The Bulletin of Symbolic Logic
Volume 24, Number 4, December 2018

CANTORIAN SET THEORY

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Abstract. Almost all set theorists pay at least lip service to Cantor's definition of a set as a collection of many things into one whole; but empty and singleton sets do not fit with it. Adapting Dana Scott's axiomatization of the cumulative theory of types, we present a 'Cantorian' system which excludes these anomalous sets. We investigate the consequences of their omission, examining their claim to a place on grounds of convenience, and asking whether their absence is an obstacle to the theory's ability to represent ordered pairs or to support the arithmetization of analysis or the development of the theory of cardinals and ordinals.

§1. Introduction.

We all know of Cantor's definition of a set as 'a collection M of definite, well-differentiated objects m ... into a whole' ([3], p. 282: all references to Cantor are to his 1932 collected works), and the great majority of set theorists pay at least lip service to it. Once we think about it, however, we can see that an empty set and singletons do not make sense in terms of the definition (see Section 2 below). In Ch. 14 of our book *Plural Logic* [19, 20] we looked at these anomalous sets, observing that Cantor did not entertain them and showing that a wide range of arguments in favour of admitting them are unsound. This encouraged us to become advocates *pro bono publico* for those who take seriously the idea of sets as collections, by developing a theory in which sets must have more than one member, if they are really to be a collection of many things into one. We called the result 'Cantorian set theory'. Since we were concerned to illustrate the use of plural language in mathematics, we based the theory on an underlying logic which is plural, in the sense that a term may denote several things at once, not just one or possibly none. That system may therefore be described as the *hybrid*—half singular, half plural—version of the Cantorian theory. Ever since Zermelo's 1908 axiomatization [28], however, set theory has been pursued as a purely singular enterprise, with plural notions never getting a look in. So the hybrid version needs to be complemented by one conforming to this singularist style, a task we did not carry out in our book. This article is therefore devoted to *singular* Cantorian set theory, which turns out to present its own issues.

Received December 20, 2016.
2010 *Mathematics Subject Classification.* 03A05, 03B20, 03E30, 03E70, 03E75.
Key words and phrases. arithmetization of analysis, Axiom of Plurality, empty set, level, ordered pair, singleton, Singular Logic, Cantor, Scott.

1079-8986/18/2404-0001
DOI:10.1017/bsl.2018.10

Our own opinion on the matter is very different from Cantor's idea of collection into a whole. We think that taking a set to be a separate thing (Cantor's *ein Ding für sich*: pp. 379, 401, 411, 419) over and above its members is a classic case of being misled by grammar (see our [20], Ch. 15). This may well be controversial, but there should be nothing controversial about the present article. Here we have two aims. One is to develop and expound a Cantorian system of set theory, restricted to multimembered sets. The other is to examine the consequences of this restriction. The literature takes for granted that empty and singleton sets can be defended on grounds of their convenience. But to our knowledge no one has yet subjected this to critical scrutiny. We also need to investigate whether the exclusion of the anomalous sets is any obstacle to the ability of the theory to support the arithmetization of analysis and the general theory of cardinal and ordinal numbers.

Our vehicle for dealing with the absence of $\varnothing$ is the Singular Logic presented in Section 3, whose salient feature is its avoidance of any existential commitment—its topic neutrality—achieved through the employment of empty terms, including empty valuations of variables. Specifically, we take a paradigm empty term, one which is empty by logical necessity, such as $\iota x(x \neq x)$, abbreviated here as O. Our contention is that all the convenience of expression gained through $\varnothing$ can equally well be gained through O. But O is not a mere shadow of $\varnothing$. For example, while $\varnothing$ is an additional object, a subset of every set, O is not even a set—it is, literally, nothing, or, as we shall frequently say, it is *zilch*.

In Section 2 we explain why empty and singleton sets do not fit the conception of sets as collections, and we summarise Cantor's own opinion on the matter. In Section 3 we outline our underlying Singular Logic for Cantorian set theory and briefly explain some of its virtues. Next come definitions in Sections 4–5, followed by axioms in Section 6. We put the development of the set theory into the Appendix, but give a précis in Section 7. In Section 8 we discuss how the theory may be strengthened, while the next three sections address applications: ordered pairs in Section 9, the arithmetization of analysis in Section 10, and cardinals and ordinals in Section 11. Finally, in Section 12 we offer an evaluation, comparing our system with orthodox theories.

§**2. The anomalous sets.** To make the article self-contained, we very briefly recapitulate material in our book. First of all we explain why our set theory is aptly called *Cantorian* set theory.

For Cantor there was no such thing as an empty set. When describing a putative point-set that turns out not to contain any points, he says that strictly speaking it does not really exist at all ('streng genommen als solchen gar nicht vorhanden ist', p. 146). Two point-sets with no point in common do not have an empty intersection; rather they have *no* intersection ('sie seien *ohne Zusammenhang*', p. 145; 'so sind sie *ohne Zusammenhang*', p. 146). A finite set does not have an empty derived set; rather it has no derived set ('keine abgeleitete Menge hat', 'und hat selbst keine Abgeleitete', p. 98). His 'finite sets' all contain a first element (p. 145), and every subset of a well-ordered set has a least member (p. 444). His cardinal and ordinal numbers start with 1 (pp. 290, 298); a cardinal or ordinal 0 could only come by abstraction from an empty set. His definition of $\beta - \alpha$ for ordinals is qualified by the assumption that α is less than β (p. 323).

Cantor says very little about singletons except in the paragraph in which he introduces the cardinal numbers:

> A single thing e_0, if we subsume it under the concept of a set $E_0 = (e_0)$, corresponds to a cardinal number which we call 'one' and symbolize by 1 . . . One can now unite another thing e_1 with E_0, calling the union set E_1, so that $E_1 = (E_0, e_1) = (e_0, e_1)$. The cardinal number of E_1 is called 'two' and symbolized by 2. (pp. 289–90)

The new ( ) notation appears to be a limiting case of his notation for the union of disjoint sets (p. 282). This presumes the treatment of e_0 as a set, and the equation $(E_0, e_1) = (e_0, e_1)$ dictates that $E_0 = e_0$. In short, although singletons do not fit Cantor's various explanations of 'set' with their plurals and 'many's and 'together's (pp. 150, 204, 282, 443), he does accept them, but only by identifying the singleton E_0 with the thing e_0 in question. This enables him to extend his grand plan to derive numbers from sets to cover 1. His solution is to generalize the definition of set to cover the collection of a single object e_0 into—what else?—e_0 itself. The number 1 is then obtained in the regular way by abstraction from individual things regarded as sets.

Alas, Frege presented a decisive argument against the general identification of singletons with their only members. Consider any two objects a, b. The singleton of their pair-set is supposed to have just one member, yet if it is identical to the pair-set, it has both a and b as members ([10], Section 10, n. 17; [11], p. 219). There is no choice, then, but to take a different line on singletons once one engages—as Cantor did not—with a full-blown set theory which deals generally with sets of sets as well as sets of ur-elements. The modern conception of a singleton as something distinct from its only member avoids Frege's reductio, but only by giving up on the idea of sets as collections. As Erik Stenius observes: 'does it make sense that a set which has just been obtained by "collecting" several objects into one whole, can be collected again into a different one?' ([24], p. 65). More recently, David Lewis complains of 'mysterious singletons':

> Here is a just cause for student protest, if ever there was one. This time, he has no 'many' . . . Rather he has just one single thing, the element, and he has another single thing, the singleton, and nothing he was told gives him the slightest guidance about what one thing has to do with the other. Nor did any of those familiar examples concern single-membered sets. His introductory lesson just does not apply. ([17], p. 30)

So the line we choose is to do without singletons altogether. We show that it is sufficient, generally if not always, to use the sole member of the singleton rather than the singleton itself, and we introduce a notation which smoothly symbolizes this procedure (see Section 4).

https://doi.org/10.1017/bsl.2018.10 Published online by Cambridge University Press

Cantor's exclusion of the empty set is entirely consonant with his most explicit description of a set of things as

> *a separate, unified thing* [*ein einheitliches Ding für sich*] in which those things are components or constitutive elements. (p. 379; see also pp. 401, 411 & 419)

Frege, too, thought that if a set is a collection of objects, no objects means no set:

> A class, in the sense in which we have so far used the word, consists of objects; it is an aggregate, a collective unity, of them; if so, it must vanish when these objects vanish. If we burn down all the trees of a wood, we thereby burn down the wood. Thus there can be no empty class. ([11], p. 212)

When Zermelo put the empty set on the map, with Axiom II of his [28], he was not dissenting from Frege's conclusion. In the very act of positing its existence he dismisses it as 'not a genuine set' (eine uneigentliche Menge). Translators who render 'uneigentliche' as 'improper', lumping it with 'improper subset' and 'improper fraction' as if it were merely a limiting or degenerate case, fail to capture the force of the adjective. In later letters to Fraenkel, Zermelo's dismissive attitude is clear:

> [The empty set is] not a genuine set and was introduced by me only for formal reasons . . . I increasingly doubt the justifiability of [the empty set]. Perhaps one can dispense with it by restricting the axiom of *separation* in a suitable way. Indeed, it serves only the purpose of *formal* simplification. (Letters cited by Ebbinghaus [6], p. 135)

Indeed by the time of Zermelo's more considered 1930 axiomatization, 'an arbitrarily chosen ur-element takes the place of the null set' ([29], p. 403), and compare [31] where he draws attention to his novel 'introduction of a basis of ur-elements instead of the null set' (p. 441). As to Zermelo's question about restricting the axiom of separation, it had already been answered by Cantor himself, with his crisp and simple formulation: 'Every sub-multitude of a set is a set [Jede Teilvielheit einer Menge ist eine Menge]' (p. 444). This Cantorian principle emerges as Theorem 18 below, and is picked out in the Zermelo-style axiomatization of Section 8.

The anomalous status of the empty set and the modern notion of singleton is therefore hardly news. So it is no surprise that several authors have changed their notion of collection in an attempt to rescue the anomalous sets. The story starts with Dedekind's idea of a set as a container containing its members like a sack (see Bernstein in [8], p. 836), which accommodates the empty set as an empty container, and helps explain why $\{a\}$ is not the same as a, since the singleton now has the container as an extra component. A recent proponent of the container conception is Michael Potter:

> Now what if we try to make something out of nothing? A container with nothing in it is still a container, and the empty collection is correspondingly a collection with no members. ([21], p. 22)

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Herbert Enderton starts by defining a set to be 'a collection of things (called its *members* or *elements*)' ([7], p. 1), but then brings in the idea of an empty container to explain why the singleton of the empty set is distinct from the empty set itself:

> the fact that $\{\varnothing\} \neq \varnothing$ is reflected in the fact that a man with an empty container is better off than a man with nothing. ([7], p. 3)

The problem for the container theory is whether there are many containers or one all-purpose container. Enderton's example is sufficient to show that there must be many, since $\{\varnothing\}$ is supposed to be a container with an empty container as its content. But $\{\varnothing\} \neq \varnothing$, so there must be two containers involved. The trouble is that once many containers are admitted, there is no reason why there cannot be many empty ones. But extensionality requires that there be just one empty set.

George Boolos and Richard Jeffrey (now endorsed by John Burgess as third co-author of *Computability and Logic*) notice this problem with regarding the empty set as an empty container and suggest a different account of it:

> By courtesy, we regard as enumerable the empty set, $\varnothing$, which has no members. (*The* empty set; there is only one. The terminology is a bit misleading. It suggests comparison of empty sets with empty containers. But sets are more aptly compared with contents, and it should be considered that all empty containers have the same null content.) ([2], p. 4)

But on their 'more apt' comparison of sets with contents, no contents means no set, making nonsense of their reference to $\varnothing$. Ian Stewart, who does liken empty sets to empty containers, runs into similar trouble when he tries to explain why 'there is only one empty set' by appealing to the fact that 'the contents of two empty bags are identical' ([25], p. 47).

These authors' confusion of the empty set with nothing ('the same null content', 'the contents of two empty bags') is surprisingly common. Thus John Barrow says that we may

> define what we mean by the natural numbers in a simple and precise way by generating them all from nothing: the empty set . . . it has enabled us to create all of the numbers from literally nothing, the set with no members. ([1], pp. 166–167)

The same idea is found in Keith Devlin's guide to the axiom of constructibility:

> in order to construct the natural numbers we need only make one basic existence assumption: namely that *nothing* exists! . . . We assumed the existence of the empty set (i.e., *nothing*), and took this to be the number 0. ([5], pp. 11–12)

Sadly Boolos and Jeffrey are no longer with us, but Burgess, Stewart, Barrow and Devlin need to be told loud and clear: if there is an empty set, it is something, not nothing.

§3. Singular Logic. We call the underlying logic set out here 'Singular Logic'. It resembles the classical predicate calculus with identity, description, function signs and constants, but is shaped by the belief that a system of logic should be *topic neutral*, i.e., applicable to any subject-matter. The classical system notoriously fails this test, since it makes it logically necessary that something exists. We return to this and other aspects of topic neutrality after sketching the syntax and semantics of our system.

The membership predicate $\in$ is the only nonlogical primitive. The predicates 'is a set' and 'is an ur-element' will both be defined in terms of $\in$. This is the language needed for the abstract version of the set theory. Each applied version will naturally add its own vocabulary of predicates, function signs and constants concerning the topic it is designed to deal with.

Syntax

We use a, b, c as schematic letters for terms of arbitrary complexity, including variables standing alone. A, B, C stand for single formulas, and Γ for any number (none or one or more) of formulas.

(i) *Logical vocabulary*

Variables, countably many

Connectives $\neg \quad \rightarrow \quad \leftrightarrow \quad \wedge \quad \vee$, plus brackets for punctuation

Universal quantifier $\forall$

Description operator ι

Identity, a two-place predicate $=$

(ii) *Nonlogical vocabulary*

Membership, a two-place predicate $\in$

(iii) *Formation rules*

Variables are terms.

If x is a variable and A a formula, ιxA is a term.

If a and b are terms, $a = b$ and $a \in b$ are formulas.

If A and B are formulas, so are $\neg A$, $(A \rightarrow B)$ etc, with the usual conventions for omitting brackets.

If x is a variable and A a formula, $\forall xA$ is a formula.

(iv) *Scope, free and bound occurrences of terms and formulas*

The scope of an occurrence of $\forall$ or ι is defined as the shortest formula or term in which it occurs. These operators always occur with a variable attached, as in $\forall xA$ or ιxA, and an occurrence of x is bound if it is within the scope of an operator whose attached variable is x; otherwise it is free. More generally, an occurrence of a term a or formula A in another term or formula is bound if it is within the

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scope of an operator whose attached variable occurs free in a or A; otherwise it is free.

Semantics

(i) *Individuals*

The individuals may be any objects; there may be none or one or more.

(ii) *Valuation and satisfaction*

For each variable x, *val* x is an individual or nothing.

val $\in$ is a two-place relation on the individuals.

val satisfies $a \in b$ iff *val* $\in$ holds between *val* a and *val* b.

val satisfies $a = b$ iff *val* a is identical to *val* b.

val satisfies ¬A iff it does not satisfy A. It satisfies $A \rightarrow B$ iff it satisfies B or does not satisfy A. Similarly for the other connectives.

val satisfies $\forall xA$ iff every x-variant (see below) of *val* satisfies A.

val ιxA is the individual *val*$'x$ if a unique x-variant *val*$'$ of *val* satisfies A; otherwise it is nothing.

(iii) *Logical truth and logical consequence*

$\models C$ iff every valuation, over no matter what individuals (none or one or more), satisfies C.

$\Gamma \models C$ iff every valuation, over no matter what individuals (none or one or more), satisfies C if it satisfies every one of Γ.

Besides the usual logical apparatus of connectives and quantification and identity, the system features the description operator ι as a primitive, producing descriptions with these denotation conditions: if A is true for some unique individual as value of x then ιxA denotes that individual. If there is no such individual, the description is empty or, as we shall say, it denotes *zilch*. Our use of 'zilch' here corresponds to the unjustly neglected use of 'nothing' as a necessarily empty term rather than a quantifier.

