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An Extremal Graph Problem with a Transcendental Solution

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We prove that the number of multigraphs with vertex set $\{1,\dots,n\}$ such that every four vertices span at most nine edges is $a^{n^2+o(n^2)}$ where a is transcendental (assuming Schanuel's conjecture from number theory). This is an easy consequence of the solution to a related problem about maximizing the product of the edge multiplicities in certain multigraphs, and appears to be the first explicit (somewhat natural) question in extremal graph theory whose solution is transcendental. These results may shed light on a question of Razborov, who asked whether there are conjectures or theorems in extremal combinatorics which cannot be proved by a certain class of finite methods that include Cauchy–Schwarz arguments.

Our proof involves a novel application of Zykov symmetrization applied to multigraphs, a rather technical progressive induction, and a straightforward use of hypergraph containers.

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1. Introduction

All logarithms in this paper are natural logarithms unless the base is explicitly written. Given a set X and a positive integer t, let

$$\binom{X}{t} = \{Y \subseteq X : |Y| = t\}.$$

A *multigraph* is a pair (V,w), where V is a set of vertices and

$$w : \binom{V}{2} \to \mathbb{N} = \{0,1,2,\dots\}.$$

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Definition 1.1. Given integers $s \geqslant 2$ and $q \geqslant 0$, a multigraph (V,w) is an (s,q)-*graph* if for every $X \in \binom{V}{s}$ we have

$$\sum_{xy\in\binom{X}{2}} w(xy) \leqslant q.$$

An (n,s,q)-graph is an (s,q)-graph with n vertices, and $F(n,s,q)$ is the set of (n,s,q)-graphs with vertex set $[n] := \{1,\ldots,n\}$.

The main goal of this paper is to prove that the maximum product of the edge multiplicities over all $(n,4,15)$-graphs is

$$2^{\gamma n^2+O(n)}, \tag{1.1}$$

where

$$\gamma = \frac{\beta^2}{2} + \beta(1-\beta)\frac{\log 3}{\log 2} \quad \text{and} \quad \beta = \frac{\log 3}{2\log 3 - \log 2}.$$

It is an easy exercise to show that both β and γ are transcendental by the Gelfond–Schneider theorem [9]. Using (1.1), we will prove that

$$|F(n,4,9)| = a^{n^2+o(n^2)}, \tag{1.2}$$

where $a = 2^\gamma$ is also transcendental (assuming Schanuel's conjecture in number theory).

Many natural extremal graph problems involving edge densities have rational solutions, and their enumerative counterparts have algebraic solutions. Specifically, for a set $\mathcal{F}$ of finite graphs, the number of $\mathcal{F}$-free graphs on $[n]$ is $2^{(\pi(\mathcal{F})+o(1))\binom{n}{2}}$, where $\pi(\mathcal{F})$ is the Turán density of $\mathcal{F}$ (which is rational due to the Erdős–Simonovits–Stone theorem [5, 6]). For example, the Erdős–Kleitman–Rothschild theorem [4] states that the number of triangle-free graphs on $[n]$ is $2^{n^2/4+o(n^2)}$ and $2^{1/4}$ is algebraic since $1/4$ is rational. For hypergraphs the situation is more complicated, and the first author and Talbot [13] proved that certain partite hypergraph Turán problems have irrational solutions (meaning irrational values of an analogue of Turán density). Going further, the question of obtaining transcendental solutions for natural extremal problems is an intriguing one. This was perhaps first explicitly posed by Fox (see [17]) in the context of Turán densities of hypergraphs. Pikhurko [17] showed that the set of hypergraph Turán densities is uncountable, thereby proving the existence of transcendental ones (see also [10]), but his list of forbidden hypergraphs was infinite. When only finitely many hypergraphs are forbidden, he obtained irrational densities. To our knowledge, (1.1) and (1.2) are the first examples of fairly natural extremal graph problems whose answer is given by (explicitly defined) transcendental numbers (modulo Schanuel's conjecture in the case of (1.2)).

Another area that (1.1) may shed light on is the general question of whether certain proof methods suffice to solve problems in extremal combinatorics. The explosion of results in extremal combinatorics using flag algebras [18] in recent years has put the spotlight on such questions, and Razborov first posed this in Question 1 of [18]. A significant result in this direction is due to Hatami and Norine [11]. They prove that the related question (due in different forms to Razborov, Lovász and Lovász–Szegedy) of whether every true linear inequality between homomorphism densities can be proved using a finite amount of manipulation with homomorphism densities

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of finitely many graphs is not decidable. While we will not attempt to state Question 1 of [18] rigorously here, its motivation is to understand whether the finite methods that are typically used in combinatorial proofs of extremal results (formalized by flag algebras and the Cauchy–Schwarz calculus) suffice for all extremal problems involving subgraph densities. Although we cannot settle this, one might speculate that these finite 'Cauchy–Schwarz methods' may not be enough to obtain (1.1). In any event, (1.1) seems to be a good test case. Curiously, our initial explorations into (1.1) were through flag algebra computations which gave the answer to several decimal places and motivated us to obtain sharp results, though our eventual proof of (1.1) uses no flag algebra machinery. Instead, it uses some novel extensions of classical methods in extremal graph theory, and we expect that these ideas will be used to solve other related problems.

As remarked earlier, (1.2) is a fairly straightforward consequence of (1.1) and since the expression in (1.1) is obtained as a product (rather than sum) of numbers, it is easier to obtain a transcendental number in this way. However, we should point out that an extremal example for (1.1) (and possibly all extremal examples, though we were not able to show this) involves partitioning the vertex set $[n]$ into two parts where one part has size approaching βn, and β is also transcendental (see Definition 2.3 and Theorem 2.4 in the next section). This might indicate the difficulty in proving (1.1) using the sort of finite methods discussed above.

Finally, we would like to mention that the problem of asymptotic enumeration of (n,s,q)-graphs is a natural extension of the work on extremal problems related to (n,s,q)-graphs by Bondy and Tuza in [3] and by Füredi and Kündgen in [8]. Further work in this direction, including a systematic investigation of extremal, stability and enumeration results for a large class of pairs (s,q), will appear in [16] (see also [14] for another example on multigraphs). Alon [1] asked whether the transcendental behaviour witnessed here is an isolated case. Although we believe that there are infinitely many such examples (see Conjecture 6.3 in Section 6) we were not able to prove this for any other pair (s,q). The infinitely many pairs for which we obtain precise extremal results in [16] have either rational or integer densities.

2. Results

Given a multigraph $G=(V,w)$, define

$$P(G) = \prod_{xy \in \binom{V}{2}} w(xy) \quad \text{and} \quad (G) = \sum_{xy \in \binom{V}{2}} w(xy).$$

Given $X \subseteq [n]$, define

$$S^G(X) = \sum_{xy \in \binom{X}{2}} w(xy).$$

We will omit the superscript when it is clear from the context.

Definition 2.1. Suppose $s \geqslant 2$ and $q \geqslant 0$ are integers. Define

$$\mathrm{ex}_\Pi(n,s,q) = \max\{P(G) : G \in F(n,s,q)\} \quad \text{and} \quad \mathrm{ex}_\Pi(s,q) = \lim_{n\to\infty} (\mathrm{ex}_\Pi(n,s,q))^{1/\binom{n}{2}}.$$

An (n,s,q)-graph G is *product-extremal* if $P(G) = \mathrm{ex}_\Pi(n,s,q)$. The limit $\mathrm{ex}_\Pi(s,q)$ (which we will show always exists) is called the *asymptotic product density*.

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The following result is an enumeration theorem for (n,s,q)-graphs in terms of $\mathrm{ex}_\Pi(s,q+\binom{s}{2})$.

Theorem 2.2. *Suppose $s \geqslant 2$ and $q \geqslant 0$ are integers. If $\mathrm{ex}_\Pi(s,q+\binom{s}{2}) > 1$, then*

$$\mathrm{ex}_\Pi\left(s,q+\binom{s}{2}\right)^{\binom{n}{2}} \leqslant |F(n,s,q)| \leqslant \mathrm{ex}_\Pi\left(s,q+\binom{s}{2}\right)^{(1+o(1))\binom{n}{2}},$$

and if $\mathrm{ex}_\Pi(s,q+\binom{s}{2}) \leqslant 1$, then $|F(n,s,q)| \leqslant 2^{o(n^2)}$.

Theorem 2.2 follows from what are now standard arguments employing the hypergraph containers method of [2, 19], and it is also a special case of Theorem 3 of [20] as well as Theorem 2.12 of [7]. For these reasons we omit the proof here, and direct the reader to [15] where the proof appears in detail. Theorem 2.2 reduces the problem of enumerating $F(n,4,9)$ to computing $\mathrm{ex}_\Pi(4,15)$. This will be the focus of the remainder of this paper.

Definition 2.3. Given n, let $W(n)$ be the set of multigraphs $G=([n],w)$ for which there is a partition L,R of $[n]$ such that $w(xy)=1$ if $xy \in \binom{L}{2}$, $w(xy)=2$ if $xy \in \binom{R}{2}$, and $w(xy)=3$ if $(x,y) \in L \times R$.

We now make a few observations about $W(n)$. First, note that $W(n) \subseteq F(n,4,15)$ for all $n \in \mathbb{N}$. Given $G \in W(n)$ with corresponding partition L,R of $[n]$, a 4-set $X \in \binom{[n]}{4}$ spans exactly 15 edges in G if and only if $1 \leqslant |X \cap L| \leqslant 2$. Straightforward calculus shows that for $G \in W(n)$, the product $P(G)$ is maximized when $|R| \approx \beta n$, where $\beta = \log 3/(2\log 3 - \log 2)$ is a transcendental number. This optimization ensures that every vertex contributes roughly the same amount to the product. Indeed, if R has size about xn, the vertices in R contribute $p = 2^{xn-1}3^{(1-x)n}$, and vertices in L contribute 3^{xn}. Making these contributions roughly equal requires $x\log 2 + (1-x)\log 3 = x\log 3$, which, when solved, yields that $x=\beta$. As β is transcendental, this might indicate the difficulty of obtaining this extremal construction using a standard induction argument. Given a family of hypergraphs $\mathcal{F}$, write $\mathcal{P}(\mathcal{F})$ for the set of $G \in \mathcal{F}$ with $P(G) = \max\{P(G') : G' \in \mathcal{F}\}$. We use the shorthand $\mathcal{P}(n,s,q)$ for $\mathcal{P}(F(n,s,q))$.

