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A Note on 4-Rank Densities

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Abstract. For certain real quadratic number fields, we prove density results concerning 4-ranks of tame kernels. We also discuss a relationship between 4-ranks of tame kernels and 4-class ranks of narrow ideal class groups. Additionally, we give a product formula for a local Hilbert symbol.

1 Introduction

Let F be a real quadratic number field and $\mathcal{O}_F$ its ring of integers. In [4], the authors gave an algorithm for computing the 4-rank of the tame kernel $K_2(\mathcal{O}_F)$. The idea of the algorithm is to consider matrices with Hilbert symbols as entries and compute matrix ranks over $\mathbb{F}_2$. Recently, the author used these matrices to obtain "density results" concerning the 4-rank of tame kernels, see [6], [7].

In this note, we consider the 4-rank of $K_2(\mathcal{O})$ for the real quadratic number fields $\mathbb{Q}(\sqrt{p_1p_2p_3})$ for primes $p_1 \equiv p_2 \equiv p_3 \equiv 1 \bmod 8$. We will see that

$$4\text{- rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p_1p_2p_3})}) = 0, 1, 2, \text{ or } 3.$$

For squarefree, odd integers d, consider the set

$$X = \{d : d = p_1p_2p_3,\, p_i \equiv 1 \bmod 8\}$$

for distinct primes p_i.

Using GP/PARI [1], we computed the following: For $50881 \le d < 2 \times 10^7$, there are 7257 d's in X. Among them, there are 2121 d's (29.23%) yielding 4-rank 0, 3977 d's (54.80%) yielding 4-rank 1, 1086 d's (14.96%) yielding 4-rank 2, and 73 d's (1.01%) yielding 4-rank 3. In fact, we prove

Theorem 1.1 *For the fields $\mathbb{Q}(\sqrt{p_1p_2p_3})$, 4-rank 0, 1, 2, and 3 appear with natural density $\frac{1}{4}$, $\frac{17}{32}$, $\frac{13}{64}$, and $\frac{1}{64}$ respectively in X.*

In the appendix we point out a beautiful result which may not be well known. It is a product formula from [11] for a certain local Hilbert symbol. This product formula both simplifies numerical computations and is a generalization of Propositions 4.6 and 4.4 in [2] and [7], respectively.

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2 Matrices

Hurrelbrink and Kolster [4] generalize Qin's approach in [8], [9] and obtain 4-rank results by computing $\mathbb{F}_2$-ranks of certain matrices of local Hilbert symbols. Specifically, let $F = \mathbb{Q}(\sqrt{d})$, $d > 1$ and squarefree. Let $p_1, p_2, \dots, p_t$ denote the odd primes dividing d. Recall 2 is a norm from F if and only if all p_i's are $\equiv \pm 1 \bmod 8$. If so, then d is a norm from $\mathbb{Q}(\sqrt{2})$, thus

$$d = u^2 - 2w^2$$

for $u, w \in \mathbb{Z}$. Now consider the matrix:

$$M_{F/\mathbb{Q}} = \begin{pmatrix} (-d, p_1)_2 & (-d, p_1)_{p_1} & \dots & (-d, p_1)_{p_t} \\ (-d, p_2)_2 & (-d, p_2)_{p_1} & \dots & (-d, p_2)_{p_t} \\ \vdots & \vdots & & \vdots \\ (-d, p_{t-1})_2 & (-d, p_{t-1})_{p_1} & \dots & (-d, p_{t-1})_{p_t} \\ (-d, v)_2 & (-d, v)_{p_1} & \dots & (-d, v)_{p_t} \\ (d, -1)_2 & (d, -1)_{p_1} & \dots & (d, -1)_{p_t} \end{pmatrix}.$$

If 2 is not a norm from F, set $v = 2$. Otherwise, set $v = u + w$. Replacing the 1's by 0's and the -1's by 1's, we calculate the matrix rank over $\mathbb{F}_2$. From [4],