Singular Logic allows that there might be nothing at all. Our method for dealing with this possibility is to permit variables, and open terms in general, to be empty. This has the great advantage of settling the logical status of open formulas without disturbing modus ponens (see Section 11.1 of our [20]). Although the semantics of formulas with free variables is thereby affected, the semantics of variable-binding is unaffected. For example, $\forall xA$ will be true just in case A is true for every assignment of an individual as value of x. When we rephrase this in terms of valuations and satisfaction, we must take care of the case where the operative variable x is empty under the given valuation. So we need the following clause:

> *val* satisfies $\forall xA$ iff every valuation that differs from *val* at most in that x has a value and in what that value may be, satisfies A.

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The valuations on the right-hand side of this biconditional are thus stipulated to assign a value to x even if *val* x is zilch. In the summary of the semantics, we used Benson Mates's [18] label 'x-variant of *val*', now understood as abbreviating 'valuation that differs from *val* at most in that x has a value and in what that value may be'. The clause for the variable-binding operator ℩ uses the same idea.

We have opted to take the universal quantifier as primitive, using it to define the existential quantifier in the usual way. Since terms may be empty, we shall want a way of expressing existence. We use $E!$ to symbolize it, and define $E!a$ in the familiar way via identity, matching the equivalence between 'a exists' and 'a is something'.

Existence $\qquad E!a =_{df} \exists x\ x = a$

In the semantics, we use 'individual' in the logical sense to cover any kind of object, sets included, not as a synonym for 'ur-element'. We have avoided singular talk of a domain of individuals, conceived as a set, resorting instead to plural talk of the individuals themselves. Consequently, in the definitions of logical truth and consequence we replace singular quantification over domains—'over no matter what domain'—with plural quantification over individuals—'over no matter what individuals (none or one or more)'. One reason for this change is that it would be something of an own goal to develop a set theory that rules out empty and singleton sets, only to find them reappearing in the semantics as domains. The second reason is topic-neutrality, now operating at the other extreme of size. We want our logic to be applicable to reasoning about kinds of things that have so many instances that they do not form a set. Indeed our set theory is a case in point, since there is no such thing as the set to which everything belongs. The reader will also note that in the definition of logical consequence we use plural language, treating the premises as a number of formulas rather than a set. Here the use of the plural is not demanded by topic neutrality, but rather serves to replace redundant and unnatural talk of sets. It also avoids invoking empty and singleton sets when there are no premises or just one.

The use of plural quantification in the semantic metalanguage means that it is expressively richer than the singular object language. It also outstrips the object language in a quite different direction, as we shall now explain. For the sake of convenience, we have used 'valuation' or *val* as an umbrella word covering the assignment of values to items of two different syntactic categories: terms and predicates. For a term a as argument, *val* a is an individual or zilch. It is thus a partial function. For the predicate $\in$ as argument, *val* $\in$ is a two-place relation on the individuals, in the sense that for any individuals x, y the relation either holds or does not hold between x (or zilch) and y (or zilch) as arguments. By contrast, standard presentations of the semantics for the predicate calculus assign set-theoretic extensions to predicates as their semantic values, e.g., a set of ordered pairs to a two-place predicate, where the ordered pairs are themselves reduced to plain sets by one or other familiar method. Unfortunately for everybody, there is no such thing as the set of the ordered pairs $<x, y>$ for which $x \in y$. That is why we reinstate the relations for which the set-theoretic extensions were at best artificial surrogates, conceiving of them, like Frege's *Begriffe*, as different from objects, and thus not as values of first-order variables (for more examples of predicates without set-theoretic extensions, see Theorem 39 in the Appendix below). Functions, such as *val*, are like relations in this respect. Thus generalizing over valuations involves second- (or higher-) order quantification in the semantic metalanguage.

The following deductive system for Singular Logic is sound and complete. The axioms are all the instances of the following schemes, both as they stand and prefaced by any number of universal quantifications.

1. A where A is tautologous
2. $\forall x(A \to B) \to (\forall x A \to \forall x B)$
3. $A \to \forall x A$ where x is not free in A
4. $\forall x A(x) \to (E!a \to A(a))$ where $A(a)$ has free a wherever $A(x)$ has free x
5. $\forall x(x = x)$
6. $a = b \to (A(a) \leftrightarrow A(b))$ where $A(b)$ has free b at zero or more places where $A(a)$ has free a
7. $\neg E!a \wedge \neg E!b \to (A(a) \leftrightarrow A(b))$ where $A(b)$ has free b at zero or more places where $A(a)$ has free a
8. $a = b \to E!a \wedge E!b$
9. $\forall y(y = \imath x A \leftrightarrow \forall x(A \leftrightarrow x = y))$ where y does not occur in $\imath x A$

Rule of inference. From A and $A \to B$ infer B.

The interested reader may consult the Appendix to Ch. 11 of our [20] for soundness and completeness proofs. It is worth remarking that our proof of completeness does without set-theoretic machinery, and so there are no empty or singleton sets needed there either. As already mentioned, we treat a premise or premises as a formula or formulas rather than a set of them. Likewise, we define a deduction as a single formula or a sequence of several, thereby avoiding the need for singletons. We also replace the construction of a maximal set of formulas with one referring to its members, and instead of invoking equivalence classes as individuals in the treatment of identity, we use representative items.

In the same chapter we prove several metatheorems that we take for granted in what follows, e.g., change of bound variables and extensionality (substitutivity of equivalents). Of particular interest is metatheorem 5 ('Open formulas and schemes')

$\Gamma(x) \vdash A(x)$ if and only if $\Gamma(a) \vdash A(a)$ for all terms a for which $\Gamma(a)$ and $A(a)$ have free a just where $\Gamma(x)$ and $A(x)$ have free x

which means that a single theorem featuring a free variable can do duty for a theorem scheme with infinitely many instances, and similarly for deducibility. We make extensive use of open formulas in what follows.

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§4. Initial definitions. Here we comment on initial definitions before discussing those needed for the theory of levels in the next section. A summary list of definitions can be found at the beginning of the Appendix. Throughout, these definitions are to be understood as including the familiar provisos to prevent unintended capture of variables. As usual, slashed two-place predicates are convenient shorthand: $a \notin b$ abbreviates $\neg(a \in b)$, etc.

As well as defining $\exists$ in terms of $\forall$, we define two more quantifiers:

'Exactly one' quantifier $\quad \exists_1 x A(x) =_{df} \exists x \forall y (A(y) \leftrightarrow x = y)$
'Many' quantifier $\quad mxA(x) =_{df} \exists x \exists y (x \neq y \wedge A(x) \wedge A(y))$

The quantifier $\exists_1$ may be read as 'there is exactly one', or simply 'one', while m may be read as 'there are many' or simply 'many', taking 'many' in its weakest sense as equivalent to 'more than one', i.e., at least two. The semantics of ι means that $E!\iota x A(x)$ is equivalent to $\exists_1 x A(\mathrm{x})$.

We call an n-place predicate F *strong* at its i-th place if it is necessary that if $Fa_1 \ldots a_n$ then a_i exists; otherwise it is *weak* at that place. We think that it is not for logic to determine the strength of places of primitive nonlogical predicates. Hence $\in$, like any two-place predicate, may be assigned a relation that holds of zilch at one or both of its places. Since, however, under its intended meaning $\in$ is strong at both places, we shall add a nonlogical axiom to ensure its strength (Axiom 1(i) in Section 6 below).

As to the sole primitive *logical* predicate $=$, we must fix its meaning and therefore need to make a choice. We opt to make $=$ strong at both places, so that $a = b$ is satisfied only if both a and b exist. This is embodied in the definition of $E!$ in Section 3. Since the corresponding notion of weak identity also proves invaluable, we use $=$ and $E!$ to define a symbol for it:

Weak identity $\quad a \equiv b =_{df} a = b \vee (\neg E!a \wedge \neg E!b)$

The identities $a = b$ and $a \equiv b$ only differ when a and b are both empty, so we can move freely between them when either or both terms are nonempty. For example, if we define a term c by d, the definition on its own only allows us to infer the weak identity $c \equiv d$, since no term is guaranteed to be nonempty. But if we are also given that d exists, we can go on to infer the strong identity $c = d$. As we explain in Section 11.4 of our [20], working the other way round is equally viable, i.e. taking weak identity as primitive and defining strong identity in terms of it.

We symbolize the paradigm empty term 'zilch' by an italic capital O. Although O may be taken as primitive, we opt to define it as a description:

Zilch $\quad O =_{df} \iota x (x \neq x)$

The description $\iota x(x \neq x)$ is necessarily empty on account of the logically unsatisfiable condition $x \neq x$ and the semantics of ι. Hence $E!O$ and $O = O$ are both logically false, while $O \equiv O$ is logically true. Also $a \equiv O$ is equivalent to $\neg E!a$, and therefore provides another way to express nonexistence.

We need to emphasize that O does not denote anything whatever, however special or recondite. It denotes zilch, that is to say, it denotes nothing. In particular, it should not be confused with $\varnothing$ as this symbol is conventionally understood, namely as standing for the empty set, which is something, not nothing. O plays a pivotal role in our set theory, where it is not simply an empty surrogate for the nonempty $\varnothing$. For more on zilch see our [20], pp. 111–114 & 120–128.

The quantifications $\forall x A(x)$ and $\exists x A(x)$ do not decide the case $A(O)$; a predicate true of everything, or true of something, may or may not be true of zilch. As a useful supplement to the standard quantifiers, then, we introduce 'inclusive' quantifiers $\forall^O$ and $\exists^O$ to cover the undecided case.

Inclusive quantifiers $\quad \forall^O x A(x) =_{df} \forall x A(x) \wedge A(O)$
$\quad \exists^O x A(x) =_{df} \exists x A(x) \vee A(O)$

The quantifier m can be used to define the notion of a set, since in the absence of the empty set and singletons, sets can be characterised as multimembered objects. We symbolize 'a is a set' by Ma, after Cantor's *Menge*.

Set $\quad Ma =_{df} mx\, x \in a$

M is a strong predicate by the definition of m and the strength of $\in$. We symbolize 'a is an ur-element' by Ua, and define it in terms of $E!$ and M:

Ur-element $\quad Ua =_{df} E!a \wedge \neg Ma$

U is thus a strong predicate too. Together, U and M provide a mutually exclusive and jointly exhaustive classification of the individuals. Next come familiar definitions of subset $\subseteq$ and proper subset $\subset$. They mean that $\subseteq$ and $\subset$ are strong at both places.

Subset $\quad a \subseteq b =_{df} Ma \wedge \forall x(x \in a \rightarrow x \in b)$
Proper subset $\quad a \subset b =_{df} a \subseteq b \wedge a \neq b$

The set abstract $\{x{:}A(x)\}$ is defined, using the description operator, in an obvious way:

Set abstraction $\quad \{x{:}A(x)\} =_{df} \iota z(Mz \wedge \forall x(x \in z \leftrightarrow A(x))$

Like any term, $\{x{:}A(x)\}$ may be empty. One cause of emptiness is when $A(x)$ is satisfied by too few things to form a set (i.e., none or one). But it may also be satisfied by too many, e.g. $\{x{:}Mx\}$ is empty because there is no set of all sets (for further examples see Theorem 39 in the Appendix).

We also define a second kind of abstract $[x{:}A(x)]$ which is designed to provide an acceptable alternative to Cantor's untenable identification of singletons with their sole members. If $A(x)$ is satisfied by a unique object, $[x{:}A(x)]$ denotes that object. As with the case of O and $\varnothing$, $[x{:}A(x)]$ is not simply a surrogate for $\{x{:}A(x)\}$ but leads a life of its own.

Modified set abstraction $\quad [x{:}A(x)] =_{df} \iota z(z = \iota x A(x) \vee z = \{x{:}A(x)\})$

This is used to define the intersection $a \cap b$ of sets a and b as the object that is either their sole common member (*not* its singleton) or the set of their many common members. If there is no common member, there is no such thing as $a \cap b$, i.e. $a \cap b \equiv O$.

Intersection $a \cap b =_{df} [x\colon x \in a \wedge x \in b]$

Definitions of other operations are given in the Appendix. It should be acknowledged that the resulting algebra of sets is decidedly more complicated than the familiar Boolean one. But to our surprise we found that the entire development in the Appendix makes no use whatever of such an algebra. We suspect that its omnipresence in the textbooks is more as an advertisement for the versatility of Boolean algebra than in providing a tool for serious work.

§5. Levels. In his 1930 article 'On boundary numbers and domains of sets', Zermelo uses a new technique to establish isomorphisms between models of his set-theoretic axioms. He describes it as the 'development' of a domain:

> its decomposition into a well-ordered sequence of separated 'layers' where the sets belonging to one layer are always 'rooted' in the preceding layers such that their elements lie in those layers, and they themselves, in turn, serve as material for subsequent layers ([29], p. 401)

Corresponding to each exclusive layer (*Schicht*), there is a cumulative segment (*Abschnitt*) or 'partial domain', which is the union of all preceding layers. The segments are none other than the cumulative types now familiar from presentations of the iterative conception of set.

In 'Axiomatizing set theory' Dana Scott picks up on Zermelo's cumulative segments, which he calls 'levels' (sometimes 'type levels' or simply 'types'), claiming that the theory of levels provides an 'intuitive justification' ([23], p. 208) for set theory in which 'the artificial, "ad hoc" axioms have been completely avoided' ([23], p. 212), and regretting that Zermelo still did not give levels the prominence they deserve. In fact, Zermelo did just that in his draft 'On the set-theoretic model' [30], which however was still unpublished at the time Scott wrote. Scott's contribution is novel in three respects. He proceeds entirely in terms of cumulative levels with no recourse to Zermelo's exclusive layers; he offers an axiomatization of the theory of levels from which Zermelo's original 'artificial ad hoc' axioms can be derived as theorems; and he does so without relying on ordinals. Later improvements on Scott's ideas were made by John Derrick in unpublished work and Michael Potter in his book *Set Theory and its Philosophy* [21]. What we shall call the Scott/Derrick/Potter theory is our starting point in what follows. Like them, we operate with a first-order framework, dropping Zermelo's appeal to second-order ideas in his 1930 papers.

When empty and singleton sets are omitted, the cumulative hierarchy of levels can be informally characterized as follows. Levels are sets and they are well-ordered by $\in$. It is convenient to reserve the variables u, v, w for levels. We say that u is *lower* than v (v is *higher* than u) if $u \in v$. The members of a level are all the ur-elements together with all the members and

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subsets of the lower levels. In short, a level is the *accumulation* of the lower levels.

Since the first or lowest level has no lower levels, it is the set of all ur-elements. In the literature, this first level is generally labelled V_0. We, however, will call it V_1 since we think it should properly be indexed with 1 for 'first' (Zermelo, as it happens, labelled his first segment P_1). This also enables us to reserve V_0 for a more principled use as standing for zilch.

The hierarchy is cumulative, since a member or subset of any level is also a member or subset of all higher levels. It is exhaustive, since every object is located within it: every set is a subset of some level and also a member of some level, while every ur-element is a member of every level. It is also infinite: for every level there is a higher one. So if there are any levels at all, there are infinitely many.

Whether there are any levels, or sets of any kind, depends on the number of ur-elements. If there are many (i.e., at least two), then levels exist. If there are none or just one, there are no sets at all. In this respect, our hierarchy differs sharply from that of Scott/Derrick/Potter. Although they rightly allow for ur-elements, they also have empty and singleton sets to start the ball rolling even without any ur-elements.

Our task is to systematize this informal conception of levels by designing suitable definitions and axioms. Scott's plan was to take 'is a level' to be primitive and govern it by an axiom ensuring that a level is the accumulation of the lower ones. But a more attractive option is to find a definition of level that makes the axiom redundant. The key is the notion of a *history*. One first defines accumulation, then history in terms of accumulation, and finally a level as the accumulation of a history. This is Derrick's idea, as subsequently improved on by Potter. We adopt this broad strategy, but in the absence of empty and singleton sets we have had to rework the whole theory of levels from scratch. It is by no means a matter of inserting an occasional restriction or making other minor adjustments. There are significant differences at every point: underlying logic, primitive vocabulary, definitions, axioms, lemmas, theorems. Consequently, the development presented in the Appendix differs substantially from Potter's own, both in global organization and locally in the strategies required to prove particular theorems. Later we comment in detail on some of these differences (see especially Section 12 and the comments on the proofs of Theorems 4 and 6 in the Appendix), but some brief comparisons are in order here.

