J. Appl. Prob. **55,** 69–81 (2018)
*doi:*10.1017/jpr.2018.6

A REMARK ON POSITIVE SOJOURN TIMES OF SYMMETRIC PROCESSES

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Abstract

We show that under some slight assumptions, the positive sojourn time of a product of symmetric processes converges towards $\frac{1}{2}$ as the number of processes increases. Monotony properties are then exhibited in the case of symmetric stable processes, and used, via a recurrence relation, to obtain upper and lower bounds on the moments of the occupation time (in the first and third quadrants) for two-dimensional Brownian motion. Explicit values are also given for the second and third moments in the n-dimensional Brownian case.

Keywords: Sojourn time; arcsine law; strong law of large numbers

2010 Mathematics Subject Classification: Primary 60F15
Secondary 60J65; 60E15

1. Introduction

In this paper, we are interested in the study of the random variables (RVs)

$$\mathcal{A}_n = \int_0^1 \mathbf{1}_{\{\prod_{i=1}^n X_u^{(i)} \geq 0\}} \, \mathrm{d}u,$$

where $X^{(1)}, \ldots, X^{(n)}$ are independent and identically distributed (i.i.d.) symmetric processes. The RV $\mathcal{A}_n$ may be interpreted as the time spent by an n-dimensional process (with independent components) in some symmetric orthants.

When $n = 1$, the RV $\mathcal{A}_1$ has been widely studied for several families of processes. The most celebrated example is the case of symmetric Lévy processes $X^{(1)} = L^{(1)}$ such that $\mathbb{P}(L_1^{(1)} = 0) = 0$, for which it is known that $\mathcal{A}_1$ follows the classic arcsine law (see [7]):

$$\mathbb{P}(\mathcal{A}_1 \in \mathrm{d}z) = \frac{1}{\pi\sqrt{z(1-z)}} \, \mathrm{d}z, \qquad z \in (0, 1).$$

When $n = 2$, the RV $\mathcal{A}_2$ corresponds to the time spent by a planar symmetric process $X^{(1)} + \mathrm{i}X^{(2)}$ in the first and third quadrant of the complex plane. In the special case of the planar Brownian motion $B^{(1)} + \mathrm{i}B^{(2)}$, a first attempt to find the law of $\mathcal{A}_2$ was undertaken by Ernst and Shepp [6] in which the authors tried to compute the double Laplace transform of $\mathcal{A}_2$. More generally, the study of the sojourn times of planar Brownian motion in a cone has already attracted much attention. In particular, it was proven by Mountford [13] that if C is a closed convex cone of magnitude θ with vertex at 0, then there exist two constants κ_1 and κ_2 such that

$$\kappa_1 t^{1/\xi} \leq \mathbb{P}\bigg(\int_0^1 \mathbf{1}_{\{(B_u^{(1)}, B_u^{(2)}) \in C\}} \, \mathrm{d}u \leq t\bigg) \leq \kappa_2 t^{1/\xi}, \qquad t \in [0, 1], \tag{1}$$

Received 6 December 2016; revision received 3 October 2017.
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with $\xi = (2/\pi)(2\pi - \theta)$. The first moments of this RV were computed by Desbois [4]; see also [2] in the special case $\theta = \pi/2$. Analogues of (1) for n-dimensional Brownian motion were obtained by Meyre and Werner [12] and Nakayama [14], in which the exponent ξ was related to the first eigenvalue of the Laplacian operator $-\Delta/2$. In addition to these bounds, the strong arcsine law (see [1]) yields the asymptotics of the sojourn time of an n-dimensional Brownian motion $(B^{(i)},\ 1 \le i \le n)$ in the positive orthant:

$$\frac{1}{\ln(t)} \int_1^t \prod_{i=1}^n \mathbf{1}_{\{B_u^{(i)} \ge 0\}} \frac{\mathrm{d}u}{u} \xrightarrow{\text{a.s.}} \frac{1}{2^n}, \qquad t \to +\infty,$$

where '$\xrightarrow{\text{a.s.}}$' denotes almost sure convergence. Observe that the summation here is logarithmic: we refer the reader to [3] for a general discussion between summability methods and limits of occupation times.

In this paper we shall first study the limit of the variables $\mathcal{A}_n$ as the dimension n goes to ∞.

Theorem 1. *Let $(X^{(i)},\ i \ge 1)$ be i.i.d. symmetric processes.*

(i) *The strong law of large numbers holds for the sequence $(\mathcal{A}_n,\ n \ge 1)$:*

$$\frac{1}{k} \sum_{n=1}^k \mathcal{A}_n \xrightarrow{\text{a.s.}} \frac{1}{2}, \qquad k \to +\infty.$$

(ii) *Assume that for almost every (a.e.) $u \in (0, 1)$, the RVs $X_u^{(1)}$ have no atoms at 0 and that for a.e. $0 < u < s < 1$,*

$$0 < \mathbb{P}(X_u^{(1)} \ge 0,\ X_s^{(1)} \ge 0) < \tfrac{1}{2}. \tag{2}$$

Then, for any $p > 0$,

$$\mathcal{A}_n \xrightarrow{L^p} \tfrac{1}{2}, \qquad n \to +\infty,$$

where '$\xrightarrow{L^p}$' denotes convergence in L^p.

(iii) *Assume furthermore that*

$$\int_0^1 \int_0^1 \frac{1}{\mathbb{P}(X_u^{(1)} \le 0,\ X_s^{(1)} \ge 0)} \,\mathrm{d}u\,\mathrm{d}s < +\infty. \tag{3}$$

Then

$$\mathcal{A}_n \xrightarrow{\text{a.s.}} \tfrac{1}{2}, \qquad n \to +\infty.$$

When thinking of symmetric Lévy processes, an interpretation of this result is as follows. The usual arcsine law essentially explains that, although $L^{(1)}$ is centered, there is a high probability that it spends more time on one side of the axis than on the other. As the number of Lévy processes increases, so do the changes of sign of the product, hence, the resulting process spends a more balanced time on each side of the abscissa axis.

Remark 1. Note that an assumption such as (2) is necessary to obtain the L^p-convergence. Indeed, let, for instance, $(X_i,\ i \ge 1)$ be a family of i.i.d. symmetric RVs admitting a density. Define the processes

$$X_t^{(i)} = tX_i \quad (t \ge 0)$$

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which do not satisfy the assumption $\mathbb{P}(X_u^{(1)} \geq 0, X_s^{(1)} \geq 0) < \frac{1}{2}$. In this case, the RVs $\mathcal{A}_n$ all have the same law:

$$\mathcal{A}_n \overset{\mathrm{D}}{=} \tfrac{1}{2}(\delta_0 + \delta_1),$$

where '$\overset{\mathrm{D}}{=}$' denotes equality in distribution, and the L^p-convergence of Theorem 1 cannot hold.

