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Cesàro mean distribution of group automata starting from measures with summable decay

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Abstract. Consider a finite Abelian group $(G, +)$, with $|G| = p^r$, p a prime number, and $\varphi : G^{\mathbb{N}} \to G^{\mathbb{N}}$ the cellular automaton given by $(\varphi x)_n = \mu x_n + \nu x_{n+1}$ for any $n \in \mathbb{N}$, where μ and ν are integers coprime to p. We prove that if $\mathbb{P}$ is a translation invariant probability measure on $G^{\mathbb{Z}}$ determining a chain with complete connections and summable decay of correlations, then for any $\underline{w} = (w_i : i < 0)$ the Cesàro mean distribution

$$\mathcal{M}_{\mathbb{P}_{\underline{w}}} = \lim_{M\to\infty} \frac{1}{M} \sum_{m=0}^{M-1} \mathbb{P}_{\underline{w}} \circ \varphi^{-m},$$

where $\mathbb{P}_{\underline{w}}$ is the measure induced by $\mathbb{P}$ on $G^{\mathbb{N}}$ conditioned by $\underline{w}$, exists and satisfies $\mathcal{M}_{\mathbb{P}_{\underline{w}}} = \lambda^{\mathbb{N}}$, the uniform product measure on $G^{\mathbb{N}}$. The proof uses a regeneration representation of $\mathbb{P}$.

1. *Introduction and main results*

Let $(G, +)$ be a finite Abelian group with $q = p^r$ elements, p being a prime number. We put $\lambda = (q^{-1}, \dots, q^{-1})$ as the uniform measure on the group. In this paper we study the measure evolution under the dynamics of the cellular automaton $\varphi : G^{\mathbb{N}} \to G^{\mathbb{N}}$, given by $(\varphi x)_n = \mu x_n + \nu x_{n+1}$ for $n \in \mathbb{N}$, where μ and ν are integers coprime to p (ℓg means $g + \cdots + g$ ℓ-times). The uniform product measure $\mathbb{P} = \lambda^{\mathbb{N}}$ is φ-invariant, $\mathbb{P} \circ \varphi^{-n} = \mathbb{P}$, but any other product measure $\mathbb{P} = \pi^{\mathbb{N}}$, with $\pi \neq \lambda$, is not φ-invariant. Moreover, even in the simplest case $G = \{0, 1\}$ and $+$ the mod 2 sum, the limit of the marginal distribution,

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$\lim_{m\to\infty} \mathbb{P}\{(\varphi^m x)_0 = g\}$ with $g \in G$, does not exist. The reason is that, for $m = 2^k$, $(\varphi^m x)_0 = x_0 + x_m$ (the other terms sum an even number of times and do not contribute to the sum) has probability $p^2 + (1-p)^2$ to be zero, while for $m = 2^k - 1$ this probability converges to $\frac{1}{2}$ because $(\varphi^m x)_0 = \sum_{\ell=0}^m x_\ell$.

Alternatively we can study the Cesàro mean distribution

$$\mathcal{M}_{\mathbb{P}} \doteq \lim_{M\to\infty} \frac{1}{M} \sum_{m=0}^{M-1} \mathbb{P} \circ \varphi^{-m}$$

for a class of initial distributions $\mathbb{P}$ on $G^{\mathbb{N}}$. In the above display and in the following $\doteq$ means 'it is defined by'.

Let $-\mathbb{N}^* = \{-i : i \in \mathbb{N} \setminus \{0\}\}$ and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. Let $\mathbb{P}$ be a translation invariant probability measure on $G^{\mathbb{Z}}$. For $\underline{w} \in G^{-\mathbb{N}^*}$ let $\mathbb{P}_{\underline{w}}$ be the measure on $G^{\mathbb{N}}$ induced by the conditional probabilities as follows. For any $m \geq 0$ and $g_0, \ldots, g_m \in G$, define

$$\mathbb{P}_{\underline{w}}\{x_0 = g_0, \ldots, x_m = g_m\} \doteq \mathbb{P}\{x_0 = g_0, \ldots, x_m = g_m \mid x_i = w_i, i < 0\}.$$

We say that $\mathbb{P}$ has *complete connections* if it satisfies

$$\forall g_0 \in G,\ \forall \underline{w} \in G^{-\mathbb{N}^*}, \quad \mathbb{P}_{\underline{w}}\{x_0 = g_0\} > 0. \tag{1.1}$$

For any $m \geq 0$ define

$$\gamma_m \doteq \sup \left\{ \left| \frac{\mathbb{P}_{\underline{w}}\{x_0 = g\}}{\mathbb{P}_{\underline{v}}\{x_0 = g\}} - 1 \right| : g \in G, \underline{v}, \underline{w} \in G^{-\mathbb{N}^*}, v_i = w_i, i \in [-m, -1] \right\}.$$

We say that $\mathbb{P}$ has *summable decay* if

$$\sum_{m=0}^{\infty} \gamma_m < \infty. \tag{1.2}$$

This is a uniform continuity condition on $\mathbb{P}_{\underline{w}}(g)$ as a function of $\underline{w}$.

The Cesàro limits have already been studied for the mod 2 sum automaton and other classes of permutative cellular automata in [**L**] and [**MM**]. In these papers it is computed mainly for Bernoulli measures, and in [**MM**] only the one site Cesàro limit is computed for a Markov measure. In the mod 2 case the limit is uniformly distributed, but for some permutative cellular automata the Cesàro mean exists but it is not necessarily uniform. In [**FMM**] the Athreya–Ney [**AN**] regeneration times representation of the r-step Markov chain was used to show the convergence of the Cesàro mean of the group automata starting with these Markov chains to the uniform Bernoulli measure.

In this paper we generalize these results for the group automaton φ and initial measures with complete connections and summable decay.

THEOREM 1.3. *Let $(G, +)$ be a finite Abelian group with $|G| = p^r$, p being a prime number. Let $\mathbb{P}$ be a translation invariant probability measure on $G^{\mathbb{Z}}$ with complete connections and summable decay. Let $\varphi : G^{\mathbb{N}} \to G^{\mathbb{N}}$ be the cellular automaton, given by $(\varphi x)_n = \mu x_n + \nu x_{n+1}$ for $n \in \mathbb{N}$, where μ and ν are integers coprime to p. Then for all $\underline{w} \in G^{-\mathbb{N}^*}$ the Cesàro mean distribution $\mathcal{M}_{\mathbb{P}_{\underline{w}}}$ exists and verifies $\mathcal{M}_{\mathbb{P}_{\underline{w}}} = \lambda^{\mathbb{N}}$, the product of uniform measures on G.*

There are two main elements in the proof: regeneration times and the distribution of Pascal triangle coefficients mod p.

