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Self-tuning proportional integral control for consensus in heterogeneous multi-agent systems

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In this paper, we present a distributed Proportional-Integral (PI) strategy with self-tuning adaptive gains for reaching asymptotic consensus in networks of non-identical linear agents under constant disturbances. Alternative adaptive strategies are presented, based on global or local measures of the agents' disagreement. The proposed approaches are validated on a representative numerical example. Preliminary analytical results further confirm the viability of the self-tuning strategies.

Key words: 91B69: Heterogeneous agent models, 05C82: Complex networks, 93C40: Adaptive control

1 Introduction

The problem of coordinating and controlling the motion of ensembles of interconnected dynamical units (or agents) has recently attracted intense research effort because of its application to diverse fields of Science and Engineering [1–3]. Examples include the rendezvous problem [4], formation control [5], frequency synchronization in power grids [6], flocking [7], and platooning problems [8]. In all these applications, a network of dynamical systems (agents) are required to coordinate their individual motions to perform a common task or exhibit a desired collective behaviour. Centralized or distributed control strategies can be used to solve this problem. Unlike centralized control where a central "station" is required for controlling the whole network, distributed control only requires local interactions among agents. Therefore, a distributed control strategy is more apt in all those situations where several constraints are present and cannot be avoided such as limited resources and energy, short wireless communication ranges, narrow bandwidths, etc [9].

The study of distributed and decentralized strategies to achieve consensus and synchronization has been the subject of much research effort in the literature. A common assumption which is often taken is that all agents in the ensemble share the same identical dynamics, but this is hardly the case in many real-world examples such as biochemical networks [11], power networks [6], or networked cyberphysical systems [12].

When the agents' dynamics are heterogeneous, proportional diffusive coupling among the agents (they adjust their behaviour proportionally to the mismatch with that of their neighbours) is unable to guarantee convergence towards a common consensus value, unless specific patterns or symmetries are present [13], or when the heterogeneous nodes share the same equilibrium point [14]. The consensus problem can be seen as a special instance within synchronization studies. Particularly, only bounded synchronization (or bounded consensus) can be achieved when nodes are characterized by non-identical dynamics and coupled through a (linear or non-linear) proportional communication protocol [15]. Recently, the use of an additional distributed integral action was considered to enhance the nodes' ability to converge towards each other. This additional action is an extension to networks of the classical PI control approach (see [16] for further information) and was shown in [17] to be effective in coping with agents' heterogeneity and external disturbances in networks of linear scalar systems (an extension to n-dimensional linear systems was later presented in [18]). Specifically, sufficient conditions for admissible consensus were obtained: conservative threshold values for the proportional and integral coupling gains were given, depending on the degree of heterogeneity of the agents and their network topology.

This paper starts from the findings in [17] and aims at overcoming the necessity of an offline estimation of the control gains proposing a set of adaptive strategies for tuning the control gains. First, two centralized strategies are presented, where the adaptive control laws are computed on the basis of a global measure of the nodes' disagreement. Then, a decentralized adaptive strategy is discussed, where each pair of nodes updates the strength (gain) of their mutual coupling on the basis of a local disagreement measure. The novel self-tuning control strategies are numerically validated on a set of representative examples, and some preliminary stability results are derived and illustrated.

The use of a distributed PI strategy has been often discussed in the literature but not always under the assumption of homogeneous node dynamics. For instance, in [19, 20] distributed integral actions are studied in the case where the agents share the same scalar dynamics and are affected by constant disturbances, while in [21, 22] the use of integral actions are discussed for networks with arbitrary homogeneous node dynamics. Previous work where networks have a certain degree of heterogeneity includes the approach based on the internal model principle [23, 24]. Moreover, distributed integral actions have been also used in numerous applications, e.g., [19,25–30]. We wish to emphasize that, in contrast to existing results, a different approach is proposed in this paper, where, uniquely, the node dynamics are heterogeneous (possibly unstable) and the control gains self-tune their states according to centralized or decentralized strategies.

The outline of the manuscript is as follows. A brief overview on control of networks is given in Section 2, while the notation and some preliminary notions on graph theory are reported in Section 3. The problem statement is given in Section 4, while in Sections 5 and 6 the centralized and decentralized control strategies are presented and illustrated via some representative examples. Conclusions are drawn in Section 7.

2 Control of networks: a brief overview

In simple terms, a multi-agent system is a collection of dynamical agents (*nodes*) interacting with each other over a network of interconnections (*links*) [9, 10]. The structure of the interconnections among nodes has been found to play a crucial role on the overall network's behaviour [31], as well as on its propensity to be controlled (or controllability) [1,32]. When controlling a network, the inputs can affect (a) the individual *node* dynamics, (b) the strength (gain) of the coupling among connected agents (network links), and/or (c) the *network topology*.

Pinning control [33] is a well-known control method aiming at steering the node dynamics towards a desired trajectory, which is prescribed by an additional and virtual node, that is called *pinner*. Specifically, a control action is only injected on a limited fraction of nodes. The optimal selection of the set of nodes to be directly controlled is the subject of much research effort, see for instance [34] and references therein. More recently, an open-loop approach based on compensatory perturbations applied to each node has been proposed to steer a network towards the basin of attraction of a desired target state [35], while in [36] parameters at nodes are manipulated to guarantee the network reaches a desired state.

Control interventions can also be considered on the gains associated to some links [37]. In this case, those links can be seen as a control input to be designed. Then, adaptive distributed control techniques can be used for self-tuning the link weights so as to steer the network onto a common synchronous solution [38,39].

Finally, control can be implemented through the adaptation of the network structure. Namely, depending on the time evolution of the node states, links are added or removed from the network. In particular, a simple dynamical model has been proposed to evolve the network topology so as to attain network synchronization [40,41].

An assumption often made in the control strategies mentioned above is that nodes share the same dynamics (homogeneous nodes). This is crucial when proving global stability of the overall network dynamics. However, in real networks the nodes do not necessarily have identical dynamics. For instance, in the electrical power grid, different type of nodes which represent consumers and multiple power generation sources such as hydroelectric, eolian, and solar, render the network highly heterogeneous.

Our control strategy affects the individual node dynamics while also adapting the coupling gains associated to each link. Contrary to previous work reported in the literature, we address the case of ensembles of heterogeneous agents interacting over a network. Specifically, we make use of proportional and integral controllers with self-tuning gains. We show that our control approach is able to adapt its gains so that the networks reaches consensus despite the presence of heterogeneous nodes and constant disturbances.