Example 1. Assumption (3) is, for instance, satisfied by symmetric α-stable Lévy processes L with $\alpha > 1$. Indeed, for $0 < u < s$, using the symmetry, independent increments, and scaling properties, we first deduce that

$$\mathbb{P}(L_s \leq 0, L_u \geq 0) = \mathbb{P}\bigg(\bigg(\frac{s}{u} - 1\bigg)^{1/\alpha} L_1 \geq Z_1, Z_1 \geq 0\bigg),$$

where Z_1 is a symmetric α-stable RV independent from L_1. Next, for $\nu > 0$ small enough, applying Fubini's theorem:

$$\begin{aligned}\int_0^{+\infty} t^{-\nu-1}\mathbb{P}(t^{1/\alpha}L_1 \geq Z_1, Z_1 \geq 0)\,\mathrm{d}t &= \frac{1}{\nu}\mathbb{E}[L_1^{\nu\alpha}\mathbf{1}_{\{L_1\geq 0\}}]\mathbb{E}[Z_1^{-\nu\alpha}\mathbf{1}_{\{Z_1\geq 0\}}] \\ &= \frac{1}{4\nu}\mathbb{E}[|L_1|^{\nu\alpha}]\mathbb{E}[|Z_1|^{-\nu\alpha}] \\ &= \frac{1}{4\alpha^2\nu}\frac{\Gamma(\nu)}{\Gamma(\nu\alpha)\cos(\nu\alpha\pi/2)}\frac{\Gamma(-\nu)}{\Gamma(-\nu\alpha)\cos(\nu\alpha\pi/2)} \\ &= \frac{1}{2\alpha\nu}\frac{\sin(\nu\alpha\pi/2)}{\sin(\nu\pi)\cos(\nu\alpha\pi/2)}.\end{aligned}$$

Therefore, using the inverse mapping for the Mellin transform (see, for example, [9]), we obtain the asymptotics

$$F(t) := \mathbb{P}(t^{1/\alpha}L_1 \geq Z_1, Z_1 \geq 0) \sim \frac{1}{\pi\alpha\sin(\pi/\alpha)}t^{1/\alpha}, \qquad t \to 0^+,$$

hence, by a change of variable

$$\begin{aligned}\int_0^1\bigg(\int_0^1 \frac{1}{\mathbb{P}(L_u \leq 0, L_s \geq 0)}\,\mathrm{d}u\bigg)\mathrm{d}s &= 2\int_0^1\bigg(\int_0^s \frac{1}{F(s/u-1)}\,\mathrm{d}u\bigg)\mathrm{d}s \\ &= \int_0^{+\infty}\frac{1}{F(t)(t+1)^2}\,\mathrm{d}t < +\infty,\end{aligned}$$

which is assumption (3).

The remainder of the paper is organized as follows. We prove Theorem 1 in Section 2, then study some monotony properties of $\mathcal{A}_n$ when dealing with stable processes in Section 3, and finally compute the first moments of $\mathcal{A}_n$ and state some bounds on $\mathcal{A}_2$ for Brownian motion in Section 4.

2. Proof of Theorem 1

Proof. (i) Let us first define the centered RVs

$$\mathcal{A}_n^* = \mathcal{A}_n - \tfrac{1}{2}$$

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and observe that these RVs are uncorrelated. Indeed, decomposing $\mathcal{A}_{k+n}$ and using the tower property of conditional expectations for $k \geq 1$,

$$\begin{aligned}
&\mathbb{E}[\mathcal{A}_n^* \mathcal{A}_{n+k}^*] \\
&\quad = \mathbb{E}[\mathcal{A}_n \mathcal{A}_{n+k}] - \tfrac{1}{4} \\
&\quad = \mathbb{E}\bigg[\mathcal{A}_n \int_0^1 \big(\mathbf{1}_{\{\prod_{i=1}^n X_u^{(i)} \geq 0\}} \mathbf{1}_{\{\prod_{i=n+1}^{n+k} X_u^{(i)} \geq 0\}} + \mathbf{1}_{\{\prod_{i=1}^n X_u^{(i)} \leq 0\}} \mathbf{1}_{\{\prod_{i=n+1}^{n+k} X_u^{(i)} \leq 0\}}\big)\, \mathrm{d}u\bigg] - \frac{1}{4} \\
&\quad = \mathbb{E}\bigg[\mathcal{A}_n \int_0^1 \bigg(\mathbf{1}_{\{\prod_{i=1}^n X_u^{(i)} \geq 0\}} \frac{1}{2} + \mathbf{1}_{\{\prod_{i=1}^n X_u^{(i)} \leq 0\}} \frac{1}{2}\bigg)\, \mathrm{d}u\bigg] - \frac{1}{4} \\
&\quad = \tfrac{1}{2}\mathbb{E}[\mathcal{A}_n] - \tfrac{1}{4} \\
&\quad = 0.
\end{aligned}$$

Now, since the RVs $(\mathcal{A}_n^*,\ n \geq 1)$ are uniformly bounded by 1, the result will follow from Theorem 1 of [10] after having checked that

$$\sum_{k \geq 1} \frac{1}{k} \mathbb{E}\bigg[\bigg(\frac{1}{k} \sum_{n=1}^k \mathcal{A}_n^*\bigg)^2\bigg] < \infty.$$

But this is immediate since developing the square and applying Fubini's theorem:

$$\sum_{k \geq 1} \frac{1}{k} \mathbb{E}\bigg[\bigg(\frac{1}{k} \sum_{n=1}^k \mathcal{A}_n^*\bigg)^2\bigg] = \sum_{k \geq 1} \frac{1}{k^3} \sum_{n=1}^k \mathbb{E}[(\mathcal{A}_n^*)^2] \leq \sum_{k \geq 1} \frac{1}{k^2} < +\infty,$$

hence, we conclude that

$$\frac{1}{k} \sum_{n=1}^k \mathcal{A}_n^* \xrightarrow{\text{a.s.}} 0, \qquad k \to +\infty,$$

which is Theorem 1(i).

