Proceedings of the Royal Society of Edinburgh, **138A**, 647–670, *2008*

Multiplicity of positive solutions for semilinear elliptic equations in $\mathbb{R}^N$

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(MS received 24 November 2006; accepted 2 May 2007)

In this paper, we study the multiplicity of positive solutions for the following semilinear elliptic equation:

$$\begin{aligned} -\Delta u+\lambda u &= f(x)u^{p-1}+h(x)u^{q-1} && \text{in } \mathbb{R}^N,\\ u &> 0 && \text{in } \mathbb{R}^N,\\ u &\in H^1(\mathbb{R}^N), \end{aligned}$$

where $1 \leqslant q < 2 < p < 2^*$ ($2^* = 2N/(N-2)$ if $N \geqslant 3$ and $2^* = \infty$ if $N = 1, 2$), $\lambda > 0, h \in L^{2/(2-q)}(\mathbb{R}^N) \setminus \{0\}$ is non-negative and $f \in C(\mathbb{R}^N)$. We will show how the shape of the graph of $f(x)$ affects the number of positive solutions.

1. Introduction

In this paper, we study the multiplicity of positive solutions for the following semilinear elliptic equation:

$$\left.\begin{aligned} -\Delta u+\lambda u &= f(x)u^{p-1}+h(x)u^{q-1} && \text{in } \mathbb{R}^N,\\ u &> 0 && \text{in } \mathbb{R}^N,\\ u &\in H^1(\mathbb{R}^N), \end{aligned}\right\} \qquad (E_\lambda)$$

where $1 \leqslant q < 2 < p < 2^*$ ($2^* = 2N/(N-2)$ if $N \geqslant 3$ and $2^* = \infty$ if $N = 1, 2$), $\lambda > 0$, $f \in C(\mathbb{R}^N)$ and $h \in L^{2/(2-q)}(\mathbb{R}^N) \setminus \{0\}$ is non-negative. Associated with equation (E_λ), we consider the energy functional:

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \lambda u^2 - \frac{1}{p}\int_{\mathbb{R}^N} f(x)|u|^p - \frac{1}{q}\int_{\mathbb{R}^N} h(x)|u|^q.$$

It is well known that the functional $J_\lambda \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ and the solutions of equation (E_λ) are the critical points of the energy functional J_λ in $H^1(\mathbb{R}^N)$.

Under the assumption $h \not\equiv 0$, our equation (E_λ) can be regarded as a perturbation problem of the following semilinear elliptic equation:

$$\left.\begin{aligned} -\Delta u+\lambda u &= f(x)u^{p-1} && \text{in } \mathbb{R}^N,\\ u &> 0 && \text{in } \mathbb{R}^N,\\ u &\in H^1(\mathbb{R}^N). \end{aligned}\right\} \qquad (1.1)$$

https://doi.org/10.1017/S0308210506001156 Published online by Cambridge University Press

It is known that the existence of positive solutions of equation (1.1) is affected by the shape of the graph of $f(x)$. This has been the focus of a great deal of research by several authors (see [6,7,9,10,18–20], etc.). Furthermore, if f is a positive constant, then equation (1.1) has a unique positive solution (see [17]).

Some progress has been made for the case when $q = 1$, as follows. Zhu [27] and Hirano [15] were mainly concerned with the following equation:

$$\left.\begin{aligned} -\Delta u + \lambda u &= u^{p-1} + h(x) \quad \text{in } \mathbb{R}^N, \\ u &> 0 \quad \text{in } \mathbb{R}^N, \\ u &\in H^1(\mathbb{R}^N), \end{aligned}\right\} \tag{1.2}$$

where $h \in L^2(\mathbb{R}^N) \setminus \{0\}$ is non-negative. They succeeded in finding that (1.2) has at least two positive solutions under $\|h\|_{L^2}$ is sufficiently small and that $h(x)$ decays faster than $\exp(-c|x|)$ for some $c > 0$. Generalizations of the result of [15,27] were made by Cao and Zhou [11], Jeanjean [16] and Adachi and Tanaka [1,2]. In [2], Adachi and Tanaka showed the existence of at least four positive solutions of the equation

$$\begin{aligned} -\Delta u + \lambda u &= f(x)u^{p-1} + h(x) \quad \text{in } \mathbb{R}^N, \\ u &> 0 \quad \text{in } \mathbb{R}^N, \\ u &\in H^1(\mathbb{R}^N), \end{aligned}$$

under the assumptions that $0 < f(x) \leqslant f^\infty = \lim_{|x|\to\infty} f(x)$, $h \in H^{-1}(\mathbb{R}^N) \setminus \{0\}$ is non-negative and $\|H\|_{H^{-1}}$ is sufficiently small. In [1,11,16], the general equations

$$\begin{aligned} -\Delta u + \lambda u &= g(x,u) + h(x) \quad \text{in } \mathbb{R}^N, \\ u &> 0 \quad \text{in } \mathbb{R}^N, \\ u &\in H^1(\mathbb{R}^N), \end{aligned}$$

were studied, where g satisfies some suitable conditions and $h \in H^{-1}(\mathbb{R}^N) \setminus \{0\}$ is non-negative, and the existence of at least two positive solutions when $\|H\|_{H^{-1}}$ sufficiently small was proved.

The main purpose of this paper is to use the shape of the graph of $f(x)$ to prove the multiplicity of positive solutions for equation (E_λ). Moreover, we extend $q \in [1,2)$ without assuming $\|H\|_{L^{2/(2-q)}}$ is sufficiently small. First, we consider the following assumptions:

(Q1) $f \in C(\mathbb{R}^N)$ and $f \geqslant 0$ in $\mathbb{R}^N$;

(Q2) $f(x) \to f^\infty > 0$ as $|x| \to \infty$;

(Q3) there exist some points $x^1, x^2, \dots, x^k$ in $\mathbb{R}^N$ such that $f(x^i)$ are strict maxima and satisfy

$$f^\infty < f(x^i) = f_{\max} \equiv \max\{f(x) \mid x \in \mathbb{R}^N\} \quad \text{for all } i = 1, 2, \dots, k.$$

Then we have the following result.

THEOREM 1.1. *Assume that conditions (Q1)–(Q3) hold. There then exists a $\lambda_0 > 0$ such that, for $\lambda > \lambda_0$, equation (E_λ) has at least $k+1$ positive solutions.*

For the other similarly problems, Ambrosetti *et al*. [4] investigated the following equation:

$$\left.\begin{aligned} -\Delta u &= u^{p-1} + \lambda u^{q-1} && \text{in } \Omega, \\ u &> 0 && \text{in } \Omega, \\ u &\in H_0^1(\Omega), \end{aligned}\right\} \tag{1.3}$$

where $1 < q < 2 < p \leqslant 2N/(N-2)$, $N \geqslant 3$, and Ω is a bounded domain in $\mathbb{R}^N$. They found that there exists a $\lambda_0 > 0$ such that equation (1.3) admits at least two positive solutions for $\lambda \in (0, \lambda_0)$, a positive solution for $\lambda = \lambda_0$ and no positive solution exists for $\lambda > \lambda_0$. Actually, Adimurthi *et al*. [3], Damascelli *et al*. [12], Ouyang and Shi [22] and Tang [24] proved that there exists a $\lambda_0 > 0$ such that there are exactly two positive solutions of equation (E_λ) in the unit ball $B^N(0;1)$ for $\lambda \in (0, \lambda_0)$, exactly one positive solution for $\lambda = \lambda_0$ and no positive solution exists for $\lambda > \lambda_0$. The result of equation (1.3) was generalized by Ambrosetti *et al*. [5], de Figueiredo *et al*. [13] and Wu [26].

This paper is organized as follows. In § 2, we give some notation and preliminaries. In § 3, we prove the existence of a local minimum. In § 4, we prove theorem 1.1.

2. Notation and preliminaries

By the change of variables $\eta = 1/\sqrt{\lambda}$, $v(x) = \eta^{2/(p-2)} u(\eta x)$, the equation (E_λ) is transformed to

$$\left.\begin{aligned} -\Delta v + v &= f_\eta v^{p-1} + \eta^{2(p-q)/(p-2)} h_\eta v^{q-1} && \text{in } \mathbb{R}^N, \\ v &> 0 && \text{in } \mathbb{R}^N, \\ v &\in H^1(\mathbb{R}^N), \end{aligned}\right\} \tag{2.1}$$

where $f_\eta = f(\eta x)$ and $h_\eta = h(\eta x)$.

For $u \in H^1(\mathbb{R}^N)$, $c \in \mathbb{R}$, non-negative bounded function $a \in C(\mathbb{R}^N)$ and non-negative function $b \in L^{2/(2-q)}(\mathbb{R}^N)$, define

$$\begin{aligned} I_{a,b}(u) &= \tfrac{1}{2}\|u\|_{H^1}^2 - \frac{1}{p}\int_{\mathbb{R}^N} a|u|^p - \eta^{2(p-q)/(p-2)}\frac{1}{q}\int_{\mathbb{R}^N} b|u|^q, \\ \boldsymbol{M}_{a,b}(c) &= \{u \in H^1(\mathbb{R}^N) \setminus \{0\} \mid \langle I'_{a,b}(u), u\rangle = c\}, \\ \alpha_{a,b}(c) &= \inf\{I_{a,b}(u) \mid u \in \boldsymbol{M}_{a,b}(c)\}, \end{aligned}$$

where

$$\|u\|_{H^1} = \left(\int_{\mathbb{R}^N} |\nabla u|^2 + u^2\right)^{1/2}$$

is a standard norm in $H^1(\mathbb{R}^N)$ and $I'_{a,b}$ denotes the Fréchet derivative of $I_{a,b}$. We will write $\boldsymbol{M}_{a,b}(0)$ and $\alpha_{a,b}(0)$ as $\boldsymbol{M}_{a,b}$ and $\alpha_{a,b}$, respectively. It is well known that the functional $I_{a,b} \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ and the solutions of equation (2.1) are the critical points of the energy functional I_{f_η,h_η} (see [23]). Moreover, we have the following result.