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2. *Regeneration times for the initial measure*

We show that under the conditions of Theorem 1.3, for all $\underline{w} \in -\mathbb{N}^*$ we can jointly construct a random sequence $\underline{x} = (x_i : i \in \mathbb{N}) \in G^{\mathbb{N}}$ with distribution $\mathbb{P}_{\underline{w}}$ and a random subsequence $(T_i : i \in \mathbb{N}^*) \subseteq \mathbb{N}$ such that $(x_{T_i} : i \in \mathbb{N}^*)$ are iid uniformly distributed in G and independent of $(x_i : i \in \mathbb{N} \setminus \{T_1, T_2, \dots\})$; furthermore $(T_i : i \in \mathbb{N}^*)$ is a stationary renewal process with finite mean inter-renewal time, independent of $\underline{w}$. A consequence of the construction is that the random vectors (of random lengths) $((x_{T_i}, \dots, x_{T_{i+1}-1}) : i \geq 1)$ are iid.

Our regeneration approach shares results with [**B**] and [**NN**]. The construction is simple: the probability space is generated by the product of iid uniform (in $[0, 1]$) random variables. It works as the well known construction and simulation of Markov chains as a function of a sequence of uniform random variables (see, for instance, [**FG**]). Bressaud *et al* [**BFG**] construct a coupling using these ideas to show the decay of correlations for measures with infinite memory.

For $\underline{w} \in G^{-\mathbb{N}^*}$ and $g \in G$ denote

$$P(g|\underline{w}) \doteq \mathbb{P}\{x_0 = g \mid x_i = w_i, i \leq -1\}.$$

Let

$$a_{-1}(g|\underline{w}) \doteq \inf\{P(z|\underline{v}) : \underline{v} \in G^{-\mathbb{N}^*}, z \in G\}. \tag{2.1}$$

Actually a_{-1} depends neither on g nor on $\underline{w}$; we keep the dependence in the notation for future (notational) convenience. Since the space $G^{-\mathbb{N}^*}$ is compact and $\mathbb{P}$ has summable decay, the infimum in (2.1) must be attained by a $g^0 \in G$ and a $\underline{w}^0 \in G^{-\mathbb{N}^*}$. Hence,

$$a_{-1}(g|\underline{w}) = P(g^0|\underline{w}^0) > 0,$$

because $\mathbb{P}$ has complete connections. For each $k \in \mathbb{N}$, $g \in G$ and $\underline{w} \in G^{-\mathbb{N}^*}$ define

$$a_k(g|\underline{w}) \doteq \inf\{P(g|w_{-1}, \dots, w_{-k}, \underline{z}) : \underline{z} \in G^{-\mathbb{N}^*}\},$$

where $(w_{-1}, \dots, w_{-k}, \underline{z}) = (w_{-1}, \dots, w_{-k}, z_{-1}, z_{-2}, \dots)$. Notice that $a_0(g|\underline{w})$ does not depend on $\underline{w}$. Let

$$b_{-1}(g|\underline{w}) \doteq a_{-1}(g|\underline{w}),$$

for $g \in G$. For $k \geq 0$,

$$b_k(g|\underline{w}) \doteq a_k(g|\underline{w}) - a_{k-1}(g|\underline{w}).$$

We construct disjoint intervals $B_k(g|\underline{w})$ for $g \in G$, $k \geq -1$, contained in $[0, 1]$, of Lebesgue measure $b_k(g|\underline{w})$, respectively, disposed in increasing order with respect to g and k: $B_{-1}(0|\underline{w}), \dots, B_{-1}(q-1|\underline{w}), B_0(0|\underline{w}), \dots, B_0(q-1|\underline{w}), B_1(0|\underline{w}), \dots, B_1(q-1|\underline{w}), \dots,$ with no intersections (we have enumerated G by $\{0, \dots, q-1\}$). The construction guarantees

$$\left| \bigcup_{k \geq -1} B_k(g|\underline{w}) \right| = P(g|\underline{w})$$

and

$$\left| \bigcup_{g \in G} \bigcup_{k \geq -1} B_k(g|\underline{w}) \right| = 1.$$

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(All the unions above are disjoint.)

Let $\underline{U} = (U_n : n \in \mathbb{Z})$ be a double infinite sequence of iid random variables uniformly distributed in $[0, 1]$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space induced by these random variables. For each $\underline{w} \in G^{-\mathbb{N}^*}$ we construct the random sequence $\underline{x}$ with distribution $\mathbb{P}_{\underline{w}}$ in Ω, as a function of $\underline{U}$, recursively: for $n \in \mathbb{N}$

$$x_n \doteq \sum_{g \in G} g \left[\sum_{\ell \geq -1} \mathbf{1}\{U_n \in B_\ell(g|x_{n-1}, \dots, x_0, \underline{w})\} \right].$$

For $\ell \geq -1$ let

$$B_\ell(\underline{w}) \doteq \bigcup_{g \in G} B_\ell(g|\underline{w}).$$

Notice that neither $B_{-1}(g|\underline{w})$ nor $B_{-1}(\underline{w})$ depend on $\underline{w}$. Furthermore,

$$\frac{|B_{-1}(g|\underline{w})|}{|B_{-1}(\underline{w})|} = |G|^{-1}. \tag{2.2}$$

For $k \in \mathbb{N}$ let

$$a_k \doteq \min_{\underline{w}} \left\{ \sum_{g \in G} a_k(g|\underline{w}) \right\}.$$

This is a non-decreasing sequence and satisfies

$$[0, a_k] \subset \bigcup_{\ell=-1}^{k} B_\ell(\underline{w}), \tag{2.3}$$

independently of $\underline{w} \in G^{-\mathbb{N}^*}$.

LEMMA 2.4. *In the event $\{U_n \leq a_k\}$ for $n \in \mathbb{N}$ we only need to look at $x_{n-1}, \dots, x_{n-k}$ to decide the value of x_n. More precisely, for $\underline{v} \in G^{\mathbb{Z}}$ such that $v_i = w_i$ for $i \leq -1$,*

$$\begin{aligned} &\mathbb{P}_{\underline{w}}\{x_n = g \mid U_n \leq a_k, x_{n-1} = v_{n-1}, \dots, x_0 = v_0\} \\ &\quad = \mathbb{P}_{\underline{w}}\{x_n = g \mid U_n \leq a_k, x_{n-1} = v_{n-1}, \dots, x_{n-k} = v_{n-k}\}. \end{aligned}$$

Proof. Follows from (2.3). □

Define times

$$\begin{aligned} T_1 &\doteq \min\{n \geq 0 : U_{n+j} \leq a_{j-1}, j \geq 0\}, \\ T_i &\doteq \min\{n > T_{i-1} : U_{n+j} \leq a_{j-1}, j \geq 0\}, \end{aligned}$$

for $i > 1$.