Lemma 2.1 *Let $F = \mathbb{Q}(\sqrt{d})$, $d > 0$ and squarefree. Then*

$$4\text{- rank } K_2(\mathcal{O}_F) = t - \operatorname{rk}(M_{F/\mathbb{Q}}) + a' - a$$

where

$$a = \begin{cases} 0 & \textit{if 2 is a norm from } F \\ 1 & \textit{otherwise} \end{cases}$$

and

$$a' = \begin{cases} 0 & \textit{if both } -1 \textit{ and 2 are norms from } F \\ 1 & \textit{if exactly one of } -1 \textit{ or 2 is a norm from } F \\ 2 & \textit{if none of } -1 \textit{ or 2 are norms from } F. \end{cases}$$

Recall that our case is $\mathbb{Q}(\sqrt{p_1 p_2 p_3})$ for primes $p_1 \equiv p_2 \equiv p_3 \equiv 1 \bmod 8$. In this case $a = a'$ and we may delete the last row of $M_{F/\mathbb{Q}}$ without changing its rank (see discussions preceding Proposition 5. 13 and Lemma 5.14 in [4]). Also note that v is an p_1-adic unit and hence

$$(-p_1 p_2 p_3, v)_{p_1} = (p_1, v)_{p_1} = \left(\frac{v}{p_1}\right).$$

Similarly, $(-p_1 p_2 p_3, v)_{p_2} = (\frac{v}{p_2})$ and $(-p_1 p_2 p_3, v)_{p_3} = (\frac{v}{p_3})$. From Lemma 2.1 we have

$$4\text{- rank } K_2(\mathcal{O}_F) = 3 - \operatorname{rk}(M_{F/\mathbb{Q}})$$

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and the matrix $M_{F/\mathbb{Q}}$ is of the form

$$\begin{pmatrix} 1 & (\frac{p_2}{p_1})(\frac{p_3}{p_1}) & (\frac{p_1}{p_2}) & (\frac{p_1}{p_3}) \\ 1 & (\frac{p_2}{p_1}) & (\frac{p_1}{p_2})(\frac{p_3}{p_2}) & (\frac{p_2}{p_3}) \\ (-d, u+w)_2 & (\frac{v}{p_1}) & (\frac{v}{p_2}) & (\frac{v}{p_3}) \end{pmatrix}.$$

Let us now prove Theorem 1.1.

Proof The idea in [6] and [7] is to first consider an appropriate normal extension N of $\mathbb{Q}$ and then relate the splitting of the primes p_i in N to their representation by certain quadratic forms. The next step is classifying 4-rank values in terms of values of the symbols $(-d, v)_2$, $(\frac{v}{p_i})$. The values of these symbols are then characterized in terms of p_i satisfying the alluded to quadratic forms. We then associate Artin symbols to the primes p_i and apply the Chebotarev density theorem. In what follows, we classify the 4-rank values in terms of the symbols $(-d, v)_2$, $(\frac{v}{p_i})$ and in parenthesis give the relevant densities in X obtained by using the above machinery. Let us consider the following four cases (see Table III in [9]).

Case 1 Suppose $(\frac{p_2}{p_1}) = (\frac{p_3}{p_1}) = (\frac{p_3}{p_2}) = 1$. Then we immediately have that

- 4- rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p_1p_2p_3})}) = 3 \Leftrightarrow \operatorname{rank}(M_{F/\mathbb{Q}}) = 0 \Leftrightarrow (-d, v)_2 = 1$ and $(\frac{v}{p_1}) = (\frac{v}{p_2}) = (\frac{v}{p_3}) = 1(\frac{1}{64})$
- 4- rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p_1p_2p_3})}) = 2 \Leftrightarrow \operatorname{rank}(M_{F/\mathbb{Q}}) = 1 \Leftrightarrow (-d, v)_2 = -1$ or $(-d, v)_2 = 1$ and $(\frac{v}{p_1}) = (\frac{v}{p_2}) = -1$ and $(\frac{v}{p_3}) = 1$ or $(-d, v)_2 = 1$ and $(\frac{v}{p_1}) = (\frac{v}{p_3}) = -1$ and $(\frac{v}{p_2}) = 1$ or $(-d, v)_2 = 1$ and $(\frac{v}{p_2}) = (\frac{v}{p_3}) = -1$ and $(\frac{v}{p_1}) = 1(\frac{7}{64})$.