Lemma 2.1. *Suppose ais a continuous bounded and non-negative function on* $\mathbb{R}^N$. *Then* $\alpha_{a,0}(c) = \frac{1}{2}c$ *for* $c > 0$ *and*

$$\alpha_{a,0} \leqslant \alpha_{a,0}(c) + \alpha_{a,0}(-c) - \frac{p-2}{2p}|c| \quad \textit{for all } c \in \mathbb{R}.$$

Proof. See [10, lemma 2.2]. □

Define

$$\psi_\eta(u) = \langle I'_{f_\eta,h_\eta}(u), u\rangle = \|u\|^2_{H^1} - \int_{\mathbb{R}^N} f_\eta|u|^p - \eta^{2(p-q)/(p-2)} \int_{\mathbb{R}^N} h_\eta|u|^q.$$

Then, for $u \in \boldsymbol{M}_{f_\eta,h_\eta}$,

$$\begin{aligned}\langle\psi'_\eta(u), u\rangle &= 2\|u\|^2_{H^1} - p\int_{\mathbb{R}^N} f_\eta|u|^p - \eta^{2(p-q)/(p-2)} q\int_{\mathbb{R}^N} h_\eta|u|^q \\ &= (2-q)\|u\|^2_{H^1} - (p-q)\int_{\mathbb{R}^N} f_\eta|u|^p.\end{aligned}$$

Using a similar method to that in [25], we split $\boldsymbol{M}_{f_\eta,h_\eta}$ into three parts:

$$\begin{aligned}\boldsymbol{M}^+_{f_\eta,h_\eta} &= \biggl\{u \in \boldsymbol{M}_{f_\eta,h_\eta} \biggm| (2-q)\|u\|^2_{H^1} - (p-q)\int_{\mathbb{R}^N} f_\eta|u|^p > 0\biggr\}, \\ \boldsymbol{M}^0_{f_\eta,h_\eta} &= \biggl\{u \in \boldsymbol{M}_{f_\eta,h_\eta} \biggm| (2-q)\|u\|^2_{H^1} - (p-q)\int_{\mathbb{R}^N} f_\eta|u|^p = 0\biggr\}, \\ \boldsymbol{M}^-_{f_\eta,h_\eta} &= \biggl\{u \in \boldsymbol{M}_{f_\eta,h_\eta} \biggm| (2-q)\|u\|^2_{H^1} - (p-q)\int_{\mathbb{R}^N} f_\eta|u|^p < 0\biggr\}.\end{aligned}$$

Then we have the following result.

Lemma 2.2. *There exists* $\eta_1 > 0$ *such that* $\boldsymbol{M}^0_{f_\eta,h_\eta} = \emptyset$ *for all* $\eta \in (0, \eta_1)$.

Proof. Assume the contrary, that is that $\boldsymbol{M}^0_{f_\eta,h_\eta} \neq \emptyset$ for all $\eta > 0$. Then for $u \in \boldsymbol{M}^0_{f_\eta,h_\eta}$, we have

$$\|u\|^2_{H^1} = \frac{p-q}{2-q}\int_{\mathbb{R}^N} f_\eta|u|^p \tag{2.2}$$

and

$$\eta^{2(p-q)/(p-2)}\int_{\mathbb{R}^N} h_\eta|u|^q = \|u\|^2_{H^1} - \int_{\mathbb{R}^N} f_\eta|u|^p = \frac{p-2}{2-q}\int_{\mathbb{R}^N} f_\eta|u|^p. \tag{2.3}$$

Moreover,

$$\begin{aligned}\biggl(\frac{p-2}{p-q}\biggr)\|u\|^2_{H^1} = \|u\|^2_{H^1} - \int_{\mathbb{R}^N} f_\eta|u|^p &\leqslant \eta^{2(p-q)/(p-2)}\|h_\eta\|_{L^{2/(2-q)}}\|u\|^q_{H^1} \\ &= \eta^{2(p-q)/(p-2)-(2-q)N/2}\|H\|_{L^{2/(2-q)}}\|u\|^q_{H^1},\end{aligned}$$

which implies

$$\|u\|_{H^1} \leqslant \biggl[\eta^{2(p-q)/(p-2)-(2-q)N/2}\biggl(\frac{p-q}{p-2}\biggr)\|H\|_{L^{2/(2-q)}}\biggr]^{1/(2-q)}. \tag{2.4}$$

Let $K_\eta : \boldsymbol{M}_{f_\eta,h_\eta} \to \mathbb{R}$ be given by

$$K_\eta(u) = C(p,q)\left(\frac{\|u\|_{H^1}^{2(p-1)}}{\int_{\mathbb{R}^N} f_\eta|u|^p}\right)^{1/(p-2)} - \eta^{2(p-q)/(p-2)}\int_{\mathbb{R}^N} h_\eta|u|^q,$$

where

$$C(p,q) = \left(\frac{2-q}{p-q}\right)^{(p-1)/(p-2)}\left(\frac{p-2}{2-q}\right).$$

Then $K_\eta(u) = 0$ for all $\eta > 0$ and $u \in \boldsymbol{M}^0_{f_\eta,h_\eta}$. Indeed, from (2.2) and (2.3) it follows that, for $u \in \boldsymbol{M}^0_{f_\eta,h_\eta}$, we have

$$\begin{aligned} K_\eta(u) &= \left(\frac{2-q}{p-q}\right)^{(p-1)/(p-2)}\left(\frac{p-2}{2-q}\right)\left(\frac{((p-q)/(2-q))^{p-1}(\int_{\mathbb{R}^N} f_\eta|u|^p)^{p-1}}{\int_{\mathbb{R}^N} f_\eta|u|^p}\right)^{1/(p-2)} \\ &\quad - \frac{p-2}{2-q}\int_{\mathbb{R}^N} f_\eta|u|^p \\ &= 0. \end{aligned} \tag{2.5}$$

However, by (2.4), the Hölder and Sobolev inequalities and

$$\left(\frac{\|u\|_{H^1}^p}{\int_{\mathbb{R}^N} f_{\max}|u|^p}\right)^{1/(p-2)} \geqslant \left(\frac{1}{f_{\max}S^p}\right)^{1/(p-2)} \quad \text{for all } u \in \boldsymbol{M}_{f_\eta,h_\eta},$$

where S is the best Sobolev constant, we have

$$\begin{aligned} &K_\eta(u) \\ &\geqslant C(p,q)\left(\frac{\|u\|_{H^1}^{2(p-1)}}{\int_{\mathbb{R}^N} f_\eta|u|^p}\right)^{1/(p-2)} - \eta^{2(p-q)/(p-2)-(2-q)N/2}\|H\|_{L^{2/(2-q)}}\|u\|_{H^1}^q \\ &> \|u\|_{H^1}^q\left(C(p,q)\left(\frac{1}{f_{\max}S^p}\right)^{1/(p-2)}\|u\|_{H^1}^{1-q} - \eta^{2(p-q)/(p-2)-(2-q)N/2}\|H\|_{L^{2/(2-q)}}\right) \\ &\geqslant \|u\|_{H^1}^q\bigg[C(p,q)\left(\frac{1}{f_{\max}S^p}\right)^{1/(p-2)}(\eta^{2(p-q)/(p-2)-(2-q)N/2})^{(1-q)/(2-q)} \\ &\qquad \times\left(\frac{p-q}{p-2}\|H\|_{L^{2/(2-q)}}\right)^{(1-q)/(2-q)} - \eta^{2(p-q)/(p-2)-(2-q)N/2}\|H\|_{L^{2/(2-q)}}\bigg] \end{aligned}$$

for all $u \in \boldsymbol{M}^0_{f_\eta,h_\eta}$. Since

$$\frac{1-q}{2-q} \leqslant 0 \quad \text{and} \quad \frac{2(p-q)}{p-2} - \frac{(2-q)N}{2} > 0$$

(see Appendix A), there exists $\eta_1 > 0$ such that, for each $\eta \in (0,\eta_1)$ and $u \in \boldsymbol{M}^0_{f_\eta,h_\eta}$, we have $K_\eta(u) > 0$, which contradicts (2.5). We can thus conclude that $\boldsymbol{M}^0_{f_\eta,h_\eta} = \emptyset$ for all $\eta \in (0,\eta_1)$. □

By lemma 2.2, for $\eta \in (0,\eta_1)$ we write $\boldsymbol{M}_{f_\eta,h_\eta} = \boldsymbol{M}^+_{f_\eta,h_\eta} \cup \boldsymbol{M}^-_{f_\eta,h_\eta}$ and define

$$\alpha^\pm_{f_\eta,h_\eta} = \inf_{u \in \boldsymbol{M}^\pm_{f_\eta,h_\eta}} I_{f_\eta,h_\eta}(u).$$

The following lemma shows that the minimizers on $\boldsymbol{M}_{f_\eta,h_\eta}$ are 'usually' critical points for I_{f_η,h_η}.

https://doi.org/10.1017/S0308210506001156 Published online by Cambridge University Press

LEMMA 2.3. *For the case when $\eta \in (0,\eta_1)$, if u_0 is a local minimizer for I_{f_η,h_η} on $\boldsymbol{M}_{f_\eta,h_\eta}$, then $I'_{f_\eta,h_\eta}(u_0)=0$ in $H^{-1}(\mathbb{R}^N)$.*

Proof. This is similar to the proof of [26, lemma 4]. □

For each $u \in H^1(\mathbb{R}^N)\setminus\{0\}$, we define

$$t_{\max} = \left(\frac{\|u\|_{H^1}^2}{(p-1)\int_{\mathbb{R}^N} f_\eta |u|^p}\right)^{1/(p-2)} > 0.$$

We then have the following lemma.