(ii) To prove the L^p-convergence, let us consider, for $n \geq 1$, the function $F_n \colon [0, 1]^2 \to [0, 1]$ defined by

$$F_n(u, s) = \mathbb{P}\bigg(\prod_{i=1}^n X_u^{(i)} \geq 0, \prod_{i=1}^n X_s^{(i)} \geq 0\bigg).$$

By symmetry and since there are no atoms at 0, we may decompose F_{n+1} as

$$\begin{aligned}
F_{n+1}(u, s) = {} & 2\mathbb{P}\bigg(\prod_{i=1}^n X_u^{(i)} \geq 0, \prod_{i=1}^n X_s^{(i)} \geq 0\bigg)\mathbb{P}(X_u^{(n+1)} \geq 0, X_s^{(n+1)} \geq 0) \\
& + 2\mathbb{P}\bigg(\prod_{i=1}^n X_u^{(i)} \leq 0, \prod_{i=1}^n X_s^{(i)} \geq 0\bigg)\mathbb{P}(X_u^{(n+1)} \leq 0, X_s^{(n+1)} \geq 0)
\end{aligned}$$

and rewrite this in the form

$$\begin{aligned}
F_{n+1}(u, s) &= 2F_n(u, s)F_1(u, s) + 2\big(\tfrac{1}{2} - F_n(u, s)\big)\big(\tfrac{1}{2} - F_1(u, s)\big) \\
&= 4\big(F_n(u, s) - \tfrac{1}{4}\big)\big(F_1(u, s) - \tfrac{1}{4}\big) + \tfrac{1}{4}.
\end{aligned}$$

In particular, by iteration, we deduce that

$$F_{n+1}(u,s) - \tfrac{1}{4} = \big(F_n(u,s) - \tfrac{1}{4}\big)(4F_1(u,s) - 1) = \tfrac{1}{4}(4F_1(u,s) - 1)^{n+1}. \tag{4}$$

Now, for a.e. $u \neq s$, we have by assumption, $-1 < 4F_1(u,s) - 1 < 1$, so we may let $n \to +\infty$ to obtain

$$F_n(u,s) \to \tfrac{1}{4}, \qquad n \to +\infty.$$

Finally, applying the dominated convergence theorem, we have

$$\mathbb{E}\Big[\Big(\mathcal{A}_n - \frac{1}{2}\Big)^2\Big] = \mathbb{E}[\mathcal{A}_n^2] - \frac{1}{4} = \int_0^1 \int_0^1 F_n(u,s)\,\mathrm{d}u\,\mathrm{d}s - \frac{1}{4} \to 0, \qquad n \to +\infty,$$

which proves the L^2-convergence of Theorem 1, hence, the L^p-convergence for any $0 < p \le 2$ by Hölder's inequality. But, since, for any $n \in \mathbb{N}$, $|\mathcal{A}_n - \frac{1}{2}| \le 1$, we further obtain, for $p \ge 2$,

$$\mathbb{E}\big[\big|\mathcal{A}_n - \tfrac{1}{2}\big|^p\big] \le \mathbb{E}\big[\big(\mathcal{A}_n - \tfrac{1}{2}\big)^2\big] \to 0, \qquad n \to +\infty,$$

which completes the proof of (ii).

(iii) Finally, to obtain the almost sure convergence of (iii), we apply Fubini's theorem to obtain the bound, thanks to (4),

$$\begin{aligned}
\sum_{n=1}^{+\infty} \mathbb{E}\Big[\Big(\mathcal{A}_n - \frac{1}{2}\Big)^2\Big] &= \sum_{n=1}^{+\infty} \int_0^1 \int_0^1 \Big(F_n(u,s) - \frac{1}{4}\Big)\,\mathrm{d}u\,\mathrm{d}s \\
&= \frac{1}{4} \int_0^1 \int_0^1 \frac{4F_1(u,s) - 1}{2 - 4F_1(u,s)}\,\mathrm{d}u\,\mathrm{d}s \\
&\le \frac{1}{16} \int_0^1 \int_0^1 \frac{1}{\mathbb{P}(X_u^{(1)} \le 0, X_s^{(1)} \ge 0)}\,\mathrm{d}u\,\mathrm{d}s \\
&< +\infty.
\end{aligned}$$

The almost sure convergence then follows from the usual application of the Bienaymé–Tchebychev inequality and the Borel–Cantelli lemma. □

3. Monotonicity for stable processes

We now assume that $(X^{(i)} = L^{(i)})_{i \ge 1}$ are independent symmetric α-stable Lévy processes with $\alpha \in (0, 2]$ defined on a probability space $(\Omega, \mathcal{F}_\infty, \mathbb{P})$. From Theorem 1(ii), we deduce that for any $p > 0$,

$$\mathbb{E}[\mathcal{A}_n^p] \to \big(\tfrac{1}{2}\big)^p, \qquad n \to +\infty.$$

When dealing with stable processes, it turns out that the sequence $(\mathbb{E}[\mathcal{A}_n^p],\ n \ge 1)$ is monotone, according to the value of p (i.e. to the convexity of the function $x \mapsto x^p$).

Proposition 1. *Let $p > 0$ be fixed. The sequence*

$$(\mathbb{E}[\mathcal{A}_n^p],\ n \ge 1) \text{ is } \begin{cases} \text{decreasing if } p > 1, \\ \text{increasing if } 0 < p < 1. \end{cases}$$

As a consequence, for any $\lambda \in \mathbb{R}$ and $n \ge 1$,

$$\mathbb{E}[\mathrm{e}^{\lambda \mathcal{A}_{n+1}}] \le \mathbb{E}[\mathrm{e}^{\lambda \mathcal{A}_n}].$$

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For symmetric stable Lévy processes, the RVs $(\mathcal{A}_n,\ n \geq 1)$ are thus ordered via moment-generating functions or Laplace transforms.

Proof of Proposition 1. We start the proof with a simple lemma.