Theorem 2.4. *For all sufficiently large n, $\mathcal{P}(W(n)) \subseteq \mathcal{P}(n,4,15)$. Consequently*

$$\mathrm{ex}_\Pi(n,4,15) = \max_{G \in W(n)} P(G) = 2^{\gamma n^2 + O(n)} \quad \textit{and} \quad \mathrm{ex}_\Pi(4,15) = 2^{2\gamma},$$

where

$$\gamma = \frac{\beta^2}{2} + \beta(1-\beta)\log_2 3 \quad \textit{and} \quad \beta = \frac{\log 3}{2\log 3 - \log 2}.$$

For reference, $\beta \approx 0.73$ and $2^\gamma \approx 1.49$. The result below follows directly from Theorems 2.2 and 2.4.

Theorem 2.5. $|F(n,4,9)| = 2^{\gamma n^2 + o(n^2)}$.

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Proof. Theorem 2.4 implies $\mathrm{ex}_\Pi(4,15) = 2^{2\gamma} > 1$. By Theorem 2.2, this implies that

$$\mathrm{ex}_\Pi(4,15)^{\binom{n}{2}} \leqslant |F(n,4,9)| \leqslant \mathrm{ex}_\Pi(4,15)^{(1+o(1))\binom{n}{2}}.$$

Consequently, $|F(n,4,9)| = 2^{2\gamma\binom{n}{2}+o(n^2)} = 2^{\gamma n^2+o(n^2)}$. □

Recall that Schanuel's conjecture from the 1960s (see [12]) states the following: if $z_1,\dots,z_n$ are complex numbers which are linearly independent over $\mathbb{Q}$, then $\mathbb{Q}(z_1,\dots,z_n,e^{z_1},\dots,e^{z_n})$ has transcendence degree at least n over $\mathbb{Q}$. As promised in the Introduction and abstract, we now show that assuming Schanuel's conjecture, 2^γ is transcendental. Observe that this implies $\mathrm{ex}_\Pi(4,15) = 2^{2\gamma}$ is also transcendental over $\mathbb{Q}$, assuming Schanuel's conjecture.

Proposition 2.6. *Assuming Schanuel's conjecture, 2^γ is transcendental.*

Proof. Assume Schanuel's Conjecture holds. It is well known that Schanuel's conjecture implies $\log 2$ and $\log 3$ are algebraically independent over $\mathbb{Q}$ (see for instance [21]). Observe $\gamma = f(\log 2,\log 3)/g(\log 2,\log 3)$ where $f(x,y) = xy^2/2 + y^2(y-x)$ and $g(x,y) = x(2y-x)^2$. Note that the coefficient of x^3 in $f(x,y)$ is 0 while in $g(x,y)$ it is 1. We now show $\log 2,\log 3,\gamma\log 2$ are linearly independent over $\mathbb{Q}$. Suppose for a contradiction that this is not the case. Then there are non-zero rationals p,q,r such that

$$p\log 2 + q\log 3 + r\gamma\log 2 = 0.$$

Replacing γ with $f(\log 2,\log 3)/g(\log 2,\log 3)$, this implies

$$p\log 2 + q\log 3 + r\frac{f(\log 2,\log 3)}{g(\log 2,\log 3)}\log 2 = 0.$$

By clearing the denominators of p,q,r and multiplying by $g(\log 2,\log 3)$, we obtain that there are integers a,b,c, not all zero, such that

$$(a\log 2 + b\log 3)g(\log 2,\log 3) + cf(\log 2,\log 3)\log 2 = 0.$$

Let $p(x,y) = (ax+by)g(x,y) + cf(x,y)x$. Observe that $p(x,y)$ is a rational polynomial such that $p(\log 2,\log 3) = 0$. Since the coefficient of x^3 is 1 in $g(x,y)$ and 0 in $f(x,y)$, the coefficient of x^4 in $p(x,y)$ is $a \neq 0$. Thus $p(x,y)$ has at least one non-zero coefficient, contradicting that $\log 2$ and $\log 3$ are algebraically independent over $\mathbb{Q}$. Thus $\log 2,\log 3,\gamma\log 2$ are linearly independent over $\mathbb{Q}$, so Schanuel's conjecture implies $\mathbb{Q}(\log 2,\log 3,\gamma\log 2,2^\gamma)$ has transcendence degree at least 3 over $\mathbb{Q}$. Suppose towards a contradiction that 2^γ is not transcendental. Then $\log 2,\log 3,\gamma\log 2$ must be algebraically independent over $\mathbb{Q}$. Let $h(x,y,z) = zg(x,y) - xf(x,y)$. Then it is clear $h(x,y,z)$ has non-zero coefficients, and

$$h(\log 2,\log 3,\gamma\log 2) = (\gamma\log 2)g(\log 2,\log 3) - (\log 2)f(\log 2,\log 3) = 0,$$

where the second equality uses the fact that $\gamma = f(\log 2,\log 3)/g(\log 2,\log 3)$. But this implies $\log 2,\log 3,\gamma\log 2$ are algebraically dependent over $\mathbb{Q}$, a contradiction. Thus 2^γ is transcendental. □

3. Extremal result for $(n,4,15)$-graphs: a two-step reduction

In this section we reduce Theorem 2.4 to two stepping-stone theorems, Theorems 3.2 and 3.3 below. These theorems focus on understanding the structure of $(4,15)$-graphs which are product-extremal subject to certain constraints. Given a set $\mathcal{F}$ of multigraphs, recall that

$$\mathcal{P}(\mathcal{F}) = \{G \in \mathcal{F} : P(G) \geqslant P(G') \text{ for all } G' \in \mathcal{F}\} \quad \text{and} \quad \mathcal{P}(n,4,15) = \mathcal{P}(F(n,4,15)).$$

The first step, Theorem 3.2, shows that a product extremal $(n,4,15)$-graph cannot have a triangle spanning more than eight edges or an edge of multiplicity greater than 3. This idea behind the proof (Section 5) is as follows. If $G \in F(n,4,15)$ contains a triangle supporting at least nine edges, then the contribution of its three vertices to the total product is at most $27 \cdot 2^{3(n-3)}$, which is much less than $3^{3\beta(n-3)}$ (as one would obtain in an optimized element of $W(n)$). Assuming now there are no triangles supporting nine edges in G, an edge of multiplicity 4 could contribute at most $4 \cdot 2^{2(n-2)}$ to the total product, which is much less than $3^{2\beta(n-2)}$ (again, as one would obtain in an optimized element of $W(n)$). Therefore, since these bad configurations imply the existence of vertices with small product-degree, they cannot exist in any product-extremal element of $F(n,4,15)$. We now state this result precisely.

Definition 3.1. Given $n \in \mathbb{N}$, define $F_{\leqslant 3}(n,4,15) = \{G \in F(n,4,15) : \mu(G) \leqslant 3\}$ and

$$D(n) = F_{\leqslant 3}(n,4,15) \cap F(n,3,8).$$

Theorem 3.2. *For all sufficiently large n,* $\mathcal{P}(n,4,15) = \mathcal{P}(D(n))$.

The second step, Theorem 3.3, shows there is a product extremal $(n,4,15)$-graph G in both $D(n)$ and $W(n)$.

Theorem 3.3. *For all sufficiently large n,* $\mathcal{P}(D(n)) \cap \mathcal{P}(W(n)) \neq \emptyset$.

The proof of Theorem 3.3 requires several lemmas, and appears in Section 4. We use the rest of this section to prove Theorem 2.4, given Theorems 3.2 and 3.3. Given $G = ([n],w) \in W(n)$, let $L(G)$ and $R(G)$ denote the parts in the partition of $[n]$ such that $w(xy) = 1$ if and only if $xy \in \binom{L(G)}{2}$. Recall the definition of γ from Theorem 2.4.

Lemma 3.4. *For all* $G \in \mathcal{P}(W(n))$, *we have* $P(G) = 2^{\gamma n^2 + O(n)}$.

Proof. Let $G = ([n],w) \in W(n)$. Set $h(y) = 2^{\binom{y}{2}} 3^{y(n-y)}$ and observe that if $|L(G)| = n-y$ and $|R(G)| = y$, then $P(G) = h(y)$. Thus it suffices to show that $\max_{y\in[n]} h(y) = 2^{\gamma n^2+O(n)}$. Basic calculus shows that $h(y)$ has a global maximum at $\tau = \beta n - (\log 2)/(2(2\log 3 - \log 2))$, where $\beta = \log 3/(2\log 3 - \log 2)$ is as in Theorem 2.4. This implies $\max_{y\in\mathbb{N}} h(y) = \max\{h(\lfloor \tau \rfloor), h(\lceil \tau \rceil)\}$. It is straightforward to check $\max\{h(\lfloor \tau \rfloor), h(\lceil \tau \rceil)\} = \max\{h(\lfloor \beta n, \rfloor), h(\lceil \beta n \rceil)\}$. By definition of γ and h, this implies $\max_{y\in[n]} h(y) = 2^{\gamma n^2+O(n)}$. □

Proof of Theorem 2.4. Fix n sufficiently large and $G_1 \in \mathcal{P}(W(n))$. By Theorem 3.3, there is some $G_2 \in \mathcal{P}(D(n)) \cap \mathcal{P}(W(n))$. Since G_1 and G_2 are both in $\mathcal{P}(W(n))$, $P(G_1) = P(G_2)$.

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Our assumption and Theorem 3.2 imply $G_2 \in \mathcal{P}(D(n)) = \mathcal{P}(n,4,15)$, so $P(G_2) = \mathrm{ex}_\Pi(n,4,15)$. Combining these facts yields $P(G_1) = P(G_2) = \mathrm{ex}_\Pi(n,4,15)$, so $G_1 \in \mathcal{P}(n,4,15)$. This shows $\mathcal{P}(W(n)) \subseteq \mathcal{P}(n,4,15)$. Since $G_1 \in \mathcal{P}(n,4,15) \cap \mathcal{P}(W(n))$, Lemma 3.4 implies $\mathrm{ex}_\Pi(n,4,15) = P(G_1) = 2^{\gamma n^2 + O(n)}$. By definition, we have that $\mathrm{ex}_\Pi(4,15) = 2^{2\gamma}$. □

4. Proof of Theorem 3.3

The goal of this section is to prove Theorem 3.3. It will require many reductions and lemmas. The general strategy is to show we can find elements in $\mathcal{P}(D(n))$ with increasingly nice properties, until we can show there is one in $W(n)$.

The proof methods can be viewed as a generalization of Zykov symmetrization to multigraphs. For graphs, Zykov symmetrization allows us to assure that any extremal graph is almost regular, since otherwise we may increase the number of edges. For example, given a triangle-free graph $G = (V,E)$, if $xy \in E$ satisfies $d(y) > d(x)+1$, then one can increase the total number of edges without creating a triangle by deleting x, replacing it with a copy y' of y, then declaring $yy' \notin E$. After repeating this process enough times, one obtains a triangle-free graph where all vertices have roughly the same degree, and where the number of edges is as large as possible. We apply a version of this idea to multigraphs. Specifically, we will show that in certain product-extremal multigraphs, all vertices will have roughly the same *product-degree*.

