, \\ q = 0 & \text{on } \Sigma, \\ q(T) = 0 & \text{in } \Omega, \end{cases} \tag{29}$$

where χ_O is characteristic function of the open set O. Similar to problem (16), the problem (29) admits a unique solution $q \in H^{2,1}(Q)$. The so-called adjoint state q depends on the unknown function w and its usefulness comes from the following observation.

First, multiplying both sides of the differential equation in (29) by $y_\tau \in C([0, T], L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ which is solution of (15), and integrating by parts in Q, we get

$$\int_0^T \int_O h_0 y_\tau \, dx \, dt + \int_0^T \int_\omega w y_\tau \, dx \, dt = \int_\Omega q(0) \hat{y}^0 \, dx \quad \forall \hat{y}^0 \in L^2(\Omega).$$

Thus, the condition (7) (or (27)) holds if and only if

$$q(0) = 0 \quad \text{in } \Omega. \tag{30}$$

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Then, multiplying both sides of the differential equation in (29) by $y_{\lambda_i} \in H^{2,1}(Q)$ which is solution of (16), and integrating by parts in Q, we have

$$\int_0^T \int_\Omega q\hat{\xi}_i \, dx\, dt = \int_0^T \int_O h_0 y_{\lambda_i} \, dx\, dt + \int_0^T \int_\omega w y_{\lambda_i} \, dx\, dt \quad 1 \leqslant i \leqslant M. \tag{31}$$

Thus, the condition (8) (or (28)) is equivalent to

$$\int_0^T \int_\Omega q\hat{\xi}_i \, dx\, dt = c_i, \quad 1 \leqslant i \leqslant M. \tag{32}$$

Therefore, the above considerations show that the existence of the sentinel defined by (6)–(9) holds if and only if the following null controllability problem with constraints on the state q holds: *Given $h_0 \in L^2(O \times (0,T))$ and $a_0 \in L^\infty$, find w of minimal norm in $L^2(\omega \times (0,T))$ such that the pair (w, q) verifies (29), (30) and (32).*

Actuall, condition (8) (or the constraints (32) on the state q) is equivalent to constraint on the control. Indeed, let Y_λ be the real vector subspace of $L^2(\omega \times (0,T))$ defined in (17). Since Y_λ is finite dimensional, there exists a unique $w_0 \in Y_\lambda$ such that

$$c_i - \int_0^T \int_O h_0 y_{\lambda_i} \, dx\, dt = \int_0^T \int_\omega w_0 y_{\lambda_i} \, dx\, dt, 1 \leqslant i \leqslant M.$$

Therefore, the condition (28) or (32) holds if and only if

$$w - w_0 = v \in Y_\lambda{}^\perp. \tag{33}$$

Consequently, replacing w by $v + w_0$ in $(29)_1$, then setting

$$h = h_0\chi_O + w_0\chi_\omega \in L^2(Q) \tag{34}$$

we finally deduce that we have the existence of the sentinel (6)–(9) if and only if null controllability with constraint on the control (20)–(23) holds. ■

2.2 An adapted Carleman inequality

The observability inequality we are looking for is a consequence of Carleman's inequality. We consider an auxiliary function $\psi \in C^2(\overline{\Omega})$ which satisfies the following conditions:

$$\psi(x) > 0 \quad \forall x \in \Omega, \quad \psi(x) = 0 \quad \forall x \in \Gamma, \quad |\nabla\psi(x)| \neq 0 \quad \forall x \in \overline{\Omega - \omega}. \tag{35}$$

Such a function ψ exists according to A. Fursikov and O. Yu. Imanuvilov [6]. We then define for any positive parameter λ the following weight functions:

$$\varphi(x,t) = \frac{e^{\lambda(m\|\psi\|_\infty + \psi(x))}}{t(T-t)}, \tag{36}$$

$$\eta(x,t) = \frac{e^{2\lambda m\|\psi\|_\infty} - e^{\lambda(m\|\psi\|_\infty + \psi(x))}}{t(T-t)}, \tag{37}$$

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for $(x,t) \in Q$ and $m > 1$. Weight functions of this kind were first introduced by O. Yu. Imanuvilov. Since φ does not vanish on Q, we set

$$\theta = \frac{e^{s\eta}}{\varphi\sqrt{\varphi}} \tag{38}$$

and we adopt the following notations

$$\begin{cases} L = \dfrac{\partial}{\partial t} - \Delta + a_0 I, \\ L^* = -\dfrac{\partial}{\partial t} - \Delta + a_0 I, \\ \mathscr{V} = \{\rho \in \mathscr{C}^\infty(\overline{Q}) \text{ such that } \rho = 0 \text{ on } \Sigma\}, \end{cases}$$

where a_0 is defined as in (18).

Lemma 2.1 *Assume that (19) holds. Let θ be the function given by (38) and P be the operator defined as in Theorem 1.2. Then there exists a positive constant $C = C(\Omega, \omega, a_0)$ such that for any $\rho \in \mathscr{V}$:*

$$\int_0^T \int_\Omega \frac{1}{\theta^2} |\rho|^2 \, dx\, dt \leqslant C\left(\int_0^T \int_\Omega |L\rho|^2 \, dx\, dt + \int_0^T \int_\omega |\rho - P\rho|^2 \, dx\, dt\right). \tag{39}$$

The proof of this lemma requires that we recall the global Carleman's inequality.