Lemma 1. *Let $n \geq 1$ and $(X_i)_{i\leq n}$ be i.i.d. symmetric RVs with common density f and let $(A_i)_{i\leq n}$ be RVs, independent from the $(X_i)_{i\leq n}$ and such that $\mathbb{P}(\prod_{i=1}^n A_i > 0) = 1$. Then the function*

$$t \mapsto \mathbb{P}\bigg(\prod_{i=1}^n (X_i + tA_i) \geq 0\bigg) \quad \textit{is increasing from } \tfrac{1}{2} \textit{ to } 1.$$

Proof. Observe first that by conditioning on the distribution of the sequence $(A_i)_{i\leq n}$,

$$\mathbb{P}\bigg(\prod_{i=1}^n (X_i + tA_i) \geq 0\bigg) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \mathbb{P}\bigg(\prod_{i=1}^n (X_i + ta_i) \geq 0\bigg) \mathbb{P}(A_1 \in da_1, \ldots, A_n \in \mathrm{d}a_n),$$

we need to prove that only the function

$$\Psi_n(t) = \mathbb{P}\bigg(\prod_{i=1}^n (X_i + ta_i) \geq 0\bigg)$$

is increasing, under the assumption that $\prod_{i=1}^n a_i > 0$. Next, for any $n \geq 1$, we have $\Psi_n(0) = \frac{1}{2}$ and $\lim_{t\to+\infty} \Psi_n(t) = 1$. We shall prove by induction on n that Ψ_n is increasing . For $n = 1$, the result is clear since in this case $a_1 > 0$. Assume now that $n \geq 2$ and that Ψ_{n-1} is increasing from $\frac{1}{2}$ to 1. Since the (X_i) are independent, we may decompose

$$\Psi_n(t) = \mathbb{P}(X_n + ta_n \geq 0)\mathbb{P}\bigg(\prod_{i=1}^{n-1} (X_i + ta_i) \geq 0\bigg) + \mathbb{P}(X_n + ta_n \leq 0)\mathbb{P}\bigg(\prod_{i=1}^{n-1} (X_i + ta_i) \leq 0\bigg).$$

We now separate two cases.

Case 1. Assume first that $a_n > 0$. Then $\prod_{i=1}^{n-1} a_i > 0$ and differentiating yields

$$\begin{aligned}
\Psi_n'(t) &= a_n f(-ta_n)\Psi_{n-1}(t) + \mathbb{P}(X_n + ta_n \geq 0)\Psi_{n-1}'(t) \\
&\quad - a_n f(-ta_n)(1 - \Psi_{n-1}(t)) - \mathbb{P}(X_n + ta_n \leq 0)\Psi_{n-1}'(t) \\
&= a_n f(-ta_n)(2\Psi_{n-1}(t) - 1) + \Psi_{n-1}'(t)(\mathbb{P}(X_n + ta_n \geq 0) - \mathbb{P}(X_n + ta_n \leq 0)).
\end{aligned}$$

Since $\Psi_{n-1}(t) \geq \frac{1}{2}$ and $\mathbb{P}(X_n + ta_n \geq 0) > \mathbb{P}(X_n + ta_n \leq 0)$, we deduce from the recursion hypothesis that $\Psi_n'(t) > 0$.

Case 2. Assume now that $a_n < 0$. Then $\prod_{i=1}^{n-1} a_i < 0$ and we deduce from the symmetry of X_1 and X_n that (with the usual convention that empty products are 1)

$$\begin{aligned}
\Psi_n(t) &= \mathbb{P}(X_n + t(-a_n) \leq 0)\mathbb{P}\bigg((X_1 + t(-a_1)) \prod_{i=2}^{n-1} (X_i + ta_i) \leq 0\bigg) \\
&\quad + \mathbb{P}(X_n + t(-a_n) \geq 0)\mathbb{P}\bigg((X_1 + t(-a_1)) \prod_{i=2}^{n-1} (X_i + ta_i) \geq 0\bigg).
\end{aligned}$$

The result then follows from case 1, since $-a_n > 0$ and $-\prod_{i=1}^{n-1} a_i > 0$. □

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We now return to the proof of Proposition 1. To simplify the notation, we set

$$P_u^{(n)} = \prod_{i=1}^{n} L_u^{(i)}.$$

Let us consider the function $F\colon \mathbb{R}^+ \to [0, 1]$ defined by

$$F(x) = \mathbb{E}\bigg[\bigg(\int_0^1 \mathbf{1}_{\{(x+Z_u)P_u^{(n)} \geq 0\}}\, \mathrm{d}u\bigg)^p\bigg],$$

where Z is another α-stable Lévy process independent from the $(L^{(i)})$. We shall prove that F is increasing on $[0, +\infty)$. By the change of variable $u = x^\alpha s$ and scaling, we have

$$F(x) = x^{\alpha p}\mathbb{E}\bigg[\bigg(\int_0^{1/x^\alpha} \mathbf{1}_{\{(1+Z_s)P_s^{(n)} \geq 0\}}\, \mathrm{d}s\bigg)^p\bigg].$$

Differentiating with respect to x and going back to the original variable, we obtain

$$F'(x) = \frac{\alpha p}{x}\mathbb{E}\bigg[\bigg(\int_0^1 \mathbf{1}_{\{(x+Z_u)P_u^{(n)} \geq 0\}}\, \mathrm{d}u\bigg)^p - \bigg(\int_0^1 \mathbf{1}_{\{(x+Z_u)P_u^{(n)} \geq 0\}}\, \mathrm{d}u\bigg)^{p-1} \mathbf{1}_{\{(x+Z_1)P_1^{(n)} \geq 0\}}\bigg].$$

Applying Fubini's theorem, we need to prove that

$$\begin{aligned}\int_0^1 \cdots \int_0^1 \mathbb{P}\bigg(\bigcap_{i=1}^{p}\{(x+Z_{u_i})P_{u_i}^{(n)} \geq 0\}\bigg)\, \mathrm{d}u_1 \cdots \mathrm{d}u_p \\ \geq \int_0^1 \cdots \int_0^1 \mathbb{P}\bigg(\bigcap_{i=1}^{p-1}\{(x+Z_{u_i})P_{u_i}^{(n)} \geq 0\} \cap \{(x+Z_1)P_1^{(n)} \geq 0\}\bigg)\, \mathrm{d}u_1 \cdots \mathrm{d}u_p.\end{aligned}$$