Definition 4.1. For a multigraph $G = (V,w)$, the *product-degree* of $x \in V$ is

$$p(x) := \prod_{u \in V \setminus \{x\}} w(ux).$$

Specifically, given $G \in F(n,4,15)$, we will replace and duplicate vertices to obtain a new multigraph $G' \in F(n,4,15)$ where all vertices have roughly the same value of $p(x)$ and where $P(G') \geqslant P(G)$. We now give some definitions which will be used in this section.

Given a multigraph $G = (V,w)$, let $\sim_G$ be the binary relation on V defined by $x \sim_G y$ if and only if either $x = y$, or $w(xy) = 1$ and for all $z \in V \setminus \{x,y\}$, $w(xz) = w(yz)$.

Proposition 4.2. *Suppose $G = (V,w)$ is a multigraph. Then $\sim_G$ forms an equivalence relation on V. Moreover, if $\tilde{V} = \{V_1,\ldots,V_t\}$ is the set of equivalence classes of V under $\sim_G$, then for each $i \neq j$, there is $w_{ij} \in \{1,2,3\}$ such that for all $(x,y) \in V_i \times V_j$, $w(xy) = w_{ij}$.*

The proof is straightforward and left to the reader. Given $G = (V,w)$ and $i,j,k \in \mathbb{N}$, an (i,j,k)-*triangle* in G is a set $\{x,y,z\} \in \binom{V}{3}$ such that $\{w(xy),w(yz),w(xz)\} = \{i,j,k\}$. Say that G *omits* (i,j,k)-*triangles* if there is no (i,j,k)-triangle in G.

Definition 4.3. A multigraph G is *neat* if $\mu(G) \leqslant 3$ and G omits (i,j,k)-triangles, for each $(i,j,k) \in \{(1,1,2),(1,1,3),(1,2,3)\}$.

In neat multigraphs, $\sim_G$ is especially well-behaved.

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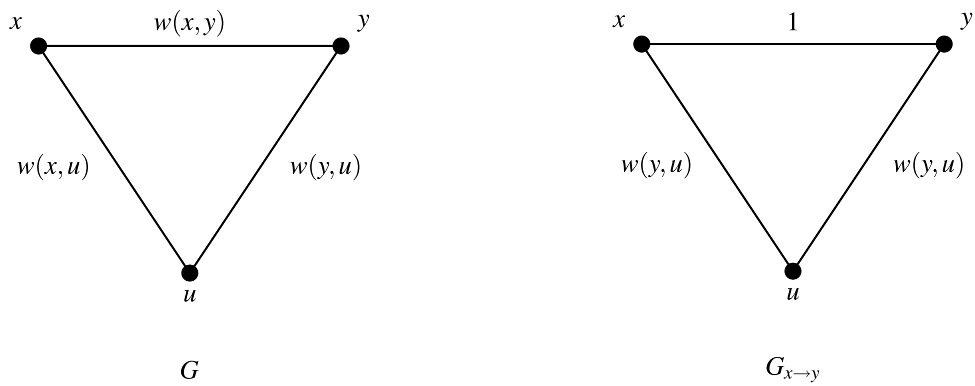

Figure 1. A triangle $\{x,y,u\}$ in G versus $G_{x\to y}$.

Proposition 4.4. *Suppose $G=(V,w)$ is a neat multigraph. Then for all $x,y\in V$, $x\sim_G y$ if and only if $w(xy)=1$. Moreover, if $\tilde{V}=\{V_1,\dots,V_t\}$ is the set of equivalence classes of V under $\sim_G$, then for each $i\neq j$, there is $w_{ij}\in\{2,3\}$ such that for all $(x,y)\in V_i\times V_j$, $w(xy)=w_{ij}$.*

Again, the proof is straightforward and left to the reader.

4.1. Finding a neat multigraph in $\mathcal{P}(D(n))$

Definition 4.5. Suppose $n\geqslant 1$. Define $C(n)$ to be the set of neat multigraphs in $D(n)$, that is,

$$C(n)=\{G\in D(n): G \text{ omits } (i,j,k)\text{-triangles, for each } (i,j,k)\in\{(1,1,2),(1,1,3),(1,2,3)\}\}.$$

Observe that for all n, $W(n)\subseteq C(n)\subseteq D(n)$. The goal of this subsection is to prove Lemma 4.8, which says that for all n, there is a product-extremal element of $D(n)$ which is also in $C(n)$. We begin with some notation.

Suppose $G=(V,w)$ is a multigraph. Given $x\neq y\in V$, define $G_{x\to y}=(V,w')$ to be the multigraph such that

- $G_{x\to y}[V\setminus\{x,y\}]=G[V\setminus\{x,y\}]$,
- $w'(xy)=1$, and
- for all $u\in V\setminus\{x,y\}$, $w'(xu)=w(yu)$.

The idea is that $G_{x\to y}$ is obtained from G by making the vertex x 'look like' the vertex y. See Figure 1 for an illustration of a triangle $\{x,y,u\}$ in G versus $G_{x\to y}$.

Observe that if $G'=G_{x\to y}$, then $x\sim_{G'} y$. This cloning operation affects $P(G)$ as follows. For any $xy\in\binom{V}{2}$,

$$P(G_{x\to y})=\frac{p(y)}{p(x)w(xy)}P(G). \tag{4.1}$$

Lemma 4.6. *Suppose $n\geqslant 1$, $G\in D(n)$, and $uv\in\binom{[n]}{2}$. Then $G_{u\to v}\in D(n)$.*

Proof. Fix $G = ([n],w) \in D(n)$ and let $G' := G_{u\to v} = ([n],w')$. We show $G' \in D(n)$. Given $X \subseteq [n]$, let $S(X) = S^G(X)$ and $S'(X) = S^{G'}(X)$. By definition of $G_{u\to v}$ and because $G \in D(n)$, $\mu(G') \leqslant 3$. We now check that $G' \in F(n,4,15)$. Suppose $X \in \binom{[n]}{4}$. If $u \notin X$, then $S'(X) = S(X) \leqslant 15$. If $X \cap \{u,v\} = \{u\}$, then $S'(X) = S((X \setminus \{u\}) \cup \{v\}) \leqslant 15$. So assume $\{u,v\} \subseteq X$, say $X = \{u,v,z,z'\}$. Because $G \in F(n,3,8)$ and by definition of $G_{u\to v}$, we have that $S'(\{v,z,z'\}) = S(\{v,z,z'\}) \leqslant 8$. Combining this with the facts that $w'(uv) = 1$ and $\mu(G') \leqslant 3$ yields

$$S'(X) = S'(\{v,x,y\}) + w'(uv) + w'(ux) + w'(uy) \leqslant 8+1+3+3 = 15.$$

We now verify that $G' \in F(n,3,8)$. Suppose $X \in \binom{[n]}{3}$. If $u \notin X$, then $S'(X) = S(X) \leqslant 8$. If $X \cap \{u,v\} = \{u\}$, then $S'(X) = S((X \setminus \{u\}) \cup \{v\}) \leqslant 8$. So assume $\{u,v\} \subseteq X$, say $X = \{u,v,z\}$. Because $\mu(G') \leqslant 3$,

$$S'(X) \leqslant w'(uv) + 3 + 3 = 1+3+3 = 7 \leqslant 8.$$

Consequently, $G' \in F_{\leqslant 3}(n,4,15) \cap F(n,3,8) = D(n)$. □

Corollary 4.7. *Suppose $n \geqslant 1$, $G = ([n],w) \in \mathcal{P}(D(n))$ and $uv \in \binom{[n]}{2}$. If $w(uv) = 1$, then $G_{u\to v} \in \mathcal{P}(D(n))$.*

Proof. By Lemma 4.6, $G_{v\to u} \in D(n)$. Combining this with (4.1) and the fact that $G \in \mathcal{P}(D(n))$ yields that

$$P(G_{v\to u}) = \frac{p(u)}{p(v)} P(G) \leqslant P(G).$$

Thus $p(u) \leqslant p(v)$. The symmetric argument shows $p(v) \leqslant p(u)$, so $p(u) = p(v)$. Therefore,

$$P(G_{u\to v}) = \frac{p(v)}{p(u)} P(G) = P(G),$$

which implies $G_{u\to v} \in \mathcal{P}(D(n))$. □

Given $G = (V,w)$, set

$$t_G = \left| \left\{ xy \in \binom{V}{2} : x \sim_G y \right\} \right|.$$

We now prove the main result of this subsection.

Lemma 4.8. *For all $n \geqslant 1$, $\mathcal{P}(D(n)) \cap C(n) \neq \emptyset$. Consequently, $\mathcal{P}(C(n)) \subseteq \mathcal{P}(D(n))$.*

Proof. Fix $G = ([n],w) \in \mathcal{P}(D(n))$ such that t_G is maximal among the elements of $\mathcal{P}(D(n))$. Let $V_1,\dots,V_k$ be the $\sim_G$-classes of G, enumerated so that $|V_1| \leqslant \cdots \leqslant |V_k|$. Suppose there is $i < j$ and $(x,y) \in V_i \times V_j$ such that $w(xy) = 1$. Let $G' = ([n],w') = G_{x\to y}$. By Corollary 4.7, $G' \in \mathcal{P}(D(n))$. It is straightforward to check that the $\sim_{G'}$ classes of G' are $V_1',\dots,V_k'$, where $V_i' = V_i \setminus \{x\}$, $V_j' = V_j \cup \{x\}$ and $V_\ell' = V_\ell$ for all $\ell \in [k] \setminus \{i,j\}$. Consequently, $t_{G'} = t_G + |V_j| - |V_i| + 1 > t_G$, contradicting the maximality of t_G. Thus, for all $i \neq j$ and $(x,y) \in V_i \times V_j$, $w(xy) \neq 1$. By Proposition 4.2, we must have that for all $i \neq j$, either $w(xy) = 2$ for all $(x,y) \in V_i \times V_j$, or $w(xy) = 3$ for all $(x,y) \in$

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$V_i \times V_j$. This implies that G omits (i,j,k)-triangles for each $(i,j,k) \in \{(1,1,2),(1,1,3),(1,2,3)\}$. Thus $G \in \mathcal{P}(D(n)) \cap C(n)$. Combining this with $C(n) \subseteq D(n)$ yields that $\mathcal{P}(C(n)) \subseteq \mathcal{P}(D(n))$. □

4.2. Acyclic multigraphs

We say two multigraphs $G = (V,w)$ and $G' = (V',w)$ are *isomorphic*, denoted $G \cong G'$, if there is a bijection $f : V \to V'$ such that $w(xy) = w'(f(x)f(y))$, for all $xy \in \binom{V}{2}$. We say that $G = (V,w)$ *contains a copy of* G' if there is $X \subseteq V$ such that $G[X] \cong G'$.