Let $\mathbf{N}$ be the counting measure on $\mathbb{N}$ induced by $(T_i : i \geq 1)$: for $A \subset \mathbb{N}$ and $n \in \mathbb{N}$,

$$\mathbf{N}(A) \doteq \sum_{i \geq 1} \mathbf{1}\{T_i \in A\}, \quad \mathbf{N}(n) \doteq \mathbf{N}(\{n\}).$$

Notice that the definitions of $(T_i : i \geq 1)$ and $\mathbf{N}$ depend only on $(U_n : n \in \mathbb{Z})$ and do not depend on $\underline{w}$.

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LEMMA 2.5. *The distribution of the counting measure* **N** *corresponds to a stationary renewal process.*

Proof. We will construct a stationary renewal process **M** in $\mathbb{Z}$ whose projection on $\mathbb{N}$ is **N**. For $k \in \mathbb{Z}$, $k' \in \mathbb{Z} \cup \{\infty\}$, define

$$H[k,k'] \doteq \begin{cases} \{U_{k+\ell} \le a_{\ell-1}, \ell = 0, \ldots, k'-k\}, & \text{if } k \le k', \\ \text{'full event'}, & \text{if } k > k'. \end{cases}$$

With this notation,

$$\mathbf{N}(n) = \mathbf{1}\{H[n,\infty]\}, \quad n \in \mathbb{N}. \tag{2.6}$$

We construct a double infinity counting process **M** using the variables $(U_n : n \in \mathbb{Z})$ by

$$\mathbf{M}(n) \doteq \mathbf{1}\{H[n,\infty]\}, \quad n \in \mathbb{Z}.$$

By construction, the distribution of **M** is translation invariant, hence **M** is stationary. Furthermore, by (2.6) it coincides with **N** in $\mathbb{N}$: $\mathbf{M}(K) = \mathbf{N}(K)$ for $K \subset \mathbb{N}$. Define T_i for $i \le 0$ as the ordered time events of **M** in the negative axis.

The (marginal) probability of a counting event at time $n \in \mathbb{Z}$ is given by

$$\mathbb{P}\{\mathbf{M}(n) = 1\} = \mathbb{P}\{U_{n+j} \le a_{j-1}, j \ge 0\} = a_{-1}a_0a_1 \cdots \doteq \beta,$$

and it is independent of n. We first show that under the hypothesis of summability of γ_k, β is strictly positive. For any $g \in G$, $w_{-1}, \ldots, w_{-k} \in G$ and $\underline{z}, \underline{v} \in G^{-\mathbb{N}^*}$

$$\left| \frac{\mathbb{P}\{g|w_{-1}\ldots w_{-k}, \underline{z}\}}{\mathbb{P}\{g|w_{-1}\ldots w_{-k}, \underline{v}\}} - 1 \right| \le \gamma_k,$$

therefore

$$\inf\{\mathbb{P}\{g|w_{-1}\ldots w_{-k}, \underline{z}\} : \underline{z} \in G^{-\mathbb{N}^*}\} \ge (1-\gamma_k)\mathbb{P}\{g|w_{-1}\ldots w_{-k}, \underline{v}\}.$$

Summing over $g \in G$ and taking a minimum on the set $\{w_{-1}, \ldots, w_{-k}\}$, we conclude that

$$a_k \ge 1 - \gamma_k.$$

Since $\sum_{k\ge 0} \gamma_k < \infty$ we deduce that $\sum_{k \ge 0}(1-a_k) < \infty$ and hence $\beta > 0$.

We now show that **M** is a renewal process on $\mathbb{Z}$. The event $\{\mathbf{M}(n) = 1\}$ depends only on $(U_k : k \ge n)$, that is, $(T_i : i \in \mathbb{Z})$ are stopping times for the process $(U_{-k} : k \in \mathbb{Z})$. Since for $k < k' < k'' \le \infty$,

$$H[k,k''] \cap H[k',k''] = H[k,k'-1] \cap H[k',k''],$$

we have that for any finite set $A = \{k_1, \ldots, k_n\}$ with $k_1 < \cdots < k_n < k'$ and for any sequence $(m_\ell : \ell > k')$ with $m_\ell \in \{0,1\}$,

$$\begin{aligned} &\mathbb{P}\{\mathbf{M}(A) = n \mid \mathbf{M}(k') = 1, \mathbf{M}(\ell) = m_\ell, \ell > k'\} \\ &\quad = \mathbb{P}\left\{ \bigcap_{i=1}^{n} H[k_i, k'-1] \,\middle|\, \mathbf{M}(k') = 1 \right\} \\ &\quad = \prod_{i=1}^{n} \mathbb{P}\{H[k_i, k_{i+1}-1]\}, \end{aligned} \tag{2.7}$$

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where $k_{n+1} \doteq k'$. The computation above could be done because $\mathbb{P}\{\mathbf{M}(k') = 1\} = \beta > 0$. Equation (2.7) means that, given a counting event at time k', the distribution of the counting events for times less than k' does not depend on the events after k'. This characterizes $\mathbf{M}$ as a renewal process. Since the density β is positive, T_1, the residual time is a honest random variable, and for $i \neq 1$, $\mathbb{E}(T_{i+1} - T_i) = \beta^{-1} < \infty$. □

LEMMA 2.8. *The variables* $(x_{T_i} : i \geq 0)$ *are iid uniformly distributed in* G.

Proof. Let us show that the marginal distribution of x_{T_i} is uniform in G. Since times $(T_i : i \in \mathbb{N}^*)$ are finite almost surely,

$$\begin{aligned}\mathbb{P}\{x_{T_i} = g\} &= \sum_{n\in\mathbb{N}} \mathbb{P}\left\{U_n \in \bigcup_{\ell\geq -1} B_\ell(g|\underline{w}), T_i = n\right\} \\ &= \sum_{n\in\mathbb{N}} \mathbb{P}\{U_n \in B_{-1}(g|\underline{w}) \mid U_n \in B_{-1}(\underline{w})\}\mathbb{P}\{T_i = n\} \\ &= |G|^{-1}.\end{aligned}$$

The second identity follows because $\{T_i = n\}$ is the intersection of $\{U_n \in B_{-1}(\underline{w})\}$ with events depending on variables $(U_{n+\ell},\ \ell \neq 0)$ which are independent of U_n. The third identity follows from (2.2). The same computation shows that for any $K \subset \mathbb{N}$, $(i(k) : k \in K) \subseteq \mathbb{N}$, and $(g_k : k \in K) \subseteq G^K$,

$$\mathbb{P}\{x_{T_{i(k)}} = g_k, k \in K\} = |G|^{-|K|},$$

so that $(x_{T_{i(k)}} : k \in K)$ are iid in G. The reason why the above computation works is that in the event $\{T_i = n\}$, $U_{n+1} \leq a_0$, hence x_{n+1} does not depend on the past. Since for all $j \geq 1$, $U_{n+j} \leq a_{n+j-1}$, x_{n+j+1} only depends on $x_{n+1}, \ldots, x_{n+j}$. □

3. *A renewal lemma*

In this section we show that a stationary discrete-time renewal process on $\mathbb{N}$ has high probability to visit sets with many points.