We shall, in fact, simply prove that the inequality holds on the integrands:

$$\begin{aligned}\mathbb{P}\bigg(\bigcap_{i=1}^{p-1}\{(x+Z_{u_i})P_{u_i}^{(n)} \geq 0\} \cap \{(x+Z_{u_p})P_{u_p}^{(n)} \geq 0\}\bigg) \\ \geq \mathbb{P}\bigg(\bigcap_{i=1}^{p-1}\{(x+Z_{u_i})P_{u_i}^{(n)} \geq 0\} \cap \{(x+Z_1)P_1^{(n)} \geq 0\}\bigg),\end{aligned} \tag{5}$$

where we may assume, up to renaming the variables, that $0 \leq u_1 \leq u_2 \leq \cdots \leq u_p \leq 1$. To simplify the notation, let us introduce the measure $\mathbb{Q}$ defined for $\Lambda \in \mathcal{F}_\infty$ by

$$\mathbb{Q}(\Lambda) = \mathbb{P}\bigg(\Lambda \;\bigg|\; \bigcap_{i=1}^{p-1}\{(x+Z_{u_i})P_{u_i}^{(n)} \geq 0\}\bigg).$$

Dividing both sides of (5) by $\mathbb{P}(\bigcap_{i=1}^{p-1}\{(x+Z_{u_i})P_{u_i}^{(n)} \geq 0\})$, we are thus led to prove that the function

$$t \to \mathbb{Q}((x+Z_{t+u_{p-1}})P_{t+u_{p-1}}^{(n)} \geq 0) \quad \text{is decreasing on } [0, 1-u_{p-1}].$$

Applying the Markov property, we may decompose

$$\mathbb{Q}((x+Z_{t+u_{p-1}})P^{(n)}_{t+u_{p-1}} \geq 0) = \mathbb{Q}\Big((x+Z_{u_{p-1}}+t^{1/\alpha}\widehat{Z}_1)\prod_{i=1}^{n}(L^{(i)}_{u_{p-1}}+t^{1/\alpha}\widehat{L}^{(i)}_1) \geq 0\Big),$$

where $\widehat{Z_1}$ and $(\widehat{L}^{(i)}_1)$ are independent symmetric α-stable RVs, independent from Z and the $(L^{(i)})$. Observe furthermore that, by the definition of $\mathbb{Q}$,

$$\mathbb{Q}\Big((x+Z_{u_{p-1}})\prod_{i=1}^{n}L^{(i)}_{u_{p-1}} > 0\Big) = \mathbb{Q}((x+Z_{u_{p-1}})P^{(n)}_{u_{p-1}} > 0) = 1.$$

Therefore, applying Lemma 1 with the sequences

$$(X_i,\ 1 \leq i \leq n) = (\widehat{L}^{(i)}_1,\ 1 \leq i \leq n), \qquad X_{n+1} = \widehat{Z}_1$$

and

$$(A_i,\ 1 \leq i \leq n) = (L^{(i)}_{u_{p-1}},\ 1 \leq i \leq n), \qquad A_{n+1} = x + Z_{u_{p-1}},$$

we deduce by composition that the function $t \to \mathbb{Q}((x+Z_{t+u_{p-1}})P^{(n)}_{t+u_{p-1}} \geq 0)$ is decreasing on $[0, 1-u_{p-1}]$ (in fact on $[0, +\infty)$), hence, the function F is increasing on $[0, +\infty)$. It remains then to apply the dominated convergence theorem, upon noting that

$$\mathbb{E}[\mathcal{A}^p_{n+1}] = F(0) \leq \lim_{x\to+\infty} F(x) = \mathbb{E}[\mathcal{A}^p_n],$$

which yields the proof for integer values.

Summing the different moments, we deduce that, for $\lambda \geq 0$ and $p \geq 1$,

$$\mathbb{E}[(\mathcal{A}_{n+1})^{\lfloor p\rfloor+1}\mathrm{e}^{\lambda\mathcal{A}_{n+1}}] \leq \mathbb{E}[(\mathcal{A}_n)^{\lfloor p\rfloor+1}\mathrm{e}^{\lambda\mathcal{A}_n}],$$

where $\lfloor p\rfloor$ denotes the integer part of p. Integrating this inequality against $\lambda^{\lfloor p\rfloor-p}$ on $(0, +\infty)$, we deduce that

$$\mathbb{E}[\mathcal{A}^p_{n+1}] \leq \mathbb{E}[\mathcal{A}^p_n] \quad (p \geq 1).$$

Next, by symmetry

$$\mathbb{E}[\mathrm{e}^{\lambda(1-\mathcal{A}_{n+1})}] \leq \mathbb{E}[\mathrm{e}^{\lambda(1-\mathcal{A}_n)}] \quad \Longleftrightarrow \quad \mathbb{E}[\mathrm{e}^{-\lambda\mathcal{A}_{n+1}}] \leq \mathbb{E}[\mathrm{e}^{-\lambda\mathcal{A}_n}],$$

hence, for $0 < p < 1$,

$$\int_0^{+\infty}\lambda^{-p-1}(1-\mathbb{E}[\mathrm{e}^{-\lambda\mathcal{A}_n}])\,\mathrm{d}\lambda \leq \int_0^{+\infty}\lambda^{-p-1}(1-\mathbb{E}[\mathrm{e}^{-\lambda\mathcal{A}_{n+1}}])\,\mathrm{d}\lambda,$$

which is exactly $\mathbb{E}[\mathcal{A}^p_n] \leq \mathbb{E}[\mathcal{A}^p_{n+1}]$ $(0 < p < 1)$. □

Remark 2. Below we provide an example of a process satisfying assumption (2), but for which the sequence $(\mathbb{E}[\mathcal{A}^2_n],\ n \geq 1)$ is not decreasing. Take, for instance,

$$X^{(1)}_t = B_t\,\mathbf{1}_{\{t\leq 1/2\}} - B_{t-1/2}\,\mathbf{1}_{\{t>1/2\}} \quad \text{and} \quad X^{(2)}_t = W_t\,\mathbf{1}_{\{t\leq 1/2\}} - W_{t-1/2}\,\mathbf{1}_{\{t>1/2\}},$$

where B and W are two independent Brownian motions started from 0. Then

$$\mathcal{A}_1 = \frac{1}{2} \quad \text{and} \quad \mathcal{A}_2 = 2\int_0^{1/2}\mathbf{1}_{\{B_tW_t\geq 0\}}\,\mathrm{d}t \overset{\mathrm{D}}{=} \int_0^1\mathbf{1}_{\{B_uW_u\geq 0\}}\,\mathrm{d}u,$$

hence (see the next section for the value of $\mathbb{E}[\mathcal{A}^2_2]$),

$$\mathbb{E}[\mathcal{A}^2_1] = \frac{1}{4} < \mathbb{E}[\mathcal{A}^2_2] = \frac{3}{8} - \frac{1}{2\pi^2}.$$

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4. A study of moments in the Brownian case