Definition 4.9. Given $t \geqslant 3$, define $C_t(3,2)$ to be the multigraph $([t],w)$ such that

$$w(12) = w(23) = \cdots = w((t-1)t) = w(t1) = 3,$$

and $w(ij) = 2$ for all other pairs $i \neq j$. For $n \geqslant 1$, set $NC(n)$ (NC = 'no cycles') to be the set of $G \in C(n)$ which do not contain a copy of $C_t(3,2)$ for any $t \geqslant 3$.

We will show in the next subsection that for large n, all product-extremal elements of $C(n)$ are in $NC(n)$. However, we must first show that we can find product-extremal elements of $NC(n)$ which are 'nice', and this is the goal of this subsection. In particular we will show that for all $n \geqslant 1$, there is a product-extremal element of $NC(n)$ which is also in $W(n)$.

We begin with some notation and definitions. If G contains a copy of $C_t(3,2)$, we will write $C_t(3,2) \subseteq G$, and if not, we will write $C_t(3,2) \nsubseteq G$. A *vertex-weighted graph* is a triple (V,E,f) where (V,E) is graph and $f : V \to \mathbb{N}^{>0}$. We now give a way of associating a vertex-weighted graph to a neat multigraph. Suppose $G = (V,w)$ is a neat multigraph, $\tilde{V} = \{V_1,\ldots,V_t\}$ is the set of equivalence classes of V under $\sim_G$, and for each $i \neq j$, $w_{ij} \in \{2,3\}$ is from Proposition 4.4. Define the *vertex-weighted graph associated with G and* $\sim_G$ to be $\tilde{G} = (\tilde{V},\tilde{E},f)$ where

$$\tilde{E} = \left\{ V_iV_j \in \binom{\tilde{V}}{2} : w_{ij} = 3 \right\}$$

and $f(V_i) = |V_i|$ for all $i \in [t]$. We will use the notation $|\cdot|^G$ to denote this vertex-weight function f, and we will drop the superscript when G is clear from context. If $H = (V,E)$ is a graph and $X \subseteq V$, then let $H[X] = (X, E \cap \binom{X}{2})$ be the subgraph of H induced by X.

Lemma 4.10. *Suppose $n \geqslant 1$ and G is a neat multigraph with vertex set $[n]$. Then $G \in NC(n)$ if and only if $\tilde{G}$ is a forest.*

Proof. Suppose $\tilde{G}$ is not a forest. Then there is $X = \{V_{i_1},\ldots,V_{i_k}\} \subseteq \tilde{V}$ such that $\tilde{G}[X]$ is a cycle of length $k \geqslant 3$. Choose some $y_j \in V_{i_j}$ for each $1 \leqslant i \leqslant k$ and let $Y = \{y_1,\ldots,y_k\}$. Then by definition of $\tilde{G}$, we must have $G[Y] \cong C_k(3,2)$. Thus $G \notin NC(n)$.

On the other hand, suppose $G \notin NC(n)$. Then either $G \notin C(n)$ or $C_t(3,2) \subseteq G$ for some $t \geqslant 3$. Suppose $G \notin C(n)$. Since G is neat, this implies $G \notin D(n)$. Since $\mu(G) \leqslant 3$ implies $G \in F(n,3,8)$, we must have $G \notin F(n,4,15)$. Thus there is some $Y \in \binom{[n]}{4}$ such that $S^G(Y) > 15$. Since $\mu(G) \leqslant 3$, this implies that either

(i) $\{w(xy) : xy \in \binom{Y}{2}\} = \{3,3,3,3,2,2\}$ or

(ii) $\{w(xy) : xy \in \binom{Y}{2}\} = \{3,3,3,3,3,j\}$, some $j \in \{1,2,3\}$.

Let X be the set of equivalence classes intersecting Y, that is, $X = \{V_i \in \tilde{V} : Y \cap V_i \neq \emptyset\}$. In case (i), because Y spans no edges of multiplicity 1 in G, the elements of Y must be in pairwise distinct equivalence classes under $\sim_G$. Thus in $\tilde{G}$, $|X| = 4$ and X spans exactly four edges. This implies $\tilde{G}[X]$ is either a 4-cycle or contains a triangle. In case (ii), if $j = 1$, then $|X| = 3$ and $\tilde{G}[X]$ is a triangle. If $j \neq 1$, then $|X| = 4$ and spans at least five edges. This implies $\tilde{G}[X]$ contains a triangle. Therefore, if $G \notin F(n,4,15)$, then $\tilde{G}$ is not a forest. Suppose now that $C_t(3,2) \subseteq G$, for some $t \geqslant 3$. Then if $X \subseteq [n]$ is such that $G[X] \cong C_t(3,2)$, $\tilde{G}[X]$ is a cycle, so consequently $\tilde{G}$ is not a forest. □

Definition 4.11. Given a vertex-weighted graph $\tilde{G} = (\tilde{V}, E, |\cdot|)$, set

$$f_\pi(\tilde{G}) = \prod_{UV \in E} 3^{|U||V|} \prod_{UV \in \binom{\tilde{V}}{2} \setminus E} 2^{|U||V|}.$$

Note that we have $P(G) = f_\pi(\tilde{G})$ for all $G \in C(n)$.

Two vertex-weighted graphs $G_1 = (V_1, E_1, f_1)$ and $G_2 = (V_2, E_2, f_2)$, are *isomorphic*, denoted $G_1 \cong G_2$, if there is a graph isomorphism $g : V_1 \to V_2$ such that for all $v \in V_1$, $f_1(v) = f_2(g(v))$.

Lemma 4.12. *Suppose $n \geqslant 1$ and $H = (\tilde{V}, E, |\cdot|)$ is a vertex-weighted forest such that*

$$\sum_{V \in \tilde{V}} |V| = n.$$

Then there is a multigraph $G \in NC(n)$ such that $\tilde{G}$ is isomorphic to H.

Proof. Let $\tilde{V} = \{V_1, \ldots, V_t\}$ and for each i, let $x_i = |V_i|$. Since $\sum_{i=1}^t x_i = n$, it is clear there exists a partition $P_1, \ldots, P_t$ of $[n]$ such that for each $i \in [t]$, $|P_i| = x_i$. Fix such a partition $P_1, \ldots, P_t$. Define $G = ([n], w)$ as follows. For each $xy \in \binom{[n]}{2}$, set

$$w(xy) = \begin{cases} 1 & \text{if } xy \in \binom{P_i}{2} \text{ for some } i \in [t], \\ 3 & \text{if } xy \in E(P_i, P_j) \text{ for some } i \neq j \text{ such that } V_iV_j \in E, \\ 2 & \text{if } xy \in E(P_i, P_j) \text{ for some } i \neq j \text{ such that } V_iV_j \notin E. \end{cases}$$

By construction, G is a neat multigraph and $\tilde{G}$ is isomorphic to H. Because $H \cong \tilde{G}$ is a forest, Lemma 4.10 implies $G \in NC(n)$. □

Given a vertex-weighted graph, $H = (\tilde{V}, E, |\cdot|)$ and $V \in \tilde{V}$, let $d^H(V)$ denote the degree of V in the graph $(\tilde{V}, E)$. Given a graph $(\tilde{V}, E)$ and disjoint subsets $\tilde{X}, \tilde{Y}$ of $\tilde{V}$, let $E(\tilde{X}) = E \cap \binom{\tilde{X}}{2}$ and $E(\tilde{X}, \tilde{Y}) = E \cap \{XY : X \in \tilde{X}, Y \in \tilde{Y}\}$.

Lemma 4.13. *Suppose $H = (\tilde{V}, E, |\cdot|)$ is a vertex-weighted forest. Then there is a vertex-weighted star $H' = (\tilde{V}, E', |\cdot|)$, with centre V satisfying $|V| = \max\{|X| : X \in \tilde{V}\}$, such that*

$$f_\pi(H') \geqslant f_\pi(H). \tag{4.2}$$

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Proof. First, without loss of generality, we may assume H is a tree, since adding edges only increases the value of f_π. Fix $V \in \tilde{V}$ such that $|V| = \max\{|X| : X \in \tilde{V}\}$. Suppose $Y \in \tilde{V}$ is a leaf, but $YV \notin E$. Say $YW \in E$. Let E' be the edge set obtained from E by deleting YW and adding YV. Then

$$f_\pi(\tilde{V}, E', |\cdot|) = (3/2)^{|V||Y|-|W||Y|} f_\pi(H) \geqslant f_\pi(H),$$

where the last inequality is because $|V| = \max\{|X| : X \in \tilde{V}\}$. Repeating this enough times yields the desired vertex-weighted star H'. □

Lemma 4.14. *Suppose $n \geqslant 1$, $G \in NC(n)$, and $\tilde{G} = (\tilde{V}, E, |\cdot|)$ is the vertex-weighted graph associated with G and $\sim_G$. Suppose $(\tilde{V}, E)$ is a star with centre V and there is $W \in \tilde{V} \setminus \{V\}$ such that $|W| > 1$. Then $G \notin \mathcal{P}(NC(n))$.*

Proof. Let $\tilde{V}' = (\tilde{V} \setminus \{W\}) \cup \{W_1, W_2\}$ and $E' = (E \setminus \{VW\}) \cup \{VW_1, VW_2\}$, where W_1, W_2 are new vertices. Let $H = (\tilde{V}', E', |\cdot|')$ where the vertex-weight function $|\cdot|'$ is defined by $|U|' = |U|$ for all $U \in \tilde{V} \setminus \{W\}$, $|W_1|' = |W| - 1$ and $|W_2|' = 1$. By definition of H,

$$\sum_{U \in \tilde{V}'} |U|' = \sum_{U \in \tilde{V}} |U| = n.$$

Since H is obtained from $\tilde{G}$ by splitting the degree-one vertex W into W_1 and W_2, and $\tilde{G}$ is a forest, H is also a forest. Thus H satisfies the hypotheses of Lemma 4.12, so there is a $G' \in NC(n)$ such that $\tilde{G}'$ is isomorphic to H. This and Definition 4.11 implies $f_\pi(H) = f_\pi(\tilde{G}') = P(G')$. Then

$$f_\pi(H) = f_\pi(\tilde{G}) 2^{|W|-1} \geqslant 2 f_\pi(\tilde{G}) > f_\pi(\tilde{G}),$$

where the last inequality is because by definition, $f_\pi(\tilde{G}) > 0$. Thus we have shown $G' \in NC(n)$ and $P(G') = f_\pi(H) > f_\pi(\tilde{G}) = P(G)$, so $G \notin \mathcal{P}(NC(n))$. □

We now prove the main result of this subsection. Observe that for all $n \geqslant 1$, $W(n) \subseteq NC(n)$.