Like us, Scott and Potter work with a first-order underlying logic, including identity as a logical constant and permitting variables to range over any objects whatever. Neither, however, pauses to articulate the logical framework, being content to operate in a decidedly informal manner without a fully specified syntax or semantics. It is clear, though, that both violate topic neutrality by requiring domains to be nonempty. Scott also rules out empty terms, while Potter allows that some kinds of term—descriptions but not constants—may be empty.

As to vocabulary, both take $\in$ as primitive. Scott takes 'is a level' as primitive, while Potter defines it. Scott also presupposes 'is a set' as a third primitive, when he uses a, b, c ... as restricted variables ranging over sets in contrast to his unrestricted variables x, y, z ... which range over all objects, ur-elements included ([23], p. 207). This notation enables him to characterize an ur-element x as a non-set $(\neg\exists a\ x = a)$, and thus distinguish it from the empty set. Potter's second primitive is the predicate U ('is an ur-element', or 'is an individual' in his terminology). He uses it to distinguish an ur-element from the empty set, noting that those who do without a second primitive 'are unable to distinguish formally between $\varnothing$ and an individual, since individuals, we may suppose, share with $\varnothing$ the property of having no members' ([21], p. 60).

In contrast, we can get by with $\in$ as our sole primitive, using it to define both 'is a set' and 'is an ur-element'. Our definitions mean that there are no individuals besides sets and ur-elements. This was Scott's assumption. Potter also allows for 'ungrounded' or 'non-well-founded' collections which lie outside the hierarchy of levels and thus are neither ur-elements nor sets. Consequently, he starts by defining the more general idea of 'collection', characterizing a set as a subcollection of some level. But even his broader notion of 'collection' is not the opposite of 'ur-element', since he also allows for a fourth possible kind of thing that is neither ur-element, set nor ungrounded collection. He finds no use for them, however, nor for ungrounded collections, and we have omitted both here. We conjecture that Potter's ultra-liberal ontology is partly motivated by a desire to economize on axioms. For example, by defining sets as grounded collections, he avoids the need for Scott's restriction axiom, which ensures that any set is a subset of some level. Potter would also need an extra axiom to rule out things that are neither collections nor ur-elements. Yet another axiom would be needed to ensure that ur-elements do not have members, but Potter opts to forgo it despite thinking it true.

We now turn to our approach to defining 'is a level'. We talked of 'the lower levels' when we characterized a level as the accumulation of all the lower ones. The plural description 'the lower levels' covers three cases: (i) zilch for those lower than the first level V_1; (ii) the single level V_1 for those lower than the second level V_2; and (iii) many levels for those lower than any other level. Since we are operating with a purely singular object language, we must find a singular replacement for the plural description 'the lower levels'. Naturally, Potter's choice is 'the set of lower levels'. For him, the set of levels lower than a level v is a history, and v is the accumulation of this history. He thereby equates a history with a set in every case: the empty set for case (i), the singleton $\{V_1\}$ for (ii), and multimembered sets for (iii). We, however, treat the three cases separately, leaving the first two as they are and employing a set only in case (iii), to replace the many lower levels with the set to which they belong. All three cases are covered by the single abstract $[w{:}w \in v]$, which (i) is empty if no level is lower than v, or (ii) denotes the single level lower than v if such there be, or (iii) denotes the set of levels lower than v if there are many such levels.

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The upshot is that we can take over Potter's definition of accumulation ([21], p. 41) as follows:

Acc function $\quad$ $\text{acc } a =_{df} \{z : (Uz \vee \exists y(y \in a \wedge (z \in y \vee z \subseteq y)))\}$

By itself, however, acc does not do the whole job. It delivers the right result for any level higher than V_2, for then the relevant a will be the set of all lower levels. But it will not work for V_2 with its sole lower level V_1. We want V_2 to be the accumulation of V_1, but the condition $y \in a$ picks out the members of V_1, not V_1 itself. Since V_1 is the set of ur-elements, and nothing is a member or subset of any ur-element, the result is that acc $V_1 = \{z{:}Uz\} = V_1$, not V_2. Potter, since he admits singletons, gets the right result with $V_2 = \text{acc } \{V_1\}$. We need to follow a different course, replacing $y \in a$ by $y = a$ in the definition, thereby introducing a second accumulation function which simplifies to

Accum function $\quad$ $\text{accum } a =_{df} \{z : (Uz \vee z \in a \vee z \subseteq a)\}$

V_2 can then be defined to give the desired result:

Second level $\quad$ $V_2 =_{df} \text{accum } V_1$

This leaves V_1, which is the accumulation of its history, namely zilch. Again we use accum to define it:

First level $\quad$ $V_1 =_{df} \text{accum } O$

Although we introduced them separately, it is worth remarking that acc can be defined in terms of accum as follows:

$$acc\ a =_{df} \{z : (Uz \vee \exists y(y \in a \wedge z \in \text{accum } y))\}$$

The functions expressed by acc and accum may be partial functions, since if there is but one ur-element, acc and accum both map it to zilch. They may also be 'co-partial' functions, mapping zilch to $\{z{:}Uz\}$ (for more see Section 5.6 of our [20]).

We say that a is an accumulation (symbolized by $\mathcal{A}a$) if it is of the form accum x or acc x. Remembering that x may be zilch, we use the inclusive quantifier $\exists^O$ to define it:

Accumulation predicate $\quad$ $\mathcal{A}a =_{df} \exists^O x(a = \text{accum } x \vee a = acc\ x)$

Next, in the spirit of Derrick, we need to define 'a is a history' (symbolized by Ha) without appealing to the notion of level:

History $\quad$ $Ha =_{df} (a \equiv O \vee Ma) \wedge (a = V_1 \vee \forall y(y \in a \to (y = V_1 \vee y = V_2 \to y = \text{accum } a \cap y) \wedge (y \neq V_1 \wedge y \neq V_2 \to y = acc\ a \cap y))$

This definition needs a little explanation. Naturally we deal separately with the possibility that y is either V_1 or V_2 by replacing acc by accum for these exceptional cases. But we also need to bring in the possibility that a is V_1. The problem is the same as before. Any member y of V_1 is an ur-element and is therefore neither V_1 nor V_2. But no ur-element y is acc $a \cap y$, since the function acc only has sets as values.

It is convenient to reserve the variables h, h_1 for histories. If $V_1 = \text{accum } h$, we say that V_1 has h has a history; similarly for V_2. If v is any level other than V_1 or V_2, and $v = \text{acc } h$, we say that v has h as a history. We want any level v to have $[w{:}w \in v]$ as its unique history, but uniqueness is liable to fail if we count an ur-element x as a history. For supposing V_1 exists, $V_1 = \text{accum } O$ and also $V_1 = \text{accum } x$. In order to rule this out, the initial conjunct in the definition requires that a history, if it exists at all, be a set.

Finally, the definition of 'a is a level' (symbolized by Va). It should be no surprise that the exceptional levels V_1 and V_2 are dealt with separately, while the rest can be characterized as acc x for some history x

Level $\quad Va =_{df} a = V_1 \vee a = V_2 \vee \exists x(Hx \wedge a = \text{acc } x)$

The definitions secure that accumulations and levels are sets, and that $\mathcal{A}$ and V, like M, are strong. But H is weak, since the history of V_1 is O.

There are three mutually exclusive and jointly exhaustive kinds of level: the first level V_1, levels next above a level, and limit levels. We define the level next above a (symbolized by a') to be the lowest level that is higher than the level a.

Level next above $\quad a' =_{df} \iota x(Va \wedge Vx \wedge a \in x \wedge \neg \exists y(Vy \wedge y \in x \wedge a \in y))$

We define 'a is a limit level' (symbolized by La) as 'a is a level that is neither V_1 nor next above any level'.

Limit level $\quad La =_{df} Va \wedge a \neq V_1 \wedge \neg \exists x\ a = x'$

§**6. Axioms.** The axioms are all the instances of the following schemes, both as they stand and prefaced by any number of universal quantifications, and are to be understood as including the familiar provisos to prevent unintended capture of variables. Recall that we use the variables u, v, w for levels, so that, e.g., $\forall u A(u)$ is short for $\forall x(Vx \rightarrow A(x))$.

1. *Membership*
   - (i) $x \in y \rightarrow E!x \wedge E!y$
   - (ii) $My \wedge Mz \rightarrow \forall x(x \in y \leftrightarrow x \in z) \rightarrow y = z$
   - (iii) $x \notin x$
   - (iv) $\exists x\ x \in y \rightarrow My$

2. *Levels*
   - (i) $mxUx \rightarrow M\{x{:}Ux\}$
   - (ii) $\exists xMx \rightarrow mxUx$
   - (iii) $M\{x{:}A(x)\} \leftrightarrow mxA(x) \wedge \exists u \forall x(A(x) \rightarrow x \in u)$
   - (iv) $\forall u \exists v\ u \in v$
   - (v) $\exists xVx \rightarrow \exists xLx$

The first group govern membership. 1(i) ensures that $\in$ is strong at both places. 1(ii) is extensionality for sets. 1(iii) rules out self-membership, while 1(iv) rules out singletons, i.e., there is no y such that $x \in y$ for just one x.

The second group are principles governing levels. Together with other axioms, 2(i) ensures that if many ur-elements exist, so does the set of them, i.e., V_1 exists. 2(ii) says that unless there are many ur-elements, there are no sets, and hence no levels. The scheme 2(iii) gives necessary and sufficient

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conditions in terms of levels for $\{x{:}A(x)\}$ to be a set, namely that there are many As and there is some level such that each A is a member of it. The $\leftarrow$ half is thus a useful principle of separation from levels. 2(iv) is Potter's 'axiom of creation'—for every level there is a higher one—from which it follows that there are infinitely many levels if there are any at all. 2(v) is a limit level axiom, which asserts the existence of a limit level if there are any levels at all.

§7. **Development of the theory.** We put the detailed development of the theory of levels into the Appendix. Unsurprisingly, given the definitional set-up, a recurrent feature of the proofs is the separate treatment of the first two levels V_1 and V_2. We prepare the ground for this by including their main peculiarities in the last of the opening lemmas.

Some preliminary theorems about accumulations and histories lead to the well-foundedness of membership on any history (Theorem 6). This is the first of three foundation principles of increasing generality. It is narrowly concerned with subsets of histories, whereas the foundation scheme for levels (Theorem 12) covers any specifiable property of levels, and foundation for sets (Theorem 17) covers sets in general. In our development, Theorem 6 is the linchpin: it is used to prove Theorem 12, which in turn is used to prove Theorem 17. In the Appendix we explain how our proof of Theorem 6 differs substantially from Potter's as a result of our restriction of separation from levels to exclude empty and singleton sets.

From Theorem 6 it follows that levels are transitive (Theorem 7) and hereditary (Theorem 8) sets. Membership between levels is irreflexive (Axiom 1(iii)), transitive (Theorem 7) and well-founded (Theorem 12), and levels are comparable under membership (Theorem 13). Hence membership well-orders the levels. So too does the proper subset relation (Theorem 27).

A level v has $[w{:}w \in v]$ as its unique history (Theorems 10, 11, and 15). Given many ur-elements, the lowest level of all is V_1—the set of ur-elements (Theorem 28). For any level v, there is a level v' next above v (Theorem 29), whose members are the ur-elements together with the subsets of v. Equivalently, v' is the set of all the members and subsets of v (Theorem 30), or as we shall say, v' is the *power-plus set* of v, symbolized by $P{+}(v)$. The third kind of level, a limit level, exists whenever there are any levels at all, i.e., provided there are many individuals (33), and is the union of its history (34).

How do objects fit into the hierarchy of levels? Every ur-element is a member of every level (Corollary (ii) of Theorem 1). Every set is a member of some level (Theorem 23) and also a subset of some level (Theorem 16(i)). If a set bears either relation to a level, it bears the same relation to every higher one (Theorem 7 and its Corollary, and Theorem 8). If a set is a member of a level, it also a subset of it (Corollary of Theorem 7). But *not* vice versa. For any set, there is a unique lowest level of which it is a subset—*the level of* the set for short (Theorem 16(ii)). For any set, there is a unique lowest level of which it is a member (Theorem 23): it is not the level of the set, but the level next above (Theorem 31).

Looking at matters from the outside, the well-ordering of levels means that they can be indexed by ordinals, with the cumulative hierarchy of levels defined by transfinite recursion:

$V_1 = \{x{:}Ux\}$
$V_{\alpha+1} = P+(V_\alpha)$ where α is any ordinal
$V_\lambda = \cup_{\beta<\alpha} V_\beta$ where λ is a limit ordinal

The position of items (their 'height') within the hierarchy is then measured by assigning ordinals to them as ranks. Thus α is the rank of a given item if $V_{\alpha+1}$ is the lowest level of which it is a member. In particular, the rank of an ur-element is 0.

Along the way we prove analogues of several axioms of Zermelo-style set theory: Foundation or Regularity (Theorem 12), Cantorian Separation (18), Union (21 and 22), Pairing (24) and Power Set (25). Although these theorems are recognizably close relatives of the conventional axioms, they are framed to suit the present context. In Foundation, the usual restriction to nonempty sets is now redundant. The Separation scheme now follows Cantor's requirement that the separated members be many. In the same vein, Pairing requires that the members of the putative pair be distinct, and Power Set requires the initial set to have many subsets, which is not true of pair sets in the absence of empty and singleton subsets.

Towards the end of the Appendix we provide a representation of ordered pairs as plain sets, which easily generalizes to n-tuples, and prove that it is adequate (Theorem 38). Finally, we prove a nonexistence Theorem (39), which means that none of the key predicates $E!$, M, V, $\mathcal{A}$ and H has a set as its extension.

§8. The Axiom of Plurality and other axioms. The abstract version of the set theory we are presenting allows for ur-elements but makes no assumptions about their number or nature, and is thus as close to being logic as it is possible to get. All the axioms are true when there are no ur-elements, and hence nothing at all, or when there is just one ur-element, and hence no sets. It follows that orthodox set theory cannot be interpreted within the Cantorian theory, since the former implies that sets exist, whereas the latter makes no existential claims. Allen Hazen [15] has claimed that reference to empty and singleton sets can be regarded as a façon de parler for talk about multimembered sets. But even apart from the vitiating artificiality of his translation of $\in$, evidently this will not work here.

The weakest additional axiom that will guarantee the existence of sets is the Axiom of Plurality, which asserts the existence of at least two things:

Axiom of Plurality $mx\ x = x$

Taken in isolation, this axiom is weaker than the more specific $mxUx$, but the two are equivalent in context. Adding the Axiom of Plurality to the rest secures a transfinite hierarchy of sets, despite its being vastly more modest than Whitehead and Russell's axiom of infinity. The reason is that it works in tandem with the limit level Axiom 2(v) to entail the existence of a limit level (Theorem 33), and therefore the existence of the lowest one V_ω (Theorem 35). Since V_ω is a set with all the finite levels among its members, the Axiom of Plurality and the limit level Axiom 2(v) function together to yield an unconditional Zermelo-style axiom of infinity. As to the other axioms of Zermelo set theory, our Axiom 1(ii) is extensionality, while Axiom 2(i) secures the existence of the set of ur-elements provided there are many of them. As we remarked in the previous section, appropriate analogues of separation, foundation, union, pairing and power set are derivable as theorems. For those more familiar with a Zermelo-style axiomatization, it may be helpful to set down our versions of the standard axioms. On the right, we indicate their location in the present development.