3 Notation and mathematical preliminaries

We denote by $\mathbf{I}_N$ the identity matrix of dimension $N \times N$, by $\mathbb{0}_{M\times N}$ a matrix of zeros of dimension $M \times N$, and by $\mathbb{1}_N$ a $N \times 1$ vector with unitary elements. A diagonal matrix, say $\mathbf{D}$, with diagonal elements $d_1, \ldots, d_N$ is indicated by $\mathbf{D} = \mathrm{diag}\{d_1, \ldots, d_N\}$. The Frobenius norm is denoted by $\|\cdot\|$ while the spectral norm by $|||\cdot|||$. Given a square symmetric matrix M, we sort its eigenvalues in ascending order as $\lambda_1(M) \leqslant \cdots \leqslant \lambda_N(M)$. Given two vectors $\zeta_1, \zeta_2 \in \mathbb{R}^{n\times 1}$ and a matrix $\mathbf{Q} \in \mathbb{R}^{n\times n}$, straightforward linear algebra implies

$$2\zeta_1^T \mathbf{Q}^T \zeta_2^T \leqslant \varepsilon \zeta_1^T \mathbf{Q}^T \mathbf{Q} \zeta_1 + \frac{1}{\varepsilon} \zeta_2^T \zeta_2, \forall \varepsilon > 0. \quad (3.1)$$

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3.1 Graph theory

An *undirected graph* $\mathscr{G}$ is a pair defined as $\mathscr{G} = (\mathscr{N}, \mathscr{E})$, where $\mathscr{N} = \{1, 2, \ldots, N\}$ is the set of nodes (vertexes), and $\mathscr{E} \subset \mathscr{N} \times \mathscr{N}$ is the set containing the P edges connecting the nodes. We assume each edge has an associated weight denoted $w_{ij} \in \mathbb{R}^+$ for all $(i, j) \in \mathscr{E}$. The weighted *Laplacian matrix* of the graph $\mathscr{G}$ is denoted $\mathscr{L}(\mathscr{G}) \in \mathbb{R}^{N \times N}$, and its ijth element $\mathscr{L}_{ij}$ is defined as

$$\mathscr{L}_{ij} := \begin{cases} \sum_{j=1, j \neq i}^{N} w_{ij}, & i = j \\ -w_{ij}, & otherwise \end{cases}. \tag{3.2}$$

Definition 3.1 ([42]) We say that an $N \times N$ symmetric matrix $\mathscr{S} = [\mathscr{S}_{ij}], i, j \in \mathscr{N}$, belongs to the set $\mathscr{M}$ if it verifies the following properties:

(1) $\mathscr{S}_{ij} \leqslant 0$, $i \neq j$, and $\mathscr{S}_{ii} = -\sum_{j=1, j \neq i}^{N} \mathscr{S}_{ij}$,

(2) $\lambda_1(\mathscr{S}) = 0$, while $\lambda_k(\mathscr{S}) > 0$ for all $k \in \{2, \ldots, N\}$.

The set of matrices $\mathscr{M}$ defined above are in fact a special instance of M-matrices as defined in [43]. Note that the Laplacian matrix $\mathscr{L}$ of an undirected graph belongs to the set $\mathscr{M}$ if its associated graph $\mathscr{G}$ is connected [44]. Next, we present a decomposition of the Laplacian matrix that, as suggested in [17], is particularly useful to prove convergence in the presence of heterogeneous nodes.

Lemma 1 ([17, 18]) *A Laplacian matrix $\mathscr{L} \in \mathscr{M}$ of an undirected and connected graph $\mathscr{G}$ can be written in block form as $\mathscr{L} = \mathbf{R}\mathbf{\Lambda}\mathbf{R}^{-1}$. Namely,*

$$\mathscr{L} := \begin{bmatrix} \mathscr{L}_{11} & \mathscr{L}_{12} \\ \mathscr{L}_{21} & \mathscr{L}_{22} \end{bmatrix} = \begin{bmatrix} 1 & N\mathbf{R}_{21}^T \\ \mathbb{1}_{N-1} & N\mathbf{R}_{22}^T \end{bmatrix} \mathbf{\Lambda} \begin{bmatrix} r_{11} & \mathbf{R}_{12} \\ \mathbf{R}_{21} & \mathbf{R}_{22} \end{bmatrix}, \tag{3.3}$$

where $\mathscr{L}_{11} \in \mathbb{R}$, $\mathscr{L}_{12} \in \mathbb{R}^{1 \times (N-1)}$, $\mathscr{L}_{21} \in \mathbb{R}^{(N-1) \times 1}$, and $\mathscr{L}_{22} \in \mathbb{R}^{(N-1) \times (N-1)}$. Also,

$$r_{11} = \frac{1}{N}, \qquad \mathbf{R}_{12} = \frac{1}{N}\mathbb{1}_{N-1}^T, \tag{3.4}$$

and $\mathbf{\Lambda} = diag\{\lambda_1(\mathscr{L}) = 0, \lambda_2(\mathscr{L}), \ldots, \lambda_N(\mathscr{L})\}$. Moreover, the blocks of appropriate dimensions $\mathbf{R}_{21} \in \mathbb{R}^{(N-1) \times 1}$ and $\mathbf{R}_{22} \in \mathbb{R}^{(N-1) \times (N-1)}$ must fulfill the following conditions:

$$\mathbf{R}_{21} + \mathbf{R}_{22}\mathbb{1}_{N-1} = \mathbb{0}_{(N-1) \times 1}, \tag{3.5}$$

$$\mathbf{R}_{21}\mathbf{R}_{21}^T + \mathbf{R}_{22}\mathbf{R}_{22}^T = \frac{1}{N}\mathbf{I}_{N-1}, \tag{3.6}$$

$$r_{11}\mathbf{R}_{21}^T + \mathbf{R}_{12}\mathbf{R}_{22}^T = \mathbb{0}_{1 \times (N-1)}, \tag{3.7}$$

$$\mathbf{R}_{21}\mathbf{R}_{21}^T = \mathbf{R}_{22}\mathbb{1}_{N-1}\mathbb{1}_{N-1}^T\mathbf{R}_{22}^T, \tag{3.8}$$

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$$|||\mathbf{R}_{22}||| \leqslant \frac{1}{\sqrt{N}}, \tag{3.9}$$

$$\|\mathbf{R}_{21}\| \leqslant \sqrt{N-1}|||\mathbf{R}_{22}||| \leqslant \sqrt{(N-1)/N}, \tag{3.10}$$

$$N\mathbf{R}_{22}^T = (\mathbf{I}_{N-1} + \mathbb{1}_{N-1}\mathbb{1}_{N-1}^T)^{-1}\mathbf{R}_{22}^{-1}, \tag{3.11}$$

$$\mathscr{L}_{22} = N\mathbf{R}_{22}^T\bar{\mathbf{\Lambda}}\mathbf{R}_{22}, \tag{3.12}$$

where $\bar{\mathbf{\Lambda}} = diag\,\{\lambda_2(\mathscr{L}),\ldots,\lambda_N(\mathscr{L})\}$.

4 Problem statement

Let us consider a multi-agent system governed by first-order linear dynamics of the form

$$\frac{dx_i}{dt} = a_i x_i(t) + b_i + u_i(t), \qquad x_i(0) = x_{i0}, \qquad i \in \{1,\ldots,N\} \tag{4.1}$$

where $x_i(t) \in \mathbb{R}$ denotes the state of the ith agent with arbitrary initial condition $x_i(0) = x_{i0}$. $a_i, b_i \in \mathbb{R}$ are constant parameters representing the damping and a constant bias for the ith agent, respectively, while $u_i(t) \in \mathbb{R}$ is a control input (or an external force) representing the exchange of information between neighbouring agents. Note that each agent is characterized by possibly different parameter values rendering their dynamics heterogeneous. Accordingly, when the agents are isolated, i.e., there is no communication between them ($u_i(t) = 0$), their dynamics can be either unstable ($a_i > 0$) or stable ($a_i < 0$) with possibly different equilibria given by $x_i^* = -b_i/a_i$. As stated in the introduction, we are interested in solving the consensus problem where the aim is to guarantee convergence of all states $x_i(t)$ asymptotically towards each other, i.e., $\lim_{t\to\infty} x_1(t) = x_2(t) = \cdots = x_N(t) = x^*$, where x^* represents the collective decision state (or consensus) of the heterogeneous multi-agent system.