Lemma 4.15. *For all $n \geqslant 1$, $\mathcal{P}(NC(n)) \cap W(n) \neq \emptyset$. Consequently, $\mathcal{P}(W(n)) \subseteq \mathcal{P}(NC(n))$.*

Proof. Fix $n \geqslant 1$, $G = ([n], w) \in \mathcal{P}(NC(n))$. Consider the vertex-weighted graph $\tilde{G} = (\tilde{V}, E, |\cdot|)$ associated with G and $\sim_G$. By Lemma 4.10, $\tilde{G} = ([n], E)$ is a forest. By Lemma 4.13, there is a vertex-weighted star $H = (\tilde{V}, E', |\cdot|)$ with centre V satisfying $|V| = \max\{|X| : X \in \tilde{V}\}$ and such that $f_\pi(H) \geqslant f_\pi(\tilde{G})$. By Lemma 4.12, there is $G' \in NC(n)$ such that $\tilde{G}' \cong H$. Thus

$$P(G') = f_\pi(H) \geqslant f_\pi(\tilde{G}) = P(G), \tag{4.3}$$

where the equalities hold by Definition 4.11. Since $G \in \mathcal{P}(NC(n))$ and $G' \in NC(n)$, (4.3) implies $G' \in \mathcal{P}(NC(n))$. Since $G' \in \mathcal{P}(NC(n))$ and $(\tilde{G}', E')$ is a star with centre V, Lemma 4.14 implies that for all $W \in \tilde{V} \setminus \{V\}$, $|W| = 1$. Consequently, by definition, $G' \in W(n)$, and thus $\mathcal{P}(NC(n)) \cap W(n) \neq \emptyset$. Since $W(n) \subseteq NC(n)$, this implies $\mathcal{P}(W(n)) \subseteq \mathcal{P}(NC(n))$. □

Although not necessary for the purposes of this paper, the arguments above can be strengthened to show that for sufficiently large n, $\mathcal{P}(W(n)) = \mathcal{P}(NC(n))$. First, one adds a clause to

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Lemma 4.13: equality holds in (4.2) if and only if $H = H'$, or H is a tree where $|V| = |W|$ for some $V \neq W$. Suppose now that n is sufficiently large and $G \in \mathcal{P}(NC(n))$. We sketch why $G \in W(n)$ (which implies $\mathcal{P}(W(n)) = \mathcal{P}(NC(n))$). Proceed as in the proof of Lemma 4.15 to find H and then $G' \in W(n)$. If $H = \tilde{G}$ then $G = G' \in W(n)$. So assume $H \neq \tilde{G}$. Since equality holds in (4.3), the new clause of Lemma 4.13 implies H is a tree and there is $W \neq V$ with $|W| = |V|$. This implies $|U| = |V| = 1$ for all $U \in \tilde{V}$, so

$$P(G) = f_\pi(G) = 3^{|E|} 2^{\binom{n}{2} - |E|} \leqslant 3^{n-1} 2^{\binom{n}{2} - n+1}$$

(where the second inequality is because H is a tree). Since n is large, this is much smaller than $2^{\binom{\lceil \beta n \rceil}{2}} 3^{\lceil \beta n \rceil (n - \lceil \beta n \rceil)}$, which can be achieved by elements of $W(n)$, contradicting $G \in \mathcal{P}(NC(n))$.

4.3. Getting rid of cycles and proving Theorem 3.3

In this subsection we prove Lemma 4.27, which shows that for large n, all product-extremal elements of $C(n)$ are in $NC(n)$. We will then prove Theorem 3.3 at the end of this subsection. Our proof uses an argument that is essentially a progressive induction.

Definition 4.16. Suppose G is a multigraph. The $(3,2)$-*girth* of G is $\min\{t \geqslant 3 : C_t(3,2) \subseteq G\}$ (where by convention, $\min \emptyset = \infty$).

Note that if $G \in C(n)$, then $C_3(3,2) \nsubseteq G$ (since $C(n) \subseteq F(n,3,8)$) and $C_4(3,2) \nsubseteq G$ (since $S(C_4(3,2)) = 16$). Consequently, no $G \in C(n)$ can have $(3,2)$-girth strictly less than 5. Therefore, to show some $G \in C(n)$ is in $NC(n)$, we only need to show $C_t(3,2) \nsubseteq G$ for $t \geqslant 5$. Given $G = (V,w)$, $X \subseteq V$ and $z \in V \setminus X$, set $P_z^G(X) = \prod_{x \in X} w(xz)$.

Lemma 4.17. *Let $5 \leqslant t \leqslant n$. Suppose $G = ([n],w) \in C(n)$ has $(3,2)$-girth t. If $X \in \binom{[n]}{t}$ is such that $G[X] \cong C_t(3,2)$, then for all $z \in [n] \setminus X$ either*

(1) $|\{x \in X : w(zx) = 3\}| \leqslant 1$ *and* $P_z^G(X) \leqslant 3 \cdot 2^{t-1}$ *or*
(2) $|\{x \in X : w(zx) = 3\}| \geqslant 2$ *and* $P_z^G(X) \leqslant 3^2 2^{t-3} < 3 \cdot 2^{t-1}$.

Proof. Let $X = \{x_1, \dots, x_t\}$ where $w(x_i x_{i+1}) = w(x_1 x_t) = 3$ for each $i \in [t-1]$ and $w(x_i x_j) = 2$ for all other pairs $ij \in \binom{[t]}{2}$. We will use throughout that $\mu(G) \leqslant 3$ (since $G \in C(n)$). Fix $z \in [n] \setminus X$ and let $Z = \{x \in X : w(zx) = 3\}$. If $|Z| \leqslant 1$, then clearly 1 holds. So assume $|Z| \geqslant 2$ and $i_1 < \dots < i_\ell$ are such that $Z = \{x_{i_1}, \dots, x_{i_\ell}\}$. Without loss of generality, assume $i_1 = 1$. Set

$$I = \{(x_{i_j}, x_{i_{j+1}}) : 1 \leqslant j \leqslant \ell - 1\} \cup \{(x_{i_1}, x_{i_\ell})\}.$$

Given $(x,y) \in I$, let

$$d(x,y) = \begin{cases} i_{j+1} - i_j & \text{if } (x,y) = (x_{i_j}, x_{i_{j+1}}) \text{ for some } 1 \leqslant j \leqslant \ell - 1, \\ t - i_\ell + 1 & \text{if } (x,y) = (x_{i_1}, x_{i_\ell}). \end{cases}$$

Note that because $C_3(3,2) \nsubseteq G$, $2 \leqslant d(x,y) \leqslant t-2$ for all $(x,y) \in I$. Suppose first that there is some $(u,v) \in I$ such that $d(u,v) = t-2$. Then since $d(x,y) \geqslant 2$ for all $(x,y) \in I$ and

$$\sum_{(x,y) \in I} d(x,y) \leqslant t,$$

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we must have that $|I|=1$ and either $(u,v)=(x_{i_1},x_{i_\ell})=(x_1,x_{t-1})$ or $(u,v)=(x_{i_1},x_{i_\ell})=(x_1,x_3)$. Without loss of generality, assume $(u,v)=(x_1,x_3)$. Then we must have that $w(zx_2)\leqslant 1$ since otherwise $G[\{z,x_1,x_2,x_3\}]\cong C_4(3,2)$, a contradiction. This shows that $P_z^G(X)\leqslant 3^2\cdot 1\cdot 2^{t-3}<3\cdot 2^{t-1}$.

Suppose now that for all $(x,y)\in I$, $d(x,y)\leqslant t-3$. Given $(x,y)\in I$, say an element x_k is *between* x and y if either $(x,y)=(x_{i_j},x_{i_{j+1}})$ and $i_j<k<i_{j+1}$ or $(x,y)=(x_{i_1},x_{i_\ell})$ and $i_\ell<k$. Then for each $(x,y)\in I$, there must be a x_k between x and y such that $w(zx_k)\leqslant 1$, since otherwise

$$\{z,x,y\}\cup\{u: u \text{ is between } x \text{ and } y\}$$

is a copy of $C_{d(x,y)+2}(3,2)$ in G, a contradiction since $d(x,y)+2<t$. This implies there are at least ℓ elements u in $X\setminus Z$ such that $w(zu)\leqslant 1$, so

$$P_z^G(X)\leqslant 3^\ell 2^{t-2\ell}\leqslant 3^2 2^{t-4}<3\cdot 2^{t-1}. \qquad \square$$

Definition 4.18. Suppose $t\leqslant n$ and $X\in\binom{[n]}{t}$. Define $\mathcal{G}_X=([n],w)$ to be the following multigraph, where $Y=[n]\setminus X$. Choose any $A\in\mathcal{P}(W(n-t))$ and $B\in W(t)$ so that $|R(B)|=\lceil\beta t\rceil$ and $|L(B)|=\lfloor(1-\beta)t\rfloor$. Define w on $\binom{Y}{2}\cup\binom{X}{2}$ to make $\mathcal{G}_X[Y]\cong A$ and $\mathcal{G}_X[X]\cong B$. Define w on the remaining pairs of vertices in the obvious way so that $\mathcal{G}_X\in W(n)$.

Observe that in the notation of Definition 4.18,

$$P(\mathcal{G}_X)=P(A)2^{\binom{|R(B)|}{2}+|R(A)||R(B)|}3^{|L(B)||R(B)|+|R(B)||L(A)|+|L(B)||R(A)|}.$$

It will be convenient to have a uniform lower bound on the exponential term in the above quantity. Towards this end, given $n,t\in\mathbb{N}$ we define

$$f(n,t)=\min\bigl\{2^{\binom{\lceil\beta t\rceil}{2}+\lceil\beta t\rceil c}3^{\lceil\beta t\rceil\lfloor(1-\beta)t\rfloor+c\lfloor(1-\beta)t\rfloor+\lceil\beta t\rceil(n-t-c)}: c\in\{\lfloor\beta(n-t)\rfloor,\lceil\beta(n-t)\rceil\}\bigr\}.$$

Lemma 4.19. *Let $t\leqslant n$ and $X\in\binom{[n]}{t}$. Then for any $A'\in\mathcal{P}(W(n-t))$, $P(\mathcal{G}_X)\geqslant P(A')f(n,t)$.*

Proof. Set $Y=[n]\setminus X$ and let $\mathcal{G}_X=([n],w)$. Let $B\in W(t)$ and $A\in\mathcal{P}(W(n-t))$ be as in the definition of $\mathcal{G}_X$ so that $\mathcal{G}_X[X]\cong B$ and $\mathcal{G}_X[Y]\cong A$. Let L_A,R_A and L_B,R_B be the partitions of Y and X respectively. By choice of B, $|L_B|=\lfloor(1-\beta)t\rfloor$ and $|R_B|=\lceil\beta t\rceil$. Let $c=|R_A|$. By definition, $|L_A|=n-t-c$, and since $A\in\mathcal{P}(W(n-t))$, $c\in\{\lfloor\beta(n-t)\rfloor,\lceil\beta(n-t)\rceil\}$ (by the proof of Lemma 3.4). Combining these observations with the definition of $f(n,t)$ implies

$$2^{\binom{|R_B|}{2}+|R_A||R_B|}3^{|L_B||R_B|+|R_B||L_A|+|L_B||R_A|}=2^{\binom{\lceil\beta t\rceil}{2}+\lceil\beta t\rceil c}3^{\lceil\beta t\rceil\lfloor(1-\beta)t\rfloor+c\lfloor(1-\beta)t\rfloor+\lceil\beta t\rceil(n-t-c)}\geqslant f(n,t).$$

Combining this with the definition of $\mathcal{G}_X$, we have

$$P(\mathcal{G}_X)=P(A)2^{\binom{|R_B|}{2}+|R_A||R_B|}3^{|L_B||R_B|+|R_B||L_A|+|L_B||R_A|}\geqslant P(A)f(n,t).$$

Since $P(A)=P(A')$ for all $A'\in\mathcal{P}(W(n-t))$, this finishes the proof. $\square$

Definition 4.20. Given $n,t\in\mathbb{N}$, let $h(n,t)=3^n 2^{\binom{t}{2}+t(n-t)-n}$.