Case 2 Suppose $(\frac{p_3}{p_2}) = (\frac{p_3}{p_1}) = 1$, $(\frac{p_2}{p_1}) = -1$. Thus

- 4- rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p_1p_2p_3})}) = 2 \Leftrightarrow \operatorname{rank}(M_{F/\mathbb{Q}}) = 1 \Leftrightarrow (-d, v)_2 = 1$ and $(\frac{v}{p_1}) = (\frac{v}{p_2}) = 1$ or $(-d, v)_2 = 1$ and $(\frac{v}{p_1}) = (\frac{v}{p_2}) = -1(\frac{3}{32})$
- 4- rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p_1p_2p_3})}) = 1 \Leftrightarrow \operatorname{rank}(M_{F/\mathbb{Q}}) = 2 \Leftrightarrow (-d, v)_2 = -1$ or $(-d, v)_2 = 1$ and $(\frac{v}{p_1}) = (\frac{v}{p_3}) = -1$ and $\left(\frac{v}{p_2}\right) = 1$ or $(-d, v)_2 = 1$ and $(\frac{v}{p_2}) = (\frac{v}{p_3}) = -1$ and $(\frac{v}{p_1}) = 1(\frac{9}{32})$.

Case 3 Suppose $(\frac{p_2}{p_1}) = (\frac{p_3}{p_1}) = -1$, $(\frac{p_3}{p_2}) = 1$. Thus

- 4- rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p_1p_2p_3})}) = 1 \Leftrightarrow \operatorname{rank}(M_{F/\mathbb{Q}}) = 2 \Leftrightarrow (-d, v)_2 = 1(\frac{3}{16})$.
- 4- rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p_1p_2p_3})}) = 0 \Leftrightarrow \operatorname{rank}(M_{F/\mathbb{Q}}) = 3 \Leftrightarrow (-d, v)_2 = -1(\frac{3}{16})$.

Case 4 Suppose $(\frac{p_2}{p_1}) = (\frac{p_3}{p_1}) = (\frac{p_3}{p_2}) = -1$. Then

- 4- rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p_1p_2p_3})}) = 1 \Leftrightarrow \operatorname{rank}(M_{F/\mathbb{Q}}) = 2 \Leftrightarrow (-d, v)_2 = 1(\frac{1}{16})$
- 4- rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p_1p_2p_3})}) = 0 \Leftrightarrow \operatorname{rank}(M_{F/\mathbb{Q}}) = 3 \Leftrightarrow (-d, v)_2 = -1(\frac{1}{16})$.

Thus 4-rank 0, 1, 2, and 3 occur with natural density $\frac{1}{16} + \frac{3}{16} = \frac{1}{4}$, $\frac{1}{16} + \frac{3}{16} + \frac{9}{32} = \frac{17}{32}$, $\frac{3}{32} + \frac{7}{64} = \frac{13}{64}$, and $\frac{1}{64}$. ∎

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Remark 2.2 The matrices in [4] are related to Rédei matrices which were used in the 1930's to study the structure of narrow ideal class groups. Namely, for $\mathbb{Q}(\sqrt{d})$, we considered the case that all odd primes divisors of d are $\equiv 1 \bmod 8$. Thus 2 is a norm from $F = \mathbb{Q}(\sqrt{d})$ and we have the representation

$$d = u^2 - 2w^2.$$

Let $d' = \prod_{i=1}^{t} p_i$. The matrix $M_{F/\mathbb{Q}}$ has the form:

$$\begin{pmatrix} 1 & & & \\ 1 & & \hat{R}_{F/\mathbb{Q}} & \\ \vdots & & & \\ 1 & & & \\ (-d,v)_2 & (-d,v)_{p_1} & \dots & (-d,v)_{p_t} \end{pmatrix}.$$