LEMMA 3.1. *Let* $\mathbf{N}$ *be a stationary renewal process with finite inter-renewal mean. Then for all* $A \subset \mathbb{N}$,

$$\mathbb{P}\{\mathbf{N}(A) = 0\} \leq \varepsilon(|A|)$$

with $\varepsilon(n) \to 0$ *as* $n \to \infty$. *Also,* $\varepsilon : \mathbb{N} \to \mathbb{R}$ *can be chosen to be decreasing.*

Proof. We are going to prove that for all $\varepsilon > 0$ there exists n_0 such that for any finite set $A \subset \mathbb{N}$ with $|A| > n_0$,

$$\mathbb{P}\{\mathbf{N}(A) = 0\} \leq \varepsilon. \tag{3.2}$$

We start with some known facts of renewal theory. Let T_i be the renewal times and $\beta = 1/\mathbb{E}(T_{i+1} - T_i)$ for some $i \geq 1$ (and hence for all $i \geq 1$). Since the inter-renewal distribution has a finite first moment, the key renewal theorem holds: we have

$$\lim_{n\to\infty} \mathbb{P}\{\mathbf{N}(n) = 1 \mid \mathbf{N}(0) = 1\} = \beta. \tag{3.3}$$

Let $S_n \doteq T_{\overline{\mathbf{N}}(n)+1} - n$ be the residual time (over jump) at n, where we have denoted $\overline{\mathbf{N}}(n) = \mathbf{N}([0, n])$, and for $k \geq 0$ let

$$\beta_k = \mathbb{P}\{\mathbf{N}(k) = 1 \mid T_1 = 0\}, \quad F(k) = \mathbb{P}\{T_2 - T_1 > k\}, \quad F_n(k) = \mathbb{P}\{S_n > k \mid T_1 = 0\}. \tag{3.4}$$

Now we have

$$F_n(k) = \sum_{j=0}^{n} F(j+k)\beta_{n-j} \leq \overline{F}(k), \tag{3.5}$$

where

$$\overline{F}(k) \doteq \sum_{j=k}^{\infty} F(j) \to 0$$

as $k \to \infty$ because we are assuming that the inter-renewal time has a finite mean.

For any subset $B \subset A$ we have

$$\mathbb{P}\{\mathbf{N}(A) = 0\} \leq \mathbb{P}\{\mathbf{N}(B) = 0\}.$$

For any A with $|A| = n$ and any $1 < \ell < n$, there exists a set

$$\{b_1^n, \ldots, b_\ell^n\} \doteq B_\ell^n \subset A$$

with

$$\left[\frac{n}{\ell}\right] \leq b_{j+1}^n - b_j^n, \quad j = 1, \ldots, \ell - 1, \tag{3.6}$$

where $[x]$ is the largest integer in x. The choice of $\{b_1^n, \ldots, b_\ell^n\}$ depends on A apart from ℓ, and (3.6) holds uniformly for all A with $|A| = n$.

Let $\varepsilon > 0$ and take any $0 < \delta < \beta$. Take n_0 such that $\beta_n > \delta$ for $n > n_0$. Let $n > \ell n_0$ and define

$$\Gamma_j^n \doteq \{S_{b_j^n} \leq [n/\ell] - n_0\},$$

as the event 'the over jump of b_j^n does not superate $[n/\ell] - n_0$'. Let

$$\Theta_j^n \doteq \{\mathbf{N}(b_j^n - b_{j-1}^n - S_{b_{j-1}^n}) = 0\}, \quad \Lambda_j^n \doteq \{\mathbf{N}(b_j^n) = 0\},$$

be the events 'starting at the over jump of b_{j-1}^n, b_j^n is not hit' and 'b_j^n is not hit', respectively. From (3.5) we get for $2 \leq j \leq \ell$

$$\mathbb{P}\{\Gamma_j^n \mid \Gamma_{j-1}^n\} \geq (1 - \overline{F}([n/\ell] - n_0)). \tag{3.7}$$

Then

$$\begin{aligned}
\mathbb{P}\{\mathbf{N}(A) = 0\} &\leq \mathbb{P}\{\mathbf{N}(B_\ell^n) = 0\} = \mathbb{P}\{\Lambda_1^n \cap \cdots \cap \Lambda_\ell^n\} \\
&\leq \mathbb{P}\{\Lambda_1^n \cap \Gamma_1^n\}\mathbb{P}\{\Lambda_2^n \cap \cdots \cap \Lambda_\ell^n \mid \Gamma_1^n\} + 1 - \mathbb{P}\{\Gamma_1^n\} \\
&\leq \prod_{j=1}^{\ell} \mathbb{P}\{\Theta_j^n \mid \Gamma_j^n\} + \sum_{j=1}^{\ell-1} (1 - \mathbb{P}\{\Gamma_j^n \mid \Gamma_{j-1}^n\}) \\
&\leq (1-\delta)^\ell + (\ell - 1)\overline{F}([n/\ell] - n_0) + \mathbb{P}\{T_1 > [n/\ell] - n_0\}
\end{aligned} \tag{3.8}$$

since $\beta_n > \delta$ for $n > n_0$ and (3.7). Now choose ℓ so that $(1-\delta)^\ell < \varepsilon/3$, then n so that $\overline{F}([n/\ell] - n_0) < \varepsilon/3(\ell - 1)$ and $\mathbb{P}\{T_1 > [n/\ell] - n_0\} \leq \varepsilon/3$, to conclude

$$\mathbb{P}\{\mathbf{N}(B_\ell^n) = 0\} \leq \varepsilon$$

for sufficiently large $n + \ell$. □

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4. *Convergence of the Cesàro limit*

To prove this theorem we shall need some results concerning walks of variables determining a chain with complete connections. For this purpose let us introduce some notation. First, $R = (r_k : k \in \mathbb{N})$ denotes an increasing sequence in $\mathbb{N}$. We put $R_n = (r_k : k \le n)$. For any subsequence $\overline{R} = (\overline{r}_k : k \in \mathbb{N})$ of R we define the index function by $f_{\overline{R}}(k) = \ell$ if $\overline{r}_k = r_\ell$. We also set $n(\overline{R}) = |\overline{R} \cap R_n|$. Let $a^R = (a_r^R : r \in R)$ be a sequence of non-negative integers. They define maps $\psi_r^R : G \to G$ such that $\psi_r^R(g) = a_r^R g = g + \cdots + g a_r^R$ times, for any $r \in R$. We associate to it the following sequence of random variables taking values in G,

$$S_n^R = \sum_{r \in R_n} a_r^R x_r, \quad n \in \mathbb{N}.$$

We will distinguish the following subsequence:

$$R^* \doteq R^*(a^R) = \{r \in R : a_r^R \ne 0 \bmod p\}.$$

Remark. Since $(G, +)$ is a finite Abelian group with $|G| = p^r$, p a prime number, then the function $\psi(g) = ag$, where $a \in \mathbb{N}$, is one-to-one whenever $a \ne 0 \bmod p$.