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Lemma 4.21. *Let $5 \leqslant t \leqslant n$, $G \in C(n)$ and $\nu > 0$. Suppose $X \in \binom{[n]}{t}$, $G[X] \cong C_t(3,2)$, and there is some $A \in \mathcal{P}(W(n-t))$ such that $P(G[[n] \setminus X]) \leqslant \nu P(A)$. Then*

$$P(G) \leqslant \nu((h(n,t))/f(n,t))P(\mathcal{G}_X).$$

Proof. Let $Y = [n] \setminus X$. Because $G[X] \cong C_t(3,2)$,

$$P(G) = P(G[Y])3^t 2^{\binom{t}{2}-t} \prod_{z \in Y} P_z^G(X).$$

By Lemma 4.17, for each $z \in Y$, $P_z^G(X) \leqslant 3 \cdot 2^{t-1}$. This implies

$$P(G) \leqslant P(G[Y])3^t 2^{\binom{t}{2}-t}(3 \cdot 2^{t-1})^{n-t} = P(G[Y])3^n 2^{\binom{t}{2}+t(n-t)-n} = P(G[Y])h(n,t). \tag{4.4}$$

By assumption, $P(G[Y]) \leqslant \nu P(A)$, so (4.4) implies $P(G) \leqslant \nu P(A)h(n,t)$. Combining this with Lemma 4.19 yields

$$P(G) \leqslant \nu P(A)h(n,t) = \nu P(A)f(n,t)(h(n,t)/f(n,t)) \leqslant \nu P(\mathcal{G}_X)(h(n,t)/f(n,t)). \qquad \square$$

The following lemma will be proved in the Appendix.

Lemma 4.22. *There are $\gamma > 0$ and $5 < K \leqslant M_1$ such that the following holds.*
(a) *For all $K \leqslant t \leqslant n$, $h(n,t) < f(n,t)$.*
(b) *For all $5 \leqslant t \leqslant K$ and $n \geqslant M_1$, $h(n,t) < 2^{-\gamma n} f(n,t)$.*

The rough idea behind the proof of Lemma 4.22 is to use certain inequalities involving β (see (I) and (II) in the Appendix) to show that the dominating terms compare to one another in the desired ways, in each case. For case (a), since t is large, the dominating terms are those involving t^2 or tn, while in case (b), since $n \gg t$, the dominating terms are only those involving n. Our next two lemmas give us upper bounds for $P(G)$ when $G \in C(n)$ has large finite girth (Lemma 4.23), or small girth (Lemma 4.25).

Lemma 4.23. *Let K be the constant from Lemma 4.22. Then for all $K \leqslant t \leqslant n$, the following holds. If $G \in C(n)$ has $(3,2)$-girth t, then for all $G_1 \in \mathcal{P}(W(n))$, $P(G) < P(G_1)$.*

Proof. Let $t \geqslant K$ and $n = t + i$. We proceed by induction on i. Suppose first that $i = 0$. Fix $G \in C(n)$ with $(3,2)$-girth t. Then $n = t$ implies $G \cong C_t(3,2)$ and so $P(G) = 3^t 2^{\binom{t}{2}-t} = h(t,t)$. Let $H \in W(n)$ have $|R(H)| \in \{\lceil \beta n \rceil, \lfloor \beta n \rfloor\}$ and $L(H) = V(H) - R(H)$. Then by definition of $f(n,t)$,

$$P(H) = 2^{\binom{|R(H)|}{2}} 3^{|L(H)||R(H)|} \geqslant f(t,t) > h(t,t) = P(G),$$

where the strict inequality is by Lemma 4.22(a). Since $H \in W(n)$, this implies $P(G) < P(G_1)$ for all $G_1 \in \mathcal{P}(W(n))$.

Suppose now that $i > 0$. Assume by induction that the conclusion of Lemma 4.23 holds for all $K \leqslant t_0 \leqslant n_0$ where $n_0 = t_0 + j$ and $0 \leqslant j < i$. Fix $G \in C(n)$ with $(3,2)$-girth t. Let $X \in \binom{[n]}{t}$ be such that $G[X] \cong C_t(3,2)$.

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Claim 4.24. *For any $A \in \mathcal{P}(W(n-t))$, $P(G[[n]\setminus X]) \leqslant P(A)$.*

Proof of claim. Note that $C_{t'}(3,2) \nsubseteq G[[n]\setminus X]$ for all $3 \leqslant t' < t$. We have two cases.

(1) If $C_{t'}(3,2) \nsubseteq G[[n]\setminus X]$ for all $t' \geqslant t$, then $G[[n]\setminus X]$ is isomorphic to some $D \in NC(n-t)$. By Lemma 4.15, for any $A \in \mathcal{P}(W(n-t))$,

$$P(G[[n]\setminus X]) = P(D) \leqslant P(A).$$

(2) If $C_{t'}(3,2) \subseteq G[[n]\setminus X]$ for some $t' \geqslant t$, then fix t_0 the smallest such t'. Set $n_0 = n-t$ and $j = n_0 - t_0$. Our assumptions imply $t_0 \geqslant t \geqslant K$, so $j < n-t = i$. Note that $G[[n]\setminus X]$ is isomorphic to some $D \in C(n-t) = C(n_0)$. Then we have that $K \leqslant t_0 \leqslant n_0$, $n_0 = t_0 + j$, $0 \leqslant j < i$, and $D \in C(n_0)$ has $(3,2)$-girth t_0. By our induction hypothesis, for any $A \in \mathcal{P}(W(n_0)) = \mathcal{P}(W(n-t))$,

$$P(G[[n]\setminus X]) = P(D) < P(A).$$

Claim 4.24 and Lemma 4.21 with $\nu = 1$ imply $P(G) \leqslant (h(n,t)/f(n,t))P(\mathcal{G}_X)$. Since $K \leqslant t \leqslant n$, Lemma 4.22(a) implies $h(n,t)/f(n,t) < 1$, so this shows $P(G) < P(\mathcal{G}_X)$. Since $\mathcal{G}_X \in W(n)$, we have $P(G) < P(\mathcal{G}_X) \leqslant P(G_1)$ for all $G_1 \in \mathcal{P}(W(n))$. □

Lemma 4.25. *Let M_1 and K be the constants from Lemma 4.22. There is M_2 such that for all $5 \leqslant t \leqslant K$ and $n \geqslant M_1 + K$, the following holds. If $G \in C(n)$ has $(3,2)$-girth t, then for all $G_1 \in \mathcal{P}(W(n))$, $P(G) \leqslant 2^{M_2}(h(n,t)/f(n,t))P(G_1)$.*

Proof. Set $M = M_1 + K$. Choose M_2 sufficiently large that for all $5 \leqslant t \leqslant K$ and $t \leqslant n \leqslant M$,

$$\mathrm{ex}_{\Pi}(n,4,15) \leqslant 2^{M_2}(h(n,t)/f(n,t)).$$

We show that the conclusions of Lemma 4.25 hold for all $n \geqslant M$ by induction. Suppose first that $n = M$. Fix $5 \leqslant t \leqslant K$ and $G \in C(n)$ with $(3,2)$-girth t. Then by our choice of M_2,

$$P(G) \leqslant \mathrm{ex}_{\Pi}(n,4,15) \leqslant 2^{M_2}(h(n,t)/f(n,t)) \leqslant 2^{M_2}(h(n,t)/f(n,t))P(G_1),$$

for all $G_1 \in \mathcal{P}(W(n))$. Suppose now that $n > M$. Assume by induction that the conclusions of Lemma 4.25 hold for all $5 \leqslant t_0 \leqslant K$ and $M \leqslant n_0 < n$. Fix $5 \leqslant t \leqslant K$ and $G \in C(n)$ with $(3,2)$-girth t. Let $X \in \binom{[n]}{t}$ be such that $G[X] \cong C_t(3,2)$ and set $n_0 = n-t$.

Claim 4.26. *For any $A \in \mathcal{P}(W(n_0))$, $P(G[[n]\setminus X]) \leqslant 2^{M_2}P(A)$,*

Proof. [Proof of claim.] Fix $A \in \mathcal{P}(W(n_0))$. Note that $C_{t'}(3,2) \nsubseteq G[[n]\setminus X]$ for all $t' < t$ and $n_0 \geqslant M_1 \geqslant K$ (since $n - t \geqslant M - K = M_1$). We will use the following observation:

$$\text{for all } 5 \leqslant t_0 \leqslant K, \quad \frac{h(n_0,t_0)}{f(n_0,t_0)} \leqslant 2^{-\gamma n_0} < 1. \tag{4.5}$$

This holds by Lemma 4.22(b) and the fact that $n_0 \geqslant M_1$. Suppose first that $n_0 < M$. Then $G[[n]\setminus X]$ is isomorphic to some $D \in F(n_0,4,15)$ and $n_0 \geqslant K$, so by our choice of M_2,

$$P(G[[n]\setminus X]) = P(D) \leqslant \mathrm{ex}_{\Pi}(n_0,4,15) \leqslant 2^{M_2}\frac{h(n_0,K)}{f(n_0,K)}P(A) \leqslant 2^{M_2}P(A),$$

where the last inequality is by (4.5). Assume now that $n_0 \geqslant M$. We have two cases.