The $(t-1)$ by t matrix $\hat{R}_{F/\mathbb{Q}}$ can be extended, without changing its rank, to a t by t matrix $R_{F/\mathbb{Q}}$ by adding the last row

$$(-d, p_t)_{p_1}, (-d, p_t)_{p_2}, \dots, (-d, p_t)_{p_t}.$$

$R_{F/\mathbb{Q}}$ is known as the Rédei matrix of the field $F' := \mathbb{Q}(\sqrt{d'})$ (see [5] or [10]). Its rank determines the 4-rank of the narrow ideal class group $C^+_{F'}$ of the field F' by

$$4\text{- rank}\, C^+_{F'} = t - 1\text{- rank}(R_{F/\mathbb{Q}}).$$

Combining this information with Lemma 2.1, we have that if $(-d, u+w)_2 = -1$, then 4-rank $K_2(\mathcal{O}_F) =$ 4-rank $C^+_{F'}$. Using Rédei matrices, Gerth in [3] derived an effective algorithm for computing densities of 4-class ranks of narrow ideal class groups of quadratic number fields. It would be interesting to see if density results concerning 4-class ranks of narrow ideal class groups (coupled with the product formula in the appendix) can be used to obtain asymptotic formulas for 4-rank densities of tame kernels.

3 Appendix: A Product Formula

Most of the local Hilbert symbols in the matrix $M_{F/\mathbb{Q}}$ are calculated directly. Difficulties arise when d is a norm from $\mathbb{Q}(\sqrt{2})$. In this case, we need to calculate the Hilbert symbols $(-d, u+w)_2$ and $(-d, u+w)_{p_k}$. The local symbol at 2 is calculated using Lemmas 5.3 and 5.4 in [4]. In this appendix we provide a product formula which allows one to calculate $(-d, u+w)_{p_k}$ using 2 factors of d at a time.

Let d be a squarefree integer and assume that all odd prime divisors of d are $\equiv \pm 1 \bmod 8$. Then d is a norm from $F = \mathbb{Q}(\sqrt{2})$ and we have the representation

$$d = u^2 - 2w^2$$

with $u > 0$. Let l be any odd prime dividing d. Note that l does not divide $u + w$ and so

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Remark 3.1 $(-d, u+w)_l = (l, u+w)_l = (\frac{u+w}{l})$.

Recall that any odd prime divisor l of d is $\equiv \pm 1 \bmod 8$. We fix x and y according to the representation:

$$(-1)^{\frac{l-1}{2}} l = N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(x + y\sqrt{2}) = x^2 - 2y^2$$

with $x \equiv 1 \bmod 4$, x, $y > 0$. Observe that mod 8, x is odd. Also we can arrange for $x \equiv 1 \bmod 4$ by multiplying $x + y\sqrt{2}$ by $(1+\sqrt{2})^2$.

For $l \equiv 1 \bmod 8$, we have $l = x^2 - 2y^2$ and so $(\frac{l}{y}) = 1$. Thus $(\frac{y}{l}) = 1$. For $l \equiv 7 \bmod 8$,

$$1 = \Big(\frac{-l}{y}\Big) = \Big(\frac{-1}{y}\Big)\Big(\frac{l}{y}\Big) = (-1)^{\frac{y-1}{2}}(-1)^{\frac{y-1}{2}}\Big(\frac{y}{l}\Big) = \Big(\frac{y}{l}\Big).$$

Now let r be an integer not divisible by l which can be represented as a norm from $\mathbb{Q}(\sqrt{2})$. Denote by $\pi_r = s + t\sqrt{2}$ an element such that $N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(\pi_r) = r$ with s, $t > 0$. Now let u_r and w_r be such that

$$u_r + w_r\sqrt{2} = (1+\sqrt{2})(x + y\sqrt{2})(s + t\sqrt{2}).$$

By the choice of x, y, s, t, we have $u_r > 0$. Note that

$$N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(u_r + w_r\sqrt{2}) = -(-1)^{\frac{l-1}{2}} lr.$$