Let $J \subseteq \mathbb{N}$ be a finite set. Consider a finite family of sequences $R^J = (R^j : j \in J)$. Associated to each sequence there is a sequence of non-negative integers $a^{R^j} = (a_r^{R^j} : r \in R^j)$ and the corresponding set of mappings $\psi^{R^j} = (\psi_r^{R^j} : r \in R^j)$. As before we consider the sequences $R^{j*} \doteq R^*(a^{R^j})$ for $j \in J$. Let $\tilde{R}^J = (\tilde{R}^j : j \in J)$ be a family of subsequences verifying the following conditions:

(H1) $\tilde{R}^j \subseteq R^{j*}$ for any $j \in J$;

(H2) $\tilde{R}^j \cap \tilde{R}^i = \emptyset$ if $i \ne j \in J$;

(H3) if $r \in \tilde{R}^j \cap R^k$ for $k < j \in J$, then $a_r^{R^k} = 0 \bmod p$.

We set

$$\tilde{n}(\tilde{R}^J) = \min\{n(\tilde{R}^j) : j \in J\}$$

and

$$\tilde{n}(R^J) = \max\{\tilde{n}(\tilde{R}^J) : \tilde{R}^J \text{ verifying (H1), (H2), (H3)}\}.$$

The proof of Theorem 1.3 is based upon the following result.

LEMMA 4.1. *Let $\mathbb{P}$ be a translation invariant measure on $G^{\mathbb{Z}}$ with complete connections such that $\sum_{m \ge 0} \gamma_m < \infty$, and let $\underline{w} \in G^{-\mathbb{N}^*}$. Then we have the following.*

(a) *$\exists \varepsilon_1 : \mathbb{N} \to \mathbb{R}$, a decreasing function with $\varepsilon_1(n) \to 0$ if $n \to \infty$, such that for any increasing sequence R in $\mathbb{N}$ and any sequence of non-negative integers a^R it is verified that*

$$|\mathbb{P}_{\underline{w}}\{S_n^R = g\} - q^{-1}| \le \varepsilon_1(n(R^*)), \quad \textit{for any } n \in \mathbb{N}, g \in G.$$

(b) *Let $J \subset \mathbb{N}$ be finite. Then there is a decreasing function $\varepsilon_J : \mathbb{N} \to \mathbb{R}$ with $\varepsilon_J(n) \to 0$ if $n \to \infty$, such that for any set of sequences $R^J = (R^j : j \in J)$ and any family of non-negative integers $(a^{R^j} : j \in J)$, it is verified that*

$$|\mathbb{P}_{\underline{w}}\{S_n^{R^j} = g_j, \textit{ for } j \in J\} - q^{-|J|}| \le \varepsilon_J(\tilde{n}(R^J)), \quad \textit{for any } n \in \mathbb{N}, (g_j : j \in J) \in G^J.$$

Before beginning the proof of Lemma 4.1 we include a useful arithmetic property. We include a proof for completeness. For $(G, +)$ a finite Abelian group with $|G| = p^r$, where p is a prime number, consider the following system of equations (S):

$$\begin{array}{lccccccc} (1) & a_{11}g_1 & + & a_{12}g_2 & + & \cdots & + & a_{1\ell}g_\ell = 0 \\ (2) & a_{21}g_1 & + & a_{22}g_2 & + & \cdots & + & a_{2\ell}g_\ell = 0 \\ & & & & & \vdots & & \\ (\ell) & a_{\ell 1}g_1 & + & a_{\ell 2}g_2 & + & \cdots & + & a_{\ell\ell}g_\ell = 0 \end{array}$$

such that

$$(\mathrm{H}') \qquad a_{ij} \in \mathbb{N}, \quad a_{ii} \neq 0 \bmod p, \quad a_{ij} = 0 \bmod p \text{ if } i < j.$$

Denote $a_{ii} = k_i p + s_i$ with $s_i \in \{1, \ldots, p-1\}$ and $a_{ij} = c_{ij} p$ for $i < j$.

LEMMA 4.2. *The system (S) has unique solution* $g_1 = g_2 = \cdots = g_\ell = 0$.

Proof. First of all we will prove that if $g_1, \ldots, g_\ell$ are solutions of (S) and for some $1 < s \leq r$, $p^s g_i = 0$, $i \in \{1, \ldots, \ell\}$, then $p^{s-1} g_i = 0$ for $i \in \{1, \ldots, \ell\}$. We prove this property by induction on $\{1, \ldots, \ell\}$. First consider equation (1),

$$(k_1 p + s_1) g_1 + \sum_{j=2}^{\ell} c_{1j} p g_j = 0.$$

If we add the equation p^{s-1} times, we obtain

$$k_1 p^s g_1 + s_1 p^{s-1} g_1 + \sum_{j=2}^{\ell} c_{1j} p^s g_j = 0,$$

then $s_1 p^{s-1} g_1 = 0$. Since the product by s_1 defines a one-to-one map we conclude that $p^{s-1} g_1 = 0$. Let us continue with the induction assuming that $p^{s-1} g_1 = 0$, $p^{s-1} g_2 = 0, \ldots, p^{s-1} g_t = 0$, for $1 \leq t < \ell$, and we prove that $p^{s-1} g_{t+1} = 0$.

Adding p^{s-1} times equation $t+1$ we get

$$\sum_{j=1}^{t} a_{t+1,j} p^{s-1} g_j + (k_{t+1} p + s_{t+1}) p^{s-1} g_{t+1} + \sum_{j=t+2}^{\ell} c_{t+1,j} p^s g_j = 0.$$

Therefore, using the induction hypothesis we obtain $s_{t+1}(p^{s-1} g_{t+1}) = 0$ and hence $p^{s-1} g_{t+1} = 0$.