4.1. Second and third moments

The first three moments are easy to compute in the Brownian case. Indeed, from (4), the second moment of $\mathcal{A}_n$ is equal to

$$\mathbb{E}[\mathcal{A}_n^2] = \frac{1}{4} + \frac{1}{2}\int_0^1 \biggl(\int_0^s (4F_1(u,s)-1)^n \,\mathrm{d}u\biggr)\,\mathrm{d}s,$$

where, from Bingham and Doney [2], the quadrant probability is given by, for $0 \le u \le s$,

$$F_1(u,s) = \frac{1}{4} + \frac{1}{2\pi}\arcsin\biggl(\sqrt{\frac{u}{s}}\biggr).$$

After some changes of variables and successive integration by parts, we deduce that

$$\begin{aligned}\mathbb{E}[\mathcal{A}_n^2] &= \frac{1}{4} + \frac{1}{8\pi^n}\int_0^\pi t^n \sin(t)\,\mathrm{d}t \\ &= \frac{1}{4} + (-1)^{\lfloor n/2\rfloor+1}\frac{n!(n-2\lfloor n/2\rfloor-1)}{8\pi^n} + \frac{1}{8}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\frac{n!}{(n-2k)!}\pi^{-2k}.\end{aligned}$$

By symmetry, since $\mathbb{E}[(1-\mathcal{A}_n)^3] = \mathbb{E}[\mathcal{A}_n^3]$, we further obtain

$$\mathbb{E}[\mathcal{A}_n^3] = \tfrac{3}{2}\mathbb{E}[\mathcal{A}_n^2] - \tfrac{1}{4}.$$

In Table 1 we provide the first values of the second and third moments, in which the decreasing property may be observed.

4.2. Higher moments for two Brownian motions

Obtaining the explicit values of higher moments seems a difficult task as outlined in the literature; see [2], [4], and [6]. We propose here a method to obtain lower and upper bounds on these moments. Recall the moments of the arcsine distribution:

$$\mathbb{E}[\mathcal{A}_1^p] = \frac{1}{2^{2p}}\binom{p}{2p} = \frac{(2p)!}{2^{2p}(p!)^2} = \frac{\Gamma(p+1/2)}{\sqrt{\pi}\,\Gamma(p+1)} \sim \frac{1}{\sqrt{\pi p}}, \qquad p \to +\infty.$$

Table 1: First values of the second and third moments.

n	Second moment $\mathbb{E}[\mathcal{A}_n^2]$	Third moment $\mathbb{E}[\mathcal{A}_n^3]$
1	$\frac{3}{8} \simeq 0.375$	$\frac{5}{16} \simeq 0.3125$
2	$\frac{3}{8} - \frac{1}{2\pi^2} \simeq 0.3243$	$\frac{5}{16} - \frac{3}{4\pi^2} \simeq 0.2365$
3	$\frac{3}{8} - \frac{3}{4\pi^2} \simeq 0.299$	$\frac{5}{16} - \frac{9}{8\pi^2} \simeq 0.1985$
4	$\frac{3}{8} - \frac{3}{2\pi^2} + \frac{6}{\pi^4} \simeq 0.2846$	$\frac{5}{16} - \frac{9}{4\pi^2} + \frac{9}{\pi^4} \simeq 0.1769$
5	$\frac{3}{8} - \frac{5}{2\pi^2} + \frac{15}{\pi^4} \simeq 0.2757$	$\frac{5}{16} - \frac{15}{4\pi^2} + \frac{45}{2\pi^4} \simeq 0.1635$
∞	0.25	0.125

https://doi.org/10.1017/jpr.2018.6 Published online by Cambridge University Press

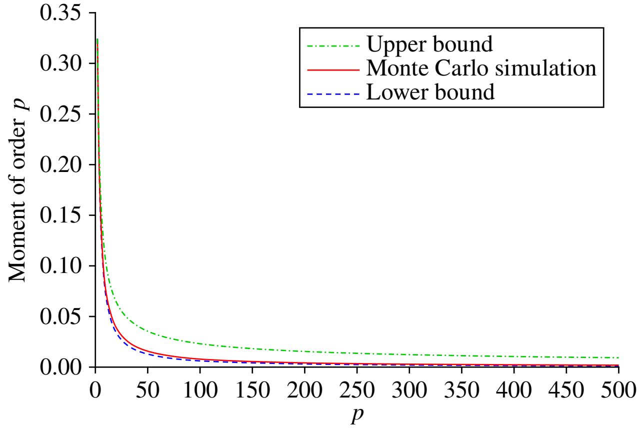

FIGURE 1: Monte Carlo simulation of $\mathbb{E}[\mathcal{A}_2^p]$ for $1 \le p \le 500$.

Proposition 2. *For any $p \ge 1$, we have*

$$\mathbb{E}[\mathcal{A}_2^p] \le \frac{1}{2p+1}\frac{8}{\pi^2}\,{}_3F_2\left[\begin{matrix}1/2, 1/2, 1\\ p+3/2, 3/2\end{matrix}; 1\right] + \frac{1}{\pi^2}\sum_{k=0}^{p-1}\frac{2}{(p-k)^2}\mathbb{E}[\mathcal{A}_1^k]$$

$$\mathbb{E}[\mathcal{A}_2^p] \ge \frac{1}{2p+1}\frac{8}{\pi^2}\,{}_3F_2\left[\begin{matrix}1/2, 1/2, 1\\ p+3/2, 3/2\end{matrix}; 1\right] + \frac{1}{\pi^2}\sum_{k=0}^{p-1}\frac{2}{(p-k)^2}\mathbb{E}[\mathcal{A}_2^k],$$

where ${}_3F_2$ denotes the usual generalized hypergeometric function; see [8, Section 9.1]. Note that both bounds are the same when p is equal to 1 and 2. Asymptotically, we further obtain

$$\frac{6}{\pi^2 p} \le \mathbb{E}[\mathcal{A}_2^p] \le \frac{1}{3\sqrt{\pi p}} \quad (p \to +\infty).$$

In particular, this implies that the RV $\mathcal{A}_2$ cannot follow a beta distribution (hence, neither a generalized arcsine distribution). Indeed, otherwise the beta distribution would be $\beta(\frac{1}{2} + 4/(\pi^2-4), \frac{1}{2} + 4/(\pi^2-4))$, since then

$$\mathbb{E}[\beta] = \frac{1}{2}, \qquad \mathbb{E}[\beta^2] = \frac{3}{8} - \frac{1}{2\pi^2}, \qquad \mathbb{E}[\beta^3] = \frac{5}{16} - \frac{3}{4\pi^2}.$$

But, as $p \to +\infty$, we would have

$$\mathbb{E}[\beta^p] = \mathcal{O}\left(\frac{1}{p^{1/2+4/(\pi^2-4)}}\right),$$

which would contradict the lower bound since $\frac{1}{2} + 4/(\pi^2-4) > 1$. Numerical computations are shown in Figure 1, in which it is seen that the lower bound is clearly the better one.