Extensionality	$My \wedge Mz \rightarrow \forall x(x \in y \leftrightarrow x \in z) \rightarrow y = z$	axiom 1(ii)
Ur-elements	$mxUx \rightarrow M\{x{:}Ux\}$	axiom 2(i)
Foundation	$Mx \rightarrow \exists y(y \in x \wedge x \cap y \equiv O)$	theorem 17
Separation	$Mx \wedge my\,(y \in x \wedge A(y)) \rightarrow M\{y{:}y \in x \wedge A(y)\}$	theorem 18
Union	$Mx \wedge \forall y(y \in x \rightarrow My) \rightarrow M(\cup x)$	theorem 22
Pairing	$E!x \wedge E!y \wedge x \neq y \rightarrow M\{x, y\}$	theorem 24
Power set	$my\; y \subseteq x \rightarrow M(P(x))$	theorem 25
Infinity	$mx\; x = x \rightarrow \exists x\,(V_1 \in x \wedge \forall v(v \in x \rightarrow v' \in x))$	thms 33, 35, 36

This leaves choice and replacement, which need to be secured by new axioms. The first may be expressed as: for any set of pairwise disjoint sets, there is a choice set that has exactly one member in common with each member of the original set. This is the same as the familiar version, except that we do not need to add that the pairwise disjoint sets are nonempty. Formalizing the principle in the current framework requires two departures from the conventional presentation. First, we express disjointness of sets using zilch ($y \cap z \equiv O$) not the empty set. Second, we spell out the character of the choice set directly, rather than take the usual detour via singleton intersections.

Axiom of Choice $\forall x((Mx \wedge \forall y(y \in x \rightarrow My) \wedge \forall y \forall z((y \in x \wedge z \in x \wedge y \neq z) \rightarrow y \cap z \equiv O)) \rightarrow \exists y \forall z(z \in x \rightarrow \exists_1 z_1(z_1 \in z \wedge z_1 \in y)))$

As to replacement, the conventional scheme needs only a slight tweak, to require that there are many members involved, just as the Cantorian separation scheme involves many members:

Axiom of Replacement $\forall x \exists_1 y A(x, y) \rightarrow \forall x((Mx \wedge mz \exists y(y \in x \wedge A(y, z))) \rightarrow E!\{z{:}\exists y(y \in x \wedge A(y, z))\})$

§9. Ordered pairs. Once one has to hand the set of natural numbers **N** (see Section 10), the arithmetization of analysis can be carried through within Cantorian set theory. Playing on the aphorism of Cantor's nemesis Kronecker, *alles andere ist Mengenwerk*. The notion of an ordered pair is essential to the project. But we cannot adopt Kuratowski's now standard

https://doi.org/10.1017/bsl.2018.10 Published online by Cambridge University Press

definition of $<a, b>$ as $\{\{a\}, \{a, b\}\}$, since it needs singletons. Hausdorff's ([14], p. 32) earlier version uses a pair-set of pair-sets equipped with 1 and 2 as markers, where 1 and 2 are arbitrarily chosen distinct objects which serve to specify a as first coordinate and b as second: $<a, b>=_{df} \{\{a, 1\}, \{b, 2\}\}$. Hausdorff sacrifices generality by requiring that neither a nor b is 1 or 2. In fact, his definition is adequate when 1 and 2 are allowed back in as coordinates, but it then requires singletons. In any case the revised definition is still not truly general, since it cannot accommodate zilch. For example, $<O, b> = \{\{O, 1\}, \{b, 2\}\} = \{\{1\}, \{b, 2\}\} = \{\{1, 1\}, \{b, 2\}\} = <1, b>$. Kuratowski's definition is likewise deficient, since $<a, O>= \{\{a\}, \{a, O\}\} = \{\{a\}, \{a\}\} = \{\{a\}, \{a, a\}\} = <a, a>$.

Allowing a or b in $<a, b>$ to be zilch is highly convenient, since it avoids the need to establish that a and b are nonempty prior to using the ordered pair notation. In Section 11 below, we include zilch as a coordinate in structures regarded as ordered pairs. Zilch also provides for representations of partial and co-partial functions as sets of ordered pairs, where the first or 'argument' coordinate may be zilch in the co-partial case, and the second or 'value' coordinate may be zilch in the partial one. These representations are convenient, since the information that and where a function is partial, for example, is determined by the set of ordered pairs itself and does not need a second, extraneous specification ('on a set X').

To give a fully general representation of ordered pairs, we rework Hausdorff's technique and use the markers 1 and 2 thrice over. We also exploit the denotational behaviour of [ ] abstracts as opposed to { } ones (see Lemma 2 in the Appendix). First, then, we define $[a, b]$ by analogy with $\{a, b\}$ as $[z{:}z = a \vee z = b]$. It follows that $[a, b] = \{a, b\}$ provided $E!a$ and $E!b$ and $a \neq b$. But $[a, O] \equiv [O, a] \equiv [a, a] \equiv a$, whereas $\{a, O\} \equiv \{O, a\} \equiv \{a, a\} \equiv O$. Next we use [ ] abstracts to define the ordered pair $<a, b>$ as $\{\{\{[a, 1], [a, 2]\}, 1\}, \{\{[b, 1], [b, 2]\}, 2\}\}$. We then prove (Theorem 38) that it is adequate for any choice of coordinates a and b—ur-element, set, or zilch. Singletons are not needed. In particular, none of $<a, a>$, $<a, O>$, $<O, a>$ reduces to $\{a\}$. The definition can be naturally generalised to cover ordered n-tuples by using markers $1, 2, \ldots, n$, thereby avoiding the conventional identification of n-tuples with iterated ordered pairs, which conflates different kinds of thing.

The markers 1 and 2 can themselves be coordinates of ordered pairs. But what are they? One option is to take them to be ur-elements, perhaps natural numbers, though in truth any pair of ur-elements will do. This course, however, must use numerals or some other kind of term to pick out the chosen ur-elements, which means that the language has to be extended. Also the proofs that ordered pairs exist and have their so-called characteristic property will depend on, say, the hypothesis $U1 \wedge U2 \wedge 1 \neq 2$, which is stronger than the Axiom of Plurality. A better option, the one we adopt, is to take the markers to be sets that can already be identified and distinguished using the resources of the original language. The obvious candidates are the first two levels, V_1 and V_2. The relevant proofs then depend only on the Axiom of Plurality.

§10. Arithmetization of analysis. With ordered pairs to hand, we can go on to define the Cartesian product $a \times b$ of sets a, b and prove its existence. Then a typical constructional path will identify (i) the set of integers **Z** with a set of infinite equivalence classes of members of $\mathbf{N} \times \mathbf{N}$ under the operation of subtraction; (ii) the set of rationals **Q** with a set of infinite equivalence classes of members of $\mathbf{Z} \times \mathbf{Z}$- under division, where **Z**- comprises the nonzero integers; (iii) the set of reals **R** with the set of lower Dedekind sections (i.e., infinite subsets of **Q**); (iv) the set of complex numbers **C** with $\mathbf{R} \times \mathbf{R}$. Given the Axiom of Plurality, the existence of **Z** and **Q** can be proved within our theory by routine applications of Cantorian separation from the power set of the relevant products, the existence of **R** by separation from the power set of **Q**, and the existence of **C** follows from the existence of products in general.

Most authorities think that it is perfectly legitimate to postulate the natural numbers (or some other number system) as ur-elements. Landau, for example, opens his classic *Foundations of Analysis* with 'We assume [the set of natural numbers] to be given' ([16], p. 1). Similarly, when Gödel is explaining the iterative conception of sets he takes it for granted that sets are obtained 'from the integers (or some other well-defined objects)' ([12], p. 180). And Paul Cohen agrees that 'a very reasonable position would be to accept the integers as primitive entities and then use sets to form higher entities' ([4], p. 50).

A minority follow Frege and *Principia* in constructing a set-theoretical representation of the natural numbers themselves, nowadays typically using the finite von Neumann ordinals $\varnothing$, $\{\varnothing\}$, $\{\varnothing, \{\varnothing\}\}$, $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$ and so on, with an axiom of infinity providing for the set to which they all belong. This particular representation is obviously not available within our theory. But, as noted in Section 8, once we add the Axiom of Plurality, it follows that V_ω exists. We say that a is inductive if (i) $V_1 \in a$ and (ii) $\forall v(v \in a \rightarrow v' \in a)$, and prove that V_ω is inductive (Theorem 36). It follows that there is an inductive set $\mathbf{N}^*$, which has as its members just those things belonging to every inductive set (Theorem 37). $\mathbf{N}^*$ is none other than the set of the finite levels, i.e. the history of V_ω. We define the set of natural numbers to be $\mathbf{N}^*$, with 0 defined as V_1 and the successor of v as v'. Given these definitions, it is straightforward to derive Peano's axioms as theorems.

§11. Cardinals and ordinals. It is conventional to define 'a is a cardinal number' as 'a is the cardinal number of b (card b for short) for some set b'. Cardinal numbers are thus ascribed to sets and count their members. In the absence of empty and singleton sets, we tweak the definition by making 0 the number of zilch and 1 the number of any ur-element, keeping the rest as numbers of members of a set.

We identify the finite cardinals with the natural numbers, taken as ur-elements or defined. Then card a, where a may be zilch or an ur-element or a finite set, is defined to be the natural number n if a and $[z\colon z \in \mathbf{N} \wedge z < n]$ are equinumerous. But the modified set abstract $[z\colon z \in \mathbf{N} \wedge z < n]$ denotes

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zilch when a is zilch and its cardinal number is 0, and it denotes 0 when a is an ur-element and its cardinal number is 1. So we tweak the standard definition of equinumerosity by adding that every ur-element is equinumerous with every ur-element and nothing else, and zilch is equinumerous only with itself.

As to infinite cardinals, we use Scott's idea [22] to represent them as equivalence classes of infinite sets under equinumerosity, à la Frege and *Principia* but avoiding problems of size. For any infinite set a, there is a unique lowest level v that has some subset equinumerous with a. So card a can be defined as the set of sets equinumerous with a whose level is v.

The standard definitions of addition, multiplication etc make use of representatives whose cardinal numbers are the arguments of the defined functions. Since these representatives now include zilch and ur-elements, the definitions require tweaking. First we define $a \underline{\in} b$ as $(a = b \wedge Ub) \vee a \in b$, and then $a \Cup b$ as $[z{:}z \underline{\in} a \vee z \underline{\in} b]$, and redefine '$a$ and b are disjoint' as $\neg\exists z(z \underline{\in} a \wedge z \underline{\in} b)$. Then, for example, we can define $\alpha + \beta$ as card $(a \Cup b)$, where card $a = \alpha$ and card $b = \beta$ and a and b are disjoint and may be zilch or ur-elements or sets.

Similar techniques are used in the treatment of ordinals. They are conventionally ascribed to well-ordered structures, where these are defined to be ordered pairs consisting of a set and a well-ordering on it. In the absence of empty and singleton sets, we need to extend the definition to cover zilch and ur-elements. It is usual to deal with these cases by making the empty set well-order the empty set as well as any singleton. We replace this artificiality with one of our own, making zilch well-order both zilch and any ur-element. The singleton of an ordered pair as the well-ordering of a pair set will be replaced by the ordered pair itself. We go on to redefine isomorphism between well-ordered structures to take account of zilch and ur-elements. Finite ordinals are identified with the natural numbers, and infinite ones constructed à la Scott. It is a routine matter to extend the definitions of addition, multiplication etc to take account of the new coordinates in structures.

§**12. Evaluation.** Our theory has a small number of simple and transparent axioms. Likewise its underlying logic. Of course, variations can be obtained by redesigning the definitions or redistributing the content of the axioms or redividing the labour between underlying logic and definitions and axioms. To take just two examples, the strength of $\in$ may be secured by the logic rather than by a nonlogical axiom, and M may be redefined so that extensionality for sets drops out as a theorem.

The theory can serve the needs of ordinary pure and applied mathematics, providing an auxiliary superstructure of sets for reasoning about any chosen ur-elements. With the natural numbers taken as ur-elements, it is also adequate for the arithmetization of analysis and the theory of transfinite numbers (Sections 9–11); the interested reader can easily check that there is no use of the anomalous sets. Alternatively, the natural numbers

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themselves can be represented by sets provided there are at least two things. They cannot, of course, be represented as *pure* sets, but we do not see this as a significant loss (see Section 14.6 of our [20]).

The techniques used to develop and apply the theory are straightforward to handle and do not introduce any new kind of individual (remember zilch is, literally, nothing). Where the conventional set theorist has the empty set, the singleton set, and the multimembered set, we can generally replace the first with zilch and the second with the sole member of the singleton. That was how we treated the histories of V_1 and V_2 in Section 5, and how we avoided empty and singleton sets as intersections. The [ ] notation allows us to deal with all three cases in one go, and provides a convenient surrogate for conventional set abstraction where necessary. We emphasize again that when we replace the empty set by zilch, we are not identifying the two, which would be to conflate something with nothing. Although there is sometimes almost a straight swap, as when $x \cap y = \varnothing$ is replaced by $x \cap y \equiv O$, their behaviour diverges starkly elsewhere. For example, it is standard to take the empty set to be a subset of every set, whereas zilch is nothing and not a subset of anything. Similarly, when we replace singletons with their sole members in certain contexts, we are not misidentifying the two. After all, $\{a\}$ is a subset of the pair-set $\{a, b\}$ in conventional set theory, but a is never a subset of $\{a, b\}$ in ours.

Our presentation is designed to facilitate comparison with rival systems. In particular, one may compare Cantorian set theory with a theory in the style of Scott/Derrick/Potter, admitting empty and singleton sets. It turns out that there are pros and cons to either theory with respect to simplicity. As to vocabulary, where we have a single, natural primitive $\in$, they need M or U as a second primitive (see Section 5). Their definitional set-up, on the other hand, can be simpler than ours. We distinguished two accumulation functions, explicitly defined the first two levels, and then treated them separately in the definitions of history and level. By contrast, Scott/Derrick/Potter can define Va more simply as $\exists x(Hx \wedge a = \text{acc}\, x)$ with Ha defined as $\forall y(y \in a \rightarrow (y = \text{acc}\, a \cap y))$, where $\cap$ is given the standard definition which admits empty and singleton intersections. As to axioms, their theory replaces the conditional 2(i) by its unconditional consequent, since $\{x{:}Ux\}$ remains a set even when there is but one ur-element or none. And similarly the conjunct $mxA(x)$ is no longer needed on the right hand side of the scheme 2(iii). The amended version delivers unrestricted separation, which makes Axiom 1(iii) redundant. Axiom 1(iv) is not needed, since they admit singletons. Axiom 2(ii) is also dropped, since the existence of sets is no longer dependent on there being many ur-elements or even any. On the other hand, the Scott/Derrick/Potter theory needs extra axioms to prevent ur-elements from having members, and to ensure that its second primitive M or U is strong. Our analogues of the traditional Zermelo-style axioms are sometimes more complex than theirs. Witness the versions of separation, pairing, power set and replacement discussed in Section 8. On the other hand, we avoid the complications caused by the empty set in the foundation and choice principles and the definition of generalized intersection.

Abraham Fraenkel argues that ‘if we do not want to state an exception—and the mathematician, in contrast to the grammarian, abhors exceptions’, we shall want the axiom of separation to cover the case where the separating property is not true of any member of the original set. The axiom thereby yields the empty set as a subset of the original ([9], p. 23). He has evidently forgotten that in Cantor’s own formulation of separation—‘every sub-multitude of a set is a set’—there is no hint of ‘stating an exception’. Subsequently Fraenkel gives an example of the utility of the empty set itself in avoiding exceptions:

> had we not introduced the null-set, we would not be able to maintain that the meet of any sequence (or set) of sets is again a set. ([9], p. 27)

As one might expect, however, plugging the exceptions by invoking an exceptional object only creates further exceptions. Take $\varnothing$ as an instance of Fraenkel’s exceptionless ‘any set of sets’. Trivially, everything is a member of every member of an empty set, so the meet of $\varnothing$ will be the universal set. But there is no such thing: his example backfires spectacularly. We think the sanest comment on the whole question of exceptions is Halmos’s:

> There is no profound problem here; it is merely a nuisance to be forced always to be making qualifications and exceptions just because some set somewhere along some construction might turn out to be empty. There is nothing to be done about this; it is just a fact of life. ([13], pp. 18–19)

In any discussion of inconvenience, the cost must be weighed against the gain in coherence. Countless authors introduce their subject by repeating Cantor’s definition of sets as collections. *Pace* von Neumann, it is simply not true that they understand a set to be ‘nothing but an object of which one knows no more and wants to know no more than what follows about it from the postulates’ ([27], p. 395). Their theories need to cohere with the informal conception that motivates them. Admitting empty and singleton sets is at odds with their Cantorian conception, and arguments from convenience do nothing to remove this tension.