To conclude we use the last property recursively beginning from the fact that $p^r g_i = 0$ for any $i \in \{1, \ldots, \ell\}$. □

Hence the transformation $A : G^\ell \to G^\ell$, $A\vec{g} = \vec{h}$, with $\vec{g}, \vec{h} \in G^\ell$ and matrix A verifying condition (H′) is a one-to-one and onto transformation. In what follows we identify $\mathbb{P}_{\underline{w}}$ with $\mathbb{P}$.

Proof of Lemma 4.1.

(a) For any increasing sequence $R = (r_k : k \in \mathbb{N})$ we put

$$\tau^R = \inf\{k \in \mathbb{N} : \mathbf{N}(r_k) = 1\}, \quad \text{where } \infty = \inf \phi,$$

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the first time that some element of the sequence R belongs to the renewal process $\mathbf{N}$ introduced in §2. Consider R^* the subsequence corresponding to mappings ψ_r^R such that $a_r^R \neq 0 \bmod p$. We denote by $n^* = n(R^*)$, $\tau^* = \tau^{R^*}$, and $f = f_{R^*}$ the corresponding index functions. First we prove

$$\mathbb{P}\{S_n^R = g \mid \tau^* \leq n^*\} = q^{-1}.$$

To see that, write

$$\begin{aligned}
&\mathbb{P}\{S_n^R = g \mid \tau^* \leq n^*\} \\
&\quad = \sum_{k=0}^{n^*} \mathbb{P}\{S_n^R = g \mid \tau^* = k\} \\
&\quad = \sum_{k=0}^{n^*} \mathbb{P}\left\{ \sum_{i=0}^{f(k)-1} \psi_{r_i}(x_{r_i}) + \psi_{r_{f(k)}}(U_{r_{f(k)}}) + \sum_{i=f(k)+1}^{n} \psi_{r_i}(x_{r_i}) = g \;\middle|\; \tau^* = k \right\} \\
&\quad = \sum_{k=0}^{n^*} \sum_{g_1, g_2 \in G} \mathbb{P}\left\{ \sum_{i=0}^{f(k)-1} \psi_{r_i}(x_{r_i}) = g_1, U_{r_{f(k)}} = \psi_{r_{f(k)}}^{-1}(g - g_1 - g_2), \right. \\
&\qquad \left. \sum_{i=f(k)+1}^{n} \psi_{r_i}(x_{r_i}) = g_2 \;\middle|\; \tau^* = k \right\} \\
&\quad = q^{-1} \sum_{k=0}^{n^*} \sum_{g_1, g_2 \in G} \mathbb{P}\left\{ \sum_{i=0}^{f(k)-1} \psi_{r_i}(x_{r_i}) = g_1, \sum_{i=f(k)+1}^{n} \psi_{r_i}(x_{r_i}) = g_2 \;\middle|\; \tau^* = k \right\} \\
&\quad = q^{-1} \sum_{k=1}^{n^*} \mathbb{P}\{\tau^* = k\} = q^{-1} \mathbb{P}\{\tau^* \leq n^*\}
\end{aligned}$$

where in the last equalities we have used that $U_{r_{f(k)}}$ is independent of the variables $(x_n : n \neq r_{f(k)})$ when $\tau^* = k$. Then,

$$\mathbb{P}\{S_n^R = g\} = q^{-1}\mathbb{P}\{\tau^* \leq n^*\} + \mathbb{P}\{S_n^R = g \mid \tau^* > n^*\}$$

and

$$\mathbb{P}\{S_n^R = g\} - q^{-1} = -q^{-1}\mathbb{P}\{\tau^* > n^*\} + \mathbb{P}\{S_n^R = g \mid \tau^* > n^*\}.$$

Using Lemma 3.1 we get

$$|\mathbb{P}\{S_n^R = g\} - q^{-1}| \leq 2\mathbb{P}\{\tau^* > n^*\} \leq 2\varepsilon(n^* + 1).$$

(b) Let $R^J = (R^j : j \in J)$ be a family of sequences, $(a^{R^j} : j \in J)$ be the family of non-negative sequences, $(\psi^{R^j} : j \in J)$ be the corresponding family of mappings, and $\tilde{R}^J$ be a family of subsequences verifying conditions (H1), (H2), and (H3). Denote $f_j = f_{\tilde{R}^j}$ and $\tau_j = \tau^{\tilde{R}^j}$ for any $j \in J$. Fix $n \in \mathbb{N}$ and put $\tilde{n} = \tilde{n}(\tilde{R}^J)$.

Take a vector $\vec{k} = (k_j : j \in J) \in \{1, \ldots, \tilde{n}\}^J$. On the set $\{\tau_j = k_j : j \in J\}$ we define the random variables

$$\rho_j(\vec{k}, n, \underline{U}) = \sum_{i \in J} \mathbf{1}\{\tilde{r}_{k_i}^i \in R_n^j\} \psi_{f_i(k_i)}(U_{f_i(k_i)}), \quad \text{for } j \in J.$$

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Consider $(g'_j : j \in J) \in G^J$. From hypotheses (H1), (H2), and (H3) the system of linear equations $\rho_j(\vec{k}, n, \underline{U}) = g'_j$, $j \in J$, defines a system of type (S). Then, by Lemma 4.2, there is a unique $(g''_j : j \in J) \in G^J$ such that

$$\rho_j(\vec{k}, n, \underline{U}) = g'_j,\ \ j \in J \Leftrightarrow U_{f_j(k_j)} = g''_j,\ \ j \in J. \tag{4.3}$$

Let $T(\vec{k}) = (\bigcup_{j\in J} R_n^j) \setminus \{f_j(k_j) : j \in J\}$. It is easy to see that variables $(S_n^{R^j} : j \in J)$ on $\{\tau_j = k_j : j \in J\}$ can be written as

$$S_n^{R^j} = \sum_{r\in T(\vec{k})\cap R_n^j} \psi_r(x_r) + \rho_j(\vec{k}, n, \underline{U}).$$

Therefore,

$$\begin{aligned}
&\mathbb{P}\{S_n^{R^j} = g_j, \tau_j = k_j : j \in J\} \\
&\quad = \sum_{h_r\in G : r\in T(\vec{k})} \mathbb{P}\Bigg\{\rho_j(\vec{k}, n, \underline{U}) = g_j - \sum_{r\in T(\vec{k})\cap R_n^j} \psi_r(h_r), x_r = h_r, \tau_j = k_j \\
&\quad\qquad : j \in J, r \in T(\vec{k})\Bigg\} \\
&\quad = \sum_{h_r\in G : r\in T(\vec{k})} \mathbb{P}\{U_{f_j(k_j)} = g''_j, x_r = h_r, \tau_j = k_j : j \in J, r \in T(\vec{k})\},
\end{aligned}$$

where $(g''_j : j \in J) \in G^J$ is given by property (4.3). By independence we conclude that

$$\mathbb{P}\{S_n^{R^j} = g_j, \tau_j = k_j : j \in J\} = q^{-|J|}\mathbb{P}\{\tau_j = k_j : j \in J\}.$$

Hence

$$\mathbb{P}\Big\{S_n^{R^j} = g_j : j \in J, \max_{j\in J} \tau_j \le \tilde{n}\Big\} = q^{-|J|}\mathbb{P}\Big\{\max_{j\in J} \tau_j \le \tilde{n}\Big\},$$

which together with Lemma 3.1 allows us to deduce that

$$|\mathbb{P}\{S_n^{R^j} = g_j : j \in J\} - q^{-|J|}| \le 2\mathbb{P}\Big\{\max_{j\in J} \tau_j > \tilde{n}\Big\} \le 2|J|\varepsilon(\tilde{n} + 1). \qquad \square$$

Now we can give the proof of the main theorem.