Proof. Let B and W be two independent Brownian motions and define

$$M_p(x) = \int_0^{+\infty} \mathrm{e}^{-t/2}\mathbb{E}\left[\left(\int_0^t \mathbf{1}_{\{(x+B_u)W_u>0\}}\,\mathrm{d}u\right)^p\right]\mathrm{d}t$$

so that

$$\mathbb{E}[\mathcal{A}_2^p] = \frac{M_p(0)}{2^{p+1}p!}.$$

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Applying first the Markov property at the stopping time $T_x = \inf\{u \geq 0,\ x + B_u = 0\}$ and Fubini's theorem, we deduce that

$$\begin{aligned}
&\mathbb{E}\bigg[\bigg(\int_0^t \mathbf{1}_{\{(x+B_u)W_u>0\}}\,\mathrm{d}u\bigg)^p\bigg] \\
&\quad = \mathbb{E}\bigg[\bigg(\int_0^t \mathbf{1}_{\{W_u>0\}}\,\mathrm{d}u\bigg)^p \mathbf{1}_{\{T_x>t\}}\bigg] \\
&\qquad + \mathbb{E}\bigg[\bigg(\int_0^{T_x} \mathbf{1}_{\{W_u>0\}}\,\mathrm{d}u + \int_{T_x}^t \mathbf{1}_{\{(x+B_u)W_u>0\}}\,\mathrm{d}u\bigg)^p \mathbf{1}_{\{T_x\leq t\}}\bigg] \\
&\quad = t^p\mathbb{E}[\mathcal{A}_1^p]\mathbb{P}(T_x>t) \\
&\qquad + \sum_{k=0}^p \binom{p}{k}\mathbb{E}\bigg[\bigg(\int_0^{T_x} \mathbf{1}_{\{W_u>0\}}\,\mathrm{d}u\bigg)^{p-k}\bigg(\int_0^{t-T_x} \mathbf{1}_{\{\widehat{B}_s(W_{T_x}+\widehat{W}_s)>0\}}\,\mathrm{d}u\bigg)^k \mathbf{1}_{\{T_x\leq t\}}\bigg],
\end{aligned}$$

where $\widehat{B}$ and $\widehat{W}$ are two independent Brownian motions, independent from B and W. We now take the Laplace transform of both sides. Applying the Fubini–Tonelli theorem and a change of variable, we obtain

$$\begin{aligned}
M_p(x) &= \mathbb{E}[\mathcal{A}_1^p]\int_0^{+\infty} \mathrm{e}^{-t/2}t^p\mathbb{P}(T_x>t)\,\mathrm{d}t \\
&\quad + \sum_{k=0}^p \binom{p}{k}\mathbb{E}\bigg[\mathrm{e}^{-T_x/2}\bigg(\int_0^{T_x}\mathbf{1}_{\{W_u>0\}}\,\mathrm{d}u\bigg)^{p-k} M_k(W_{T_x})\bigg] \\
&= R_{p-1}(x) + \mathbb{E}[\mathrm{e}^{-T_x/2}M_p(W_{T_x})],
\end{aligned}$$

where, by scaling, R_{p-1} is defined by

$$\begin{aligned}
R_{p-1}(x) &= \mathbb{E}[\mathcal{A}_1^p]\int_0^{+\infty} \mathrm{e}^{-t/2}t^p\mathbb{P}(T_x>t)\,\mathrm{d}t \\
&\quad + \sum_{k=0}^{p-1} \binom{p}{k}\mathbb{E}\bigg[\mathrm{e}^{-T_x/2}T_x^{p-k}\bigg(\int_0^1\mathbf{1}_{\{W_u>0\}}\,\mathrm{d}u\bigg)^{p-k} M_k(\sqrt{T_x}W_1)\bigg].
\end{aligned}$$

Thus, we obtain the relation, since $\mathbb{E}[\mathrm{e}^{-T_x/2}] = \mathrm{e}^{-|x|}$,

$$M_p(x) - M_p(0) = R_{p-1}(x) - (1-\mathrm{e}^{-|x|})M_p(0) + \mathbb{E}[\mathrm{e}^{-T_x/2}(M_p(W_{T_x}) - M_p(0))]. \tag{6}$$

The expectation on the right-hand side may be computed as

$$\begin{aligned}
&\mathbb{E}[\mathrm{e}^{-T_x/2}(M_p(W_{T_x}) - M_n(0))] \\
&\quad = 2\int_0^{+\infty} \frac{x}{\sqrt{2\pi t^3}}\mathrm{e}^{-x^2/2t-t/2}\,\mathrm{d}t\int_0^{+\infty}\frac{1}{\sqrt{2\pi t}}\mathrm{e}^{-z^2/2t}(M_p(z)-M_p(0))\,\mathrm{d}z \\
&\quad = 2\int_0^{+\infty}\frac{x}{\pi}\frac{K_1(\sqrt{x^2+z^2})}{\sqrt{x^2+z^2}}(M_p(z)-M_p(0))\,\mathrm{d}z,
\end{aligned}$$

where K_ν denotes the modified Bessel function of the second kind of order ν. Integrating (6) with respect to $K_0(x)(\mathrm{d}x/x)$, we deduce from [5, Equation (33)] and [8, Equation (9)],

$$\int_0^{+\infty}\frac{K_1(\sqrt{x^2+z^2})}{\sqrt{x^2+z^2}}K_0(x)\,\mathrm{d}x = \frac{\pi}{2z}K_0(z) \quad\text{and}\quad \int_0^{+\infty}(1-\mathrm{e}^{-x})K_0(x)\frac{\mathrm{d}x}{x} = \frac{\pi^2}{8}$$

that

$$\int_0^{+\infty} (M_p(x) - M_p(0))K_0(x)\frac{\mathrm{d}x}{x}$$
$$= \int_0^{+\infty} (R_{p-1}(x) - (1-\mathrm{e}^{-|x|})M_p(0))K_0(x)\frac{\mathrm{d}x}{x} + \int_0^{+\infty} K_0(z)(M_p(z) - M_p(0))\frac{\mathrm{d}z}{z};$$

hence,

$$M_p(0) = \frac{8}{\pi^2}\int_0^{+\infty} R_{p-1}(x)K_0(x)\frac{\mathrm{d}x}{x}. \tag{7}$$