Now fix $\mathfrak{l} = \langle x - y\sqrt{2}\rangle$ a prime ideal above l in $\mathbb{Q}(\sqrt{2})$. As l splits in $\mathbb{Q}(\sqrt{2})$, $\mathbb{Z}[\sqrt{2}]/\mathfrak{l} \cong \mathbb{Z}/l\mathbb{Z}$. This allows us to work mod $\mathfrak{l}$ as opposed to mod l. From the above, $u_r + w_r = 2xs + 3tx + 3sy + 4yt$. Modulo $\mathfrak{l}$, we have

$$\begin{aligned} u_r + w_r &\equiv 2sy\sqrt{2} + 3ty\sqrt{2} + 3sy + 4yt \\ &\equiv y(3 + 2\sqrt{2})(s + t\sqrt{2}). \end{aligned}$$

As $(\frac{y}{l}) = 1$, $(\frac{u_r+w_r}{l}) = (\frac{y}{l})(\frac{\pi_r}{\mathfrak{l}}) = (\frac{\pi_r}{\mathfrak{l}})$ where

$$\Big(\frac{\pi_r}{\mathfrak{l}}\Big) = \begin{cases} 1 & \text{if } x^2 \equiv \pi_r \bmod \mathfrak{l} \text{ is solvable,} \\ -1 & \text{otherwise.} \end{cases}$$

In the case $r = \prod_i^{t-1} p_i$ where $p_i \equiv \pm 1 \bmod 8$, we obtain for each p_i an element $\pi_i \in \mathbb{Q}(\sqrt{2})$ of norm $(-1)^{\frac{p_i-1}{2}} p_i$. Let c be the number of primes dividing r which are congruent to 7 modulo 8. Then we have (up to squares of units) $\pi_r = (1+\sqrt{2})^c \prod_i^{t-1} \pi_i$. This yields

$$\Big(\frac{u_r + w_r}{l}\Big) = \Big(\frac{(1+\sqrt{2})^c}{\mathfrak{l}}\Big) \prod_i^{t-1} \Big(\frac{\pi_i}{\mathfrak{l}}\Big),$$

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and so

$$\Big(\frac{u_r + w_r}{l}\Big) = \Big(\frac{u_{-1} + w_{-1}}{l}\Big)^c \prod_i^{t-1} \Big(\frac{u_{p_i} + w_{p_i}}{l}\Big).$$

As -1 and 2 are also norms from $\mathbb{Q}(\sqrt{2})$, we can include r's having factors -1 or ± 2. Thus for $r = (-1)^n(2)^m \prod_i^{t-1} p_i$ with $m, n = 0$, or 1, and each $p_i \equiv \pm 1 \bmod 8$ and $l \neq p_i$ for any i, we have

Remark 3.2

$$\Big(\frac{u_r + w_r}{l}\Big) = \Big(\frac{u_{-1} + w_{-1}}{l}\Big)^{n+c} \Big(\frac{u_2 + w_2}{l}\Big)^m \prod_i^{t-1} \Big(\frac{u_{p_i} + w_{p_i}}{l}\Big).$$

Setting $r = \frac{d}{l}$, we have $-(-1)^{\frac{l-1}{2}} d = N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(u_r + w_r\sqrt{2})$. So for any prime $l \equiv 7 \bmod 8$, we have $N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(u_r + w_r\sqrt{2}) = d = N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(u + w\sqrt{2})$. Then, up to squares, $(\frac{u_r+w_r}{l}) = (\frac{u+w}{l})$. For prime divisors $l \equiv 1 \bmod 8$, we have $-d = N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(u_r + w_r\sqrt{2})$ and so we include $(\frac{u_{-1}+w_{-1}}{l})$. To summarize,