LEMMA 2.4. *For each $u \in H^1(\mathbb{R}^N)\setminus\{0\}$,*

(i) *there exists a unique $t^- = t^-(u) > t_{\max} > 0$ such that*

$$t^- u \in \boldsymbol{M}^-_{f_\eta,h_\eta} \quad \text{and} \quad I_{f_\eta,h_\eta}(t^- u) = \max_{t \geqslant t_{\max}} I_{f_\eta,h_\eta}(tu),$$

(ii) *$t^-(u)$ is a continuous function for non-zero u,*

(iii) $\boldsymbol{M}^-_{f_\eta,h_\eta} = \left\{ u \in H^1(\mathbb{R}^N)\setminus\{0\} \;\middle|\; \dfrac{1}{\|u\|_{H^1}} t^-\left(\dfrac{u}{\|u\|_{H^1}}\right) = 1 \right\},$

(iv) *if*

$$\int_{\mathbb{R}^N} h|u|^q > 0,$$

then there exists a unique $0 < t^+ = t^+(u) < t_{\max}$ such that $t^+u \in \boldsymbol{M}^+_{f_\eta,h_\eta}$ and $I_{f_\eta,h_\eta}(t^+u) = \min_{0\leqslant t \leqslant t^-} I_{f_\eta,h_\eta}(tu)$.

Proof. This is similar to the proof of [26, lemma 5]. □

3. Existence of a local minimum

In this section, we will establish the existence of a local minimum for I_{f_η,h_η} on $\boldsymbol{M}_{f_\eta,h_\eta}$. Let

$$d = \frac{2(p-q)}{p-2} - \frac{(2-q)N}{2} > 0$$

(see Appendix A). Then we have the following results.

LEMMA 3.1.

(i) *For each $u \in \boldsymbol{M}^+_{f_\eta,h_\eta}$ we have*

$$\int_{\mathbb{R}^N} h_\eta |u|^q > 0$$

and $I_{f_\eta,h_\eta}(u) < 0$. In particular, $\alpha_{f_\eta,h_\eta} \leqslant \alpha^+_{f_\eta,h_\eta} < 0$.

(ii) *I_{f_η,h_η} is coercive and bounded below on $\boldsymbol{M}_{f_\eta,h_\eta}$ for all*

$$\eta \in \left(0, \left(\frac{p-2}{p-q}\right)^{1/d}\right).$$

Proof. (i) For each $u \in \boldsymbol{M}^+_{f_\eta,h_\eta}$, $(2-q)\|u\|^2_{H^1} - (p-q)\int_{\mathbb{R}^N} f_\eta |u|^p > 0$ and

$$\|u\|^2_{H^1} = \int_{\mathbb{R}^N} f_\eta |u|^p + \eta^{2(p-q)/(p-2)} \int_{\mathbb{R}^N} h_\eta |u|^q,$$

we have

$$\eta^{2(p-q)/(p-2)} \int_{\mathbb{R}^N} h_\eta |u|^q = \|u\|^2_{H^1} - \int_{\mathbb{R}^N} f_\eta |u|^p > \frac{p-2}{2-q} \int_{\mathbb{R}^N} f_\eta |u|^p > 0$$

and

$$\begin{aligned} I_{f_\eta,h_\eta}(u) &= \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^N} f_\eta |u|^p - \left(\frac{1}{q} - \frac{1}{2}\right) \eta^{2(p-q)/(p-2)} \int_{\mathbb{R}^N} h_\eta |u|^q \\ &< -\frac{(p-q)(p-2)}{2pq} \int_{\mathbb{R}^N} f_\eta |u|^p < 0. \end{aligned}$$

(ii) For $u \in \boldsymbol{M}_{f_\eta,h_\eta}$, we have

$$\|u\|^2_{H^1} = \int_{\mathbb{R}^N} f_\eta |u|^p + \eta^{2(p-q)/(p-2)} \int_{\mathbb{R}^N} h_\eta |u|^q.$$

Then, by the Hölder and Young inequalities,

$$\begin{aligned} I_{f_\eta,h_\eta}(u) &\geqslant \left(\frac{p-2}{2p}\right) \|u\|^2_{H^1} - \left(\frac{p-q}{pq}\right) \eta^d \|H\|_{L^{2/(2-q)}} \|u\|^q_{H^1} \\ &\geqslant \left[\frac{p-2}{2p} - \eta^d \left(\frac{p-q}{2p}\right)\right] \|u\|^2_{H^1} - \eta^d \left(\frac{(p-q)(2-q)}{2pq}\right) \|H\|^{2/(2-q)}_{L^{2/(2-q)}} \\ &= \frac{1}{2p}[(p-2) - \eta^d (p-q)] \|u\|^2_{H^1} - \eta^d \left(\frac{(p-q)(2-q)}{2pq}\right) \|H\|^{2/(2-q)}_{L^{2/(2-q)}}. \end{aligned}$$

Thus, I_{f_η,h_η} is coercive and bounded below on

$$\boldsymbol{M}_{f_\eta,h_\eta} \quad \text{for all } \eta \in \left(0, \left(\frac{p-2}{p-q}\right)^{1/d}\right).$$

□

Furthermore, we have the following theorem.

Theorem 3.2. *For each positive number*

$$\eta < \eta_* = \min\left\{\eta_1, \left(\frac{p-2}{p-q}\right)^{1/d}\right\}$$

equation (2.1) has a positive solution $u_\eta \in \boldsymbol{M}^+_{f_\eta,h_\eta}$ *such that* $I_{f_\eta,h_\eta}(u_\eta) = \alpha^+_{f_\eta,h_\eta} = \alpha_{f_\eta,h_\eta}$ *and* $I_{f_\eta,h_\eta}(u_\eta) \to 0$ *as* $\eta \to 0$.

Proof. This is similar to the proof of [26, proposition 9]. There exists a sequence $\{u_n\} \subset \boldsymbol{M}_{f_\eta,h_\eta}$ such that

$$\begin{aligned} I_{f_\eta,h_\eta}(u_n) &= \alpha_{f_\eta,h_\eta} + o(1), \\ I'_{f_\eta,h_\eta}(u_n) &= o(1) \text{ in } H^{-1}(\mathbb{R}^N). \end{aligned}$$

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Then by lemma 3.1(ii), there exist a subsequence $\{u_n\}$ and $u_\eta \in H^1(\mathbb{R}^N)$ is a solution of equation (2.1) such that

$$u_n \rightharpoonup u_\eta \text{ weakly in } H^1(\mathbb{R}^N) \text{ and } u_n \to u_\eta \text{ a.e. in } \mathbb{R}^N.$$

Moreover, by $h \in L^{2/(2-q)}(\mathbb{R}^N)$, the Egorov theorem and the Hölder inequality, we have

$$\int_{\mathbb{R}^N} h_\eta |u_n|^q \to \int_{\mathbb{R}^N} h_\eta |u_\eta|^q.$$

Now we prove that

$$\int_{\mathbb{R}^N} h_\eta |u_\eta|^q \neq 0.$$

If we suppose otherwise, then

$$\|u_n\|_{H^1}^2 = \int_{\mathbb{R}^N} f_\eta |u_n|^p + o(1)$$

and

$$\begin{aligned}\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^N} f_\eta |u_n|^p &= \tfrac{1}{2}\|u_n\|_{H^1}^2 - \frac{1}{p}\int_{\mathbb{R}^N} f_\eta |u_n|^p \\ &\quad - \eta^{2(p-q)/(p-2)}\frac{1}{q}\int_{\mathbb{R}^N} h_\eta |u_n|^q + o(1) \\ &= \alpha_{f_\eta,h_\eta} + o(1),\end{aligned}$$

which contradicts the condition $\alpha_{f_\eta,h_\eta} < 0$. Thus,

$$\int_{\mathbb{R}^N} h_\eta |u_\eta|^q \neq 0.$$

In particular, u_η is a non-trivial solution of equation (2.1). Now we prove that $u_n \to u_\eta$ strongly in $H^1(\mathbb{R}^N)$. Otherwise $\|u_\eta\|_{H^1} < \liminf_{n\to\infty} \|u_n\|_{H^1}$ and so

$$\begin{aligned}\alpha_{f_\eta,h_\eta} \leqslant I_{f_\eta,h_\eta}(u_\eta) &= \left(\frac{1}{2}-\frac{1}{p}\right)\|u_\eta\|_{H^1}^2 - \left(\frac{1}{q}-\frac{1}{p}\right)\eta^{2(p-q)/(p-2)}\int_{\mathbb{R}^N} h_\eta |u_\eta|^q \\ &< \lim_{n\to\infty} I_{f_\eta,h_\eta}(u_n) \\ &= \alpha_{f_\eta,h_\eta},\end{aligned}$$

which is a contradiction. Consequently, $u_n \to u_\eta$ strongly in $H^1(\mathbb{R}^N)$ and

$$I_{f_\eta,h_\eta}(u_\eta) = \alpha_{f_\eta,h_\eta}.$$

Moreover, we have $u_\eta \in \boldsymbol{M}^+_{f_\eta,h_\eta}$. In fact, if $u_\eta \in \boldsymbol{M}^-_{f_\eta,h_\eta}$, by lemma 2.4, there exist unique t_0^+ and t_0^- such that $t_0^+ u_\eta \in \boldsymbol{M}^+_{f_\eta,h_\eta}$ and $t_0^- u_\eta \in \boldsymbol{M}^-_{f_\eta,h_\eta}$, and we have $t_0^+ < t_0^- = 1$. Since

$$\frac{\mathrm{d}}{\mathrm{d}t} I_{f_\eta,h_\eta}(t_0^+ u_\eta) = 0 \quad \text{and} \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} I_{f_\eta,h_\eta}(t_0^+ u_\eta) > 0,$$

there exists $\bar{t} \in (t_0^+, t_0^-]$ such that $I_{f_\eta,h_\eta}(t_0^+ u_\eta) < I_{f_\eta,h_\eta}(\bar{t} u_\eta)$. By lemma 2.4,

$$I_{f_\eta,h_\eta}(t_0^+ u_\eta) < I_{f_\eta,h_\eta}(\bar{t} u_\eta) \leqslant I_{f_\eta,h_\eta}(t_0^- u_\eta) = I_{f_\eta,h_\eta}(u_\eta),$$

which is a contradiction. Thus,

$$I_{f_\eta,h_\eta}(u_\eta) = \alpha_{f_\eta,h_\eta} = \alpha^+_{f_\eta,h_\eta},$$