Proposition 2.1 (*Global Carleman's inequality*) *Let ψ, φ and η be the functions defined respectively as in (35)–(37). Then, there exist numbers $\lambda_0 = \lambda_0(\Omega, \omega, a_0) > 1$ and $s_0 = s_0(\Omega, \omega, a_0, \lambda, T) > 1$ and there exists some number $C = C(\Omega, \omega, a_0) > 0$ such that, for any $\lambda \geqslant \lambda_0$, for any $s \geqslant s_0$ and for any $\rho \in \mathscr{V}$, the following inequality holds:*

$$\begin{aligned} &\int_0^T \int_\Omega \frac{e^{-2s\eta}}{s\varphi}\left(\left|\frac{\partial \rho}{\partial t}\right|^2 + |\Delta\rho|^2\right) dx\, dt + \int_0^T \int_\Omega s\lambda^2 \varphi e^{-2s\eta} |\nabla\rho|^2 \, dx\, dt \\ &\quad + \int_0^T \int_\Omega s^3\lambda^4\varphi^3 e^{-2s\eta} |\rho|^2 \, dx\, dt \\ &\leqslant C\left(\int_0^T \int_\Omega e^{-2s\eta} |L\rho|^2 \, dx\, dt + \int_0^T \int_\omega s^3\lambda^4\varphi^3 e^{-2s\eta} |\rho|^2 \, dx\, dt\right). \end{aligned} \tag{40}$$

Proof We refer to [6, 7]. ■

According to the definition of φ and η given respectively by (36) and (37), the function θ given by (38) is positive and $1/\theta = \varphi\sqrt{\varphi}e^{-s\eta}$ is bounded. So, replacing $e^{s\eta}/\varphi\sqrt{\varphi}$ by θ in (40) the following inequality holds:

$$\int_0^T \int_\Omega \frac{1}{\theta^2}|\rho|^2 \, dx\, dt \leqslant C\left(\int_0^T \int_\Omega \frac{1}{\theta^2\varphi^3 s^3\lambda^4}|L\rho|^2 \, dx\, dt + \int_0^T \int_\omega \frac{1}{\theta^2}|\rho|^2 \, dx\, dt\right).$$

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As a consequence of the boundedness of $1/\theta$ and $1/\varphi^3 s^3 \lambda^4$, we get the next observability inequality:

$$\int_0^T \int_\Omega \frac{1}{\theta^2} |\rho|^2 \, dx \, dt \leqslant C \left(\int_0^T \int_\Omega |L\rho|^2 \, dx \, dt + \int_0^T \int_\omega |\rho|^2 \, dx \, dt \right). \tag{41}$$

Proof of Lemma 2.1 The proof uses a well-known compactness-uniqueness argument and the inequality (41). Indeed, suppose that (39) does not hold. Then

$$\begin{cases} \forall n \in N^*, \exists \rho_n \in \mathscr{V}, \int_0^T \int_\Omega \frac{1}{\theta^2} |\rho_n|^2 \, dx \, dt = 1, \\ \int_0^T \int_\Omega |L\rho_n|^2 \, dx \, dt \leqslant \frac{1}{n} \text{ and } \int_0^T \int_\omega |\rho_n - P\rho_n|^2 \, dx \, dt \leqslant \frac{1}{n}. \end{cases} \tag{42}$$

Now, the rest of the proof consists in showing that (42) yields a contradiction. We do it in four steps.

Step 1. We have

$$\int_0^T \int_\omega \frac{1}{\theta^2} |P\rho_n|^2 \, dx \, dt \leqslant \int_0^T \int_\omega \frac{1}{\theta^2} |\rho_n|^2 \, dx \, dt + \int_0^T \int_\omega \frac{1}{\theta^2} |\rho_n - P\rho_n|^2 \, dx \, dt.$$

Since $1/\theta^2$ is bounded, it follows from (42) that

$$\int_0^T \int_\omega \frac{1}{\theta^2} |P\rho_n|^2 \, dx \, dt \leqslant C. \tag{43}$$

Since $P\rho_n \in Y_\lambda$ and Y_λ is finite dimensional, $(P\rho_n)_n$ (and so $(\rho_n)_n$) is bounded in $L^2(\omega \times (0, T))$.

Step 2. We can extract a subsequence, still denoted $(\rho_n)_n$, such that on the one hand

$$\rho_n \rightharpoonup g \text{ weakly in } L^2(\omega \times (0, T)), \tag{44}$$

and on the other hand,

$$\rho_n - P\rho_n \to 0 \text{ strongly in } L^2(\omega \times (0, T)). \tag{45}$$

Next, we deduce from the compactness of P (because Y_λ is finite dimensional) that there exists $\sigma \in Y_\lambda$ such that

$$P\rho_n \to \sigma \text{ strongly in } L^2(\omega \times (0, T)). \tag{46}$$

We deduce from (45) and (46) that $\rho_n \to g = \sigma$ strongly in $L^2(\omega \times (0, T))$. Thanks to the continuity of P, we have $P\rho_n \to Pg$ strongly in $L^2(\omega \times (0, T))$. Therefore, $Pg = g$ and $g \in Y_\lambda$.