Appendix

For ease of reference we summarise the definitions given so far and also repeat the set-theoretic axioms. Like the definitions and axioms, the following results and proofs are to be understood as including the familiar provisos to prevent unintended capture of variables.

Our proofs are often longer than those in Potter’s [21]. For the most part, the difference is only superficial, since it results from our relatively slower and more cautious approach to deduction. There is some relative inconvenience in developing the theory of levels, since we have to deal with the first two levels separately from the rest, particularly in the early stages. But now that we have done the work the reader is spared the trouble.

This inconvenience is not intrinsic to the project of excluding the anomalous sets. It is peculiar to the singular version of Cantorian set theory. The version based on plural logic which we presented in our [20] does not suffer this trouble.

Definitions

Existential quantifier	$\exists x A =_{df} \neg \forall x \neg A$
'Exactly one' quantifier	$\exists_1 x A(x) =_{df} \exists x \forall y (A(y) \leftrightarrow x = y)$
'Many' quantifier	$mxA(x) =_{df} \exists x \exists y (x \neq y \wedge A(x) \wedge A(y))$
Existence	$E!a =_{df} \exists x\ x = a$
Weak identity	$a \equiv b =_{df} a = b \vee (\neg E!a \wedge \neg E!b)$
Zilch	$O =_{df} \iota x (x \neq x)$
Inclusive quantifiers	$\forall^O x A(x) =_{df} \forall x A(x) \wedge A(O)$
	$\exists^O x A(x) =_{df} \exists x A(x) \vee A(O)$
Set	$Ma =_{df} mx\, x \in a$
Ur-element	$Ua =_{df} E!a \wedge \neg Ma$
Subset	$a \subseteq b =_{df} Ma \wedge \forall x (x \in a \rightarrow x \in b)$
Proper subset	$a \subset b =_{df} a \subseteq b \wedge a \neq b$
Set abstraction	$\{x{:}A(x)\} =_{df} \iota z (Mz \wedge \forall x (x \in z \leftrightarrow A(x))$
	$[x{:}A(x)] =_{df} \iota z (z = \iota x A(x) \vee z = \{x{:}A(x)\})$
Intersection	$a \cap b =_{df} [x{:}\ x \in a \wedge x \in b]$
Accumulation	$\text{accum}\ a =_{df} \{z{:}(Uz \vee z \in a \vee z \subseteq a)\}$
	$\text{acc}\ a =_{df} \{z{:}(Uz \vee \exists y (y \in a \wedge (z \in y \vee z \subseteq y)))\}$
	$\mathcal{A}a =_{df} \exists^O x\ (a = \text{accum}\ x \vee a = \text{acc}\ x)$
First level	$V_1 =_{df} \text{accum}\ O$
Second level	$V_2 =_{df} \text{accum}\ V_1$
History	$Ha =_{df} (a \equiv O \vee Ma) \wedge (a = V_1 \vee \forall y (y \in a \rightarrow$
	$(y = V_1 \vee y = V_2 \rightarrow y = \text{accum}\ a \cap y)$
	$\wedge (y \neq V_1 \wedge y \neq V_2 \rightarrow y = \text{acc}\ a \cap y)))$
Level	$Va =_{df} a = V_1 \vee a = V_2 \vee \exists x (Hx \wedge a = \text{acc}\ x)$
Level next above	$a' =_{df} \iota x (Va \wedge Vx \wedge a \in x \wedge \neg \exists y (Vy \wedge y \in x \wedge a \in y))$
Limit level	$La =_{df} Va \wedge a \neq V_1 \wedge \neg \exists x\ a = x'$

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Axioms

1. *Membership*
(i) $x \in y \to E!x \wedge E!y$
(ii) $My \wedge Mz \to \forall x(x \in y \leftrightarrow x \in z) \to y = z$
(iii) $x \notin x$
(iv) $\exists x\, x \in y \to My$

2. *Levels*
(i) $mxUx \to M\{x{:}Ux\}$
(ii) $\exists xMx \to mxUx$
(iii) $M\{x{:}A(x)\} \leftrightarrow mxA(x) \wedge \exists u\forall x(A(x) \to x \in u)$
(iv) $\forall u\exists v\, u \in v$
(v) $\exists xVx \to \exists xLx$

Lemmas

Lemma 1. *Extensionality for abstracts*
(i) $\forall x(A(x) \leftrightarrow B(x)) \to \{x{:}A(x)\} \equiv \{x{:}B(x)\}$
(ii) $\forall x(A(x) \leftrightarrow B(x)) \to [x{:}A(x)] \equiv [x{:}B(x)]$

Proof of (i). Suppose $\forall x(A(x) \leftrightarrow B(x))$. Then $\imath z(Mz \wedge \forall x(x \in z \leftrightarrow A(x)) \equiv \imath z(Mz \wedge \forall x(x \in z \leftrightarrow B(x))$. So by the definition of the abstracts, $\{x{:}A(x)\} \equiv \{x{:}B(x)\}$.

Proof of (ii). Suppose $\forall x(A(x) \leftrightarrow B(x))$. Then $\imath y(y = \imath xA(x) \vee y = \imath z(Mz \wedge \forall x(x \in z \leftrightarrow A(x))) \equiv \imath y(y = \imath xB(x) \vee y = \imath z(Mz \wedge \forall x(x \in z \leftrightarrow B(x)))$. So by the definition of the abstracts, $[x{:}A(x)] \equiv [x{:}B(x)]$.

Lemma 2. [ ] *abstracts*
(i) $\neg\exists xA(x) \to [x{:}A(x)] \equiv O$
(ii) $\exists_1 xA(x) \to [x{:}A(x)] = \imath xA(x)$
(iii) $mxA(x) \to [x{:}A(x)] \equiv \{x{:}A(x)\}$

This lemma spells out the denotation of an abstract $[x{:}A(x)]$ according to the number of things—none, one or more—satisfying the formula $A(x)$.

Proof of (i). Suppose $\neg\exists xA(x)$. Then $\neg E!\imath xA(x)$. Also from $\neg\exists xA(x)$ it follows that $\neg mxA(x)$, whence $\neg E!\{x{:}A(x)\}$ by the definitions of the abstract and M. Hence $\neg\exists z(z = \imath xA(x) \vee z = \{x{:}A(x)\})$, whence $\neg E!\imath z(z = \imath xA(x) \vee z = \{x{:}A(x)\})$. Hence $[x{:}A(x)] \equiv O$ by the definition of the abstract.

Proof of (ii). Suppose $\exists_1 xA(x)$. Then $\exists_1 y\; y = \imath xA(x)$. Now, for a reductio suppose $E!\{x{:}A(x)\}$. Then by the definitions of the abstract and M, $mxA(x)$. Contradiction. Hence $\neg E!\{x{:}A(x)\}$. Hence by the strength of identity $\exists_1 z(z = \imath xA(x) \vee z = \{x{:}A(x)\})$, whence $E!\imath z(z = \imath xA(x) \vee z = \{x{:}A(x)\})$. Also by the strength of identity $(z = \imath xA(x) \vee z = \{x{:}A(x)\}) \leftrightarrow z = \imath xA(x)$. Hence $\imath z(z = \imath xA(x) \vee z = \{x{:}A(x)\}) =$

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$\iota z(z = \iota x A(x)) = \iota x A(x)$, whence $[x{:}A(x)] = \iota x A(x)$ by the definition of the abstract.

Proof of (iii). Suppose $mxA(x)$. Then $\neg \exists_1 x A(x)$, whence $\neg E!\iota x A(x)$. Hence by the strength of identity $(y = \iota x A(x) \vee y = \{x{:}A(x)\}) \leftrightarrow y = \{x{:}A(x)\}$, whence $\iota y(y = \iota x A(x) \vee y = \{x{:}A(x)\}) \equiv \iota y(y = \{x{:}A(x)\}) \equiv \{x{:}A(x)\}$. Hence $[x{:}A(x)] \equiv \{x{:}A(x)\}$ by the definition of the abstract.

Lemma 3. *Existence* $\quad E!a \leftrightarrow Ua \vee Ma$

The $\rightarrow$ half means that the individuals divide exhaustively into ur-elements and sets. The $\leftarrow$ half expresses the strength of the predicates U and M.

Proof.

1. For the $\rightarrow$ half, suppose $E!a$. Also suppose $\neg Ma$. Then by definition Ua, a fortiori $Ua \vee Ma$.
2. For the $\leftarrow$ half, suppose $Ua \vee Ma$. By definition, if Ua then $E!a$, and if Ma then $mx\, x \in a$, whence by Axiom 1(i) $E!a$ again.

Lemma 4. *Abstraction and Reduction*

(i) $a = \{x{:}A(x)\} \leftrightarrow Ma \wedge \forall y(y \in a \leftrightarrow A(y))$

(ii) $mxA(x) \wedge a = [x{:}A(x)] \leftrightarrow Ma \wedge \forall y(y \in a \leftrightarrow A(y))$

Proof of (i).

1. For the $\rightarrow$ half, suppose $a = \{x{:}A(x)\}$. Then by the definition of the abstract, $a = \iota z(Mz \wedge \forall y(y \in z \leftrightarrow A(y))$, whence $Ma \wedge \forall y(y \in a \leftrightarrow A(y))$.
2. For the $\leftarrow$ half, suppose $Ma \wedge \forall y(y \in a \leftrightarrow A(y))$. Then by Lemma 3 $E!a$. For a reductio suppose $Mx \wedge \forall y(y \in x \leftrightarrow A(y))$ for some $x \neq a$. Then $\forall y(y \in x \leftrightarrow y \in a)$, whence $x = a$ by Axiom 1(ii). Contradiction. Hence $a = \iota z(Mz \wedge \forall y(y \in z \leftrightarrow A(y)))$. So $a = \{x{:}A(x)\}$ by the definition of the abstract.

Proof of (ii).

1. For the $\rightarrow$ half, suppose $mxA(x) \wedge a = [x{:}A(x)]$. Then by Lemma 2(iii) $a = \{x{:}A(x)\}$, whence by Lemma 4(i), $Ma \wedge \forall y(y \in a \leftrightarrow A(y))$.
2. For the $\leftarrow$ half, suppose $Ma \wedge \forall y(y \in a \leftrightarrow A(y))$. Then by Lemma 4(i) $a = \{x{:}A(x)\}$, whence $E!\{x{:}A(x)\}$ by the strength of identity. Hence $mxA(x)$ by the definitions of the abstract and M, whence $a = [x{:}A(x)]$ by Lemma 2(iii).

Lemma 5. *Membership* (i) $a \in b \rightarrow Mb$

(ii) $a = \{x{:}x \in a\} \leftrightarrow Ma$

Part (ii) allows for movement between different expressions for a set, with $\{x{:}x \in a\}$ sometimes being the most convenient form.

Proof of (i). Suppose $a \in b$. Then by Axiom 1(i) $E!a$. Hence $\exists x\, x \in b$, whence Mb by Axiom 1(iv).

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PROOF OF (ii).

1. For the $\rightarrow$ half, suppose $a = \{x{:}x \in a\}$. Then Ma by Lemma 4(i).

2. For the $\leftarrow$ half, suppose Ma. Since $\forall x(x \in a \leftrightarrow x \in a)$, it follows that $a = \{x{:}x \in a\}$ by Lemma 4(i).

LEMMA 6. *Subset*

(i) $a \subseteq b \rightarrow Mb$
(ii) $Ma \leftrightarrow a \subseteq a$
(iii) $a \subset b \rightarrow \exists x(x \notin a \wedge x \in b)$

PROOF OF (i). Suppose $a \subseteq b$. Then Ma by the definition of $\subseteq$, whence $mx\ x \in a$ by the definition of M. Also $\forall x(x \in a \rightarrow x \in b)$ by the definition of $\subseteq$. Hence $mx\ x \in b$, whence Mb by the definition of M.

PROOF OF (ii).

1. For the $\rightarrow$ half suppose Ma. Since $\forall x(x \in a \rightarrow x \in a)$, it follows by the definition of $\subseteq$ that $a \subseteq a$.

2. For the $\leftarrow$ half suppose $a \subseteq a$. Then Ma by the definition of $\subseteq$.

PROOF OF (iii). Suppose $a \subset b$. Then $a \subseteq b$ and $a \neq b$ by the definition of $\subset$, whence Ma and $\forall x(x \in a \rightarrow x \in b)$ by the definition of $\subseteq$. By Lemma 6(i), Mb. For a reductio suppose that $\forall x(x \in b \rightarrow x \in a)$. Then by Axiom 1(ii), $a = b$. Contradiction. Hence $\exists x(x \notin a \wedge x \in b)$.

LEMMA 7. *The first two levels*

(i) $V_1 \equiv \{z{:}Uz\} \equiv \text{acc } O \equiv \text{acc } V_1$
(ii) $E!V_1 \rightarrow MV_1 \wedge \forall y(y \in V_1 \leftrightarrow Uy)$
(iii) $Ma \rightarrow V_1 = \{z{:}Uz\}$
(iv) $V_2 \equiv \{z{:}Uz \vee z \subseteq V_1\}$
(v) $E!V_2 \rightarrow MV_2 \wedge \forall y(y \in V_2 \leftrightarrow (Uy \vee y \subseteq V_1))$
(vi) $E!V_2 \rightarrow E!V_1$
(vii) $E!V_2 \rightarrow V_1 \in V_2$
(viii) $V_1 \neq V_2$

PROOF OF (i).

1. By definition $V_1 \equiv \text{accum } O \equiv \{z{:}Uz \vee z \in O \vee z \subseteq O\}$. Since $\neg E!O$, it follows by Lemmas 3, 5(i) and 6(i) that $\neg \exists x(x \in O \vee x \subseteq O)$. Hence $(Uz \vee z \in O \vee z \subseteq O) \leftrightarrow Uz$, whence $V_1 \equiv \{z{:}Uz\}$ by Lemma 1(i).

2. By the definition, $\text{acc } O \equiv \{z{:}(Uz \vee \exists y(y \in O \wedge (z \in y \vee z \subseteq y)))\}$. Since $\neg \exists y\ y \in O$, it follows that $(Uz \vee \exists y(y \in O \wedge (z \in y \vee z \subseteq y))) \leftrightarrow Uz$, whence $\text{acc } O \equiv \{z{:}Uz\}$ by Lemma 1(i).

3. By the definition, $\text{acc } V_1 \equiv \{z{:}(Uz \vee \exists y(y \in V_1 \wedge (z \in y \vee z \subseteq y)))\}$. If $y \in V_1$ then $E!V_1$ by Axiom 1(i), whence Uy by Lemma 4(i). Hence $\neg My$ by the definition of U, whence $\neg \exists z(z \in y \vee z \subseteq y)$ by Lemmas 5(i) and 6(i). Hence $(Uz \vee \exists y(y \in V_1 \wedge (z \in y \vee z \subseteq y)))\} \leftrightarrow Uz$, It follows that $\text{acc } V_1 \equiv \{z{:}Uz\}$ by Lemma 1(i).

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Proof of (ii). Suppose $E!V_1$. Then $V_1 = \{z{:}Uz\}$ by Lemma 7(i). Hence $MV_1 \wedge \forall y(y \in V_1 \leftrightarrow Uy)$ by Lemma 4(i).

Proof of (iii). Suppose Ma. Then $E!a$ by Lemma 3, whence $\exists xMx$. Hence $M\{z{:}Uz\}$ by Axioms 2(i) and 2(ii). Hence $E!\{z{:}Uz\}$ by Lemma 3, whence $V_1 = \{z{:}Uz\}$ by Lemma 7(i).