Proof of Theorem 1.3. First, let us introduce some notation. The p-expansion of $m \in \mathbb{N}$ is $m = \sum_{i\ge 0} m_i p^i$ with $m_i \in \mathbb{Z}_p$. We denote by $\mathcal{I}(m) = \{i \in \mathbb{N} : m_i \ne 0\}$ its support and we denote its elements in decreasing order, $\mathcal{I}(m) = \{\delta_{1,m} > \cdots > \delta_{s_m,m}\}$, where $s_m = |\mathcal{I}(m)|$. Now put $m^{(i)} = m_{\delta_{i,m}}$, so $m = \sum_{i=1}^{s_m} m^{(i)} p^{\delta_{i,m}}$. Observe that $\delta_{1,m} =$ integer part $(\log m)$, where we take $\log m$ in base p.

Since p is a prime number, the Lucas' theorem [**Lu**] asserts that

$$\left[\binom{m}{k}\right]_p = \left[\prod_{i\ge 0} \binom{m_i}{k_i}\right]_p,$$

where $\binom{r}{s} = 0$ if $r < s$. In particular, $\left[\binom{m}{k}\right]_p > 0$ if and only if $k_i \le m_i$ for all $i \ge 0$.

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Let us return to the automaton φ. Since G is Abelian, a simple recurrence implies

$$(\varphi^m x)_i = \sum_{k\le m} \binom{m}{k} \mu^{m-k} \nu^k x_{k+i}.$$

Observe that this expression has the form of variables S_n^R defined before. In this case the mapping has the shape $\binom{m}{k}\mu^{m-k}\nu^k\ g$ which is one-to-one if $\left[\binom{m}{k}\right]_p \neq 0$ since μ and ν are coprime to p. Then our computations are devoted to showing that we have enough one-to-one mappings.

In order to clarify the proof we shall first prove that the Cesàro mean of the marginal distribution exists and it is uniform, that means

$$\pi(g) \doteq \lim_{M\to\infty} \frac{1}{M} \sum_{m=0}^{M-1} \mathbb{P}_{\underline{w}}\{(\varphi^m x)_0 = g\} \text{ exists and verifies } \pi(g) = q^{-1}, \text{ for any } g \in G.$$

Let us fix $\alpha \in (0, \frac{1}{2})$. For $M > 0$ consider the set $\mathcal{R}_M = \{m \le M : |\mathcal{I}(m)| \ge \alpha \log\log M\}$. We will prove that $(\mathcal{R}_M : M \in \mathbb{N})$ is a sequence of sets of density one, which means $|\{m \le M\} \setminus \mathcal{R}_M|/M \xrightarrow[M\to\infty]{} 0$. For that purpose we make the decomposition $\{m \le M\} = \bigcup_{1\le s\le s_M+1} A_{s,M}$ with

$$\begin{aligned}
A_{1,M} &= \{m \le M : \delta_{1,m} < \delta_{1,M}\},\\
A_{s,M} &= \{m \le M : \delta_{r,m} = \delta_{r,M} \text{ for } r < s \text{ and } \delta_{s,m} < \delta_{s,M}\} \quad \text{for } 1 \le s \le s_M,\\
A_{s_M+1,M} &= \{M\}.
\end{aligned}$$

Observe that $|A_{s,M}| = M^{(s)} p^{\delta_{s,M}}$ for $1 \le s \le s_M$. Take $s_M^* = \sup\{s : \delta_{s,M} \ge \log\log M\}$. Since $\delta_{1,M} =$ integer part $(\log M)$, we have $s_M^* \ge 1$. Now,

$$\begin{aligned}
|\{m \in A_{s,M} : |\mathcal{I}(m)| \le \alpha\delta_{s,M}\}| &\le \sum_{t\le\alpha\delta_{s,M}} (p-1)^t \binom{\delta_{s,M}}{t}\\
&\le (p-1)^{\alpha\delta_{s,M}} 2^{\delta_{s,M}} e^{-2(\alpha-\frac{1}{2})^2\delta_{s,M}}.
\end{aligned}$$

Hence,

$$|\{m \le M\} \setminus \mathcal{R}_M| \le \sum_{1\le s\le s_M^*} (2(p-1)^\alpha)^{\delta_{s,M}} e^{-2(\alpha-\frac{1}{2})^2\delta_{s,M}} + \sum_{s_M^* < s \le s_M} M^{(s)} p^{\delta_{s,M}} + 1.$$

We have

$$\sum_{s_M^* < s\le s_M} M^{(s)} p^{\delta_{s,M}} + 1 \le (\log M)^2 + 1.$$

Take $\alpha < \frac{1}{2} p (\log(p-1))^{-1}$, then $p' \doteq 2(p-1)^\alpha e^{-2(\alpha-\frac{1}{2})^2} < p$. Therefore,

$$\begin{aligned}
\frac{1}{M} \sum_{1\le s\le s_M^*} (2(p-1)^\alpha)^{\delta_{s,M}} e^{-2(\alpha-\frac{1}{2})^2\delta_{s,M}} &\le \frac{1}{M} \sum_{1\le s\le s_M^*} p'^{\delta_{s,M}}\\
&\le \sum_{1\le s\le s_M^*} \left(\frac{p'}{p}\right)^{\delta_{s,M}} \le \frac{p}{p-p'} \left(\frac{p'}{p}\right)^{\log\log M}.
\end{aligned}$$

Hence $|\{m \leq M\} \setminus \mathcal{R}_M|/M \longrightarrow_{M\to\infty} 0$. So $(\mathcal{R}_M : M \in \mathbb{N})$ is a sequence of sets of density one. Hence,

$$\pi(g) = \lim_{M\to\infty} \frac{1}{M} \sum_{m\in\mathcal{R}_M} \frac{1}{M}\mathbb{P}_{\underline{w}}\{(\varphi^m x)_0 = g\}$$
$$= \lim_{M\to\infty} \frac{1}{M} \sum_{m\in\mathcal{R}_M} \mathbb{P}_{\underline{w}}\bigg\{\sum_{k\leq m}\binom{m}{k}\mu^{m-k}\nu^k x_k = g\bigg\}.$$