Definition 4.1 (*admissible consensus*) [17] The multi-agent system (4.1) is said to achieve admissible consensus if, for any set of initial conditions $x_i(0) = x_{i0}$, we have

$$\lim_{t\to\infty} x_i(t) = x^*, \qquad |u_i(t)| \leqslant W < +\infty, \forall t \geqslant 0, \tag{4.2}$$

for all $i \in \{1,\ldots,N\}$, with W being a non-negative constant.

Based on the recent findings in [17], in this paper we make use of a distributed PI controller to guarantee admissible consensus of network (4.1) despite the presence of heterogeneities and constant drifts. Namely, we choose

$$u_i(t) = \alpha(t)\sum_{i=1}^{N} w_{ij}(x_j(t) - x_i(t)) + \beta(t)\sum_{i=1}^{N} w_{ij}\int_0^t (x_j(\tau) - x_i(\tau))d\tau, \tag{4.3}$$

where $w_{ij} \in \mathbb{R}^+$ are positive constant gains, and $\alpha(t), \beta(t)$ are two time-varying adaptive control parameters that modulate the relative strength of the proportional and integral actions, respectively.

The heterogeneous multi-agent system (4.1), complemented with the distributed control strategy (4.3), is the closed-loop network that will be the subject of our investigation. This network is represented by an undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, where the set of nodes $\mathcal{N}$ denotes the index of each agent state $x_i(t)$, while an edge $(i, j) \in \mathcal{E}$ denotes a bidirectional communication between nodes i and j with an associated edge weight w_{ij} for any $(i, j) \in \mathcal{E}$.

The crucial problem becomes now that of selecting how to vary $\alpha(t)$ and $\beta(t)$ in order for the closed-loop network to achieve admissible consensus. In the case, where both $\alpha(t)$ and $\beta(t)$ are chosen to be constant, sufficient conditions were derived in [17]. Specifically, the following result was proven

Theorem 4.2 ([17,45]) *Under the distributed adaptive PI control* (4.3) *with constant gains, i.e.,* $\alpha(t) = \alpha$ *and* $\beta(t) = \beta$, *where* $\alpha, \beta > 0$, *the heterogeneous multi-agent system* (4.1) *reaches admissible consensus if*

$$\hat{a} := (1/N) \sum_{k=1}^{N} a_k < 0, \tag{4.4a}$$

$$\alpha \lambda_2 N > \max_i a_i + \frac{\sum_{k=2}^{N} (a_k - a_1)^2}{|\hat{a}|}, \tag{4.4b}$$

$$\beta > 0. \tag{4.4c}$$

Moreover, the consensus value x^* *can be computed as* $x^* = -\sum_{k=1}^{N} b_k / \sum_{k=1}^{N} a_k$.

Note that the conditions for admissible consensus given in Theorem 4.2 are independent of the gain value β, provided that is positive; therefore, it could be arbitrarily picked by the control designer, while the gain α should be appropriately tuned based on the node dynamics (a_i) and network structure (λ_2). To overcome the need for this off-line tuning of the control gains α and β, we propose three self-tuning adaptive strategies of increasing complexity as listed below.

(i) A *centralized* self-tuning strategy for $\alpha(t)$, with $\beta(t) = \beta$ being a positive (and arbitrary) constant (*Adaptive Strategy 1* (AS1), Section 5.1).

(ii) A *centralized* self-tuning strategy for both control gains $\alpha(t)$ and $\beta(t)$ (*Adaptive Strategy 2* (AS2), Section 5.2).

(iii) A *decentralized* self-tuning strategy in which every node adapts its couplings on the basis of only a local measure of the disagreement with their neighbours (*Adaptive Strategy 3* (AS3), Section 6).

We present each of the above strategies and illustrate their effectiveness via numerical examples. We give also a proof of convergence of Adaptive Strategy 1. The convergence analysis of AS2 and AS3 is currently under investigation and will be presented elsewhere.

5 Centralized self-tuning PI consensus

5.1 Adaptive Strategy 1

From Theorem 4.2, a sufficient condition for guaranteeing admissible consensus is given and it is independent of the β value. Therefore, based on this observation, our first strategy only adapts the proportional control gain $\alpha(t)$, while the integral control gain is a positive kept constant that can be arbitrarily selected, that is, $\beta(t) = \beta > 0$. Specifically, $\alpha(t)$ is updated according to the following self-adapting rule:

$$\frac{d\alpha}{dt} = \frac{\kappa_P}{N}\boldsymbol{\delta}^T(t)\mathscr{L}_{22}\boldsymbol{\delta}(t), \qquad \alpha(0) = 0, \tag{5.1}$$

where $\mathscr{L}_{22}$ is the main squared block of the Laplacian matrix associated to the graph $\mathscr{G}$ as shown in (3.3), $\boldsymbol{\delta}(t) := [x_2(t)-x_1(t),\ldots,x_N(t)-x_1(t)]^T$ is the *global disagreement vector*, and $\kappa_P \in \mathbb{R}^+$ is an arbitrary constant parameter modulating the rate of growth of $\alpha(t)$. In what follows, we prove that Adaptive Strategy 1 drives all states x_i of the heterogeneous multi-agent systems towards a common consensus value x^*, which corresponds to the unique equilibrium of the closed-loop multi-agent system, and that the proportional gain $\alpha(t)$ asymptotically converges towards a finite steady-state value. Before giving our main result, we will first compute the consensus value, to then recast the error dynamics using an appropriate transformation.

Remark 5.1 *The stability analysis of the Adaptive Strategy 1 is not influenced by the selection of the initial conditions $\alpha(0)$, that can be arbitrarily picked. However, we set $\alpha(0) = 0$ as the most natural choice in adaptive control. Indeed, we exclude negative initial conditions that might slow down convergence and induce large overshoots. Also, we avoid taking positive values of $\alpha(0)$ since they may induce unnecessarily large steady-state values and control effort.*

5.1.1 *Consensus value*

Let us define $\mathbf{A} := \text{diag}\{a_1,\ldots,a_N\}$, $\mathbf{B} := [b_1,\ldots,b_N]^T$, let $\mathbf{x}(t) := [x_1(t),\ldots,x_N(t)]^T$ be the stack vector of all node states and

$$\mathbf{z}(t) = [z_1(t),\ldots,z_N(t)]^T := -\beta\mathscr{L}\int_0^t \mathbf{x}(\tau)d\tau, \tag{5.2}$$

be the stack vector of all integral states. We can recast the overall dynamics of the closed-loop network (4.1), (4.3) as

$$\begin{bmatrix} d\mathbf{x}/dt \\ d\mathbf{z}/dt \end{bmatrix} = \begin{bmatrix} \mathbf{A}-\alpha(t)\mathscr{L} & \mathbf{I}_N \\ -\beta\mathscr{L} & \mathbb{0}_{N\times N} \end{bmatrix}\begin{bmatrix} \mathbf{x}(t) \\ \mathbf{z}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbb{0}_{N\times 1} \end{bmatrix}. \tag{5.3}$$

Proposition 5.1 *Given an undirected, weighted, and connected graph $\mathscr{G}$, the network of heterogeneous agents (5.3) has an equilibrium $(\mathbf{x}^*, \mathbf{z}^*, \alpha^*)$, where*

$$\mathbf{x}^* := x^*\mathbb{1}_N, \qquad \mathbf{z}^* := -(x^*\mathbf{A}\mathbb{1}_N + \mathbf{B}), \qquad \alpha^* = \alpha_c, \tag{5.4}$$

with α_c being a positive constant, and

$$x^* := -\frac{\sum_{k=1}^{N} b_k}{\sum_{k=1}^{N} a_k}, \tag{5.5}$$

being the collective decision of the network.