Proof of (iv). By definition $V_2 \equiv \text{accum } V_1 \equiv \{z{:}(Uz \vee z \in V_1 \vee z \subseteq V_1)\}$. If $\neg\, E!V_1$ then $\neg\exists y\; y \in V_1$ by Axiom 1(i). If $E!V_1$ then $z \in V_1 \leftrightarrow Uz$ by Lemma 7(ii). So either way $(Uz \vee z \in V_1 \vee\; z \subseteq V_1) \leftrightarrow (Uz \vee z \subseteq V_1)$. Hence $V_2 \equiv \{z{:}\; Uz \vee z \subseteq V_1\}$ by Lemma 1(i).

Proof of (v). Suppose $E!V_2$. Then $V_2 = \{z{:}(Uz \vee z \subseteq V_1)\}$ by Lemma 7(iv). Hence $MV_2 \wedge \forall y(y \in V_2 \leftrightarrow (Uy \vee y \subseteq V_1))$ by Lemma 4(i).

Proof of (vi). Suppose $E!V_2$. Then MV_2 by Lemma 7(v), whence $V_1 = \{z{:}Uz\}$ by Lemma 7(iii), whence $E!V_1$ by the strength of identity.

Proof of (vii). Suppose $E!V_2$. Then $E!V_1$ by Lemma 7(vi), whence MV_1 by Lemma 7(ii). Hence $V_1 \subseteq V_1$ by Lemma 6(ii), whence $V_1 \in V_2$ by Lemma 7(v).

Proof of (viii). For a reductio suppose $V_1 = V_2$. Then $E!V_1$ and $E!V_2$ by the strength of identity. Hence $V_1 \in V_2$ by Lemma 7(vii), whence $V_1 \in V_1$. By Lemma 7(ii) UV_1, whence $\neg\, MV_1$ by the definition of U. But MV_1 also by Lemma 7(ii). Contradiction. Hence $V_1 \neq V_2$.

Theorems

Theorem 1. *Accumulations* (i) $\mathcal{A}x \rightarrow Mx$
(ii) $\mathcal{A}x \rightarrow \forall z(Uz \rightarrow z \in x)$
(iii) $Vx \rightarrow \mathcal{A}x$

Proof of (i). Suppose $\mathcal{A}x$. If $x = \text{accum } y$, then $x = \{z{:}(Uz \vee z \in y \vee z \subseteq y)\}$ by the definition of accum, whence Mx by Lemma 4(i). Similarly for the other case $x = \text{acc } y$.

Proof of (ii). Suppose $\mathcal{A}x$. If $x = \text{accum } y$, then $x = \{z{:}(Uz \vee z \in y \vee z \subseteq y)\}$ by the definition of accum, whence $\forall z(Uz \rightarrow z \in x)$ by Lemma 4(i). Similarly for the other case $x = \text{acc } y$.

Proof of (iii). Suppose Vx. Then $x = V_1 \vee x = V_2 \vee \exists y\; x = \text{acc } y$ by the definition of V, whence $\mathcal{A}x$ by the definitions of $\mathcal{A}$, V_1 and V_2.

Corollaries. *Let Vx. Then* (i) Mx *and* (ii) $\forall z(Uz \rightarrow z \in x)$.

Proof of (i). Immediate by Theorems 1(i) and (iii).

Proof of (ii). Immediate by Theorem 1(ii) and (iii).

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Recall that we reserve u, v, w for levels, and h, h_1 for histories.

THEOREM 2. *Histories I* *Let* $v = acc\ h$, *then* (i) $x \in h \to x \in v$
(ii) $v \neq V_1 \to Mh \wedge h \neq V_1$.

PROOF OF (i). Suppose $x \in h$. By Axiom 1(i) it follows that $E!x$, whence $Ux \vee Mx$ by Lemma 3. Suppose Ux. Then a fortiori $Ux \vee \exists y(y \in h \wedge (x \in y \vee x \subseteq y))$. Suppose instead that Mx. Then by Lemma 6(ii), $x \subseteq x$. Since $x \in h$, it follows that $Ux \vee \exists y(y \in h \wedge (x \in y \vee x \subseteq y))$. So either way $Ux \vee \exists y(y \in h \wedge (x \in y \vee x \subseteq y))$. By definition, $v = \text{acc}\ h = \{x{:}(Ux \vee \exists y(y \in h \wedge (x \in y \vee x \subseteq y)))\}$, so $x \in v$ by Lemma 4(i).

PROOF OF (ii). Suppose $v \neq V_1$. For a reductio suppose $h \equiv O$. Then $v = \text{acc}\ h = \text{acc}\ O = V_1$ by Lemma 7(i). Contradiction. Hence Mh by the definition of H. For a reductio suppose $h = V_1$. Then $v = \text{acc}\ h = \text{acc}\ V_1 = V_1$ by Lemma 7(i). Contradiction. Hence $h \neq V_1$.

THEOREM 3. *Histories II* *Let* $h \neq V_1$ *and* $x \in h$, *then* Mx.

PROOF. Since by hypothesis $h \neq V_1$ and $x \in h$, it follows that $\mathcal{A}x$ by the definitions of H and $\mathcal{A}$, whence Mx by Theorem 1(i).

THEOREM 4. *Histories III* *Let* $h \neq V_1$, $x \in h$, $x \neq V_1$ *and* $x \neq V_2$. *Then*
(i) $x = acc\ h \cap x$
(ii) $E!\,h \cap x$
(iii) $mz(z \in h \wedge z \in x)$.

Part (iii) of the theorem plays an important role in several subsequent proofs, where we shall need to infer $y \in h \cap x$ from $y \in h \wedge y \in x$, or vice versa, or to infer that $h \cap x$ is the set $\{y{:}\ y \in h \wedge y \in x\}$. Given our definition of $\cap$, these inferences may fail, since when there is but one common member y of h and x, the intersection $h \cap x$ is y itself (not y's singleton as per the orthodox definition of $\cap$). This is the one place that doing without singletons presents a serious challenge. In order for the inferences to go through, we need to have established that there are many common members of h and x. Part (iii) does this, subject to the conditions laid down in the hypothesis. In step 3 of its proof, Axiom 1(iii) $x \notin x$ is used for the first time, to argue for the distinctness of a set from any of its members.

PROOF OF (i). Since by hypothesis $h \neq V_1$ and $x \in h$ and $x \neq V_1$ and $x \neq V_2$, it follows by the definition of H that $x = \text{acc}\ h \cap x$.

PROOF OF (ii). For a reductio suppose $h \cap x \equiv O$. Then by Theorem 4(i) $x = \text{acc}\ O$, whence $x = V_1$ by Lemma 7(i). Contradiction. So $E!\,h \cap x$.

PROOF OF (iii).

1. By Theorem 4(ii) $E!\,h \cap x$, and so by the definition of $\cap$ and Lemma 2(i), either $\exists_1 z(z \in h \wedge z \in x)$ or $mz(z \in h \wedge z \in x)$. For a reductio suppose that $z_1 \in h \wedge z_1 \in x$ for some unique z_1. Then by the definition of $\cap$ and Lemma 2(ii), $h \cap x = z_1$. Hence by Theorem 4(i) $x = \text{acc}\ z_1$.
2. For a reductio suppose $h \cap z_1 \equiv O$. Since $h \neq V_1$ and $z_1 \in h$, it follows by the definition of H that $z_1 = \text{accum}\ h \cap z_1$ or $z_1 = \text{acc}\ h \cap z_1$. Suppose $z_1 = \text{accum}\ h \cap z_1$. Then $z_1 = \text{accum}\ O = V_1$ by the definition

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of V_1. Suppose $z_1 = \text{acc}\ h \cap z_1$. Then $z_1 = \text{acc}\ O = V_1$ by Lemma 7(i). So either way $z_1 = V_1$. Hence $x = \text{acc}\ z_1 = \text{acc}\ V_1$, whence by Lemma 7(i), $x = V_1$. Contradiction. Hence $E!\,h \cap z_1$.

3. Returning to the reductio initiated in step 1, since $E!\,h \cap z_1$ it follows by the definition of $\cap$ and Lemma 2(i) that $z_2 \in h \wedge z_2 \in z_1$ for some z_2. By Axiom 1(iii), $z_2 \neq z_1$, and by the definition of acc, $x = \text{acc}\ z_1 = \{z{:}(Uz \vee \exists y(y \in z_1 \wedge (z \in y \vee z \subseteq y)))\}$. Since $h \neq V_1$ and $z_2 \in h$, it follows by Theorem 3 that Mz_2. Hence by Lemma 6(ii), $z_2 \subseteq z_2$. Since $z_2 \in z_1$ and $z_2 \subseteq z_2$, it follows that $z_2 \in x$ by Lemma 4(i). Hence $z_1 \in h$ and $z_1 \in x$ and $z_2 \in h$ and $z_2 \in x$ and $z_2 \neq z_1$. Contradiction. Hence $mz(z \in h \wedge z \in x)$.

Theorem 5. *Histories IV Let $h \neq V_1$ and $x \in h$, then $H(h \cap x)$ and Vx.*

Proof. By hypothesis $x \in h$, whence $E!x$ by Axiom 1(i), and also $V_1 = \{z{:}Uz\}$ by Lemmas 5(i) and 7(iii). We consider three cases separately: (i) $x = V_1$, (ii) $x = V_2$, (iii) $x \neq V_1$ and $x \neq V_2$.

Case (i) $x = V_1$

Since $x = V_1$ it follows that Vx by the definition of V, and also that $x = \{z{:}Uz\}$. Hence $y \in x \rightarrow Uy$ by Lemma 4(i). Also $y \in h \rightarrow \neg Uy$ by Theorem 3 and the definition of U, whence $\neg\exists z(z \in h \wedge z \in x)$. Hence by Lemma 2(i) and the definition of $\cap$, $h \cap x \equiv O$, whence $H(h \cap x)$ by the definition of H.

Case (ii) $x = V_2$

1. Since $x = V_2$ it follows that Vx by the definition of V. Since $h \neq V_1$ and $x \in h$, then $x = \text{accum}\ h \cap x$ by the definition of H.
2. For a reductio suppose that $h \cap x \equiv O$. Then $x = \text{accum}\ O = V_1$ by the definition of V_1. But by Lemma 7(viii), $V_1 \neq V_2$. Contradiction. Hence $E!\,h \cap x$.
3. Since $h \cap x =_{df} [y{:}\ y \in h \wedge y \in x]$, it follows that $E![y{:}\ y \in h \wedge y \in x]$, whence by Lemma 2(i), $y \in h \wedge y \in x$ for some y. We shall prove that there is exactly one such y, namely V_1.
4. Since $x = V_2$, it follows by Lemma 7(iv) that $x = \{z{:}\ Uz \vee z \subseteq V_1\} = \{z{:}\ Uz \vee z \subseteq \{z_1{:}Uz_1\}\}$. Since $y \in x$, it follows by Lemma 4(i) that $Uy \vee y \subseteq \{z_1{:}Uz_1\}$. Since $y \in h$, it follows by Theorem 3 and the definition of U that $\neg\ Uy$. Hence $y \subseteq \{z_1{:}Uz_1\}$.
5. For a reductio suppose that $y \subset \{z_1{:}Uz_1\}$. Then by Lemma 6(iii), $y_1 \in \{z_1{:}Uz_1\}$ and $y_1 \notin y$ for some y_1, whence by Lemma 4(i), Uy_1 and $y_1 \notin y$. By Lemmas 7(ii) and (v), $\forall x(Ux \rightarrow x \in V_1)$ and $\forall x(Ux \rightarrow x \in V_2)$, whence $y \neq V_1$ and $y \neq V_2$. Since $h \neq V_1$ and $y \in h$ and $y \neq V_1$ and $y \neq V_2$, it follows that $y = \text{acc}\ h \cap y = \{x{:}(Ux \vee \exists y(y \in h \cap y \wedge (x \in y \vee x \subseteq y)))\}$ by the definitions of H and acc. Since Uy_1, it follows by Lemma 4(i) that $y_1 \in y$. Contradiction. Hence $y \not\subset \{z_1{:}Uz_1\}$. Since $y \subseteq \{z_1{:}Uz_1\}$, it follows by the definition of $\subset$ that $y = \{z_1{:}Uz_1\} = V_1$, whence $V_1 = \imath y(y \in h \wedge y \in x)$. Hence $\exists_1 y(y \in h \wedge y \in x)$, whence by Lemma 2(ii) and the definition of $\cap$, $h \cap x = V_1$. Hence $H(h \cap x)$ by the definition of H.

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CASE (iii) $x \neq V_1$ and $x \neq V_2$

1. By Theorems 4(i), (ii) and (iii), $x = \text{acc}\, h \cap x$, $E!\, h \cap x$, and $mz(z \in h \wedge z \in x)$. By the definition of $\cap$ and by Lemma 2(iii), $h \cap x = \{z\colon z \in h \wedge z \in x\}$. Hence $M(h \cap x)$ by Lemma 4(i), so $mz\; z \in h \cap x$ by the definition of M.
2. Consider an arbitrary z_1 such that $z_1 \in h \cap x$. Then $z \in z_1 \rightarrow \exists y(y \in h \cap x \wedge z \in y)$, a fortiori $z \in z_1 \rightarrow (Uz \vee \exists y(y \in h \cap x \wedge (z \in y \vee z \subseteq y)))$. By the definition of acc, $x = \text{acc}\, h \cap x = \{z\colon (Uz \vee \exists y(y \in h \cap x \wedge (z \in y \vee z \subseteq y)))\}$. Hence by Lemma 4(i), $z \in z_1 \rightarrow z \in x$. By the definition of $\cap$, $(h \cap x) \cap z_1 \equiv [y\colon y \in [y_1\colon y_1 \in h \wedge y_1 \in x] \wedge y \in z_1]$, whence $(h \cap x) \cap z_1 \equiv [y\colon y \in h \wedge y \in x \wedge y \in z_1]$ by Lemmas 1(ii) and 4(ii) and (from step 1) $mz(z \in h \wedge z \in x)$. Since $z \in z_1 \rightarrow z \in x$, it follows by Lemma 1(ii) that $(h \cap x) \cap z_1 \equiv [y\colon y \in h \wedge y \in z_1]$, whence $(h \cap x) \cap z_1 \equiv h \cap z_1$ by the definition of $\cap$.
3. Since $z_1 \in h \cap x$, it follows by the definition of $\cap$ that $z_1 \in [z\colon z \in h \wedge z \in x]$, whence $z_1 \in h$ by Lemma 4(ii) and (from step 1) $mz(z \in h \wedge z \in x)$. Suppose $z_1 = V_1 \vee z_1 = V_2$. Since $h \neq V_1$ and $z_1 \in h$, then $z_1 = \text{accum}\; h \cap z_1$ by the definition of H, whence $z_1 = \text{accum}\; (h \cap x) \cap z_1$. Suppose instead that $z_1 \neq V_1 \wedge z_1 \neq V_2$. Then by the definition of H it follows that $z_1 = \text{acc}\; h \cap z_1$, whence $z_1 = \text{acc}\; (h \cap x) \cap z_1$. Since z_1 was arbitrary, we can generalize to get $\forall y(y \in h \cap x \rightarrow (y = V_1 \vee y = V_2 \rightarrow y = \text{accum}\; (h \cap x) \cap y) \wedge (y \neq V_1 \wedge y \neq V_2 \rightarrow y = \text{acc}\; (h \cap x) \cap y))$. Hence $H(h \cap x)$ by the definition of H.
4. Since $E!\, h \cap x$, $H(h \cap x)$, and $x = \text{acc}\, h \cap x$, it follows that Vx by the definition of V.

THEOREM 6. *$\in$ is well-founded on any history Let $x \subseteq h$, then $\exists y(y \in x \wedge x \cap y \equiv O)$.*

In English: any subset of a history has a member whose intersection with that subset is zilch.