since $I_{f_\eta,h_\eta}(u_\eta) = I_{f_\eta,h_\eta}(|u_\eta|)$ and $|u_\eta| \in \boldsymbol{M}^+_{f_\eta,h_\eta}$. By lemma 2.3 and the maximum principle, we may assume that u_η is a positive solution of equation (2.1). Moreover, by lemma 3.1 we have

$$0 > I_{f_\eta,h_\eta}(u_\eta) \geqslant -\eta^d \biggl(\frac{(p-q)(2-q)}{2pq}\biggr) \|H\|^{2/(2-q)}_{L^{2/(2-q)}},$$

since $d > 0$. We obtain $I_{f_\eta,h_\eta}(u_\eta) \to 0$ as $\eta \to 0$. □

4. Proof of theorem 1.1

First, we use the graph of the coefficient f to find some Palais–Smale sequences which are used to prove theorem 1.1. For $a > 0$, let $C_a(x^i)$ denote the hypercube

$$\prod_{j=1}^{N} (x^i_j - a, x^i_j + a)$$

centred at $x^i = (x^i_1, x^i_2, \dots, x^i_N)$ for $i = 1, 2, \dots, k$. Let $\overline{C_a(x^i)}$ and $\partial C_a(x^i)$ denote the closure and the boundary of $C_a(x^i)$, respectively. By conditions (Q1) and (Q3), we can choose numbers $K, l > 0$ such that $C_l(x^i)$ are disjoint, $f(x) < f(x^i)$ for $x \in \partial C_l(x^i)$ for all $i = 1, 2, \dots, k$ and $\bigcup_{i=1}^k C_l(x^i) \subset \prod_{i=1}^N (-K, K)$.

Define $\phi_\eta \in C(\mathbb{R}, \mathbb{R})$, $g_\eta \in C(H^1(\mathbb{R}^N), \mathbb{R}^N)$ by

$$\phi_\eta(t) = \begin{cases} \dfrac{2K}{\eta}, & t > \dfrac{2K}{\eta}, \\[2ex] t, & -\dfrac{2K}{\eta} \leqslant t \leqslant \dfrac{2K}{\eta}, \\[2ex] -\dfrac{2K}{\eta}, & t < -\dfrac{2K}{\eta}, \end{cases}$$

$$g^j_\eta(u) = \frac{\int_{\mathbb{R}^N} \phi_\eta(x_j)|u|^p}{\int_{\mathbb{R}^N} |u|^p} \quad \text{for } j = 1, 2, \dots, N$$

and

$$g_\eta(u) = (g^1_\eta(u), g^2_\eta(u), \dots, g^N_\eta(u)).$$

Let $C^i_{l/\eta} \equiv C_{l/\eta}(x^i/\eta)$,

$$N^i_\eta = \{u \in \boldsymbol{M}^-_{f_\eta,h_\eta} \mid u \geqslant 0 \text{ and } g_\eta(u) \in C^i_{l/\eta}\},$$
$$\partial N^i_\eta = \{u \in \boldsymbol{M}^-_{f_\eta,h_\eta} \mid u \geqslant 0 \text{ and } g_\eta(u) \in \partial C^i_{l/\eta}\}$$

for $i = 1, 2, \dots, k$. It is easy to verify that N^i_η and ∂N^i_η are non-empty sets for all $i = 1, 2, \dots, k$. For $i = 1, 2, \dots, k$, consider the minimization problems in N^i_η and ∂N^i_η for I_{f_η,h_η},

$$\gamma^i_\eta = \inf_{u \in N^i_\eta} I_{f_\eta,h_\eta}(u), \qquad \tilde{\gamma}^i_\eta = \inf_{u \in \partial N^i_\eta} I_{f_\eta,h_\eta}(u).$$

Let w be a unique positive radial solution of

$$\begin{aligned} -\Delta u + u &= f_{\max} u^{p-1} \quad \text{in}\mathbb{R}^N, \\ u &> 0 \quad \text{in } \mathbb{R}^N, \\ u &\in H^1(\mathbb{R}^N), \end{aligned}$$

such that $I_{f_{\max},0}(w) = \alpha_{f_{\max},0}$. By condition (Q3) and routine computations, we have

$$\alpha_{f_{\max},0} < \alpha_{f^\infty,0}. \tag{4.1}$$

For small $\eta > 0$ satisfying $2\sqrt{\eta} < 1$, we define a function $\psi_\eta \in C^1(\mathbb{R}^N, [0,1])$ such that

$$\psi_\eta(x) = \begin{cases} 1, & |x| < \dfrac{1}{2\sqrt{\eta}} - 1, \\ 0, & |x| > \dfrac{1}{2\sqrt{\eta}} - 1, \end{cases}$$

and $|\nabla \psi_\eta| \leqslant 2$ in $\mathbb{R}^N$. Let

$$x^\eta = \frac{1}{2\sqrt{\eta}}(1, 1, \dots, 1) \in \mathbb{R}^N \quad \text{and} \quad w_\eta(x) = t_\eta^- w\left(x - \frac{x^i}{\eta} + x^\eta\right)\psi_\eta\left(x - \frac{x^i}{\eta} + x^\eta\right),$$

where $t_\eta^- > 0$ are selected such that $w_\eta \in \boldsymbol{M}^-_{f_\eta,h_\eta}$. We then have the following results.

Lemma 4.1. *We have*

(i) $\eta^{2(p-q)/(p-2)} \displaystyle\int_{\mathbb{R}^N} h_\eta w^q\left(x - \frac{x^i}{\eta} + x^\eta\right)\psi_\eta^q\left(x - \frac{x^i}{\eta} + x^\eta\right) \to 0$ *as* $\eta \to 0$.

(ii) $t_\eta^- \to 1$ *as* $\eta \to 0$.

Proof. (i) Since

$$\frac{2(p-q)}{p-2} - \frac{(2-q)N}{2} > 0,$$

we have

$$\begin{aligned} 0 &\leqslant \eta^{2(p-q)/(p-2)} \int_{\mathbb{R}^N} h_\eta w^q\left(x - \frac{x^i}{\eta} + x^\eta\right)\psi_\eta^q\left(x - \frac{x^i}{\eta} + x^\eta\right) \\ &\leqslant \eta^{2(p-q)/(p-2)-(2-q)N/2} \|H\|_{L^{2/(2-q)}} \left\| w\left(x - \frac{x^i}{\eta} + x^\eta\right)\psi_\eta\left(x - \frac{x^i}{\eta} + x^\eta\right)\right\|_{H^1}^q \end{aligned}$$

and

$$\left\| w\left(x - \frac{x^i}{\eta} + x^\eta\right)\psi_\eta\left(x - \frac{x^i}{\eta} + x^\eta\right)\right\|_{H^1}^2 \to \frac{2p}{p-2}\alpha_{f_{\max},0} \quad \text{as } \eta \to 0.$$

Thus,

$$\eta^{2(p-q)/(p-2)} \int_{\mathbb{R}^N} h_\eta w^q\left(x - \frac{x^i}{\eta} + x^\eta\right)\psi_\eta^q\left(x - \frac{x^i}{\eta} + x^\eta\right) \to 0 \quad \text{as } \eta \to 0.$$

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(ii) Since $w_\eta \in \boldsymbol{M}^-_{f_\eta,h_\eta}$, we have

$$(t_\eta^-)^2\biggl[\int_{\mathbb{R}^N}\biggl|\nabla\biggl(w\biggl(x-\frac{x^i}{\eta}+x^\eta\biggr)\psi_\eta\biggl(x-\frac{x^i}{\eta}+x^\eta\biggr)\biggr)\biggr|^2 + \biggl(w\biggl(x-\frac{x^i}{\eta}+x^\eta\biggr)\psi_\eta\biggl(x-\frac{x^i}{\eta}+x^\eta\biggr)\biggr)^2\biggr]$$
$$= (t_\eta^-)^p\int_{\mathbb{R}^N} f_\eta w^p\biggl(x-\frac{x^i}{\eta}+x^\eta\biggr)\psi_\eta^p\biggl(x-\frac{x^i}{\eta}+x^\eta\biggr) + \eta^{2(p-q)/(p-2)}(t_\eta^-)^q\int_{\mathbb{R}^N} h_\eta w^q\biggl(x-\frac{x^i}{\eta}+x^\eta\biggr)\psi_\eta^q\biggl(x-\frac{x^i}{\eta}+x^\eta\biggr).$$

Since $\|w\|_{H^1}^2 = \int_{\mathbb{R}^N} f_{\max} w^p$, from (i) we have that

$$(t_\eta^-)^2\|w\|_{H^1}^2 = (t_\eta^-)^2\biggl\|w\biggl(x-\frac{x^i}{\eta}+x^\eta\biggr)\psi_\eta\biggl(x-\frac{x^i}{\eta}+x^\eta\biggr)\biggr\|_{H^1}^2$$
$$= (t_\eta^-)^p\int_{\mathbb{R}^N} f_\eta w^p\biggl(x-\frac{x^i}{\eta}+x^\eta\biggr)\psi_\eta^p\biggl(x-\frac{x^i}{\eta}+x^\eta\biggr)+o(\eta)$$
$$= (t_\eta^-)^p\int_{\mathbb{R}^N} f(\eta x + x^i - \eta x^\eta)w^p + o(\eta),$$

where $o(\eta)\to 0$ as $\eta\to 0$. Moreover, $\eta x^\eta \to 0$ as $\eta\to 0$ and

$$t_\eta^- > t_{\max} = \biggl(\frac{\|w(x-(x^i/\eta)+x^\eta)\psi_\eta(x-(x^i/\eta)+x^\eta)\|_{H^1}^2}{(p-1)\int_{\mathbb{R}^N} f_\eta|w(x-(x^i/\eta)+x^\eta)\psi_\eta(x-(x^i/\eta)+x^\eta)|^p}\biggr)^{1/(p-2)} = (p-1)^{1/(2-p)}+o(\eta).$$

Thus, $t_\eta^- \to 1$ as $\eta\to 0$. □

Let

$$\eta_* = \min\biggl\{\eta_1, \biggl(\frac{p-2}{p-q}\biggr)^{1/d}\biggr\}$$

as in theorem 3.2. We then have the following result.