Remark 3. Note that we might have used the Kontorovitch–Lebedev transform (see, for example, [11]) to obtain a recurrence relation between $M_p(x)$ and $R_{p-1}(x)$:

$$M_p(x) = \frac{2}{\pi^2}\int_0^{+\infty} \frac{\cosh(\pi\gamma/2)}{\cosh(\pi\gamma/2)-1}K_{i\gamma}(x)\bigg(\int_0^{+\infty} K_{i\gamma}(z)R_{p-1}(z)\frac{\mathrm{d}z}{z}\bigg)\gamma\sinh(\pi\gamma)\,\mathrm{d}\gamma$$

but this leads to quite complicated calculations, even for $p = 3$.

Rather, we shall obtain bounds on R_{p-1}. Substituting the expression of R_{p-1} into (7), we first need to compute

$$\begin{aligned}
&\int_0^{+\infty}\bigg(\int_0^{+\infty} \mathrm{e}^{-t/2}t^p\mathbb{P}(T_x > t)\,\mathrm{d}t\bigg)K_0(x)\frac{\mathrm{d}x}{x}\\
&= \int_0^{+\infty} \mathrm{e}^{-t/2}t^p\,\mathrm{d}t\int_t^{+\infty}\frac{1}{4s}\mathrm{e}^{s/4}K_0\Big(\frac{s}{4}\Big)\,\mathrm{d}s\\
&= \int_1^{+\infty}\frac{\mathrm{d}u}{4u}\int_0^{+\infty}\mathrm{e}^{-t(2-u)/4}K_0\Big(\frac{ut}{4}\Big)\,\mathrm{d}t\\
&= 2^{p-1}\sqrt{\pi}\frac{(\Gamma(p+1))^2}{\Gamma(p+3/2)}\int_1^{+\infty} {}_2F_1\left[\begin{matrix}p+1, 1/2\\ p+3/2\end{matrix}; 1-u\right]\frac{\mathrm{d}u}{u} \qquad \text{(see [8, p. 700])}\\
&= 2^{p-1}\sqrt{\pi}\frac{(\Gamma(p+1))^2}{\Gamma(p+3/2)}\int_0^{1} {}_2F_1\left[\begin{matrix}p+1, 1/2\\ p+3/2\end{matrix}; 1-\frac{1}{x}\right]\frac{\mathrm{d}x}{x}\\
&= 2^{p-1}\sqrt{\pi}\frac{(\Gamma(p+1))^2}{\Gamma(p+3/2)}\int_0^{1} {}_2F_1\left[\begin{matrix}1/2, 1/2\\ p+3/2\end{matrix}; 1-x\right]\frac{\mathrm{d}x}{\sqrt{x}} \qquad \text{(using Pfaff's formula)}\\
&= 2^{p}\sqrt{\pi}\frac{(\Gamma(p+1))^2}{\Gamma(p+3/2)}\,{}_3F_2\left[\begin{matrix}1/2, 1/2, 1\\ p+3/2, 3/2\end{matrix}; 1\right] \qquad \text{(see [8, p. 813]).}
\end{aligned}$$

Next, from Section 3 and the proof of monotony of the moments, we deduce that

$$M_k(0) \le M_k(\sqrt{T_x}W_1) \le M_k(+\infty);$$

hence, returning to the expression of R_{p-1}, it remains to compute

$$\begin{aligned}
\int_0^{+\infty}\mathbb{E}[\mathrm{e}^{-T_x/2}T_x^{p-k}]K_0(x)\frac{\mathrm{d}x}{x} &= \int_0^{+\infty}K_0(x)\frac{\mathrm{d}x}{x}\int_0^{+\infty}\frac{x}{\sqrt{2\pi t^3}}t^{p-k}\mathrm{e}^{-t/2}\mathrm{e}^{-x^2/2t}\,\mathrm{d}t\\
&= 4^{p-k-1}\int_0^{+\infty}t^{p-k-1}\mathrm{e}^{-t}K_0(t)\,\mathrm{d}t\\
&= \frac{\sqrt{\pi}}{4}2^{p-k}\frac{(\Gamma(p-k))^2}{\Gamma(p-k+1/2)}.
\end{aligned}$$

Therefore, we obtain the lower bound

$$\begin{aligned}\mathbb{E}[\mathcal{A}_2^p] &= \frac{1}{2^{p+1}p!}\frac{8}{\pi^2}\int_0^{+\infty} R_{p-1}(x)K_0(x)\frac{\mathrm{d}x}{x} \\ &\geq \frac{\mathbb{E}[\mathcal{A}_1^p]}{2p!}\frac{8}{\pi^2}\sqrt{\pi}\frac{(\Gamma(p+1))^2}{\Gamma(p+3/2)}\,{}_3F_2\left[\begin{matrix}1/2, 1/2, 1\\ p+3/2, 3/2\end{matrix};\ 1\right] \\ &\quad + \sum_{k=0}^{p-1}\sqrt{\pi}2^{-k}\frac{(\Gamma(p-k))^2}{\Gamma(p-k+1/2)}\binom{p}{k}\mathbb{E}[\mathcal{A}_1^{p-k}]2^{k+1}k!\,\mathbb{E}[\mathcal{A}_2^k].\end{aligned}$$

Then, using the explicit value of the moments of the arcsine distribution, we finally obtain, after some simplifications,

$$\mathbb{E}[\mathcal{A}_2^p] \geq \frac{1}{2p+1}\frac{8}{\pi^2}\,{}_3F_2\left[\begin{matrix}1/2, 1/2, 1\\ p+3/2, 3/2\end{matrix};\ 1\right] + \frac{1}{\pi^2}\sum_{k=0}^{p-1}\frac{2}{(p-k)^2}\mathbb{E}[\mathcal{A}_2^k],$$

which is the announced result. The computations for the upper bound are similar. □

Acknowledgements

We thank the anonymous referees for pointing out the existence of the strong arcsine law and suggesting various improvements in the presentation of the proofs of the paper. This research benefited from the support of the Chaire Marchés en Mutation, Fédération Bancaire Française.

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