Remark 3.3

$$(-d, u + w)_l = \begin{cases} (\frac{u_r+w_r}{l}) & \text{if } l \equiv 7 \bmod 8 \\ (\frac{u_{-1}+w_{-1}}{l})(\frac{u_r+w_r}{l}) & \text{if } l \equiv 1 \bmod 8. \end{cases}$$

We may now reduce to the following $d = rl$: $d = -l$, $d = 2l$, and $d = pl$, *i.e.* calculate the symbols $(\frac{u_{-1}+w_{-1}}{l})$, $(\frac{u_2+w_2}{l})$, and $(\frac{u_p+w_p}{l})$. The first two symbols can be calculated using the following two elementary lemmas.

Lemma 3.4 *$(\frac{u_{-1}+w_{-1}}{l}) = 1 \Leftrightarrow (-1)^{\frac{l-1}{2}} l = a^2 - 32b^2$ for some $a, b \in \mathbb{Z}$ with $a \equiv 1 \bmod 4$.*

Lemma 3.5 *$(\frac{u_2+w_2}{l}) = 1 \Leftrightarrow l \equiv \pm 1 \bmod 16$.*

A little care is necessary in computing $(\frac{u_p+w_p}{l})$. If $\Big(\frac{(-1)^{\frac{p-1}{2}} p}{l}\Big) = 1$, then the symbol $(\frac{\pi}{l})$ is well defined (see discussion preceding Proposition 3.5 in [2]) and can be computed using

Lemma 3.6 *For $\mathcal{K} = \mathbb{Q}\big(\sqrt{(-1)^{\frac{p-1}{2}} 2p}\big)$ with $p \equiv \pm 1 \bmod 8$ and $h^+(\mathcal{K})$ the narrow class number of $\mathcal{K}$, we have*

$$\Big(\frac{\pi}{l}\Big) = 1 \Leftrightarrow l^{\frac{h^+(\mathcal{K})}{4}} = n^2 - 2pm^2 \quad \textit{for some } n, m \in \mathbb{Z} \textit{ with } m \not\equiv 0 \bmod l.$$

For $\mathcal{K} = \mathbb{Q}(\sqrt{-2p})$ with $p \equiv 7 \bmod 8$, $(\frac{\pi}{l}) = -1 \Leftrightarrow l^{\frac{h^+(\mathcal{K})}{4}} = 2n^2 + pm^2$ for some $n, m \in \mathbb{Z}$ with $m \not\equiv 0 \bmod l$.

For $\mathcal{K} = \mathbb{Q}(\sqrt{2p})$ with $p \equiv 1 \bmod 8$, $(\frac{\pi}{l}) = -1 \Leftrightarrow l^{\frac{h^+(\mathcal{K})}{4}} = pn^2 - 2m^2$ for some $n, m \in \mathbb{Z}$ with $m \not\equiv 0 \bmod l$.

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In fact,

Lemma 3.7 *If* $\left(\frac{(-1)^{\frac{p-1}{2}}p}{l}\right) = 1$, *then*

$$\left(\frac{u_p + w_p}{l}\right) = \begin{cases} \left(\frac{\pi}{l}\right) & \text{for } p \equiv 1 \bmod 8 \\ \left(\frac{u_{-1}+w_{-1}}{l}\right)\left(\frac{\pi}{l}\right) & \text{for } p \equiv 7 \bmod 8. \end{cases}$$

The case where $\left(\frac{(-1)^{\frac{p-1}{2}}p}{l}\right) = -1$ can be done by finding u_p and w_p from the presentation

$$N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(u_p + w_p\sqrt{2}) = -(-1)^{\frac{p-1}{2}}pl.$$

Combining Remarks 3.1, 3.2, and 3.3, we have

Theorem 3.8 *For* $d = (-1)^n(2)^m \prod_{i=1}^{t} p_i$, *with each* $p_i \equiv \pm 1 \bmod 8$, *we have*

$$(-d, u+w)_{p_k} = \left(\frac{u_{-1} + w_{-1}}{p_k}\right)^{n+(-1)^{\frac{p_k+1}{2}}} \left(\frac{u_2 + w_2}{p_k}\right)^m \prod_{i \neq k} \left(\frac{u_{p_i} + w_{p_i}}{p_k}\right).$$