Lemma 4.2. *For each $\varepsilon>0$, there exists $\eta_\varepsilon\in(0,\eta_*]$ such that*

$$\alpha^-_{f_\eta,h_\eta} \leqslant \gamma^i_\eta < \min\{\alpha_{f_{\max},0}+\varepsilon, \alpha_{f_\eta,h_\eta}+\alpha_{f^\infty,0}\} \quad \textit{for } i=1,2,\dots,k \textit{ and } \eta\in(0,\eta_\varepsilon).$$

Proof. First, we show that $g_\eta(w_\eta)\in C^i_{l/\eta}$, since

$$g^j_\eta(w_\eta) = \frac{\int_{\mathbb{R}^N}\phi_\eta(x_j)w^p(x-(x^i/\eta)+x^\eta)\psi^p_\eta(x-(x^i/\eta)+x^\eta)}{\int_{\mathbb{R}^N} w^p(x-(x^i/\eta)+x^\eta)\psi^p_\eta(x-(x^i/\eta)+x^\eta)}$$

and

$$\psi_\eta\biggl(x-\frac{x^i}{\eta}+x^\eta\biggr)=0 \quad \text{if } \biggl|x_j-\frac{x^i_j}{\eta}\biggr| > \frac{1}{\sqrt{\eta}}.$$

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By the definition of ψ_η, we have

$$g_\eta^j(w_\eta) = \frac{\int_{C^i_{l/\eta}} \phi_\eta(x_j) w^p(x-(x^i/\eta)+x^\eta)\psi_\eta^p(x-(x^i/\eta)+x^\eta)}{\int_{C^i_{l/\eta}} w^p(x-(x^i/\eta)+x^\eta)\psi_\eta^p(x-(x^i/\eta)+x^\eta)}$$

provided that $1/\sqrt{\eta} < l/\eta$. From the definition of ϕ_η we conclude that $g_\eta(w_\eta) \in C^i_{l/\eta}$. Thus, $w_\eta \in N^i_\eta$. Moreover, by lemma 4.1,

$$\begin{aligned} I_{f_\eta,h_\eta}(w_\eta) = \frac{(t_\eta^-)^2}{2}\biggl[\int_{\mathbb{R}^N}\biggl|\nabla\biggl(w\biggl(x-\frac{x^i}{\eta}+x^\eta\biggr)\psi_\eta\biggl(x-\frac{x^i}{\eta}+x^\eta\biggr)\biggr)\biggr|^2 \\ + \biggl(w\biggl(x-\frac{x^i}{\eta}+x^\eta\biggr)\psi_\eta\biggl(x-\frac{x^i}{\eta}+x^\eta\biggr)\biggr)^2\biggr] \\ - \frac{(t_\eta^-)^p}{p}\int_{\mathbb{R}^N} f_\eta w^p\biggl(x-\frac{x^i}{\eta}+x^\eta\biggr)\psi_\eta^p\biggl(x-\frac{x^i}{\eta}+x^\eta\biggr) \\ - \eta^{2(p-q)/(p-2)}(t_\eta^-)^q\int_{\mathbb{R}^N} h_\eta w^q\biggl(x-\frac{x^i}{\eta}+x^\eta\biggr)\psi_\eta^q\biggl(x-\frac{x^i}{\eta}+x^\eta\biggr) \\ = \frac{1}{2}\int_{\mathbb{R}^N}|\nabla w|^2+w^2-\frac{1}{p}\int_{\mathbb{R}^N} f(\eta x+x^i-\eta x^\eta)w^p+o(\eta), \end{aligned} \tag{4.2}$$

where $o(\eta)\to 0$ as $\eta\to 0$. Since $\eta x^\eta \to 0$ as $\eta\to 0$, from (4.2), we have

$$I_{f_\eta,h_\eta}(w_\eta) = I_{f_{\max},0}(w)+o(\eta) = \alpha_{f_{\max},0}+o(\eta).$$

Therefore, for any $\varepsilon>0$ there exists $\eta_2>0$ such that

$$\gamma_\eta^i < \alpha_{f_{\max},0}+\varepsilon \quad \text{for } i=1,2,\dots,k \text{ and } \eta\in(0,\eta_2).$$

Moreover, if $\alpha_{f_{\max},0} < \alpha_{f^\infty,0}$ and $\alpha_{f_\eta,h_\eta}\to 0$ as $\eta\to 0$, then there exists $\eta_3>0$ such that

$$\gamma_\eta^i < \alpha_{f_\eta,h_\eta}+\alpha_{f^\infty,0} \quad \text{for } i=1,2,\dots,k \text{ and } \eta\in(0,\eta_3).$$

We take $\eta_\varepsilon=\min\{\eta_2,\eta_3\}$. This implies that

$$\gamma_\eta^i < \min\{\alpha_{f_{\max},0}+\varepsilon, \alpha_{f_\eta,h_\eta}+\alpha_{f^\infty,0}\} \quad \text{for } i=1,2,\dots,k \text{ and } \eta\in(0,\eta_\varepsilon).$$

This completes the proof. □

LEMMA 4.3. *There are positive numbers δ and $\eta_\delta\in(0,\eta_*]$ such that, for $i=1,2,\dots,k$,*

$$\tilde\gamma_\eta^i > \alpha_{f_{\max},0}+\delta \quad \textit{for all } \eta\in(0,\eta_\delta).$$

Proof. Fix $i\in\{1,2,\dots,k\}$. Assume the contrary. There then exists a sequence $\{\eta_n\}$ with $\eta_n\to 0$ as $n\to\infty$ such that $\tilde\gamma^i_{\eta_n}\to c\leqslant \alpha_{f_{\max},0}$. Consequently, there exists a sequence $\{u_n\}\subset \partial N^i_{\eta_n}$ such that $g_{\eta_n}(u_n)\in\partial C^i_{l/\eta_n}$,

$$\int_{\mathbb{R}^N}|\nabla u_n|^2+u_n^2 = \int_{\mathbb{R}^N} f_{\eta_n}|u_n|^p+\eta_n^{2(p-q)/(p-2)}\int_{\mathbb{R}^N} h_{\eta_n}|u_n|^q \tag{4.3}$$

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and

$$I_{f_{\eta_n},h_{\eta_n}}(u_n) \to c \leqslant \alpha_{f_{\max},0} \quad \text{as } n \to \infty.$$

It follows that $\{u_n\}$ is uniformly bounded in $H^1(\mathbb{R}^N)$. Since $u_n \in \boldsymbol{M}^-_{f_{\eta_n},h_{\eta_n}}$, we deduce from the Sobolev imbedding theorem that $\|u_n\|_{H^1} > \nu > 0$ for some constant ν and for all n. Applying the concentration-compactness principle of Lions [19, 20] to $|u_n|^p$, there exist positive constants R, θ and $\{y_n\} \subset \mathbb{R}^N$ such that

$$\int_{B^N(y_n;R)} |u_n|^p \geqslant \theta \quad \text{for all } n,$$

where $B^N(y_n;R) = \{x \in \mathbb{R}^N \mid |x - y_n| < R\}$. Let $\tilde{u}_n = u_n(x + y_n)$. Then there exists a non-zero $u_0 \in H^1(\mathbb{R}^N)$ such that

$$\begin{gathered} \tilde{u}_n \rightharpoonup u_0 \quad \text{in } H^1(\mathbb{R}^N), \\ \tilde{u}_n \to u_0 \quad \text{a.e. in } \mathbb{R}^N, \\ \int_{B^N(0;R)} |\tilde{u}_n|^p \to \int_{B^N(0;R)} |u_0|^p \geqslant \theta. \end{gathered}$$

Set $w_n = \tilde{u}_n - u_0$. By the Brézis–Lieb lemma [8] we obtain

$$\int_{\mathbb{R}^N} f(\eta_n x + \eta_n y_n)|\tilde{u}_n|^p = \int_{\mathbb{R}^N} f(\eta_n x + \eta_n y_n)|u_0|^p + \int_{\mathbb{R}^N} f(\eta_n x + \eta_n y_n)|w_n|^p + o(1). \tag{4.4}$$

Since $\{u_n\}$ is uniformly bounded and $\tilde{u}_n \rightharpoonup u_0$, we have

$$\eta_n^{2(p-q)/(p-2)} \int_{\mathbb{R}^N} h_{\eta_n} |u_n|^q \to 0 \quad \text{as } n \to \infty \tag{4.5}$$

and

$$\int_{\mathbb{R}^N} |\nabla \tilde{u}_n|^2 + \tilde{u}_n^2 = \int_{\mathbb{R}^N} |\nabla u_0|^2 + u_0^2 + \int_{\mathbb{R}^N} |\nabla w_n|^2 + w_n^2 + o(1). \tag{4.6}$$

Moreover, from (4.3) and (4.5) we have that

$$\int_{\mathbb{R}^N} |\nabla \tilde{u}_n|^2 + \tilde{u}_n^2 = \int_{\mathbb{R}^N} f(\eta_n x + \eta_n y_n)|\tilde{u}_n|^p + o(1). \tag{4.7}$$

Combining (4.4), (4.6) and (4.7), we have

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla w_n|^2 + w_n^2 - \int_{\mathbb{R}^N} f(\eta_n x + \eta_n y_n)|w_n|^p \\ &\qquad = -\biggl(\int_{\mathbb{R}^N} |\nabla u_0|^2 + u_0^2 - \int_{\mathbb{R}^N} f(\eta_n x + \eta_n y_n)|u_0|^p \biggr) + o(1). \end{aligned} \tag{4.8}$$

We distinguish the following cases:

(I) $\|w_n\|_{H^1} \to 0$;

(II) $\|w_n\|_{H^1} \to c > 0$.