In the proof we need to deal with two cases separately. When the history is V_1 by itself, we argue that every member of h, and so every member of any subset x of h, is an ur-element and so does not have any members. It trivially follows that x has some member with which it shares no members. When $h \neq V_1$, on the other hand, we assume for the sake of a reductio that h has a subset x that misbehaves, i.e., every member of x shares a member with x. Then we show that there is a set b consisting precisely of the common members of the members of x. A contradiction follows.

It is instructive to compare our proof with Potter's proof of the corresponding theorem (3.6.4) in his [21]. He avoids the need to proceed by cases, since for him a history is always a set of levels. So instead of $h = V_1$, he has $h = \{V_1\}$, which can be dealt with alongside the other possibilities for h. In broad outline, his reductio resembles ours, but the details are quite different. As we shall explain, the principal differences can be traced to Potter's adoption of unrestricted separation from levels, which allows for empty and singleton sets, unlike the restricted version embodied in our Axiom 2(iii).

Potter first uses unrestricted separation from levels to show that the set b exists. He argues that there is a level to which the common members of the members of x all belong, since any member of h and therefore any member of its subset x is a level.

But we have only the restricted version of separation from levels to work with and so we need in addition to show there are many common members of the members of x. Here we employ Axiom 2(ii) for the first time to infer the existence of many ur-elements from x's being a set. Since every member of x is a level, and every ur-element is a member of every level, it follows that each of the many ur-elements is a member of every member of x.

The second difference between our proof and Potter's turns on the nature of the ensuing contradiction. Potter shows that the set b has each of its subsets as a member, contrary to his proposition (3.6.1) which implies that every set has a subset not among its members. But his proof of (3.6.1) makes essential use of unrestricted separation from levels, which is not available to us. Hence we follow a different route. We show more specifically that $b \in b$, contrary to Axiom 1(iii). In fact, the same axiom is needed in the reductio to show that $b \in b$. For at a crucial point (see step 12 of the proof of the second case below), we need to infer $w \in h \cap v$ from $w \in h \wedge w \in v$, which relies on Theorem 4(iii) and therefore on Axiom 1(iii).

Our proof, then, depends on the two Axioms 1(iii) and 2(ii), which do not appear in Potter's list. We shall see more of these axioms in subsequent proofs. Potter would, of course, count Axiom 2(ii) as false, since he thinks that even when there are no ur-elements, there are still the so-called pure sets. On the other hand, he seems to think Axiom 1(iii) is true. Certainly, he derives the proposition that no *set* is a member of itself (3.7.3) even before he has reached foundation for sets. As to ur-elements, although he does not include an axiom forcing them to be memberless, he notes that 'this might be added for the sake of tidiness' ([21], p. 30).

Proof. The hypothesis $x \subseteq h$ entails Mx by the definition of $\subseteq$, whence $mzUz$ by Lemma 3 and Axiom 2(ii). Also from Mx it follows that $V_1 = \{z{:}Uz\}$ by Lemma 7(iii), whence $E!V_1$ by the strength of identity. We consider two cases separately: (i) $h = V_1$ and (ii) $h \neq V_1$.

Case (i) $h = V_1$

Since by hypothesis $x \subseteq h$, it follows that $y \in x \to y \in h$ by the definition of $\subseteq$. Hence $y \in x \to y \in V_1$, whence $y \in x \to Uy$ by Lemma 4(i). Hence $y \in x \to \neg\exists z\ z \in y$ by the definition of U and Axiom 1(iv). Since Mx, it follows by the definition of M that $my\ y \in x$, whence $\exists y\ y \in x$. Hence $\exists y(y \in x \wedge \neg\exists z(z \in x \wedge z \in y))$, whence $\exists y(y \in x \wedge x \cap y \equiv O)$ by Lemma 2(i) and the definition of $\cap$.

Case (ii) $h \neq V_1$

1. Since by hypothesis $x \subseteq h$, it follows that $y \in x \to y \in h$, by the definition of $\subseteq$. Since $h \neq V_1$, it follows by Theorem 5 that $y \in x \to Vy$. For a reductio suppose that $\neg\exists y(y \in x \wedge \neg\exists z(z \in x$

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$\wedge\ z \in y)$, whence $\forall y(y \in x \rightarrow \exists z(z \in x \wedge z \in y))$. Since Mx, it follows by the definition of M that $my\ y \in x$, whence $\exists y\ y \in x$. Consider an arbitrary level v such that $v \in x$. Then $w \in x \wedge w \in v$ for some level w. By the same reasoning applied to w, it follows from $w \in x$ that $w_1 \in x \wedge w_1 \in w$ for some level w_1. And similarly $w_2 \in x \wedge w_2 \in w_1$ for some level w_2. Also since $y \in x \rightarrow y \in h$, it follows that $v \in h$ and $w \in h$.

2. Let b be short for $\{z{:}\forall y(y \in x \rightarrow z \in y)\}$. We prove that (i) Mb, (ii) $b \subseteq w$, (iii) $v = \text{acc}\ h \cap v$, (iv) $E!\,h \cap v$, (v) $mz(z \in h \wedge z \in v)$, and (vi) $\forall y(y \in x \rightarrow b \in y)$.
3. For (i), with a view to using Axiom 2(iii) we shall prove that (ia) $mz\ \forall y(y \in x \rightarrow z \in y)$ and (ib) $\exists u \forall z(\forall y(y \in x \rightarrow z \in y) \rightarrow z \in u)$.
4. For (ia), since $y \in x \rightarrow Vy$ (from step 1), and $Vy \rightarrow \forall z(Uz \rightarrow z \in y)$ by Corollary (ii) of Theorem 1, it follows that $\forall z(Uz\rightarrow \forall y(y \in x \rightarrow z \in y))$. Since $mz Uz$, it follows that $mz\ \forall y(y \in x \rightarrow z \in y)$.
5. For (ib), since $v \in x$, it follows that $\forall z(\forall y(y \in x \rightarrow z \in y) \rightarrow z \in v)$, whence $\exists u \forall z(\forall y(y \in x \rightarrow z \in y) \rightarrow z \in u)$.
6. From (ia) and (ib), it follows that Mb by Axiom 2(iii).
7. For (ii), since $w \in x$, it follows that $\forall z(\forall y(y \in x \rightarrow z \in y) \rightarrow z \in w)$. Since Mb, it follows by Lemma 3 that $E!b$, whence $b = \{z{:}\forall y(y \in x \rightarrow z \in y)\}$. Hence by Lemma 4(i), $z \in b \leftrightarrow \forall y(y \in x \rightarrow z \in y)$, whence $\forall z(z \in b \rightarrow z \in w)$. Hence $b \subseteq w$ by the definition of $\subseteq$.
8. For (iii), (iv) and (v), we shall first prove (iiia) $v \neq V_1$ and (iiib) $v \neq V_2$.
9. For (iiia), for a reductio suppose that $v = V_1 = \{z{:}Uz\}$. Since $w \in v$, it follows by Lemma 4(i) that Uw. But by Corollary (i) of Theorem 1, Mw, whence by the definition of U, $\neg\ Uw$. Contradiction. Hence $v \neq V_1$.
10. For (iiib), for a reductio suppose $v = V_2$. Then $v = \{z{:}Uz \vee z \subseteq V_1\}$ by Lemma 7(iv). Since $w \in v$, it follows by Lemma 4(i) that $Uw \vee w \subseteq V_1$. Since $\neg Uw$ it follows that $w \subseteq V_1$, whence $w_1 \in V_1$ by the definition of $\subseteq$, whence Uw_1 by Lemma 4(i). But by Corollary (i) of Theorem 1, Mw_1, whence $\neg Uw_1$ by the definition of U. Contradiction. Hence $v \neq V_2$.
11. Since $h \neq V_1$ and $v \in h$ and $v \neq V_1$ and $v \neq V_2$, it follows by Theorems 4(i), (ii), and (iii) that $v = \text{acc}\ h \cap v$ and $E!\,h \cap v$ and $mz(z \in h \wedge z \in v)$.
12. For (vi), since $E!\,h \cap v$, it follows that $h \cap v = [z{:}\ z \in h \wedge z \in v]$ by the definition of $\cap$. Since $mz(z \in h \wedge z \in v)$ and $w \in h \wedge w \in v$, it follows by Lemma 4(ii) that $w \in h \cap v$. By the definition of acc, $v = \text{acc}\ h \cap v = \{z{:}(Uz \vee \exists y(y \in h \cap v \wedge (z \in y \vee z \subseteq y)))\}$. Since $w \in h \cap v$ and $b \subseteq w$, it follows by Lemma 4(i) that $b \in \text{acc}\ h \cap v$, whence $b \in v$. Since $b \in v$ for arbitrary $v \in x$, we can generalize to get $\forall y(y \in x \rightarrow b \in y)$.
13. We can now proceed to the reductio started in step 1. Since $\forall y(y \in x \rightarrow z \in y) \rightarrow z \in b$ then in particular $\forall y(y \in x \rightarrow b \in y) \rightarrow b \in b$.

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Since $\forall y(y \in x \to b \in y)$, it follows that $b \in b$, contrary to Axiom 1(iii). Hence $\exists y(y \in x \wedge \neg \exists z(z \in x \wedge z \in y)$, whence $\exists y(y \in x \wedge x \cap y \equiv O)$ by Lemma 2(i) and the definition of $\cap$.

THEOREM 7. *Levels are transitive sets* *Let $x \in y$ and $y \in v$, then $x \in v$.* In particular, membership between levels is transitive. It also follows that if a set is a member of a level, it is a member of all higher levels. The corollary below tells us that it is a subset of all those levels too, the original included.

PROOF. From the hypothesis $x \in y$ it follows that My by Lemma 5(i), whence by the definition of U, $\neg Uy$. Since My it follows that $V_1 = \{z{:}Uz\}$ by Lemma 7(iii). From the hypothesis $y \in v$ it follows that $E!v$ by Axiom 1(i). We tackle three cases separately: (i) $v = V_1$, (ii) $v = V_2$, and (iii) $v \neq V_1$ and $v \neq V_2$.

CASE (i) $v = V_1$
From the hypothesis $y \in v$ it follows by Lemma 4(i) that Uy. But also $\neg Uy$. Hence $x \in v$ by the tautology $A \wedge \neg A \to B$.

CASE (ii) $v = V_2$
By the strength of identity, $E!V_2$. Since $y \in v$, it follows by Lemma 7(v) that $Uy \vee y \subseteq V_1$. Since $\neg Uy$, it follows that $y \subseteq V_1$. Since $x \in y$, it follows that $x \in V_1$ by the definition of $\subseteq$. Hence Ux by Lemma 4(i). Hence $x \in V_2$ by Lemma 7(v), whence $x \in v$.

CASE (iii) $v \neq V_1$ and $v \neq V_2$

1. By the definition of V, $v = \text{acc}\ h$ for some history h. For a reductio suppose that $h = V_1$. Then $v = \text{acc}\ V_1 = V_1$ by Lemma 7(i). Contradiction. Hence $h \neq V_1$, whence by the definition of acc, Theorem 5 and Lemma 1(i), $v = \{z{:}(Uz \vee \exists u(u \in h \wedge (z \in u \vee z \subseteq u)))\}$. Since $y \in v$, it follows by Lemma 4(i) that $Uy \vee \exists u(u \in h \wedge (y \in u \vee y \subseteq u))$. Since $\neg Uy$, it follows that $y \in w \vee y \subseteq w$ for some $w \in h$. We tackle three cases separately: (a) $w = V_1$, (b) $w = V_2$, and (c) $w \neq V_1$ and $w \neq V_2$.
2. For case (a), for a reductio suppose $y \in w$. Then $y \in V_1$, whence by Lemma 4(i), Uy. Contradiction. Hence $y \subseteq w$. Since $x \in y$, it follows by the definition of $\subseteq$ that $x \in w$. Since $v = \{z{:}(Uz \vee \exists u(u \in h \wedge (z \in u \vee z \subseteq u)))\}$, it follows by Lemma 4(i) that $x \in v$.
3. For case (b), suppose $y \in w$. Then $y \in V_2$, whence $Uy \vee y \subseteq V_1$ by the strength of identity and Lemma 7(v). Since $\neg Uy$, it follows that $y \subseteq V_1$. Since $x \in y$, it follows by the definition of $\subseteq$ that $x \in V_1$, whence Ux by Lemma 4(i). Hence by Corollary (ii) of Theorem 1, $x \in v$. Suppose instead that $y \subseteq w$. Since $x \in y$, it follows by the definition of $\subseteq$ that $x \in w$, whence $x \in v$ by Lemma 4(i).
4. For case (c), we deal with two subcases separately: (ci) $\exists_1 u(u \in h \wedge (y \in u \vee y \subseteq u))$, and (cii) $mu(u \in h \wedge (y \in u \vee y \subseteq u))$.
5. For case (ci), $u \in h \wedge (y \in u \vee y \subseteq u)$ for some unique level u, namely w. From $h \neq V_1, w \in h, w \neq V_1, w \neq V_2$, it follows that $w = \text{acc}\ h \cap w$, $E!\,h \cap w$, and $mz(z \in h \wedge z \in w)$ by Theorems 4(i), 4(ii), and 4(iii). Hence by the definition of acc, $w = \{z{:}(Uz \vee \exists z_1(z_1 \in h \cap w$

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$\wedge (z \in z_1 \vee z \subseteq z_1)))\}$. For a reductio suppose $y \in w$. Then by Lemma 4(i), $Uy \vee \exists z_1(z_1 \in h \cap w \wedge (y \in z_1 \vee y \subseteq z_1))$. Since My, it follows that $\exists z_1(z_1 \in h \cap w \wedge (y \in z_1 \vee y \subseteq z_1))$ by the definition of U. Since $E!\, h \cap w$, it follows that $h \cap w = h \cap w$. Since $mz(z \in h \wedge z \in w)$, it follows by the definition of $h \cap w$ and Lemma 4(ii) that for some z_1, $z_1 \in h \wedge z_1 \in w \wedge (y \in z_1 \vee y \subseteq z_1)$. Since $z_1 \in h$ and $h \neq V_1$, it follows by Theorem 5 that Vz_1. Since $z_1 \in w$, it follows by Axiom 1(iii) that $z_1 \neq w$. But $z_1 \in h \wedge (y \in z_1 \vee y \subseteq z_1)$ and Vz_1 and $z_1 \neq w$ are together contrary to $\exists_1 u(u \in h \wedge (y \in u \vee y \subseteq u))$. Hence $y \subseteq w$. Since $x \in y$, it follows by the definition of $\subseteq$ that $x \in w$, and so by Lemma 4(i) $x \in v$.