Step 3. In fact, we have $g = 0$. Indeed, from (42), we also have $L\rho_n \to 0$ strongly in $L^2(Q)$. Thus $L\rho_n \to 0$ strongly in $L^2(\omega \times (0, T))$. We conclude that $L\rho_n \rightharpoonup 0$ weakly in $\mathscr{D}'(\omega \times (0, T))$ and so $Lg = 0$. The assumption (19) implies $g = 0$ on $\omega \times (0, T)$. Finally, $\rho_n \to 0$ strongly in $L^2(\omega \times (0, T))$.

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Step 4. Since $\rho_n \in \mathscr{V}$, it follows from the observability inequality (41) that

$$\int_0^T \int_\Omega \frac{1}{\theta^2} |\rho_n|^2 \, dx\, dt \leqslant C \left(\int_0^T \int_\Omega |L\rho_n|^2 \, dx\, dt + \int_0^T \int_\omega |\rho_n|^2 \, dx\, dt \right).$$

Then, the conclusions in the third step yield that $\int_0^T \int_\Omega \frac{1}{\theta^2} |\rho_n|^2 \, dx\, dt \to 0$ when $n \to +\infty$. The contradiction occurs thanks to the first condition in (42), where $\int_0^T \int_\Omega \frac{1}{\theta^2} |\rho_n|^2 \, dx\, dt = 1$. The proof of (39) is complete. ■

3 Null controllability with constraint on the control

The main tool used is the observability inequality (39) – adapted to the constraint.

3.1 Existence of optimal control variable for null controllability

Let us consider the following symmetric bi-linear form:

$$a(\rho, \hat{\rho}) \quad = \quad \int_0^T \int_\Omega L\rho L\hat{\rho} \, dx\, dt + \int_0^T \int_\omega (\rho - P\rho)(\hat{\rho} - P\hat{\rho}) \, dx\, dt. \tag{47}$$

According to Lemma 2.1, this symmetric bi-linear form is a scalar product on $\mathscr{V}$. Let V be the completion of $\mathscr{V}$ with respect to the norm:

$$\rho \mapsto \|\rho\|_V = \sqrt{a(\rho, \rho)}. \tag{48}$$

The closure of $\mathscr{V}$ is the Hilbert space V.

Remark 7 1. The norm $\|.\|_V$ is related to the right side of the inequality (39) while the left member of (39) leads to the norm

$$\|\rho\|_\theta = \left(\int_0^T \int_\Omega \frac{1}{\theta^2} |\rho|^2 \, dx\, dt \right)^{\frac{1}{2}}.$$

2. The completion of $\mathscr{V}$ is the weighted Hilbert space usually denoted by $L^2_{\frac{1}{\theta}}$.
3. The inequality (39) shows that

$$\|\rho\|_\theta \leqslant C\|\rho\|_V. \tag{49}$$

Let θ be defined by (38) and $h \in L^2(Q)$ be such that $\theta h \in L^2(Q)$. Then, thanks to Cauchy-Schwarz's inequality and (39), the following linear form defined on V by

$$\rho \to \int_0^T \int_\Omega h\rho \, dx\, dt$$

is continuous. Therefore, Lax-Milgram's Theorem allows us to say that, for every function $h \in L^2(Q)$ such that $\theta h \in L^2(Q)$, there exists one and only one solution ρ_θ in V of the

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variational equation:

$$a(\rho_\theta,\rho) = \int_0^T\int_\Omega L\rho L\rho_\theta \,dx\,dt + \int_0^T\int_\omega (\rho - P\rho)(\rho_\theta - P\rho_\theta)\,dx\,dt \\ = \int_0^T\int_\Omega h\rho\,dx\,dt\,\forall\rho\in V. \tag{50}$$

Proposition 3.1 *Assume (19) holds. For $h \in L^2(Q)$ such that $\theta h \in L^2(Q)$, let ρ_θ be the unique solution of (51),*

$$v_\theta = -(\rho_\theta\chi_\omega - P\rho_\theta) \tag{51}$$

and

$$q_\theta = L\rho_\theta. \tag{52}$$

Then, the pair (v_θ, q_θ) is such that (20)–(22) hold. Moreover, we have

$$\|\rho_\theta\|_V \leqslant C\,\|\theta h\|_{L^2(Q)}, \tag{53}$$

$$\|v_\theta\|_{L^2(\omega\times(0,T))} \leqslant C\|\theta h\|_{L^2(Q)}, \tag{54}$$

$$\|q_\theta\|_{H^{2,1}(Q)} \leqslant C\|\theta h\|_{L^2(Q)}, \tag{55}$$

where C represents different positive constants.

Proof We proceed in two steps.

Step 1. We prove that (v_θ, q_θ) is a solution of (20)–(22).