From the remark, $\binom{m}{k} \neq 0 \bmod p$ implies that the mapping $\psi(g) = \binom{m}{k}\mu^{m-k}\nu^k g$ is one-to-one. Therefore, from Lucas' theorem and Lemma 4.1(a) we get that for any $m \in \mathcal{R}_M$

$$\bigg|\bigg\{k \leq m : \binom{m}{k} \bmod p \neq 0\bigg\}\bigg| \geq 2^{\alpha \log\log M}$$

and then

$$\bigg|\mathbb{P}_{\underline{w}}\bigg\{\sum_{k\leq m}\binom{m}{k}\mu^{m-k}\nu^k x_k = g\bigg\} - q^{-1}\bigg| \leq \varepsilon_1(2^{\alpha\log\log M}).$$

Then $\pi(g) = q^{-1}$.

Now we are ready to prove the result. Notice that for every $(g_j : j < s) \in G^s$ there exists a $(g'_j : j < s) \in G^s$ such that

$$\{x \in G^{\mathbb{N}} : (\varphi^n x)_j = g_j \text{ for } j < s\} = \{x \in G^{\mathbb{N}} : (\varphi^{n+j} x)_0 = g'_j \text{ for } j < s\}.$$

Then it suffices to show that for any finite set $J \subseteq \mathbb{N}$ with $0 \in J$ and $(g_j : j \in J) \in G^J$ it is verified that

$$\lim_{M\to\infty} \frac{1}{M} \sum_{m\leq M} \mathbb{P}_{\underline{w}}\{(\varphi^{m+j}x)_0 = g_j : j \in J\} = q^{-|J|}.$$

Introduce the following notation. We put $G_m = |\{n \leq \delta_{1,m} : m_n < p-1\}|$ and we denote

$$\{n \leq \delta_{1,m} : m_n < p-1\} = \{\beta_{1,m} < \beta_{2,m} < \cdots < \beta_{G_m,m}\}.$$

Fix $\alpha \in (0, \frac{1}{2})$, $\varepsilon \in (0, \alpha)$, and $\varepsilon' \in (0, \frac{1}{2}(\alpha - \varepsilon))$. Denote $\ell = \max J$ and define

$$\mathcal{R}'_M = \{m \leq M : \log(2(\ell+1)) \leq G_m \text{ and } \beta_{[\log 2(\ell+1)],m} \leq \varepsilon \log\log M\}$$
$$\mathcal{R}''_M = \{m \leq M : \delta_{1,m} > \varepsilon\log\log M, |\mathcal{I}(m) \cap \{\varepsilon\log\log M \leq n \leq \delta_{1,m}\}| \geq \varepsilon'\log\log M\}.$$

Both families of sets $(\mathcal{R}'_M : M \in \mathbb{N})$ and $(\mathcal{R}''_M : M \in \mathbb{N})$ are of density one.

Now for any family of sets $(\tilde{\mathcal{R}}_M : M \in \mathbb{N})$ with $\tilde{\mathcal{R}}_M \subseteq \{m \leq M\}$, we put $\tilde{\mathcal{R}}_{M,J} = \{m \leq M : m + j \in \tilde{\mathcal{R}}_M \text{ for } j \in J\}$. If $(\tilde{\mathcal{R}}_M : M \in \mathbb{N})$ is of density one then also $(\tilde{\mathcal{R}}_{M,J} : M \in \mathbb{N})$ is of density one. Hence $(\mathcal{R}_{M,J} : M \in \mathbb{N})$, $(\mathcal{R}'_{M,J} : M \in \mathbb{N})$, and $(\mathcal{R}''_{M,J} : M \in \mathbb{N})$ are sequences of density one.

Let $m \in \mathcal{R}'_{M,J} \cap \mathcal{R}''_{M,J}$. We denote $\mathcal{I}_+(m+j) = \mathcal{I}(m+j) \cap \{n > \varepsilon\log\log M\}$, and $\mathcal{I}_-(m+j) = \mathcal{I}(m+j) \cap \{n \leq \varepsilon\log\log M\}$. From the definition of $\mathcal{R}'_M$ we have that $\mathcal{I}_+(m+j) = \mathcal{I}_+(m)$ for $j \in J$. Put $\mathcal{C}_+(m+j) = \{(m+j)_i : i \in \mathcal{I}_+(m+j)\}$ and $\mathcal{C}_-(m+j) = \{(m+j)_i : i \in \mathcal{I}_-(m+j)\}$ for $j \in J$. We have $\mathcal{C}_+(m+j) = \mathcal{C}_+(m)$ for $j \in J$, and the sets $(\mathcal{C}_-(m+j) : j \in J)$ are all different between them. Define for $j \in J$

$$\tilde{\mathcal{R}}^j = \{k \leq m+j : \mathcal{I}(k) \subseteq \mathcal{I}(m+j), k_i \leq m_i \text{ for } i \in \mathcal{I}_+(m),$$
$$k_i = (m+j)_i \text{ for } i \in \mathcal{I}_-(m+j)\}.$$

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The family $(\tilde{\mathcal{R}}^j : j \in J)$ is disjoint because the sets $(\mathcal{C}_-(m+j) : j \in J)$ are different. Moreover, $|\tilde{\mathcal{R}}^j| \geq 2^{\varepsilon' \log\log M}$.

From Lemma 4.1(b) and the remark we get the result. In fact, for every $m \in \mathcal{R}'_{M,J} \cap \mathcal{R}''_{M,J}$ and $j \in J$ we have that

$$(\varphi^{m+j}x)_0 = \sum_{k=0}^{m+j} \binom{m+j}{k} \mu^{m+j-k} \nu^k x_k$$

and the sequences $(\tilde{\mathcal{R}}^j : j \in J)$ satisfy conditions (H1), (H2), and (H3). Indeed, property (H1) follows from $\tilde{\mathcal{R}}^j \subset \{k \leq m+j : \binom{m+j}{k} \bmod p > 0\}$, since they are disjoint (H2) holds, and if $k \in \tilde{R}^j$ then $\binom{m+j'}{k} \bmod p = 0$ for every $j' < j$ in J which shows property (H3). Then, from Lemma 4.1(b), for any such m

$$|\mathbb{P}_{\underline{w}}\{x : (\varphi^{m+j}x)_0 = g_j, j \in J\} - q^{-|J|}| \leq \varepsilon_J(2^{\varepsilon' \log\log M}).$$

Then the theorem is shown. □

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