Proof By setting the left-hand side of (5.3) to zero, and from the fact that $\mathscr{L} \in \mathscr{M}$, it follows that the equilibrium $(\mathbf{x}^*, \mathbf{z}^*, \alpha^*)$ of system (5.1)–(5.3) must satisfy the following conditions:

$$\mathbf{x}^* = p\mathbb{1}_N, \qquad \mathbf{z}^* = -\left(a\mathbf{A}\mathbb{1}_N + \mathbf{B}\right), \qquad \alpha^* = \alpha_c,$$

where $p \in \mathbb{R}$. From the definition of $\mathbf{z}(t)$ in (5.2), we also have $\mathbb{1}_N^T\mathbf{z}(t) = 0 \Rightarrow \mathbb{1}_N^T\mathbf{z}^* = 0$, yielding

$$p = -\mathbb{1}_N^T\mathbf{B}/\mathbb{1}_N^T\mathbf{A}\mathbb{1}_N = -\left(\sum_{k=1}^{N} b_k\right)\left(\sum_{k=1}^{N} a_k\right)^{-1} = x^*.$$

□

Now, shifting the state-space origin via the transformation $\mathbf{y}(t) := \mathbf{z}(t) + \mathbf{B}$, one has

$$\begin{bmatrix} d\mathbf{x}/dt \\ d\mathbf{y}/dt \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \alpha(t)\mathscr{L} & \mathbf{I}_N \\ -\beta\mathscr{L} & \mathbb{0}_{N\times N} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \end{bmatrix}. \tag{5.6}$$

5.1.2 *Error dynamics*

Assuming the graph $\mathscr{G}$ to be connected, from Lemma 1, we can write $\mathscr{L} = \mathbf{R}\boldsymbol{\Lambda}\mathbf{R}^{-1}$. Next, we define the error dynamics given by the state transformation $\mathbf{e}(t) = [e_1, \ldots, e_N] = \mathbf{R}^{-1}\mathbf{x}(t)$; therefore, using the block representation of $\mathbf{R}^{-1}$ and letting $\bar{\mathbf{e}}(t) = [e_2(t), \ldots, e_N(t)]^T$, $\bar{\mathbf{x}}(t) = [x_2(t), \ldots, x_N(t)]^T$, we obtain

$$e_1(t) = r_{11}x_1(t) + \mathbf{R}_{12}\bar{\mathbf{x}}(t), \tag{5.7a}$$

$$\bar{\mathbf{e}}(t) = \mathbf{R}_{21}x_1(t) + \mathbf{R}_{22}\bar{\mathbf{x}}(t). \tag{5.7b}$$

From (3.5), we can express $\mathbf{R}_{21} = -\mathbf{R}_{22}\mathbb{1}_{N-1}$ and substituting in (5.7*b*) yields

$$\bar{\mathbf{e}}(t) = \mathbf{R}_{22}\left(\bar{\mathbf{x}}(t) - x_1(t)\mathbb{1}_{N-1}\right). \tag{5.8}$$

Hence, $\bar{\mathbf{e}}(t) = \mathbb{0}$ if and only if $\bar{\mathbf{x}}(t) - x_1(t)\mathbb{1}_{N-1} = \mathbb{0}$ since the matrix $\mathbf{R}_{22}$ is full rank [17]. Then, admissible consensus is achieved if $\lim_{t\to\infty} \bar{\mathbf{e}}(t) = \mathbb{0}$ and $\|\mathbf{y}(t)\| \leqslant W < +\infty, |\alpha(t)| \leqslant \bar{\alpha} < +\infty, \forall t > 0$.

Transforming the state vector $\mathbf{y}$, we can now recast (5.6) in the new coordinates $\mathbf{e}(t)$ and $\mathbf{w}(t)$ as

$$\begin{aligned} d\mathbf{e}/dt &= \left(\Psi - \begin{bmatrix} 0 & \mathbb{0}_{1\times N} \\ \mathbb{0}_{N\times 1} & \alpha(t)\bar{\boldsymbol{\Lambda}} \end{bmatrix}\right)\mathbf{e}(t) + \begin{bmatrix} 0 \\ \bar{\mathbf{w}}(t) \end{bmatrix}, \\ d\bar{\mathbf{w}}/dt &= -\beta\bar{\boldsymbol{\Lambda}}\bar{\mathbf{e}}(t), \end{aligned} \tag{5.9}$$

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where $\bar{\mathbf{w}}(t) := [w_2(t), \ldots, w_N(t)]^T$. Note that the equation for $w_1(t)$ can be neglected as it has trivial dynamics with null initial conditions and represents an uncontrollable and unobservable state. Moreover, it is important to note that matrix Ψ is a block matrix defined as [17]

$$\begin{aligned}\Psi &:= \mathbf{R}^{-1}\mathbf{A}\mathbf{R} = \begin{bmatrix} \psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix}, \\ &= \mathbf{R}^{-1} \begin{bmatrix} a_1 & \mathbb{0}_{1\times(N-1)} \\ \mathbb{0}_{(N-1)\times 1} & \bar{\mathbf{A}} \end{bmatrix} \mathbf{R}\end{aligned} \tag{5.10}$$

with $\bar{\mathbf{A}} := \operatorname{diag}\{a_2, \ldots, a_N\}$. Now, using properties (3.4)–(3.7), we can write [17]

$$\psi_{11} := \hat{a}, \tag{5.11}$$
$$\Psi_{12} := \bar{\boldsymbol{\rho}}^T \mathbf{R}_{22}^T, \tag{5.12}$$
$$\Psi_{21} := \mathbf{R}_{22}\bar{\boldsymbol{\rho}}, \tag{5.13}$$
$$\Psi_{22} := N\mathbf{R}_{22}\left(\bar{\mathbf{A}} + a_1 \mathbb{1}_{N-1}\mathbb{1}_{N-1}^T\right)\mathbf{R}_{22}^T, \tag{5.14}$$

where

$$\bar{\boldsymbol{\rho}} := [a_2 - a_1, \ldots, a_N - a_1]^T. \tag{5.15}$$

5.1.3 *Proof of convergence*

We can now state our main stability result.