According to (51), we have $\rho_\theta \in V$. Consequently $q_\theta \in L^2(Q)$ and since $P\rho_\theta \in Y_\lambda$, the function $v_\theta = -(\rho_\theta\chi_\omega - P\rho_\theta) \in {Y_\lambda}^\perp$. Next, replacing $L\rho_\theta$ by q_θ and $(-\rho_\theta\chi_\omega + P\rho_\theta)$ by v_θ in (51), we obtain

$$\int_0^T\int_\Omega q_\theta L\rho\,dx\,dt - \int_0^T\int_\omega v_\theta(\rho - P\rho)\,dx\,dt = \int_0^T\int_\Omega h\rho\,dx\,dt, \quad \forall\rho\in V.$$

Since $P\rho \in Y_\lambda$ and $v_\theta \in Y_\lambda^\perp$, this latter equality is reduced to

$$\int_0^T\int_\Omega q_\theta L\rho\,dx\,dt = \int_0^T\int_\Omega h\rho\,dx\,dt + \int_0^T\int_\omega v_\theta\rho\,dx\,dt, \quad \forall\rho\in V. \tag{56}$$

Now, for $\phi \in L^2(Q)$, let p be the solution of the system

$$\begin{cases} p' - \Delta p + a_0 p = \phi & \text{in } Q, \\ p = 0 & \text{on } \Sigma, \\ p(0) = 0 & \text{in } \Omega. \end{cases} \tag{57}$$

Then $p \in V$. Consequently, replacing ρ by p in (56), we have

$$\int_0^T\int_\Omega q_\theta\phi\,dx\,dt = \int_0^T\int_\Omega hp\,dx\,dt + \int_0^T\int_\omega v_\theta p\,dx\,dt.$$

Thus, $q_\theta \in H^{2,1}(Q)$ is the solution of the problem (21) with $v = v_\theta$ (*cf.* [10]). In other words q_θ is the solution of the following problem:

$$\begin{cases} -q'_\theta - \Delta q_\theta + a_0 q_\theta = h + v_\theta \chi_\omega & \text{in } Q, \\ q_\theta = 0 & \text{on } \Sigma, \\ q_\theta(T) = 0 & \text{in } \Omega. \end{cases} \tag{58}$$

Since $q_\theta \in H^{2,1}(Q)$, we also have $q_\theta \in C([0,T], L^2(\Omega))$.

Multiplying $(58)_1$ by $\rho \in \mathscr{V}$ and integrating by parts in Q, it follows

$$\int_\Omega q_\theta(0)\rho(0)\,dx + \int_0^T \int_\Omega q_\theta L\rho\,dx\,dt = \int_0^T \int_\Omega h\rho\,dx\,dt + \int_0^T \int_\omega v_\theta \rho\,dx\,dt, \quad \forall \rho \in \mathscr{V}.$$

As $\rho \in \mathscr{V}$, we deduce from (56) that

$$\int_\Omega q_\theta(0)\rho(0)\,dx = 0 \quad \forall \rho \in \mathscr{V}.$$

Hence, $q_\theta(0) = 0$ in Ω.

Step 2. Let us prove the estimates (53)–(55). Replacing ρ by ρ_θ in (51) and using the Cauchy-Schwarz's inequality, we get from (39), (48),

$$\begin{aligned} a(\rho_\theta, \rho_\theta) &= \|q_\theta\|^2_{L^2(Q)} + \|v_\theta\|^2_{L^2(\omega\times(0,T))} \\ &\leqslant \|\theta h\|_{L^2(Q)} \|\rho_\theta\|_\theta \\ &\leqslant C\|\theta h\|_{L^2(Q)} \|\rho_\theta\|_V. \end{aligned}$$

Therefore, from (48), we obtain (53) and then (54). Estimate (55) is a consequence of (54) and of the properties of the heat equation. ■

Proposition 3.2 *Under the assumptions of Proposition 3.1, there exists a control variable v such that the pair (v, q) satisfies (20)–(22). Moreover, we can obtain a unique control $\hat{v}_\theta$ such that (23) holds.*

Proof We have proved in Proposition 3.1 that (v_θ, q_θ) satisfies (20)–(22). Consequently, the set $\mathscr{E}$ of the control variables $v \in L^2(\omega \times (0,T))$ such that $(v, q(x,t,v))$ verifies (20)–(22) is non-empty. Moreover, adapted observability inequality (39) shows that the choice of the scalar product on $\mathscr{V}$ is not unique. Thus, proceeding as in Proposition 3.1, we can construct infinitely many control functions v which belong to $\mathscr{E}$. It is then clear that $\mathscr{E}$ is a non-empty closed convex subset of $L^2(\omega \times (0,T))$. Therefore, there exists a unique *control variable* $\widehat{v}_\theta$ of minimal norm in $L^2(\omega \times (0,T))$ such that $(\widehat{v}_\theta, \widehat{q}_\theta = q(x,t,\widehat{v}_\theta))$ solves (20)–(23).

3.2 Proof of Theorem 1.2

In this subsection, we are concerned with the proof of Theorem 1.2. That is, the optimality system for the control $\hat{v}_\theta$ such that the pair $(\hat{v}_\theta, \hat{q}_\theta)$ satisfies (20)–(23). As a classical way to derive this optimality system is the method of penalization due to J. L. Lions [10], the proof of Theorem 1.2 requires some preliminary results.