(1) If $C_{t'}(3,2) \nsubseteq G[[n]\setminus X]$ for all $t' \geqslant t$, then $G[[n]\setminus X]$ is isomorphic to some $D \in NC(n_0)$. By Lemma 4.15,

$$P(G[[n]\setminus X]) = P(D) \leqslant P(A) \leqslant 2^{M_2}P(A).$$

(2) If $C_{t'}(3,2) \subseteq G[[n]\setminus X]$ for some $t' \geqslant t$, choose t_0 the smallest such t', and let $D \in C(n_0)$ be such that $G[[n]\setminus X] \cong D$. Suppose first that $t_0 \leqslant K$. Then we have $5 \leqslant t \leqslant t_0 \leqslant K$, $M \leqslant n_0 < n$, and $D \in C(n_0)$ has $(3,2)$-girth t_0. Therefore our induction hypothesis implies the conclusions of Lemma 4.25 hold for D, n_0, t_0. In other words, since $A \in \mathcal{P}(W(n_0))$,

$$P(G[[n]\setminus X]) = P(D) \leqslant 2^{M_2}\frac{h(n_0,t_0)}{f(n_0,t_0)}P(A) \leqslant 2^{M_2}P(A),$$

where the last inequality is by (4.5). Suppose finally that $t_0 > K$. Then $K \leqslant t_0 \leqslant n_0$ and $D \in C(n_0)$ has $(3,2)$-girth t_0. Thus we have by Lemma 4.23 that

$$P(G[[n]\setminus X]) = P(D) < P(A) \leqslant 2^{M_2}P(A).$$

Claim 4.26 and Lemma 4.21 with $\nu = 2^{M_2}$ imply $P(G) \leqslant 2^{M_2}(h(n,t)/f(n,t))P(\mathcal{G}_X)$. Since $\mathcal{G}_X$ is in $W(n)$, we have that

$$P(G) \leqslant 2^{M_2}(h(n,t)/f(n,t))P(\mathcal{G}_X) \leqslant 2^{M_2}(h(n,t)/f(n,t))P(G_1),$$

for all $G_1 \in \mathcal{P}(W(n))$. □

We can now prove that for large n, all product-extremal elements of $C(n)$ are in $NC(n)$.

Lemma 4.27. *For all sufficiently large n, $\mathcal{P}(C(n)) \subseteq NC(n)$. Therefore, $\mathcal{P}(C(n)) = \mathcal{P}(NC(n))$.*

Proof. Let γ, K and M_1 be as in Lemma 4.22 and let M_2 be as in Lemma 4.25. Choose $M \geqslant M_1 + K$ sufficiently large that $2^{M_2-\gamma n} < 1$ for all $n \geqslant M$. Suppose $n > M$ and $G \in C(n)\setminus NC(n)$. We show $G \notin \mathcal{P}(C(n))$. Since $W(n) \subseteq C(n)$, it suffices to show there is $G_1 \in W(n)$ such that $P(G_1) > P(G)$. Since $G \notin NC(n)$, there is $5 \leqslant t \leqslant n$ such that G has $(3,2)$-girth t. If $t \geqslant K$, then Lemma 4.23 implies that for any $G_1 \in \mathcal{P}(W(n))$, $P(G) < P(G_1)$. If $5 \leqslant t < K$, then Lemma 4.25 implies that for any $G_1 \in \mathcal{P}(W(n))$,

$$P(G) \leqslant 2^{M_2}(h(n,t)/f(n,t))P(G_1) \leqslant 2^{M_2-\gamma n}P(G_1),$$

where the second inequality is because of Lemma 4.22(b). By our choice of M, this implies that for all $G_1 \in W(n)$, $P(G) < P(G_1)$. This shows $\mathcal{P}(C(n)) \subseteq NC(n)$. Since $NC(n) \subseteq C(n)$, this implies $\mathcal{P}(C(n)) = \mathcal{P}(NC(n))$. □

We can prove the main result of this section, Theorem 3.3.

Proof of Theorem 3.3. Assume n sufficiently large. By Lemma 4.8, we can choose some G in $\mathcal{P}(D(n)) \cap C(n) = \mathcal{P}(C(n))$. By Lemma 4.27, $\mathcal{P}(C(n)) = \mathcal{P}(NC(n))$, so $G \in \mathcal{P}(NC(n))$. By Lemma 4.15, there is some $G' \in \mathcal{P}(NC(n)) \cap W(n) = \mathcal{P}(W(n))$. Since G and G' are both in

$\mathcal{P}(NC(n))$, $P(G) = P(G')$. Since $G \in \mathcal{P}(D(n))$ and $W(n) \subseteq D(n)$, this implies that $G' \in \mathcal{P}(D(n))$. Thus we have shown $G' \in \mathcal{P}(D(n)) \cap \mathcal{P}(W(n))$. □

5. Proof of Theorem 3.2

In this section we prove Theorem 3.2. We will need the following computational lemma, which is proved in the Appendix. Given n, t, let $k(n,t) = 15^t 2^{\binom{t}{2}+t(n-t)-t}$.

Lemma 5.1. *There is M such that for all $n \geqslant M$ and $2 \leqslant t \leqslant n$, $k(n,t) < f(n,t)$.*

Lemma 5.1 is proved via a similar strategy to Lemma 4.22. Specifically, when t is small, one analyses only the terms containing n to obtain the inequality. When t is large, one more carefully compares the terms containing tn, t^2 and $n-t$. The following can be checked easily by hand and is left to the reader.

Lemma 5.2. *Suppose a, b and c are non-negative integers. If $a+b \leqslant 4$, then $a \cdot b \leqslant 2^2$. If $a+b+c \leqslant 6$, then $a \cdot b \cdot c \leqslant 2^3$.*

Proof of Theorem 3.2. Let n_0 be such that Lemmas 4.27 and 5.1 hold for all $n > n_0$, and fix $n \gg n_0$ sufficiently large. It suffices to show $\mathcal{P}(n,4,15) \subseteq D(n)$. Suppose towards a contradiction there is $G = ([n], w) \in \mathcal{P}(F(n,4,15)) \setminus D(n)$. Given $X \subseteq [n]$ and $z \in [n] \setminus X$, let $S(X) = S(G[X])$ and $S_z(X) = \sum_{x \in X} w(xz)$. If $G \notin F(n,3,8)$, let $D_1, \dots, D_k$ be a maximal collection of pairwise disjoint elements of $\binom{[n]}{3}$ such that $S(D_i) \geqslant 9$ for each i, and set $D = \bigcup_{i=1}^k D_i$. If $G \in F(n,3,8)$, set $D = \emptyset$. If $\mu(G[[n] \setminus D]) > 3$, choose $e_1, \dots, e_m$ a maximal collection of pairwise disjoint elements of $\binom{[n]\setminus D}{2}$ such that $S(e_i) \geqslant 4$ for each i and set $C = \bigcup_{i=1}^m e_i$. If $\mu(G[[n] \setminus D]) \leqslant 3$, set $C = \emptyset$. Let $X = D \cup C$ and $\ell = |X| = 3k + 2m$. Note that by assumption X is non-empty, so we must have $\ell \geqslant 2$. We now make a few observations. If $D \neq \emptyset$, then for each D_i and $z \in [n] \setminus D_i$,

$$S_z(D_i) \leqslant S(D_i \cup \{z\}) - S(D_i) \leqslant 15 - 9 = 6 = 2 \cdot 3,$$

which implies by Lemma 5.2 that $P_z^G(D_i) \leqslant 2^3$. By maximality of the collection $D_1, \dots, D_k$, $G[[n] \setminus D]$ is a $(3,8)$-graph. Thus if $C \neq \emptyset$, then for each i and $z \in [n] \setminus (D \cup e_i)$,

$$S_z(e_i) \leqslant S(e_i \cup \{z\}) - 4 \leqslant 8 - 4 = 4 = 2 \cdot 2,$$

which implies by Lemma 5.2 that $P_z^G(e_i) \leqslant 2^2$. Since $\mu(G) \leqslant 15$, for each D_i and e_j, $P(D_i) \leqslant 5^3$ and $P(e_j) \leqslant 15$. Let $Y = [n] \setminus X$ and write $P(Y)$ for $P(G[Y])$. Our observations imply that $P(G)$ is at most

$$P(Y)\left(\prod_{i=1}^{k} P(D_i)\right)\left(\prod_{i=1}^{m} P(e_i)\right) 2^{\binom{\ell}{2}+\ell(n-\ell)-\ell+m} \leqslant P(Y) 15^{\ell-m} 2^{\binom{\ell}{2}+\ell(n-\ell)-\ell+m} \leqslant P(Y) k(n,\ell). \tag{5.1}$$

Note that $G[Y]$ is isomorphic to an element of $D(n-\ell)$. We partition the argument into two cases.

Case 1. $n - \ell \leqslant n_0$. In this case we can use the crude bounds

$$P(G) < 15^{\binom{n_0}{2}} 15^{\ell-m} 2^{\binom{\ell}{2}+\ell n_0} < 2^{\binom{\ell}{2}+4\ell+2n_0^2+\ell n_0} < \mathrm{ex}_{\Pi}(n,4,15),$$

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where the second inequality holds since $\ell \geqslant n - n_0 \gg n_0$ since $n \gg n_0$. The last inequality holds because $n \gg n_0$, $\mathrm{ex}_\Pi(n,4,15) = 2^{\gamma n^2 + o(n^2)}$ and $2^\gamma > \sqrt{2}$. This contradicts the fact that $G \in \mathcal{P}(n,4,15)$.

Case 2. $n - \ell > n_0$. In this case we may apply Lemma 4.27 to $G[Y]$ as $|Y| = n - \ell > n_0$. Fix $A \in \mathcal{P}(W(n-\ell))$. By Lemmas 4.8, 4.27 and 4.15, $\mathcal{P}(W(n-\ell)) \subseteq \mathcal{P}(D(n-\ell))$, which implies that $P(Y) \leqslant P(A)$. Combining this with Lemma 4.19 yields $P(\mathcal{G}_X) \geqslant P(A)f(n,\ell) \geqslant P(Y)f(n,\ell)$. This along with the bound on $P(G)$ in (5.1) implies

$$\frac{P(G)}{P(\mathcal{G}_X)} \leqslant \frac{P(Y)k(n,\ell)}{P(Y)f(n,\ell)} = \frac{k(n,\ell)}{f(n,\ell)} < 1,$$

where the last inequality is by choice of $n_0 < n$ and Lemma 5.1. So $P(G) < P(\mathcal{G}_X)$, a contradiction. □

6. Concluding remarks

The arguments used to prove Theorem 2.4 can be adapted to prove a version for sums. If $G = (V,w)$, let $S(G) = \sum_{xy \in \binom{V}{2}} w(xy)$. Given integers $s \geqslant 2$ and $q \geqslant 0$, set

$$\mathrm{ex}_\Sigma(n,s,q) = \max\{S(G) : G \in F(n,s,q)\}.$$

An (n,s,q)-graph G is *sum-extremal* if $S(G) = \mathrm{ex}_\Sigma(n,s,q)$. Let $\mathcal{S}(n,s,q)$ denote the set of sum-extremal (n,s,q)-graphs with vertex set $[n]$, and let $\mathcal{S}(W(n))$ denote the set of $G \in W(n)$ such that $S(G) \geqslant S(G')$ for all $G' \in W(n)$. Straightforward calculus shows that for $G \in W(n)$, the sum $S(G)$ is maximized when $|L(G)| \approx (2/3)n$. Then the ideas in our proofs can be adapted to the sum-setting to yield the following theorem.