Theorem 5.1 *The heterogeneous multi-agent system* (4.1), (4.3), *with* $\alpha(t)$ *varying according to the Adaptive Strategy 1* (5.1) *and* $\beta(t) = \beta > 0$, *achieves admissible consensus on the consensus value* x^* *given in* (5.5), *i.e.,* $\lim_{t\to\infty} x_i(t) = x^*$ *with* $\lim_{t\to\infty} u_i(t) = u_i^* := -(a_i x^* + b_i), i \in \mathcal{N}$, *if* $\hat{a} = (1/N)\sum_{k=1}^N a_k < 0$. *Moreover, the gain* $\alpha(t)$ *asymptotically converges to a positive finite value.*

Proof Consider the Lyapunov candidate function (in what follows we omit the time dependence of the state variables $e_1(t)$, $\bar{\mathbf{e}}(t)$, and $\bar{\mathbf{w}}(t)$ to simplify the notation)

$$V = \frac{1}{2}\left(e_1^2 + \bar{\mathbf{e}}^T\bar{\mathbf{e}} + \frac{1}{\beta}\bar{\mathbf{w}}^T\bar{\mathbf{\Lambda}}^{-1}\bar{\mathbf{w}} + \frac{1}{\kappa_P}(\alpha(t) - c)^2\right), \tag{5.16}$$

where $\bar{\mathbf{\Lambda}}^{-1} := \operatorname{diag}\{1/\lambda_2, \ldots, 1/\lambda_N\}$ is a positive definite matrix and c is an arbitrary positive constant. From the hypotheses, κ_P and β are both positive; therefore, V is a positive definite and radially unbounded function. Next, calculating the time derivative of V along trajectories of (5.9) yields

$$\frac{dV}{dt} = e_1\frac{de_1}{dt} + \bar{\mathbf{e}}^T\frac{d\bar{\mathbf{e}}}{dt} + \frac{1}{\beta}\left(\bar{\mathbf{w}}^T\bar{\mathbf{\Lambda}}^{-1}\frac{d\bar{\mathbf{w}}}{dt}\right) + \frac{1}{\kappa_P}(\alpha(t) - c)\frac{d\alpha}{dt}.$$

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By using property (3.12) and from (5.1), we have

$$\frac{d\alpha}{dt} = \frac{\kappa_P}{N}\boldsymbol{\delta}^T(t)N\mathbf{R}_{22}^T\bar{\boldsymbol{\Lambda}}\mathbf{R}_{22}\boldsymbol{\delta}(t).$$

Note that, from (5.8), we have $\bar{\mathbf{e}} = \mathbf{R}_{22}\boldsymbol{\delta}$, and therefore

$$\frac{d\alpha}{dt} = \kappa_P \bar{\mathbf{e}}^T\bar{\boldsymbol{\Lambda}}\bar{\mathbf{e}}. \tag{5.17}$$

Moreover, (5.12) and (5.13) imply $\Psi_{12}=\Psi_{21}^T$, and then one gets

$$\frac{dV}{dt} = \psi_{11}e_1^2 + 2e_1\Psi_{12}\bar{\mathbf{e}} + \bar{\mathbf{e}}^T\Psi_{22}\bar{\mathbf{e}} - \frac{c}{\kappa_P}\bar{\mathbf{e}}^T\bar{\boldsymbol{\Lambda}}\bar{\mathbf{e}}. \tag{5.18}$$

Now, using (3.1), one has

$$2e_1\bar{\mathbf{e}}^T\mathbf{R}_{22}\bar{\boldsymbol{\rho}} \leqslant \varepsilon\bar{\mathbf{e}}^T\mathbf{R}_{22}\mathbf{R}_{22}^T\bar{\mathbf{e}} + \frac{1}{\varepsilon}\bar{\boldsymbol{\rho}}^T\bar{\boldsymbol{\rho}}e_1^2,$$

where ε is an arbitrary positive scalar. Then, we can rewrite (5.18) as

$$\frac{dV}{dt} \leqslant (\psi_{11} + \frac{1}{\varepsilon}\bar{\boldsymbol{\rho}}^T\bar{\boldsymbol{\rho}})e_1^2 + \bar{\mathbf{e}}^T\Psi_{22}\bar{\mathbf{e}} - \frac{c}{\kappa_P}\bar{\mathbf{e}}^T\bar{\boldsymbol{\Lambda}}\bar{\mathbf{e}} + \varepsilon\bar{\mathbf{e}}^T\mathbf{R}_{22}\mathbf{R}_{22}^T\bar{\mathbf{e}}. \tag{5.19}$$

From linear algebra it is possible to obtain an upper-bound for each term of (5.19). Thus, by definition we know that $\Psi = \mathbf{R}^{-1}\mathbf{A}\mathbf{R}$ is a symmetric matrix with eigenvalues $a_i, \forall i \in \mathscr{N}$, and $\lambda_{\max}(\Psi_{22}) \leqslant \lambda_{\max}(\Psi) = \max_i a_i$ (see Theorem 8.4.5 in [46]). Furthermore, $-c\bar{\mathbf{e}}^T\bar{\boldsymbol{\Lambda}}\bar{\mathbf{e}} \leqslant -c\lambda_2(\mathscr{L})\bar{\mathbf{e}}^T\bar{\mathbf{e}}$. Therefore, following similar algebraic steps as reported in [18], we find

$$\frac{dV}{dt} \leqslant g(\mathbf{e}) := (\psi_{11} + \frac{1}{\varepsilon}\bar{\boldsymbol{\rho}}^T\bar{\boldsymbol{\rho}})e_1^2 + \left(\max_i a_i + \varepsilon|||\mathbf{R}_{22}|||^2 - \frac{c}{\kappa_P}\lambda_2(\mathscr{L})\right)\bar{\mathbf{e}}^T\bar{\mathbf{e}}. \tag{5.20}$$

Next, using property (3.9) we have that $|||\mathbf{R}_{22}|||^2 \leqslant 1/N$; therefore, $dV/dt \leqslant 0$ if $\xi_1 := \psi_{11} + 1/\varepsilon\bar{\boldsymbol{\rho}}^T\bar{\boldsymbol{\rho}} < 0$ and $\xi_2 := \max_i a_i + \varepsilon/N - c\lambda_2(\mathscr{L}) < 0$. From the hypotheses, we have $\hat{a} = \psi_{11} < 0$. Therefore, by setting $\varepsilon = \bar{\varepsilon} > \bar{\boldsymbol{\rho}}^T\bar{\boldsymbol{\rho}}/|\psi_{11}|$, and $c > \kappa_P(\max_i \rho_i + \bar{\varepsilon}/N)/\lambda_2(\mathscr{L})$, one has $\xi_1, \xi_2 < 0$. Hence, we can conclude that $dV/dt \leqslant g(\mathbf{e}) \leqslant 0$, which implies the boundedness of $\mathbf{e}$, $\bar{\mathbf{w}}$, and $\alpha(t)$. Moreover, as $g(\mathbf{e}) = 0 \iff \mathbf{e} = 0$, from the Krasovskii–LaSalle invariance principle [47] follows that $\lim_{t\to\infty}\mathbf{e}(t) = 0$. Together with (5.1), this also implies that $\lim_{t\to\infty} d\alpha(t)/dt = 0$. Therefore, in the original variables the equilibrium $(\mathbf{x}^*, \mathbf{z}^*, \alpha^*)$ in (5.4) is globally asymptotically stable. Finally, from (4.3) and (5.2), we can write

$$\mathbf{u}^* := [u_1^*, \ldots, u_N^*] = -\alpha(t)\mathscr{L}\mathbf{x}^* + \mathbf{z}^* = \mathbf{z}^*,$$

yielding $\mathbf{u}^* = -(x^*\mathbf{A}\mathbb{1}_N + \mathbf{B})$, which completes the proof. □

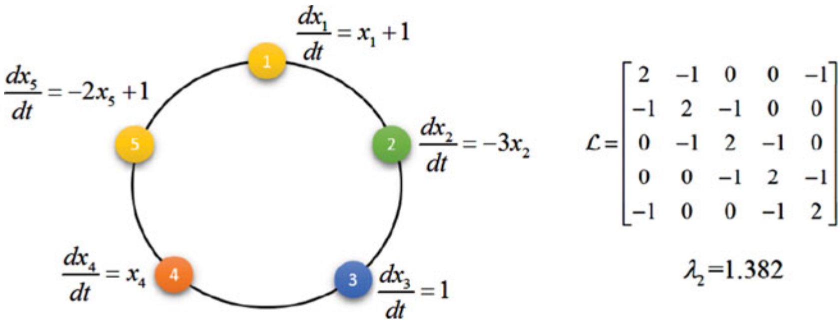

FIGURE 1. Numerical example: agents' own dynamics and network topology.