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Let $\epsilon > 0$. For any pair (v, q) such that

$$\begin{cases} v \in Y_\lambda{}^\perp, \\ -\dfrac{\partial q}{\partial t} - \Delta q + a_0 q \in L^2(Q), \\ q = 0 \text{ on } \Sigma, \quad q(0) = q(T) = 0 \text{ in } \Omega, \end{cases} \tag{59}$$

we define the functional

$$J_\epsilon(v, q) = \frac{1}{2}\|v\|^2_{L^2(\omega\times(0,T))} + \frac{1}{2\epsilon}\| - \frac{\partial q}{\partial t} - \Delta q + a_0 q - h - v\chi_\omega\|^2_{L^2(Q)} \tag{60}$$

and we consider the minimization problem

$$\min J_\epsilon(v, q), \quad (v, q) \text{ subject to (59)}. \tag{61}$$

Proposition 3.3 *Under the assumptions of Proposition 3.1, the problem (61) has an optimal solution. In other words , there exists a unique pair (v_ϵ, q_ϵ) such that*

$$J_\epsilon(v_\epsilon, q_\epsilon) = \min\{J_\epsilon(v, q), (v, q) \textit{ subject to } (59)\} \tag{62}$$

Proof Since (v_θ, q_θ) defined in Proposition 3.1 satisfies (59) and $J_\epsilon(v, q) \geqslant 0$, we can define the real d_ϵ such that

$$d_\epsilon = \min\{J_\epsilon(v, q), (v, q) \text{ subject to (59)}\}.$$

Let (v_n, q_n) be a minimizing sequence satisfying (59) and such that

$$d_\epsilon \leqslant J_\epsilon(v_n, q_n) < d_\epsilon + \frac{1}{n} < d_\epsilon + 1.$$

In particular,

$$0 < d_\epsilon \leqslant J_\epsilon(v_\theta, q_\theta) = \frac{1}{2}|v_\theta|^2_{L^2(\omega\times(0,T))}.$$

It follows from the estimates (54) that there exists a constant C, independent of n such that

$$J_\epsilon(v_n, q_n) \leqslant C^2.$$

Therefore, from the form of J_ϵ, we get

$$\left\| -\frac{\partial q_n}{\partial t} - \Delta q_n + a_0 q_n - h - v_n\chi_\omega \right\|_{L^2(Q)} \leqslant C\sqrt{\epsilon}, \tag{63}$$

$$\|v_n\|_{L^2(\omega\times(0,T))} \leqslant C. \tag{64}$$

According to (64), there exists v_ϵ in $L^2(\omega \times (0, T))$ and a subsequence extracted from (v_n) (still called (v_n)) such that

$$v_n \rightharpoonup v_\epsilon \quad \text{weakly in} \quad L^2(\omega \times (0, T)). \tag{65}$$

Since v_n belongs to $Y_\lambda^\perp$ which is a closed vector subspace of $L^2(\omega \times (0, T))$, we have

$$v_\epsilon \in Y_\lambda^\perp. \tag{66}$$

As a consequence of (59) and (63),

$$\|q_n\|_{H^{2,1}(Q)} \leqslant C.$$

Thus, there exists $q_\epsilon \in H^{2,1}(Q)$ and a subsequence extracted from (q_n) (still called (q_n)) such that

$$q_n \rightharpoonup q_\epsilon \quad \text{weakly in} \quad H^{2,1}(Q). \tag{67}$$

Moreover, q_ϵ verifies

$$\begin{cases} -\dfrac{\partial q_\epsilon}{\partial t} - \Delta q_\epsilon + a_0 q_\epsilon \in L^2(Q), \\ q_\epsilon = 0 \text{ on } \Sigma, \quad q_\epsilon(T) = q_\epsilon(0) = 0 \text{ in } \Omega. \end{cases} \tag{68}$$

Finally, from (66) and (68), we deduce that (v_ϵ, q_ϵ) satisfies (59). Combining (65), (67) and the weak lower semicontinuous of J_ϵ, we obtain that $J_\epsilon(v_\epsilon, q_\epsilon) \leqslant \liminf_{n\to+\infty} J_\epsilon(v_n, q_n) = d_\epsilon$. In other words (v_ϵ, q_ϵ) is the optimal control. The uniqueness of (v_ϵ, q_ϵ) is the immediate consequence of the strict convexity of J_ϵ. ■

Proposition 3.4 *The assumptions are as in Proposition 3.1. Then, the pair (v_ϵ, q_ϵ) is optimal solution of the problem (62) if and only if there exists a function ρ_ϵ such that $(v_\epsilon, q_\epsilon, \rho_\epsilon) \in L^2(\omega \times (0,T)) \times H^{2,1}(Q) \times V$ satisfies the following approximate optimality system:*

$$v_\epsilon = -(\rho_\epsilon \chi_\omega - P\rho_\epsilon) \in Y_\lambda^\perp, \tag{69}$$

$$\begin{cases} -\dfrac{\partial q_\epsilon}{\partial t} - \Delta q_\epsilon + a_0 q_\epsilon = h + v_\epsilon \chi_\omega - \epsilon \rho_\epsilon & \text{in } Q, \\ q_\epsilon = 0 & \text{on } \Sigma, \\ q_\epsilon(T) = 0 & \text{in } \Omega, \end{cases} \tag{70}$$

$$q_\epsilon(0) = 0 \text{ in } \Omega, \tag{71}$$

$$\begin{cases} \dfrac{\partial \rho_\epsilon}{\partial t} - \Delta \rho_\epsilon + a_0 \rho_\epsilon = 0 & \text{in } Q, \\ \rho_\epsilon = 0 & \text{on } \Sigma. \end{cases} \tag{72}$$