Theorem 6.1. *For all sufficiently large n, $\mathcal{S}(W(n)) \subseteq \mathcal{S}(n,4,15)$. Consequently*

$$\begin{aligned}\mathrm{ex}_\Sigma&(n,4,15)\\ &= \max\left\{2\binom{\lfloor (2n)/3 \rfloor}{2} + 3\left(\left\lfloor \frac{2n}{3}\right\rfloor\right)\left(\left\lceil \frac{n}{3}\right\rceil\right), 2\binom{\lceil (2n)/3 \rceil}{2} + 3\left(\left\lceil \frac{2n}{3}\right\rceil\right)\left(\left\lfloor \frac{n}{3}\right\rfloor\right)\right\}\\ &= \frac{8}{3}\binom{n}{2} + O(n).\end{aligned}$$

We would like to point out that the asymptotic value for $\mathrm{ex}_\Sigma(n,4,15)$ was already known as a consequence of [8]. Our contribution is in showing $\mathcal{S}(W(n)) \subseteq \mathcal{S}(n,4,15)$. The following result shows that product-extremal $(n,4,15)$-graphs are far from sum-extremal ones. Given two multigraphs $G = (V,w)$ and $G' = (V,w')$, set

$$\Delta(G,G') = \left\{xy \in \binom{V}{2} : w(xy) \neq w'(xy)\right\}.$$

We say G and G' are δ*-close* if $|\Delta(G,G')| \leqslant \delta n^2$, otherwise they are δ*-far*.

Corollary 6.2. *There is $\delta > 0$ such that for all sufficiently large n, the following holds. Suppose $G \in \mathcal{P}(n,4,15)$ and $G' \in \mathcal{S}(n,4,15)$. Then G and G' are δ-far from one another.*

The proof of Corollary 6.2 is straightforward and left to the reader. Details appear in [15]. Given $a \geqslant 2$, let $W_a(n)$ be the set of multigraphs $([n], w)$ such that there is a partition L, R of $[n]$ with $w(xy) = a-1$ for all $xy \in \binom{L}{2}$, $w(xy) = a$ for all $xy \in \binom{R}{2}$, and $w(xy) = a+1$ for all $x \in L$, $y \in R$. Basic calculus shows that for $G \in W_a(n)$, $P(G)$ is maximized when $|R| \approx \beta_a n$, where

$$\beta_a = \frac{\log(a+2) - \log(a-1)}{2\log(a+2) - \log a - \log(a-1)}.$$

Note that the $W(n) = W_2(n)$. Based on our results for $(4, 15)$, we make the following conjecture.

Conjecture 6.3. *For all $a \geqslant 2$, $\mathcal{P}(W_a(n)) \subseteq \mathcal{P}(n, 4, 6a+3)$. Consequently,*

$$\mathrm{ex}_{\Pi}(n, 4, 6a+3) = 2^{\gamma_a n^2 + O(n)},$$

where

$$\gamma_a = \frac{(1-\beta_a)^2}{2}\log_2(a-1) + \frac{\beta_a^2}{2}\log_2 a + \beta_a(1-\beta_a)\log_2(a+2).$$

When $a = 2$, this is Theorem 2.4. However, at least some of the arguments used in this paper will not transfer immediately to cases with $a > 2$. For instance, the proofs of Lemma 4.6 and Corollary 4.7 use the fact that $a = 2$ in a non-trivial way (in particular it is key there that the smallest multiplicity appearing in $W(n)$ is 1). Further, when $a > 2$, one must contend with 'small' edge multiplicities, that is, those in $\{i : 1 \leqslant i < a-2\}$. This is not an issue for $(4, 15)$ since this set is empty.

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Appendix

We begin by stating two inequalities. See [15] for formal proofs.
(I) $2^{1-\beta^2} < 3^{1.5\beta(1-\beta)}$.
(II) $5(1-\beta^2 - 2\beta(1-\beta)\log_2 3) + \log_2 3 - 1 < 0$.

For any $r \in \mathbb{R}$,

$$\binom{r}{2} = \frac{r^2 - r}{2}.$$

Given $2 \leqslant t \leqslant n$, let

$$f_*(n,t) = 2^{\binom{\beta t}{2} + \beta^2 t(n-t)} 3^{2\beta t(1-\beta)(n-t) + \beta(1-\beta)t^2}.$$

The next lemma allows us to work with $f^*(n,t)$ instead of $f(n,t)$.

Proposition A.1. *For all $2 \leqslant t \leqslant n$, if $m = n-t$ then $f(n,t) \geqslant f_*(n,t) 2^{-3t-m} 3^{-2t-m}$.*

Proof. Define $\eta(u,v,z,w) = 2^{\binom{u}{2}+uz}3^{uv+zv+uw}$ and observe that

$$f(n,t) \geqslant \eta(\beta t-1,(1-\beta)t-1,\beta m-1,(1-\beta)m-1).$$

We leave it to the reader to verify that

$$\eta(\beta t-1,(1-\beta)t-1,\beta m-1,(1-\beta)m-1) = f_*(n,t)2^{2-2\beta t-\beta m}3^{3-2t-m}.$$

Thus $f(n,t) \geqslant f_*(n,t)2^{2-2\beta t-\beta m}3^{3-2t-m} \geqslant f_*(n,t)2^{-2t-m}3^{-2t-m}$. □

Proof of Lemma 4.22. Choose K sufficiently large that $3 < \beta(1-\beta)K/2$ and

$$3^{3K+K(\beta+1)/2} < 3^{\beta(1-\beta)K^2/8}.$$

We now prove part (a) for this K. Fix $K \leqslant t \leqslant n$ and set $m = n-t$. Proposition A.1 along with the definitions of $h(n,t)$ and $f_*(n,t)$ implies

$$\frac{h(n,t)}{f(n,t)} \leqslant \frac{3^{m+t}2^{\binom{t}{2}+tm-(m+t)}}{f_*(n,t)2^{-2t-m}3^{-2t-m}} = \left(\frac{2^{1-\beta^2}}{3^{2\beta(1-\beta)}}\right)^{mt}\left(\frac{2^{(1-\beta^2)/2}}{3^{\beta(1-\beta)}}\right)^{t^2} 2^{t(1+\beta)/2+m}3^{2m+3t}. \tag{A.1}$$

Combining A.1 with (I) yields that

$$\frac{h(n,t)}{f(n,t)} \leqslant (3^{-\beta(1-\beta)/2})^{tm}(3^{-\beta(1-\beta)/4})^{t^2}2^{t(1+\beta)/2+m}3^{2m+3t}.$$

Combining this with our assumptions on K and the fact that $t \geqslant K$, we obtain

$$\frac{h(n,t)}{f(n,t)} < 3^{(3-\beta(1-\beta)t/2)m-\beta(1-\beta)t^2/4+t(1+\beta)/2+3t} \leqslant 3^{-\beta(1-\beta)t^2/8} < 1. \tag{A.2}$$

This finishes the proof of part (a). For part (b), fix $5 \leqslant t \leqslant K$. Observe that for $n \gg K$, we have $f(n,t) = 2^{\beta^2 nt}3^{2\beta(1-\beta)t+o(n)}$ and $h(n,t) = 2^{nt-n}3^{n+o(n)}$, so

$$\frac{h(n,t)}{f(n,t)} = \frac{2^{n(t-1+\log_2(3))+o(n)}}{2^{\beta^2 nt+2\beta(1-\beta)tn\log_2(3)+o(n)}} = 2^{n(t(1-\beta^2-2\beta(1-\beta)\log_2(3))-1+\log_2(3))+o(n)}. \tag{A.3}$$

Set

$$\alpha = 5(1-\beta^2-2\beta(1-\beta)\log_2(3))-1+\log_2(3)$$

and note that (II) implies $\alpha < 0$. Since $t \geqslant 5$, (A.3) implies $h(n,t)/f(n,t) \leqslant 2^{\alpha n+o(n)}$. Thus for sufficiently large n, $h(n,t) \leqslant f(n,t)2^{-\gamma n}$ where $\gamma := -\alpha/2 > 0$. Since the definition of α did not depend on t, this finishes the proof of part (b). □

Proof of Lemma 5.1. Recall we want to show there is M such that for all $n \geqslant M$ and $2 \leqslant t \leqslant n$, $k(n,t) < f(n,t)$, where $k(n,t) = 15^t 2^{\binom{t}{2}+t(n-t)-t}$. Let K be from Lemma 4.22 and recall the proof of Lemma 4.22 (see inequality (A.2)) showed that for all $K \leqslant t \leqslant n$,

$$h(n,t)/f(n,t) \leqslant 3^{-\beta(1-\beta)t^2/8}.$$

Choose $K' \geqslant K$ sufficiently large that $(15/2)^{K'}3^{-\beta(1-\beta)(K')^2/8} < 1$. Suppose now that $K' \leqslant t \leqslant n$. Then by definition of $k(n,t)$, and our choice of $t \geqslant K'$,

$$\frac{k(n,t)}{f(n,t)} = \frac{(15/2)^t(2/3)^n h(n,t)}{f(n,t)} \leqslant (15/2)^t(2/3)^n 3^{-\beta(1-\beta)t^2/8} < (2/3)^n < 1.$$

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Thus $k(n,t) < f(n,t)$ for all $K' \leqslant t \leqslant n$. Suppose now that $2 \leqslant t \leqslant K'$ and $n \gg K'$. Then since $t \ll n$, $f(n,t) \geqslant (2^{\beta^2}3^{2\beta(1-\beta)})^{nt-o(n)}$ and $k(n,t) \leqslant 2^{nt+o(n)}$, we have

$$\frac{k(n,t)}{f(n,t)} \leqslant \left(\frac{2}{2^{\beta^2}3^{2\beta(1-\beta)}}\right)^{nt+o(n)} = \left(\frac{2^{1-\beta^2}}{3^{2\beta(1-\beta)}}\right)^{nt+o(n)} < 3^{-2\beta(1-\beta)n+o(n)},$$

where the last inequality is by (I) and because $t \geqslant 2$. Thus we can choose M sufficiently large that if $n > M$ and $2 \leqslant t \leqslant K'$, then $k(n,t)/f(n,t) < 1$. Combining our two cases yields that $k(n,t)/f(n,t) < 1$ for all $n \geqslant \max\{M, K'\}$ and $2 \leqslant t \leqslant n$. □

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