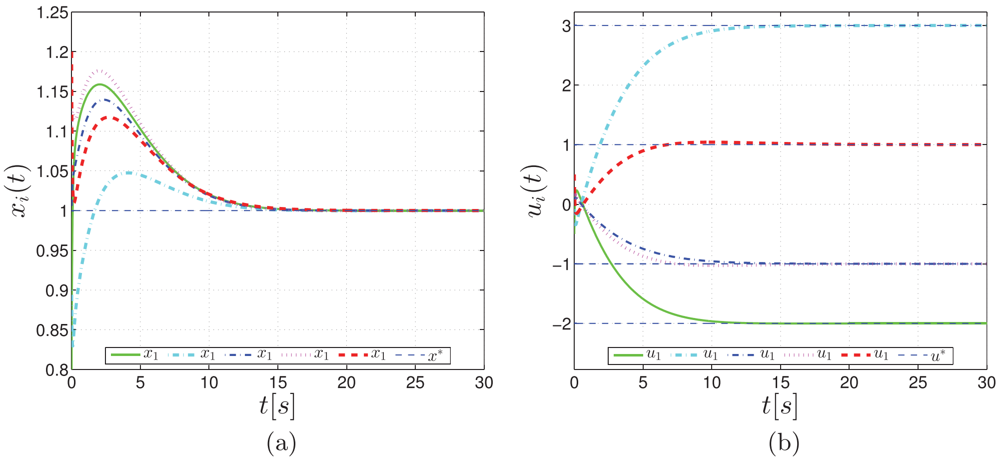

FIGURE 2. Time evolution of an heterogeneous network controlled by distributed PI with constant control gains $\alpha = 6.5$ and $\beta = 2$. The blue dashed lines represent the steady-state values.

5.1.4 *Example*

Consider the heterogeneous network of five agents shown in Figure 1. For the sake of simplicity, we consider all link weights to be unitary ($w_{ij} = 1$ for all $(i, j) \in \mathscr{E}$). The initial conditions are chosen to be $\mathbf{x}(0) = [0.8, 0.9, 1, 1.1, 1.2]^T$.

We first consider the case when the control parameters $\alpha(t)$ and $\beta(t)$ are assumed to be constant; this is, $\alpha(t) = \alpha$ and $\beta(t) = \beta$. From Theorem 4.2, as $\hat{a} = -0.6 < 0$, we have that admissible consensus is guaranteed if $\alpha > 6.42$ and $\beta > 0$ onto $x^* = \sum_{k=1}^{N} b_k / \sum_{k=1}^{N} a_k = 1$ as shown in Figure 2. Next, we consider the case where the control parameter $\alpha(t)$ self-tunes its value according to Adaptive Strategy 1 with $\kappa_P = 0.9$. From Theorem 5.1, admissible consensus is expected on $x^* = 1$, while the control action asymptotically converges to $\mathbf{u}^* = [u_1^*, u_2^*, u_3^*, u_4^*, u_5^*] = -(\mathbf{A}\mathbf{x}^* + \mathbf{D}) = [0, 2, -1, -1, 0]^T$. This is confirmed in Figure 3. Notice that the adaptive gain $\alpha(t)$ converges onto a value of approximately 0.66 at steady state which is considerably lower than the conservative estimate given by Theorem 4.2 when α is set to be constant.

Without loss of generality, we set $\kappa_P = \gamma$ with γ taking values in the open interval $]0, 1[$. Then, the evolution of $\alpha(t)$ is computed for four different values of γ as shown

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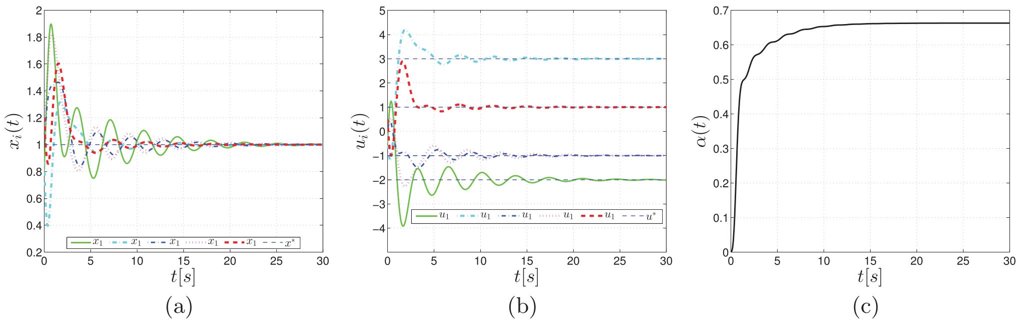

FIGURE 3. Heterogeneous network controlled by distributed PI control with AS1 ($\kappa_P = 0.9$ and $\beta = 2$): time evolution of the state $\mathbf{x}$ (a), the control input $\mathbf{u}$ (b), and the proportional gain α (c). The blue dashed lines in panels (a) and (b) represent the convergence values x^* and $\mathbf{u}^*$, respectively.

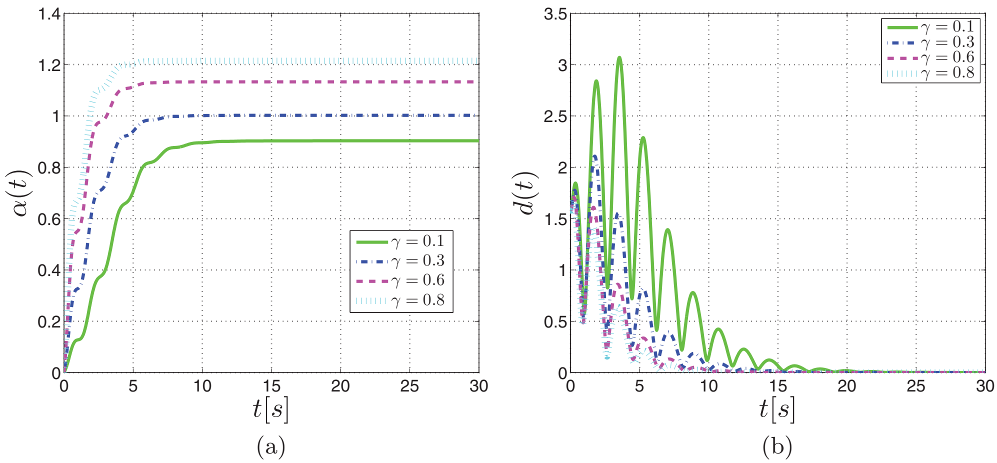

FIGURE 4. Heterogeneous network controlled by distributed PI with AS1: Time evolution of the coupling gain $\alpha(t)$ (a), and disagreement dynamics (b) for different values of $\kappa_P = \gamma$.

in Figure 4(a), where the initial conditions are set to $\mathbf{x}(0) = [2, 1.5, 1, 0.5, 0]^T$. We also compute the network *disagreement dynamics* denoted by $d(t)$, and defined as

$$d(t) := \left\| \mathbf{x}(t) - (1/N)\left(\mathbb{1}_N \mathbb{1}_N^T\right)\mathbf{x}(t) \right\|, \tag{5.21}$$

where $d(t) = 0$ indicates that all the states $x_i(t)$, for all $i \in \mathcal{N}$ coincide at time t. The time evolution of $d(t)$ for different values of γ is shown in Figure 4(b). Note that higher asymptotic values of $\alpha(t)$ are reached when the sensitivity parameter γ increases (see Figure 4(a)). We observe that the steady-state values of $\alpha(t)$ are far below the bound 6.42 computed for the case of constant control parameters. Also, note that when γ decreases, the convergence time onto admissible consensus increases. This can be clearly seen in Figure 4(b), where admissible consensus is attained in 20 s when $\gamma = 0.2$, while the convergence time reduces to 10 s when $\gamma = 0.9$. This indicates a trade-off between the rate of convergence and lower asymptotic values of $\alpha(t)$ when γ varies.