Proof We express the Euler-Lagrange optimality conditions which characterize $(v_\varepsilon, q_\varepsilon)$:

$$\begin{cases} \dfrac{d}{d\mu} J_\epsilon(v_\epsilon, q_\epsilon + \mu\varphi)|_{\mu=0} = 0, \text{ for all } \varphi \in C^\infty(\overline{Q}) \text{ such that} \\ \varphi = 0 \text{ on } \Sigma, \quad \varphi(0) = \varphi(T) = 0 \text{ in } \Omega, \end{cases}$$

$$\frac{d}{d\mu} J_\epsilon(v_\epsilon + \mu v, q_\epsilon)|_{\mu=0} = 0, \quad \forall v \in Y_\lambda^\perp.$$

After calculations, we have

$$\begin{cases} \displaystyle\int_0^T \int_\Omega \frac{1}{\epsilon} \left(-\frac{\partial q_\epsilon}{\partial t} - \Delta q_\epsilon + a_0 q_\epsilon - h - v_\epsilon \chi_\omega \right) \left(-\frac{\partial \varphi}{\partial t} - \Delta\varphi + a_0\varphi \right) dx\, dt = 0, \\ \text{for all } \varphi \in C^\infty(\overline{Q}) \text{ such that } \varphi = 0 \text{ on } \Sigma, \quad \varphi(0) = 0, \quad \varphi(T) = 0 \text{ in } \Omega \end{cases} \tag{73}$$

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and

$$\int_0^T \int_\omega v_\epsilon v \, dx \, dt - \int_0^T \int_\Omega \frac{1}{\epsilon} \left(-\frac{\partial q_\epsilon}{\partial t} - \Delta q_\epsilon + a_0 z_\epsilon - h - v_\epsilon \chi_\omega \right) v \, dx \, dt = 0, \quad \forall v \in {Y_\lambda}^\perp. \quad (74)$$

Then we define the adjoint state

$$\rho_\epsilon = -\frac{1}{\epsilon} \left(-\frac{\partial q_\epsilon}{\partial t} - \Delta q_\epsilon + a_0 q_\epsilon - h - v_\epsilon \chi_\omega \right). \quad (75)$$

Hence, we deduce that $-\frac{\partial q_\epsilon}{\partial t} - \Delta q_\epsilon + a_0 q_\epsilon = h + v_\epsilon \chi_\omega - \epsilon \rho_\epsilon$ in Q. And since (v_ϵ, q_ϵ) verifies (59), we have $q_\epsilon = 0$ on Σ, $q_\epsilon(T) = q_\epsilon(0) = 0$ in Ω. Thus $(q_\epsilon, v_\epsilon, \rho_\epsilon)$ is such that (70) holds.

Now, replacing $-\frac{1}{\epsilon}(-\frac{\partial q_\epsilon}{\partial t} - \Delta q_\epsilon + a_0 q_\epsilon - h - v_\epsilon \chi_\omega)$ by ρ_ϵ in (73) and (74), we respectively obtain

$$\begin{cases} \int_0^T \int_\Omega \rho_\epsilon (-\frac{\partial \varphi}{\partial t} - \Delta \varphi + a_0 \varphi) \, dx \, dt = 0, \text{ for all } \varphi \in C^\infty(\overline{Q}) \text{ such that} \\ \varphi = 0 \text{ on } \Sigma, \quad \varphi(0) = \varphi(T) = 0 \text{ in } \Omega, \end{cases} \quad (76)$$

and

$$\int_0^T \int_\omega v_\epsilon v \, dx \, dt + \int_0^T \int_\Omega \rho_\epsilon v \, dx \, dt = 0, \quad \forall v \in {Y_\lambda}^\perp. \quad (77)$$

Therefore, from (76), we derive

$$\frac{\partial \rho_\epsilon}{\partial t} - \Delta \rho_\epsilon + a_0 \rho_\epsilon = 0 \text{ in } Q.$$

Thus, $\rho_\epsilon \in L^2(Q)$ and $L\rho_\epsilon \in L^2(Q)$. Consequently, we can define ρ_ϵ on Σ and prove that $\rho_\epsilon = 0$ on Σ.

¿From (77), we have

$$\int_0^T \int_\omega (v_\epsilon + \rho_\epsilon) v \, dx \, dt = 0, \quad \forall v \in {Y_\lambda}^\perp.$$

Hence, $v_\epsilon + \rho_\epsilon \chi_\omega \in Y_\lambda$. Since $v_\epsilon \in {Y_\lambda}^\perp$, we have $v_\epsilon + \rho_\epsilon \chi_\omega = P(v_\epsilon + \rho_\epsilon \chi_\omega) = P\rho_\epsilon$. Thus, $v_\epsilon = -(\rho_\epsilon \chi_\omega - P\rho_\epsilon)$. ∎

Remark 8 Let us mention that there is no information concerning $\rho_\epsilon(0)$ and $\rho_\epsilon(T)$.