Case I. By condition (Q3) we can choose $s > 0$ such that

$$f(x) < f_{\max} \quad \text{for } x \in \bar{C}^i_{l+s} \setminus C^i_{l-s}.$$

We complete the proof by establishing the contradiction

$$\lim_{n\to\infty} I_{f_{\eta_n},h_{\eta_n}}(u_n) > \alpha_{f_{\max},0}.$$

Choose the sequence $\{\eta_n y_n\}$. By passing to a subsequence if necessary, we may assume that one of the following cases occurs:

(I_1) $\{\eta_n y_n\} \subset \bar{C}^i_{l+s} \setminus C^i_{l-s}$;

(I_2) $\{\eta_n y_n\} \subset \bar{C}^i_{l-s}$;

(I_3) $\{\eta_n y_n\} \subset \mathbb{R}^N \setminus C^i_{l+s}$, and $\{\eta_n y_n\}$ is bounded;

(I_4) $\{\eta_n y_n\}$ is unbounded.

Let $\epsilon > 0$ and $R_\epsilon > 0$ be such that

$$\frac{\int_{|x|\geqslant R_\epsilon} |\tilde{u}_n|^p}{\int_{\mathbb{R}^N} |\tilde{u}_n|^p} \leqslant \epsilon. \tag{4.9}$$

In case (I$_1$), we may assume that $\eta_n y_n \to \tilde{y} \in \bar{C}^i_{l+s} \setminus C^i_{l-s}$ and $f(\tilde{y}) < f_{\max}$. Consequently,

$$\begin{aligned}\lim_{n\to\infty} I_{f_{\eta_n},h_{\eta_n}}(u_n) &= \lim_{n\to\infty} \biggl\{\frac{1}{2}\int_{\mathbb{R}^N} |\nabla \tilde{u}_n|^2 + \tilde{u}_n^2 - \frac{1}{p}\int_{\mathbb{R}^N} f(\eta_n x + \eta_n y_n)|\tilde{u}_n|^p \\ &\qquad - \eta_n^{2(p-q)/(p-2)} \int_{\mathbb{R}^N} h_{\eta_n}|u_n|^q\biggr\} \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_0|^2 + u_0^2 - \frac{1}{p}\int_{\mathbb{R}^N} f(\tilde{y})|u_0|^p \\ &> \alpha_{f_{\max},0},\end{aligned}$$

which is a contradiction.

In case (I$_2$),

$$\begin{aligned} g^j_{\eta_n}(u_n) &= \frac{\int_{\mathbb{R}^N} \phi_{\eta_n}(x_j + (y_n)_j)|\tilde{u}_n|^p}{\int_{\mathbb{R}^N} |\tilde{u}|^p} \\ &= \frac{\int_{|x|\leqslant R_\epsilon} \phi_{\eta_n}(x_j + (y_n)_j)|\tilde{u}_n|^p + \int_{|x|\geqslant R_\epsilon} \phi_{\eta_n}(x_j + (y_n)_j)|\tilde{u}_n|^p}{\int_{\mathbb{R}^N} |\tilde{u}_n|^p}.\end{aligned}$$

In the region $|x_j| \leqslant R_\epsilon$, we have

$$\begin{aligned} x_j + (y_n)_j &\in \biggl(\frac{x^i_j - (l-s)}{\eta_n} - R_\epsilon, \frac{x^i_j + (l-s)}{\eta_n} + R_\epsilon\biggr) \\ &\subset \biggl(-\frac{2K}{\eta_n}, \frac{2K}{\eta_n}\biggr) \quad \text{for } n \text{ sufficiently large.}\end{aligned}$$

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It then follows from (4.9) and the definition of ϕ_{η_n} that

$$g^j_{\eta_n}(u_n) > \left(\frac{x^i_j - (l-s)}{\eta_n} - R_\epsilon\right)(1-\epsilon) - \frac{2K}{\eta_n}\epsilon,$$
$$g^j_{\eta_n}(u_n) < \left(\frac{x^i_j + (l-s)}{\eta_n} + R_\epsilon\right)(1-\epsilon) + \frac{2K}{\eta_n}\epsilon.$$

From the above inequalities it is clear that we can choose $s > \epsilon > 0$, sufficiently small such that

$$g^j_{\eta_n}(u_n) \in \left(\frac{x^i_j - l}{\eta_n}, \frac{x^i_j + l}{\eta_n}\right) \quad \text{for } n \text{ sufficiently large.}$$

This contradicts $g_{\eta_n}(u_n) \in \partial C^i_{l/\eta_n}$.

In the case (I_3), we may assume that $\eta_n y_n \to \tilde{y} \in \bar{C}^i_{l+s}$ as $n \to \infty$, $\tilde{y}_i \geqslant x^i_j + l + s$ for some i and

$$(y_n)_j > \frac{x^i_j + l + s/2}{\eta_n} \quad \text{for all } n.$$

For $|x_j| \leqslant R_\epsilon$ we have

$$x_j + (y_n)_j > \frac{x^i_j + l + s/2}{\eta_n} - R_\epsilon$$

and

$$g^j_{\eta_n}(u_n) > \left(\frac{x^i_j - (l-s)}{\eta_n} - R_\epsilon\right)(1-\epsilon) - \frac{2K}{\eta_n}\epsilon$$

for sufficiently small $\epsilon > 0$, $s < \epsilon$ and sufficiently large n. This contradicts $g_{\eta_n}(u_n) \in \partial C^i_{l/\eta_n}$.

The case (I_4) is excluded by assuming $\eta_n y_n \to \infty$ as $n \to \infty$, and using a similar argument to case (I_1).

CASE II. Set

$$\int_{\mathbb{R}^N} |\nabla u_0|^2 + u_0^2 - \int_{\mathbb{R}^N} f(\eta_n x + \eta_n y_n)|u_0|^p = A + o(1).$$

Then, by (4.8),

$$\int_{\mathbb{R}^N} |\nabla w_n|^2 + w_n^2 - \int_{\mathbb{R}^N} f(\eta_n x + \eta_n y_n)|w_n|^p = -A + o(1).$$

Without loss of generality, we may assume that $A > 0$ ($A < 0$ can be considered similarly). We can find a sequence $\{t_n\}$ with $t_n \to 1$ as $n \to \infty$ such that $v_n = t_n w_n$ satisfies

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 + v_n^2 - \int_{\mathbb{R}^N} f(\eta_n x + \eta_n y_n)|v_n|^p = -A.$$

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Since $u_0 \in \boldsymbol{M}_{f(\eta_n x+\eta_n y_n),0}(A+o(1))$, by (4.4)–(4.6) and lemma 2.1 we have

$$\begin{aligned}
I_{f_{\eta_n},h_{\eta_n}}(u_n) &= \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u_0|^2+u_0^2-\frac{1}{p}\int_{\mathbb{R}^N}f(\eta_n x+\eta_n y_n)|u_0|^p \\
&\quad+\frac{1}{2}\int_{\mathbb{R}^N}|\nabla w_n|^2+w_n^2-\frac{1}{p}\int_{\mathbb{R}^N}f(\eta_n x+\eta_n y_n)|w_n|^p+o(1) \\
&\geqslant \frac{A+o(1)}{2}+\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v_n|^2+v_n^2 \\
&\quad-\frac{1}{p}\int_{\mathbb{R}^N}f(\eta_n x+\eta_n y_n)|v_n|^p+o(1) \\
&= \alpha_{f(\eta_n x+\eta_n y_n),0}(A)+\alpha_{f(\eta_n x+\eta_n y_n),0}(-A)+o(1) \\
&> \alpha_{f(\eta_n x+\eta_n y_n),0}+\left(\frac{p-2}{4p}\right)A+o(1) \\
&\geqslant \alpha_{f_{\max},0}+\left(\frac{p-2}{4p}\right)A+o(1),
\end{aligned}$$

which is a contradiction. If $A=0$, we can find $s_n,t_n>0$, $s_n\to 1$ as $n\to\infty$ such that $\bar{w}_n=t_n w_n$, $\bar{v}_n=s_n u_0$ satisfy

$$\int_{\mathbb{R}^N}|\nabla\bar{w}_n|^2+\bar{w}_n^2=\int_{\mathbb{R}^N}f(\eta_n x+\eta_n y_n)|\bar{w}_n|^p,$$
$$\int_{\mathbb{R}^N}|\nabla\bar{v}_n|^2+\bar{v}_n^2=\int_{\mathbb{R}^N}f(\eta_n x+\eta_n y_n)|\bar{v}_n|^p.$$

Hence,

$$\begin{aligned}
\lim_{n\to\infty}I_{f_{\eta_n},h_{\eta_n}}(u_n) &= \lim_{n\to\infty}\biggl[\frac{1}{2}\int_{\mathbb{R}^N}|\nabla\bar{v}_n|^2+\bar{v}_n^2-\frac{1}{p}\int_{\mathbb{R}^N}f(\eta_n x+\eta_n y_n)|\bar{v}_n|^p \\
&\qquad+\frac{1}{2}\int_{\mathbb{R}^N}|\nabla\bar{w}_n|^2+\bar{w}_n^2-\frac{1}{p}\int_{\mathbb{R}^N}f(\eta_n x+\eta_n y_n)|\bar{w}_n|^p\biggr] \\
&> \alpha_{f_{\max},0}.
\end{aligned}$$

This completes the proof. □

Throughout this section, take $\eta_0=\min\{\eta_\varepsilon,\eta_\delta\}$; η_ε and η_δ are as in lemmas 4.2 and 4.3. Using the idea of Ni and Takagi [21] and Wu [26], we have the following result.

Lemma 4.4. *For each $\eta\in(0,\eta_0)$ and $u\in N_\eta^i$, there exist $\epsilon>0$ and a differentiable function $t^*: B(0;\epsilon)\subset H^1(\mathbb{R}^N)\to\mathbb{R}^+$ such that $t^*(0)=1$, $t^*(v)(u-v)\in N_\eta^i$ for all $v\in B(0;\epsilon)$ and*

$$\langle (t^*)'(0),v\rangle=\frac{2\int_{\mathbb{R}^N}\nabla u\nabla v+uv-p\int_{\mathbb{R}^N}f_\eta|u|^{p-2}uv-\eta^{2(p-q)/(p-2)}\int_{\mathbb{R}^N}h_\eta|u|^{q-2}uv}{\int_{\mathbb{R}^N}|\nabla u|^2+u^2-(p-1)\int_{\mathbb{R}^N}f_\eta|u|^p}$$

for all $v\in H^1(\mathbb{R}^N)$.