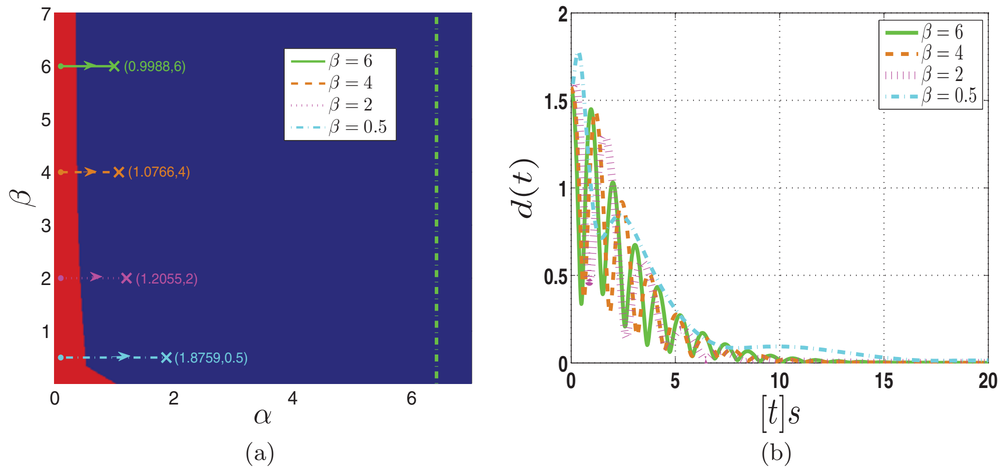

FIGURE 5. Heterogeneous network controlled by distributed PI control with AS1: (a) Two-dimensional stability diagram in the control parameter space (α, β): the blue and red areas represent the stable and unstable regions, respectively, the $\alpha(t)$ trajectories for different values of β are represented by the four coloured lines, while the green dashed line represent the stability threshold given by Theorem 4.2; (b) Time evolution of disagreement dynamics $d(t)$ for different values of β.

Moreover, to further emphasize the differences among the adaptive and static design of the gain α, in Figure 5(a) we report the numerical two-dimensional stability diagram in the control parameter space (α, β), where both $\alpha(t) = \alpha$ and $\beta(t) = \beta$ are assumed to be two positive constants.

To obtain the diagram, at each point in the (α, β) space, we computed the maximum eigenvalue of the error system dynamics (5.9), depicting in blue those points where the eigenvalue is negative (consensus is achieved) and in red those where it is positive (convergence is not attained). The region where convergence is guaranteed independently from β ($\alpha > 6.42$ from Theorem 4.2) is to the right of the green dashed line. On the same plot, we reported the trajectory of the gain $\alpha(t)$ under AS1 with different values of β, and we observed that its asymptotic values are in the stability region, but are much lower than the value of 6.42 computed from the sufficient condition for the static gain case.

Next, we compute the time evolution of the network disagreement dynamics $d(t)$, for different values of β as shown in Figure 5(b). Note that the selection of a larger value of β corresponds to a faster convergence, an increase of oscillations at transient state (Figure 5(b)), and a lower steady-state value for $\alpha(t)$ (Figure 5(a)).

5.2 Adaptive Strategy 2

Differently from the adaptive strategy (5.1), here we explore the case when the integral gain $\beta(t)$ also self-tune its value; so that, the evolution of both gains contribute to the control action.

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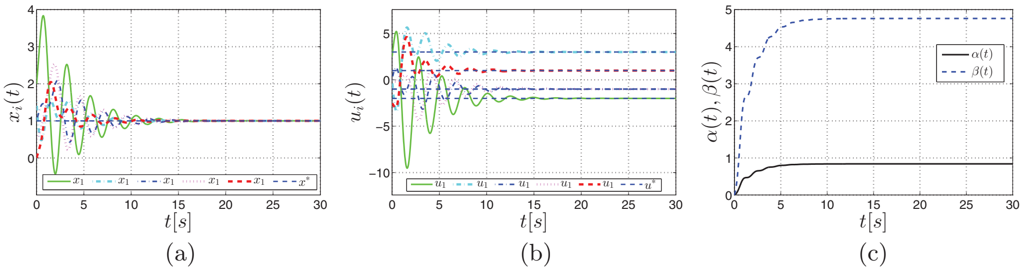

FIGURE 6. Heterogeneous network controlled by distributed PI with AS2: Time evolution of the heterogeneous network controlled by distributed PI control with AS2, $\kappa_P = 0.15$ and $\kappa_I = 0.85$. The blue dot-dashed lines represent the convergence values x^* and $\mathbf{u}^*$. The initial conditions were chosen as $\mathbf{x}(0) = [2, 1.5, 1, 0.5, 0]^T$.

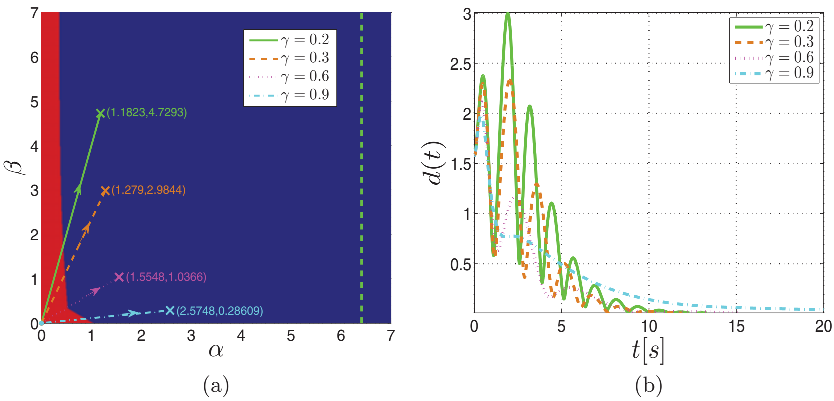

FIGURE 7. (a) Two-dimensional stability diagram in the control parameter space (α, β), the $\alpha(t)$ and $\beta(t)$ trajectories are represented by the four coloured lines with their respective steady-state points. (b) Time evolution of disagreement dynamics $d(t)$.