Proposition 3.5 *Let $(v_\epsilon, q_\epsilon, \rho_\epsilon)$ be defined as in Proposition 3.4. Then there exists a constant $C > 0$ independent on ϵ such that*

$$\|q_\epsilon\|_{H^{2,1}(Q)} \leqslant C, \quad (78)$$

$$\|\rho_\epsilon - P\rho_\epsilon\|_{L^2(\omega \times (0,T))} \leqslant C, \quad (79)$$

$$\|\rho_\epsilon\|_{L^2(\omega \times (0,T))} \leqslant C, \quad (80)$$

$$\|\rho_\epsilon\|_V \leqslant C. \quad (81)$$

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Proof From (63) and (64), we have

$$\left\| -\frac{\partial q_\epsilon}{\partial t} - \Delta q_\epsilon + a_0 q_\epsilon - h - v_\epsilon \chi_\omega \right\|_{L^2(Q)} \leqslant C\sqrt{\epsilon}, \tag{82}$$

$$\|v_\epsilon\|_{L^2(\omega\times(0,T))} \leqslant C. \tag{83}$$

Since q_ϵ verifies (59), we derive from (82) the relation (78). From (69) and (83), we obtain (79). Then as $L\rho_\epsilon = 0$, using the definition of the norm on V given by (48), we have (81).

On the other hand, since $\rho_\epsilon \in \mathscr{V}$, applying the observability inequality (39) to ρ_ϵ, we have $\|\frac{1}{\theta}\rho_\epsilon\|_{L^2(\omega\times(0,T))} \leqslant C$. Therefore, using (79) and the fact that $1/\theta$ is in $L^\infty(Q)$, we deduce that $\|\frac{1}{\theta}P\rho_\epsilon\|_{L^2(\omega\times(0,T))} \leqslant C$. Since $P\rho_\epsilon$ is in Y_λ which is finite dimensional, we have $\|P\rho_\epsilon\|_{L^2(\omega\times(0,T))} \leqslant C$. Hence using again (79), we obtain estimate (80). ∎

Proof of Theorem 1.2 We proceed in three steps:

Step 1. We study the convergence of $(v_\epsilon, q_\epsilon)_\epsilon$.
According to (83) and (78) we can extract subsequences, still denoted $(q_\epsilon)_\epsilon$ and $(v_\epsilon)_\epsilon$ such that

$$v_\epsilon \rightharpoonup v_0 \text{weakly in } L^2(\omega\times(0,T)), \tag{84}$$

$$q_\epsilon \rightharpoonup q_0 \text{ weakly in } H^{2,1}(Q). \tag{85}$$

And, as v_ϵ belongs to $Y_\lambda{}^\perp$ which is a closed vector subspace of $L^2(\omega\times(0,T))$, we have

$$v_0 \in Y_\lambda{}^\perp. \tag{86}$$

Since the injection of $H^{2,1}(Q)$ into $L^2(Q)$ is compact, the pair (v_0, q_0) is such that

$$\begin{cases} -\dfrac{\partial q_0}{\partial t} - \Delta q_0 + a_0 q_0 = h + v_0\chi_\omega & \text{in } Q, \\ q_0 = 0 & \text{on } \Sigma, \\ q_0(T) = 0 & \text{in } \Omega, \end{cases} \tag{87}$$

$$q_0(0) = 0 \text{ in } \quad \Omega. \tag{88}$$

Step 2. We prove that $(v_0, q_0 = q(x,t,v_0)) = (\hat{v}_\theta, \hat{q}_\theta = q(x,t,\hat{v}_\theta))$.
From the expression of J_ϵ given by (60), we can write

$$\frac{1}{2}\|v_\epsilon\|^2_{L^2(\omega\times(0,T))} \leqslant J_\epsilon(v_\epsilon, q_\epsilon).$$

Since $(\hat{v}_\theta, \hat{q}_\theta)$ satisfies (20)–(22) (or equivalently verifies (59)), this latter inequality becomes

$$\frac{1}{2}\|v_\epsilon\|^2_{L^2(\omega\times(0,T))} \leqslant J_\epsilon(v_\epsilon, q_\epsilon) \leqslant \frac{1}{2}\|\hat{v}_\theta\|^2_{L^2(\omega\times(0,T))}. \tag{89}$$

Then using (84) while passing to the limit in (89), we obtain

$$\frac{1}{2}\|v_0\|^2_{L^2(\omega\times(0,T))} \leqslant \liminf_{\epsilon\to 0} J_\epsilon(v_\epsilon, q_\epsilon) \leqslant \frac{1}{2}\|\hat{v}_\theta\|^2_{L^2(\omega\times(0,T))}.$$

Consequently,

$$\|v_0\|_{L^2(\omega\times(0,T))} \leqslant \|\hat{v}_\theta\|_{L^2(\omega\times(0,T))}$$

and thus,

$$\|v_0\|_{L^2(\omega\times(0,T))} = \|\hat{v}_\theta\|_{L^2(\omega\times(0,T))}.$$

Hence, $v_0 = \hat{v}_\theta$ and since (87) has a unique solution, it follows that $q_0 = \hat{q}_\theta$.