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Proof. For $u \in N^i_\eta$, define a function $F_u : \mathbb{R} \times H^1(\mathbb{R}^N) \to \mathbb{R}$ by

$$\begin{aligned} F_u(t,w) &= \langle I'_{f_\eta,h_\eta}(t(u-w)), t(u-w)\rangle \\ &= t^2 \int_{\mathbb{R}^N} |\nabla(u-w)|^2 + (u-w)^2 - |t|^p \int_{\mathbb{R}^N} f_\eta |u-w|^p \\ &\quad - \eta^{2(p-q)/(p-2)}|t|^q \int_{\mathbb{R}^N} h_\eta |u-w|^q. \end{aligned}$$

Then $F_u(1,0) = \langle I'_{f_\eta,h_\eta}(u), u\rangle = 0$ and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} F_u(1,0) &= 2\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 - p\int_{\mathbb{R}^N} f_\eta |u|^p - \eta^{2(p-q)/(p-2)} q \int_{\mathbb{R}^N} h_\eta |u|^q \\ &= \int_{\mathbb{R}^N} |\nabla u|^2 + u^2 - (p-1)\int_{\mathbb{R}^N} f_\eta |u|^p < 0. \end{aligned}$$

According to the implicit function theorem, there exist $\epsilon > 0$ and a differentiable function $t^* : B(0;\epsilon) \subset H^1(\mathbb{R}^N) \to \mathbb{R}$ such that $t^*(0) = 1$,

$$\langle (t^*)'(0), v\rangle = \frac{2\int_{\mathbb{R}^N} \nabla u \nabla v + uv - p\int_{\mathbb{R}^N} f_\eta |u|^{p-2}uv - \eta^{2(p-q)/(p-2)}\int_{\mathbb{R}^N} h_\eta |u|^{q-2}uv}{\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 - (p-1)\int_{\mathbb{R}^N} f_\eta |u|^p}$$

and

$$F_u(t^*(v), v) = 0 \quad \text{for all } v \in B(0;\epsilon),$$

which is equivalent to

$$\langle I'_{f_\eta,h_\eta}(t^*(v)(u-v)), t^*(v)(u-v)\rangle = 0 \quad \text{for all } v \in B(0;\epsilon).$$

Furthermore,

$$\int_{\mathbb{R}^N} |\nabla t^*(v)(u-v)|^2 + [t^*(v)(u-v)]^2 - (p-1)\int_{\mathbb{R}^N} f_\eta |t^*(v)(u-v)|^p < 0$$

and

$$g_\eta(t^*(v)(u-v)) \in C^i_{l/\eta}$$

still holds if ϵ is sufficiently small by the continuity of the maps g_η and t^*. □

PROPOSITION 4.5. *For each $\eta \in (0, \eta_0)$ we have*

$$\alpha^-_{f_\eta,h_\eta} \leqslant \gamma^i_\eta < \min\{\alpha_{f_\eta,h_\eta} + \alpha_{f^\infty,0}, \tilde{\gamma}^i_\eta\}$$

and there exists a sequence $\{u_n\} \subset N^i_\eta$ such that

$$\begin{aligned} I_{f_\eta,h_\eta}(u_n) &= \gamma^i_\eta + o(1), \\ I'_{f_\eta,h_\eta}(u_n) &= o(1) \in H^{-1}(\mathbb{R}^N) \end{aligned}$$

for all $i = 1, 2, \dots, k$.

Proof. If $\bar{N}^i_\eta$ denotes the closure of N^i_η, then first we note that $\bar{N}^i_\eta = N^i_\eta \cup \partial N^i_\eta$ for each $i = 1, 2, \dots, k$. It then follows from lemmas 4.2 and 4.3 that, for a positive number $\varepsilon \leqslant \delta$ and taking $\eta_0 = \min\{\eta_\varepsilon, \eta_\delta\}$, we obtain

$$\gamma^i_\eta < \min\{\alpha_{f_\eta,h_\eta} + \alpha_{f^\infty,0}, \tilde{\gamma}^i_\eta\} \quad \text{for } i = 1, 2, \dots, k,\ \eta \in (0, \eta_0). \tag{4.10}$$

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Hence,

$$\gamma_\eta^i = \inf\{I_{f_\eta,h_\eta}(u) \mid u \in \bar{N}_\eta^i\} \quad \text{for } i = 1, 2, \dots, k. \tag{4.11}$$

Now we fix $i \in \{1, 2, \dots, k\}$. Applying the Ekeland variational principle [14], there exists a minimizing sequence $\{u_n\} \subset \bar{N}_\eta^i$ such that

$$I_{f_\eta,h_\eta}(u_n) < \gamma_\eta^i + \frac{1}{n} \tag{4.12}$$

and

$$I_{f_\eta,h_\eta}(u_n) \leqslant I_{f_\eta,h_\eta}(w) + \frac{1}{n}\|w - u_n\|_{H^1} \quad \text{for all } w \in \bar{N}_\eta^i. \tag{4.13}$$

Using (4.10) we may assume that $u_n \in N_\eta^i$ for n sufficiently large. Applying lemma 4.4 with $u = u_n$, we obtain the function $t_n^* : B(0; \epsilon_n) \to \mathbb{R}$ for some $\epsilon_n > 0$ such that $t_n^*(w)(u_n - w) \in N_\eta^i$. Let $0 < \delta < \epsilon_n$ and $u < H^1(\mathbb{R}^N)$ with $u \not\equiv 0$. We set

$$w_\delta = \frac{\delta u}{\|u\|_{H^1}}$$

and $z_\delta = t_n^*(w_\delta)(u_n - w_\delta)$. Since $z_\delta \in N_\eta^i$, we deduce from (4.13) that

$$I_{f_\eta,h_\eta}(z_\delta) - I_{f_\eta,h_\eta}(u_n) \geqslant -\frac{1}{n}\|z_\delta - u_n\|_{H^1}.$$

By the mean-value theorem, we obtain

$$\langle I'_{f_\eta,h_\eta}(u_n), z_\delta - u_n\rangle + o(\|z_\delta - u_n\|) \geqslant -\frac{1}{n}\|z_\delta - u_n\|_{H^1}.$$

Therefore,

$$\begin{aligned}\langle I'_{f_\eta,h_\eta}(u_n), -w_\delta\rangle + (t_n^*(w_\delta) - 1)\langle I'_{f_\eta,h_\eta}(u_n), (u_n - w_\delta)\rangle \\ \geqslant -\frac{1}{n}\|z_\delta - u_n\|_{H^1} + o(\|z_\delta - u_n\|).\end{aligned} \tag{4.14}$$

Now we observe that $t_n^*(w_\delta)(u_n - w_\delta) \in N_\eta^i$ and consequently we get from (4.14) that

$$\begin{aligned}-\delta\left\langle I'_{f_\eta,h_\eta}(u_n), \frac{u}{\|u\|_{H^1}}\right\rangle + \frac{(t_n^*(w_\delta) - 1)}{t_n^*(w_\delta)}\langle I'_{f_\eta,h_\eta}(z_\delta), t_n^*(w_\delta)(u_n - w_\delta)\rangle \\ + (t_n^*(w_\delta) - 1)\langle I'_{f_\eta,h_\eta}(u_n) - I'_{f_\eta,h_\eta}(z_\delta), (u_n - w_\delta)\rangle \\ \geqslant -\frac{1}{n}\|z_\delta - u_n\|_{H^1} + o(\|z_\delta - u_n\|).\end{aligned}$$

Then we write the above inequality in the following form

$$\begin{aligned}\left\langle I'_{f_\eta,h_\eta}(u_n), \frac{u}{\|u\|_{H^1}}\right\rangle \leqslant \frac{\|z_\delta - u_n\|_{H^1}}{\delta n} + \frac{o(\|z_\delta - u_n\|_{H^1})}{\delta} \\ + \frac{(t_n^*(w_\delta) - 1)}{\delta}\langle I'_{f_\eta,h_\eta}(u_n) - I'_{f_\eta,h_\eta}(z_\delta), (u_n - w_\delta)\rangle.\end{aligned} \tag{4.15}$$

Since we can find a constant $C > 0$, independent of δ such that

$$\|z_\delta - u_n\|_{H^1} \leqslant \delta + C(|t_n^*(w_\delta) - 1|)$$

and

$$\lim_{\delta\to 0} \frac{|t_n^*(w_\delta) - 1|}{\delta} \leqslant \|(t_n^*)'(0)\| \leqslant C.$$

For a fixed n, let $\delta \to 0$ in (4.15). Using the fact that

$$\lim_{\delta\to 0} \|z_\delta - u_n\|_{H^1} = 0,$$

we obtain

$$\left\langle I'_{f_\eta,h_\eta}(u_n), \frac{u}{\|u\|_{H^1}} \right\rangle \leqslant \frac{C}{n}.$$

This implies

$$I_{f_\eta,h_\eta}(u_n) = \gamma_\eta^i + o(1)$$

and

$$I'_{f_\eta,h_\eta}(u_n) = o(1) \quad \text{in } H^{-1}(\mathbb{R}^N).$$

□

We need the following proposition to provide the precise description of the Palais–Smale sequences for I_{f_η,h_η}.

PROPOSITION 4.6. *Assume that* $\{u_n\} \subset \boldsymbol{M}^-_{f_\eta,h_\eta}$ *is a sequence satisfying*

$$I_{f_\eta,h_\eta}(u_n) = \beta + o(1),$$
$$I'_{f_\eta,h_\eta}(u_n) = o(1) \text{ in } H^{-1}(\mathbb{R}^N),$$

where $\beta < \alpha_{f_\eta,h_\eta} + \alpha_{f^\infty,0}$. *There then exist a subsequence* $\{u_n\}$ *and* u_0 *in* $H^1(\mathbb{R}^N)$ *such that* $u_n \to u_0$ *strongly in* $H^1(\mathbb{R}^N)$ *and* $I_{f_\eta,h_\eta}(u_0) = \beta$.