In particular, under Adaptive Strategy 2, the proportional gain $\alpha(t)$ is updated according to (5.1), while $\beta(t)$ evolves according to

$$\frac{d\beta}{dt} = \frac{\kappa_I}{N}\boldsymbol{\delta}(t)\mathscr{L}_{22}\boldsymbol{\delta}(t), \qquad \beta(0) = 0. \tag{5.22}$$

Note that $d\beta/dt$ is a strictly positive function and its derivative $d\beta/dt = 0$ when admissible consensus is achieved. As an illustrative example, we consider again the ring network of heterogeneous agents described in Section 5.1.4. In the following simulations, the proportional and integral gains are updated according to AS2, with $\kappa_P = 0.15$ and $\kappa_I = 0.85$. The time evolution of the heterogeneous network with self-tuning gains is depicted in Figure 6 where, consensus is achieved on $x^* = 1$. Note also that the control parameters $\alpha(t)$ and $\beta(t)$ converge to the constant values 0.83 and 4.76, respectively. We find these values to be dependent on the value of the constant gain κ_I as can be seen from Figure 7, where, without loss of generality, we set $\kappa_P = \gamma$ and $\kappa_I = (1 - \gamma)$ with $\gamma \in]0, 1[$. In this particular case, we assumed both constants to be linearly related, so that changing

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γ varies the share of the control effort taken by the proportional and the integral terms in the control strategy (5.1) and (5.22).

Note that with our self-tuning strategy, and by choosing different values for the gain γ, we can explore the (α, β) parameter space and obtain different solutions as can be seen in Figure 7(a). These variations also affect the rate of convergence of the agents in the network onto the consensus value. This point is illustrated in Figure 7(b), where $d(t)$ is the disagreement index defined in (5.21). Note that for $\gamma = 0.2$ and $\gamma = 0.3$ the rate of convergence is approximately 12s, while for $\gamma = 0.6$ and $\gamma = 0.9$ is about 15s and 27s, respectively.

6 Decentralized self-tuning PI consensus

In this section, we consider the case in which the global disagreement vector is not available to each agent. Specifically, we assume that each agent can only measure a *local disagreement*, which is defined as the following function of the state of all its neighbours:

$$\delta_i^l(t) = x_i(t) - \hat{x}_i(t), \qquad \hat{x}_i(t) := \frac{1}{|\mathscr{D}_i|} \sum_{i \in \mathscr{D}_i}^{N} x_j(t), \qquad i = 1, \dots, N,$$

where $\mathscr{D}_i$ is the set of neighbours of the ith vertex, while $|\mathscr{D}_i|$ represents its cardinality. In this case, we propose the use of Adaptive Strategy 3, where the proportional as well as the integral control gains self-tune their values in a distributed manner. Namely, we choose

$$u_i(t) = \alpha_i(t) \sum_{i=1}^{N} w_{ij}(x_j(t) - x_i(t)) + \beta_i(t) \sum_{i=1}^{N} w_{ij} \int_0^t (x_j(\tau) - x_i(\tau)) d\tau, \tag{6.1}$$

where $\alpha_i(t)$ and $\beta_i(t)$ are the distributed control gains that need to be determined for reaching admissible consensus. The control gains are assumed to be different at each node i and their evolution is given by

$$\begin{aligned} d\alpha_i/dt &= \kappa_P \left|\delta_i^l(t)\right|, \qquad \alpha_i(0) = 0 \\ d\beta_i/dt &= \kappa_I \left|\delta_i^l(t)\right|, \qquad \beta_i(0) = 0 \end{aligned} \tag{6.2}$$

where κ_P and κ_I are arbitrary positive constants, modulating the rate of growth for the proportional and integral gains, respectively. Note that the ith control gains are updated according to the difference of the node state $x_i(t)$ and the average state $\hat{x}_i$ of its neighbouring nodes.

Differently from the Adaptive Strategy 2 (5.1)–(5.22), where a complete knowledge of the network is required, here each $\alpha_i(t)$ and $\beta_i(t)$ evolves by only considering relative information based on the states of their neighbours.

To illustrate the effectiveness of AS3, we consider again the ring network of Figure 1 controlled now by the distributed PI control strategy (6.1), where the control gains evolve according to (6.2) for all $i \in \mathscr{N}$ with $\kappa_P = \gamma$ and $\kappa_I = 1 - \gamma$, $\gamma \in]0, 1[$. The time evolution of the node states, control inputs, and control gains is shown in Figure 8, where admissible consensus is achieved and the control gains asymptotically converge to constant positive values.

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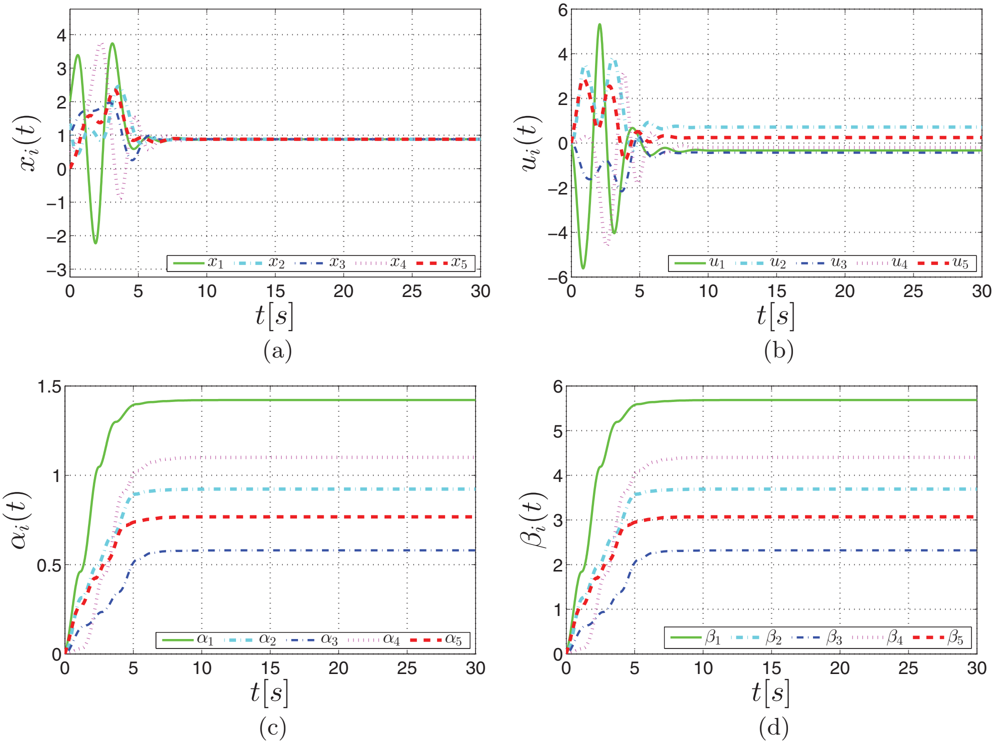

Figure 8. Time evolution of the heterogeneous network under distributed adaptation of the control gain with $\gamma = 0.2$.

7 Conclusions and future work

In this paper, we presented novel self-tuning PI strategies for consensus of heterogeneous linear multi-agent systems. Our approach starts from the findings in [17], and is based on the dynamic adaptation of the proportional and integral gains characterizing the distributed PI strategy of interest. In particular, both centralized and decentralized adaptive laws are considered. Adaptive Strategies 1 and 2 assume that the gains are updated on the basis of a global disagreement measure available to all nodes, while Adaptive Strategy 3 makes each node able to independently tune the strength of the proportional and integral gains characterizing the interconnections with its neighbouring nodes on the basis of their local disagreement. The effectiveness of the approaches was illustrated on a representative example of 5 agents and complemented with a rigorous proof of convergence for Adaptive Strategy 1. A complete stability analysis of all the proposed approaches is currently under investigation and will be presented elsewhere.

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