Step 3. According to the inequalities (80) and (81), we can extract a subsequence, still denoted $(\rho_\epsilon)_\epsilon$, such that

$$\rho_\epsilon \rightharpoonup \hat{\rho}_\theta \text{ weakly in } L^2(\omega\times(0,T)), \tag{90}$$

$$\rho_\epsilon \rightharpoonup \hat{\rho}_\theta \text{ weakly in } V. \tag{91}$$

As P is a compact operator, we deduce from (90) that

$$P\rho_\epsilon \to P\hat{\rho}_\theta \text{ strongly in } L^2(\omega\times(0,T)). \tag{92}$$

Therefore, combining (90) and (92), we get

$$v_\epsilon = \rho_\epsilon\chi_\omega - P\rho_\epsilon \rightharpoonup \hat{v}_\theta = \hat{\rho}_\theta\chi_\omega - P\hat{\rho}_\theta \text{ weakly in } L^2(\omega\times(0,T)).$$

Thus, we have proved that there exists θ given by (38) such that for a given $h \in L^2(Q)$ with $\theta h \in L^2(Q)$, the unique pair $(\hat{v}_\theta, \hat{q}_\theta)$ satisfies (20)–(23) with $\hat{v}_\theta = \hat{\rho}_\theta\chi_\omega - P\hat{\rho}_\theta$, and where $\hat{\rho}_\theta$ is a solution of (26). Since the function h defined by (34) belongs to $L^2(Q)$ if $\theta h \in L^2(Q)$, the proof of Theorem 1.2 is complete. ■

4 Expression of the sentinel with given sensitivity and identification of parameter λ_i

We are now able to give the expression of the sentinel S defined by (6)–(9) and identify the parameter λ_i.

4.1 Expression of the sentinel with given sensitivity

We consider the results obtained in the previous sections and we assume that h given by (34) and θ given by (38) are such that $\theta h \in L^2(O\times(0,T))$. Let $(\hat{v}_\theta, \hat{\rho}_\theta)$ be defined as in Theorem 1.2. Since $\hat{v}_\theta = -(\hat{\rho}_\theta\chi_\omega - P\hat{\rho}_\theta)$ realizes the minimum in $L^2(\omega\times(0,T))$ among all controls v such that the pair (v,q) satisfies (20)–(23), using (33), we deduce that $w = w_0 + \hat{v}_\theta = w_0 - (\hat{\rho}_\theta\chi_\omega - P\hat{\rho}_\theta)$. Consequently, replacing w by its expression in (6), the function S becomes

$$S(\lambda,\tau) = \int_0^T\int_O h_0 y(\lambda,\tau)\,dx\,dt + \int_0^T\int_\omega (w_0 - (\hat{\rho}_\theta - P\hat{\rho}_\theta\chi_\omega) y(\lambda,\tau)\,dx\,dt \tag{93}$$

and (w,S) is such that (7)–(9) hold.

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4.2 Identification of the parameter λ_i

According to (14), y_0 which is the solution of the problem (1) when $\lambda = 0$ and $\tau = 0$ is identically zero in Q. Hence, from (93) we have

$$S(0,0) = \int_0^T \int_O h_0 y_0 \, dx \, dt + \int_0^T \int_\omega (w_0 - (\hat{\rho}_\theta - P\hat{\rho}_\theta \chi_\omega) y_0 \, dx \, dt = 0.$$

Next, using (7), we obtain

$$S(\lambda,\tau) - S(0,0) \approx \sum_{i=1}^M \lambda_i \frac{\partial S}{\partial \lambda_i}(0,0) \text{ for } \lambda_i \text{ and } \tau \text{ small.}$$

Since we have at our disposal the observation y_{obs} which is the measure of the concentration of the pollutant in the river, we have

$$S(\lambda,\tau) - S(0,0) = \int_0^T\int_O h_0(y_{\text{obs}} - y_0) \, dx \, dt + \int_0^T\int_\omega w(y_{\text{obs}} - y_0) \, dx \, dt.$$

Thus, we also have the following information:

$$\sum_{i=1}^M \lambda_i \frac{\partial S}{\partial \lambda_i}(0,0) \approx \int_0^T \int_O h_0(y_{\text{obs}} - y_0) \, dx \, dt + \int_0^T \int_\omega w(y_{\text{obs}} - y_0) \, dx \, dt$$

which, using (8), gives

$$\sum_{i=1}^M \lambda_i c_i \approx \int_0^T \int_O h_0(y_{\text{obs}} - y_0) \, dx \, dt + \int_0^T \int_\omega w(y_{\text{obs}} - y_0) \, dx \, dt.$$

Now, fixing $i \in \{1, M\}$ and choosing $c_i \neq 0$ and $c_j = 0$, for all j in $\{1, M\}$ with $j \neq i$, we get this estimate of the parameter λ_i

$$\begin{aligned} \lambda_i &\approx \frac{1}{c_i} \left\{ \int_0^T \int_O h_0(y_{\text{obs}} - y_0) \, dx \, dt + \int_0^T \int_\omega w(y_{\text{obs}} - y_0) \, dx \, dt \right\} \\ &\approx \frac{1}{c_i} \left\{ \int_0^T \int_O h_0 y_{\text{obs}} \, dx \, dt + \int_0^T \int_\omega w y_{\text{obs}} \, dx \, dt \right\} \end{aligned}$$

since $y_0 = 0$ in Q.

5 Concluding remarks

The concept of sentinels of J. L. Lions that we revisited allows not only the identification of parameters with given sensitivity, but also the detection of parameters by distinguishing between them the missing data. The method is general and can be applied to other types of problems governed by evolution equations.

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