Proof. By lemma 3.1(ii), there exist a subsequence $\{u_n\}$ and u_0 in $H^1(\mathbb{R}^N)$ such that

$$u_n \rightharpoonup u_0 \quad \text{weakly in } H^1(\mathbb{R}^N).$$

First, we claim that $u_0 \not\equiv 0$. Otherwise, by $h \in L^{2/(2-q)}(\mathbb{R}^N)$, the Egorov theorem and the Hölder inequality, we have

$$\|u_n\|^2_{H^1} = \int_{\mathbb{R}^N} f^\infty |u_n|^p + o(1) \tag{4.16}$$

and

$$\begin{aligned}\left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^N} f^\infty |u_n|^p &= \tfrac{1}{2}\|u_n\|^2_{H^1} - \frac{1}{p} \int_{\mathbb{R}^N} f_\eta |u_n|^p \\ &\quad - \frac{1}{q}\eta^{2(p-q)/(p-2)} \int_{\mathbb{R}^N} h_\eta |u_n|^q + o(1) \\ &= \beta + o(1).\end{aligned}$$

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Moreover, $\{u_n\} \subset \boldsymbol{M}^-_{f_\eta,h_\eta}$ and

$$\|u_n\|_{H^1} > c \quad \text{for some } c > 0.$$

We get $\beta \geqslant \alpha_{f^\infty,0}$, this contradicts the condition $\beta < \alpha_{f_\eta,h_\eta} + \alpha_{f^\infty,0}$. Thus, u_0 is a non-trivial solution of equation (2.1) and $I_{f_\eta,h_\eta}(u_0) \geqslant \alpha_{f_\eta,h_\eta}$. We write $u_n = u_0 + v_n$ with $v_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$. By the Brézis–Lieb lemma [8], we have

$$\begin{aligned}\int_{\mathbb{R}^N} f_\eta |u_n|^p &= \int_{\mathbb{R}^N} f_\eta |u_0|^p + \int_{\mathbb{R}^N} f_\eta |v_n|^p + o(1)\\ &= \int_{\mathbb{R}^N} f_\eta |u_0|^p + \int_{\mathbb{R}^N} f^\infty |v_n|^p + o(1).\end{aligned}$$

Since $\{u_n\}$ is a bounded sequence in $H^1(\mathbb{R}^N)$ and so $\{v_n\}$ is also a bounded sequence in $H^1(\mathbb{R}^N)$. Moreover, by $h \in L^{2/(2-q)}(\mathbb{R}^N)$, the Egorov theorem and the Hölder inequality, we have

$$\int_{\mathbb{R}^N} h_\eta |v_n|^q = \int_{\mathbb{R}^N} h_\eta |u_n|^q - \int_{\mathbb{R}^N} h_\eta |u_0|^q + o(1) = o(1).$$

Hence, for n large enough, we can conclude that

$$\begin{aligned}\alpha_{f_\eta,h_\eta} + \alpha_{f^\infty,0} &> I_{f_\eta,h_\eta}(u_0 + v_n)\\ &= I_{f_\eta,h_\eta}(u_0) + \tfrac{1}{2}\|v_n\|^2_{H^1} - \frac{1}{p}\int_{\mathbb{R}^N} f^\infty |v_n|^p + o(1)\\ &\geqslant \alpha_{f_\eta,h_\eta} + \tfrac{1}{2}\|v_n\|^2_{H^1} - \frac{1}{p}\int_{\mathbb{R}^N} f^\infty |v_n|^p + o(1)\end{aligned}$$

or

$$\tfrac{1}{2}\|v_n\|^2_{H^1} - \frac{1}{p}\int_{\mathbb{R}^N} f^\infty |v_n|^p < \alpha_{f^\infty,0} + o(1). \tag{4.17}$$

Also, from $I'_{f_\eta,h_\eta}(u_n) = o(1)$ in $H^{-1}(\mathbb{R}^N)$, with $\{u_n\}$ uniformly bounded and u_0 a solution of equation (2.1), we obtain

$$o(1) = \langle I'_{f_\eta,h_\eta}(u_n), u_n\rangle = \|v_n\|^2_{H^1} - \int_{\mathbb{R}^N} f^\infty |v_n|^p + o(1). \tag{4.18}$$

We claim that (4.17) and (4.18) can hold simultaneously only if $\{v_n\}$ admits a subsequence $\{v_{n_i}\}$ which converges strongly to zero. Otherwise, the $\|v_n\|_{H^1}$ is bounded away from zero, that is

$$\|v_n\|_{H^1} \geqslant c \quad \text{for some } c > 0.$$

From (4.18), it follows that

$$\int_{\mathbb{R}^N} f^\infty |v_n|^p \geqslant \left(\frac{2p}{p-2}\right)\alpha_{f^\infty,0} + o(1).$$

By (4.17) and (4.18), for n large enough,

$$\begin{aligned}\alpha_{f^\infty,0} &\leqslant \left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^N} f^\infty |v_n|^p + o(1)\\ &= \tfrac{1}{2}\|v_n\|_{H^1}^2 - \frac{1}{p}\int_{\mathbb{R}^N} f^\infty |v_n|^p + o(1)\\ &< \alpha_{f^\infty,0},\end{aligned}$$

which is a contradiction. Therefore, $u_n \to u_0$ strongly in $H^1(\mathbb{R}^N)$ and $I_{f_\eta,h_\eta}(u_0) = \beta$. □

Proof of theorem 1.1. By propositions 4.5 and 4.6 for each $\eta \in (0,\eta_0)$, there exist sequences $\{u_n^i\} \subset N_\eta^i$ and $u_0^i \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that

$$\begin{aligned}I_{f_\eta,h_\eta}(u_n^i) &= \gamma_\eta^i + o(1),\\ I'_{f_\eta,h_\eta}(u_n^i) &= o(1) \quad \text{in } H^{-1}(\mathbb{R}^N),\end{aligned}$$

and

$$u_n^i \to u_0^i \text{ strongly in } H^1(\mathbb{R}^N).$$

Obviously, the function u_0^i is a solution of the equation (2.1) and $I_{f_\eta,h_\eta}(u_0^i) = \gamma_\eta^i$. It is clear that u_0^i is non-negative, by the maximum principle u_0^i is positive. Since $g_\eta^i(u_0^i) \in \overline{C_{l/\eta}(x^i)}$,

$$u_\eta \in \boldsymbol{M}^+_{f_\eta,h_\eta} \quad \text{and} \quad u_0^i \in \boldsymbol{M}^-_{f_\eta,h_\eta},$$

where u_η is a positive solution of equation (2.1) as in theorem 3.2. This implies that u_η and u_0^i are different. Letting $\lambda_0 = \eta_0^{-2}, U_\eta(x) = \lambda^{1/(p-2)} u_\eta(\sqrt{\lambda}x)$ and $U_i(x) = \lambda^{1/(p-2)} u_0^i(\sqrt{\lambda}x)$, we find that U_η and U_i are positive solutions of the equation (E_λ). □

Appendix A.

Lemma A.1.

$$\frac{2(p-q)}{p-2} - \frac{(2-q)N}{2} > 0,$$

where $1 \leqslant q < 2 < p < 2^*$ *and* $N \geqslant 1$.

Proof.

Case I. $1 \leqslant q < 2 < p < 2^*$ and $N = 1$. Since $q < 2 < p$ we have

$$\frac{6q-4}{2+q} < 2 < p.$$

Thus,

$$2(p-q) > \frac{(p-2)(2-q)}{2}$$

and so

$$\frac{2(p-q)}{p-2} - \frac{(2-q)}{2} > 0.$$

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Case II. $1 \leqslant q < 2 < p < 2^*$ and $N = 2$. Since $4 - (4/q) < 2 < p$ we have

$$4q - 4 < pq.$$

Thus,

$$2(p - q) > (p - 2)(2 - q)$$

and so

$$\frac{2(p-q)}{p-2} - (2 - q) > 0.$$

Case III. $1 \leqslant q < 2 < p < 2^*$ and $N \geqslant 3$. We need only to show that

$$p[4 - N(2 - q)] > 4q - 2N(2 - q), \tag{A 1}$$

since it is equivalent to

$$\frac{2(p-q)}{p-2} - \frac{(2-q)N}{2} > 0.$$

(a) $q = 1$ and $N \geqslant 3$. Then (A 1) becomes

$$p(4 - N) > 4 - 2N. \tag{A 2}$$

Clearly, (A 2) holds for $N = 3, 4$. Since

$$p < \frac{2N}{N-2} < \frac{2N-4}{N-4} \quad \text{for } N \geqslant 5,$$

(A 2) holds for $N \geqslant 5$.

(b) $1 < q < 2 < p < 2^*$ and $N = 3, 4$. Since $q < 2$, we have

$$4q - 2N(2 - q) < 8 - 4N + 2qN.$$

Moreover,

$$q > 1 \geqslant 2 - \frac{4}{N} \quad \text{for } N = 3, 4.$$

Thus,

$$\frac{2q - 2N(2-q)}{4 - N(2-q)} < 2 < p \quad \text{for } N = 3, 4.$$

(c) $q = 2 - (4/N)$ and $N \geqslant 5$. Since

$$4q - 2N(2 - q) = 4\left(2 - \frac{4}{N}\right) - 8 < 0$$

and $4 - (2 - q)N = 0$, we have

$$p(4 - N(2 - q)) = 0 < 4q - 2N(2 - q).$$

(d) $q \in (1, 2 - (4/N))$ and $N \geqslant 5$. Since

$$2N[N(2 - q) - 4] < (N - 2)[2N(2 - q) - 4q]$$

and $N(2 - q) - 4 > 0$, we have

$$p < \frac{2N}{N-2} < \frac{2N(2-q) - 4q}{N(2-q) - 4} \quad \text{for } N \geqslant 5.$$

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(e) $q \in (2 - (4/N), 2)$ and $N \geqslant 5$. Since

$$2[4 - N(2 - q)] > 4q - 2N(2 - q)$$

and $4 - N(2 - q) > 0$, we have

$$p > 2 > \frac{4q - 2N(2 - q)}{4 - N(2 - q)}.$$

This completes the proof. □

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(*Issued 13 June 2008*)