Example 3.9 Consider the cases $d = \pm pl, \pm 2pl$ with $p \equiv 7 \bmod 8$, $l \equiv 1 \bmod 8$, and $\left(\frac{l}{p}\right) = 1$ (see Proposition 4.6 in [2]). Note that $\left(\frac{\pi}{l}\right)$ is well defined and so Lemma 3.7 is applicable.

For $d = pl$, we have $n = 0$, $m = 0$ and so

$$\begin{aligned} (-d, u+w)_l &= \left(\frac{u_{-1} + w_{-1}}{l}\right)^{-1} \left(\frac{u_2 + w_2}{l}\right)^0 \left(\frac{u_{-1} + w_{-1}}{l}\right) \left(\frac{\pi}{l}\right) \\ &= \left(\frac{\pi}{l}\right). \end{aligned}$$

For $d = 2pl$, we have $n = 0$, $m = 1$. Thus

$$\begin{aligned} (-d, u+w)_l &= \left(\frac{u_{-1} + w_{-1}}{l}\right)^{-1} \left(\frac{u_2 + w_2}{l}\right)^1 \left(\frac{u_{-1} + w_{-1}}{l}\right) \left(\frac{\pi}{l}\right) \\ &= \left(\frac{2 + \sqrt{2}}{l}\right) \left(\frac{\pi}{l}\right). \end{aligned}$$

For $d = -pl$, we have $n = 1$, $m = 0$. This yields

$$\begin{aligned} (-d, u+w)_l &= \left(\frac{u_{-1} + w_{-1}}{l}\right)^0 \left(\frac{u_2 + w_2}{l}\right)^0 \left(\frac{u_{-1} + w_{-1}}{l}\right) \left(\frac{\pi}{l}\right) \\ &= \left(\frac{1 + \sqrt{2}}{l}\right) \left(\frac{\pi}{l}\right). \end{aligned}$$

Finally, for $d = -2pl$, we have $n = 1$, $m = 1$. So

$$\begin{aligned} (-d, u+w)_l &= \left(\frac{u_{-1} + w_{-1}}{l}\right)^0 \left(\frac{u_2 + w_2}{l}\right)^1 \left(\frac{u_{-1} + w_{-1}}{l}\right) \left(\frac{\pi}{l}\right) \\ &= \left(\frac{2 + \sqrt{2}}{l}\right) \left(\frac{1 + \sqrt{2}}{l}\right) \left(\frac{\pi}{l}\right). \end{aligned}$$

Example 3.10 Consider the cases $d = \pm pl$ with $p \equiv l \equiv 1 \bmod 8$, and $(\frac{l}{p}) = 1$ (see Proposition 4.4 in [7]). Again $(\frac{\pi}{l})$ is well defined and so Lemma 3.7 is applicable.

For $d = pl$, we have $n = 0$, $m = 0$, and so

$$\begin{aligned}(-d, u+w)_l &= \Big(\frac{u_{-1}+w_{-1}}{l}\Big)^{-1}\Big(\frac{u_2+w_2}{l}\Big)^{0}\Big(\frac{u_p+w_p}{l}\Big)\\ &= \Big(\frac{1+\sqrt{2}}{l}\Big)\Big(\frac{\pi}{l}\Big).\end{aligned}$$

For $d = -pl$, we have $n = 1$, $m = 0$. Thus

$$\begin{aligned}(-d, u+w)_l &= \Big(\frac{u_{-1}+w_{-1}}{l}\Big)^{0}\Big(\frac{u_2+w_2}{l}\Big)^{0}\Big(\frac{u_p+w_p}{l}\Big)\\ &= \Big(\frac{\pi}{l}\Big).\end{aligned}$$

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