6. For case (cii), let b be short for $\{u\colon u \in h \wedge (y \in u \vee y \subseteq u)\}$. Since $v = \text{acc}\ h$, it follows by Theorem 2(i) that $z \in h \to z \in v$. Hence $\forall u((u \in h \wedge (y \in u \vee y \subseteq u)) \to u \in v)$, whence $\exists v_1 \forall u(u \in h \wedge (y \in u \vee y \subseteq u)) \to u \in v_1)$. This along with $mu(u \in h \wedge (y \in u \vee y \subseteq u))$ entails Mb by Axiom 2(iii). Hence by Lemma 3 $E!b$, whence $b = \{u\colon u \in h \wedge (y \in u \vee y \subseteq u)\}$. Hence by Lemma 4(i), $z \in b \to z \in h$, whence by the definition of $\subseteq$, $b \subseteq h$. Hence by Theorem 6, $\neg\exists z_2(z_2 \in b \wedge z_2 \in z_1)$ for some $z_1 \in b$. Hence by Lemma 4(i), $z_1 \in h \wedge (y \in z_1 \vee y \subseteq z_1)$. We deal with three subcases separately: (ciiα) $z_1 = V_1$, (ciiβ) $z_1 = V_2$, and (ciiγ) $z_1 \neq V_1 \wedge z_1 \neq V_2$.
7. In case (ciiα), it follows that $x \in v$ by the reasoning in step 2.
8. In case (ciiβ), it follows that $x \in v$ by the reasoning in step 3.
9. In case (ciiγ), since $h \neq V_1$, $z_1 \in h$, $z_1 \neq V_1$ and $z_1 \neq V_2$, it follows that $z_1 = \text{acc}\ h \cap z_1$, $E!\, h \cap z_1$ and $mz(z \in h \wedge z \in z_1)$ by Theorems 4(i), (ii), and (iii). For a reductio suppose $y \in z_1$. Then $y \in \text{acc}\ h \cap z_1$, whence by the definition of acc, $y \in \{z\colon (Uz \vee \exists z_3(z_3 \in h \cap z_1 \wedge (z \in z_3 \vee z \subseteq z_3)))\}$. Since $\neg\, Uy$, it follows by Lemma 4(i) that $y \in z_3 \vee y \subseteq z_3$ for some $z_3 \in h \cap z_1$. Since $E!\, h \cap z_1$, it follows that $h \cap z_1 = h \cap z_1$, whence by the definition of $\cap$ and Lemma 4(ii), $z_3 \in h \wedge z_3 \in z_1$. Since $h \neq V_1$ and $z_3 \in h$, it follows by Theorem 5 that Vz_3. Since Vz_3 and $z_3 \in h \wedge (y \in z_3 \vee y \subseteq z_3)$, it follows by Lemma 4(i) that $z_3 \in b$. But $z_3 \in b$ and $z_3 \in z_1$ are together contrary to $\neg\exists z_2(z_2 \in b \wedge z_2 \in z_1)$. Hence $y \notin z_1$, whence $y \subseteq z_1$. Since $x \in y$, it follows by the definition of $\subseteq$ that $x \in z_1$, whence $x \in v$ by Lemma 4(i).

Corollary. *Let My and $y \in v$, then $y \subseteq v$.*

Proof. Since $y \in v$, it follows that $x \in y \to x \in v$ by Theorem 7, which together with My entails $y \subseteq v$ by the definition of $\subseteq$.

Theorem 8. *Levels are hereditary sets Let $x \subseteq y$ and $y \in v$, then $x \in v$.*

For this sense of 'hereditary set', see Tarski [26], p. 177. In particular, if a set is a subset of a level, it is a member of all higher levels. The proof of Theorem 8 only differs from the proof of Theorem 7 in half a dozen places, but to help the reader we give it in full.

Proof. From the hypothesis $x \subseteq y$ it follows that My by Lemma 6(i), whence by the definition of U, $\neg Uy$. Since My it follows that $V_1 = \{z\colon Uz\}$

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by Lemma 7(iii). From the hypothesis $y \in v$ it follows that $E!v$ by Axiom 1(i). We tackle three cases separately: (i) $v = V_1$, (ii) $v = V_2$, and (iii) $v \neq V_1$ and $v \neq V_2$.

Case (i) $v = V_1$

From the hypothesis $y \in v$ it follows by Lemma 4(i) that Uy. But also $\neg Uy$. Hence $x \in v$ by the tautology $A \wedge \neg A \rightarrow B$.

Case (ii) $v = V_2$

By the strength of identity, $E!V_2$. Since $y \in v$, it follows by Lemma 7(v) that $Uy \vee y \subseteq V_1$. Since $\neg Uy$, it follows that $y \subseteq V_1$. Since $x \subseteq y$, it follows that $x \subseteq V_1$ by the definition of $\subseteq$. Hence $x \in V_2$ by Lemma 7(v), whence $x \in v$.

Case (iii) $v \neq V_1$ and $v \neq V_2$

1. By the definition of V, $v = \text{acc } h$ for some history h. For a reductio suppose that $h = V_1$. Then $v = \text{acc } V_1 = V_1$ by Lemma 7(i). Contradiction. Hence $h \neq V_1$, whence by the definition of acc, Theorem 5 and Lemma 1(i), $v = \{z{:}(Uz \vee \exists u(u \in h \wedge (z \in u \vee z \subseteq u)))\}$. Since $y \in v$, it follows by Lemma 4(i) that $Uy \vee \exists u(u \in h \wedge (y \in u \vee y \subseteq u))$. Since $\neg Uy$, it follows that $y \in w \vee y \subseteq w$ for some $w \in h$. We tackle three cases separately: (a) $w = V_1$, (b) $w = V_2$, and (c) $w \neq V_1$ and $w \neq V_2$.
2. For case (a), for a reductio suppose $y \in w$. Then $y \in V_1$, whence by Lemma 4(i), Uy. Contradiction. Hence $y \subseteq w$. Since $x \subseteq y$, it follows by the definition of $\subseteq$ that $x \subseteq w$. Since $v = \{z{:}(Uz \vee \exists u(u \in h \wedge (z \in u \vee z \subseteq u)))\}$, it follows by Lemma 4(i) that $x \in v$.
3. For case (b), suppose $y \in w$. Then $y \in V_2$, whence $Uy \vee y \subseteq V_1$ by the strength of identity and Lemma 7(v). Since $\neg Uy$, it follows that $y \subseteq V_1$. Since $x \subseteq y$, it follows by the definition of $\subseteq$ that $x \subseteq V_1$. Hence by Lemma 7(v), $x \in V_2$, whence $x \in w$. So by Lemma 4(i) $x \in v$. Suppose instead that $y \subseteq w$. Since $x \subseteq y$, it follows by the definition of $\subseteq$ that $x \subseteq w$, whence $x \in v$ by Lemma 4(i).
4. For case (c), we deal with two subcases separately: (ci) $\exists_1 u(u \in h \wedge (y \in u \vee y \subseteq u))$, and (cii) $mu(u \in h \wedge (y \in u \vee y \subseteq u))$.
5. For case (ci), $u \in h \wedge (y \in u \vee y \subseteq u)$ for some unique level u, namely w. From $h \neq V_1, w \in h, w \neq V_1, w \neq V_2$, it follows that $w = \text{acc } h \cap w$, $E!\, h \cap w$, and $mz(z \in h \wedge z \in w)$ by Theorems 4(i), 4(ii), and 4(iii). Hence by the definition of acc, $w = \{z{:}(Uz \vee \exists z_1(z_1 \in h \cap w \wedge (z \in z_1 \vee z \subseteq z_1)))\}$. For a reductio suppose $y \in w$. Then by Lemma 4(i), $Uy \vee \exists z_1(z_1 \in h \cap w \wedge (y \in z_1 \vee y \subseteq z_1))$. Since My, it follows that $\exists z_1(z_1 \in h \cap w \wedge (y \in z_1 \vee y \subseteq z_1))$ by the definition of U. Since $E!\, h \cap w$, it follows that $h \cap w = h \cap w$. Since $mz(z \in h \wedge z \in w)$, it follows by the definition of $h \cap w$ and Lemma 4(ii) that for some z_1, $z_1 \in h \wedge z_1 \in w \wedge (y \in z_1 \vee y \subseteq z_1)$. Since $z_1 \in h$ and $h \neq V_1$, it follows by Theorem 5 that Vz_1. Since $z_1 \in w$, it follows by Axiom 1(iii) that $z_1 \neq w$. But $z_1 \in h \wedge (y \in z_1 \vee y \subseteq z_1)$ and Vz_1 and $z_1 \neq w$ are together contrary to $\exists_1 u(u \in h \wedge (y \in u \vee y \subseteq u))$. Hence $y \subseteq w$.

Since $x \subseteq y$, it follows by the definition of $\subseteq$ that $x \subseteq w$, and so by Lemma 4(i) $x \in v$.

6. For case (cii), let b be short for $\{u: u \in h \wedge (y \in u \vee y \subseteq u)\}$. Since $v =$ acc h, it follows by Theorem 2 that $z \in h \rightarrow z \in v$. Hence $\forall u((u \in h \wedge (y \in u \vee y \subseteq u)) \rightarrow u \in v)$, so $\exists v_1 \forall u(u \in h \wedge (y \in u \vee y \subseteq u)) \rightarrow u \in v_1)$. This together with $mu(u \in h \wedge (y \in u \vee y \subseteq u))$ entails Mb by Axiom 2(iii). Hence by Lemma 3 $E!b$, whence $b = \{u: u \in h \wedge (y \in u \vee y \subseteq u)\}$. Hence by Lemma 4(i), $z \in b \rightarrow z \in h$, whence by the definition of $\subseteq$, $b \subseteq h$. Hence by Theorem 6, $\neg\exists z_2(z_2 \in b \wedge z_2 \in z_1)$ for some $z_1 \in b$. Hence by Lemma 4(i), $z_1 \in h \wedge (y \in z_1 \vee y \subseteq z_1)$. We deal with three subcases separately: (ciiα) $z_1 = V_1$, (ciiβ) $z_1 = V_2$, and (ciiγ) $z_1 \neq V_1 \wedge z_1 \neq V_2$.
7. In case (ciiα), it follows that $x \in v$ by the reasoning in step 2.
8. In case (ciiβ), it follows that $x \in v$ by the reasoning in step 3.
9. In case (ciiγ), since $h \neq V_1$, $z_1 \in h$, $z_1 \neq V_1$ and $z_1 \neq V_2$, it follows that $z_1 =$ acc $h \cap z_1$, $E!\,h \cap z_1$ and $mz(z \in h \wedge z \in z_1)$ by Theorems 4(i), (ii), and (iii). For a reductio suppose $y \in z_1$. Then $y \in$ acc $h \cap z_1$, whence by the definition of acc, $y \in \{z:(Uz \vee \exists z_3(z_3 \in h \cap z_1 \wedge (z \in z_3 \vee z \subseteq z_3)))\}$. Since $\neg Uy$, we have by Lemma 4(i) that $y \in z_3 \vee y \subseteq z_3$ for some $z_3 \in h \cap z_1$. Since $E!\,h \cap z_1$, it follows that $h \cap z_1 = h \cap z_1$, whence by the definition of $\cap$ and Lemma 4(ii), $z_3 \in h \wedge z_3 \in z_1$. Since $h \neq V_1$ and $z_3 \in h$, it follows by Theorem 5 that Vz_3. Since Vz_3 and $z_3 \in h \wedge (y \in z_3 \vee y \subseteq z_3)$, it follows by Lemma 4(i) that $z_3 \in b$. But $z_3 \in b$ and $z_3 \in z_1$ are together contrary to $\neg\,\exists z_2(z_2 \in b \wedge z_2 \in z_1)$. Hence $y \notin z_1$, whence $y \subseteq z_1$. Since $x \subseteq y$, it follows by the definition of $\subseteq$ that $x \subseteq z_1$, whence $x \in v$ by Lemma 4(i).

THEOREM 9. *Lower levels I*

(i) $\neg\exists w\ w \in V_1$ *and* $[w{:}w \in V_1] \equiv O$.

(ii) *Let* $E!V_2$, *then* $\exists_1 w\ w \in V_2$ *and* $[w{:}w \in V_2] = \imath w(w \in V_2) = V_1$.

(iii) *Let* $E!v$ *and* $v \neq V_1$ *and* $v \neq V_2$, *then* $mw\ w \in v$ *and* $[w{:}w \in v] = \{w{:}w \in v\}$.

The parts cover the three cases for the number of lower levels: (i) V_1 has none; (ii) V_2 has V_1 as its sole lower level; (iii) any other level has many lower levels. Modified set abstraction is used to express each case as an identity.

PROOF OF (i). For a reductio suppose that $w \in V_1$ for some level w. Then by Corollary (i) of Theorem 1, Mw, and by Axiom 1(i), $E!V_1$. Hence by Lemma 7(ii), Uw, whence $\neg Mw$ by the definition of U. Contradiction. Hence $\neg\exists w\ w \in V_1$, whence by Lemma 2(i), $[w{:}w \in V_1] \equiv O$.

PROOF OF (ii).

1. It follows from the hypothesis $E!V_2$ that $V_2 = \{z{:}Uz \vee z \subseteq V_1\}$ by Lemma 7(iv), and that $E!V_1$ by Lemma 7(vi). Hence by Lemma 7(i), $V_1 = \{z{:}Uz\}$. By Lemma 7(vii), $V_1 \in V_2$, and by the definition of V, $V(V_1)$.
2. For a reductio suppose $w \neq V_1 \wedge w \in V_2$ for some level w. Then by Axiom 1(iii), $w \neq V_2$. Also by Lemma 4(i), $Uw \vee w \subseteq V_1$. Since

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Vw, it follows by Corollary (i) of Theorem 1 that Mw, whence by the definition of U, $\neg Uw$. Hence $w \subseteq V_1$. Since $w \neq V_1$, it follows by the definition of $\subset$ that $w \subset V_1$, whence $w \subset \{z{:}Uz\}$. Hence by Lemma 6(iii), $x \notin w \wedge x \in \{z{:}Uz\}$ for some x, whence by Lemma 4(i), $x \notin w \wedge Ux$, contrary to Corollary (ii) of Theorem 1. Hence $\neg \exists w(w \neq V_1 \wedge w \in V_2)$.

3. Since $V(V_1)$ and $V_1 \in V_2$ and $\neg\exists w(w \neq V_1 \wedge w \in V_2)$, it follows that $\exists_1 w\ w \in V_2$ and $\imath w(w \in V_2) = V_1$, whence $[w{:}w \in V_2] = \imath w(w \in V_2) = V_1$ by Lemma 2(ii).

Proof of (iii).

1. Since by hypothesis $E!v$ and $v \neq V_1$ and $v \neq V_2$, it follows by the definition of V that $v = \text{acc } h$ for some history h. By the definition of acc, $v = \text{acc } h = \{z{:}(Uz \vee \exists y(y \in h \wedge (z \in y \vee z \subseteq y)))\}$. Hence by Lemma 4(i), $z \in v \leftrightarrow (Uz \vee \exists y(y \in h \wedge (z \in y \vee z \subseteq y)))$.
2. By Theorem 2(ii), $Mh \wedge h \neq V_1$. Hence $my\ y \in h$ by the definition of M, whence $my(y \in h \wedge Vy)$ by Theorem 5, and so $mw\ w \in h$. Consider an arbitrary level $u \in h$. By Corollary (i) of Theorem 1, Mu. Hence by Lemma 6(ii), $u \subseteq u$, whence $u \in v$ by $z \in v \leftrightarrow (Uz \vee \exists y(y \in h \wedge (z \in y \vee z \subseteq y)))$. Hence $mw\ w \in v$, whence by Lemma 2(iii), $[w{:}w \in v] \equiv \{w{:}w \in v\}$. Since $\forall w(w \in v \rightarrow w \in v)$, by existential generalization $\exists v_1 \forall w(w \in v \rightarrow w \in v_1)$. Hence by Axiom 2(iii), $M\{w{:}w \in v\}$, whence $E!\{w{:}w \in v\}$ by Lemma 3. Hence $[w{:}w \in v] = \{w{:}w \in v\}$.

Theorem 10. *Lower levels II*

(i) *Let* $v = V_1$ *or* $v = V_2$, *then* $v = accum\, [w{:}w \in v]$.

(ii) *Let* $E!v$ *and* $v \neq V_1$ *and* $v \neq V_2$, *then* $v = acc\, [w{:}w \in v]$.

Proof of (i). Suppose $v = V_1$. Then by Theorem 9(i) $[w{:}w \in v] \equiv O$, whence $v = \text{accum } [w{:}w \in v]$ by the definition of V_1. Suppose instead that $v = V_2$. Then $E!V_2$ by the strength of identity, whence by Theorem 9(ii) $[w{:}w \in v] = V_1$. Hence $v = \text{accum } [w{:}w \in v]$ by the definition of V_2.

Proof of (ii).

1. Since by hypothesis $E!v$ and $v \neq V_1$ and $v \neq V_2$, it follows that $[w{:}w \in v] = \{w{:}w \in v\}$ by Theorem 9(iii), whence by Lemma 4(i) $y \in \{w{:}w \in v\} \leftrightarrow (y \in v \wedge Vy)$. It also follows from the hypothesis by the definition of V that $v = \text{acc } h$ for some history h. By the definition of acc, $v = \text{acc } h = \{z{:}(Uz \vee \exists y(y \in h \wedge (z \in y \vee z \subseteq y)))\}$. Hence by Lemma 4(i), $z \in v \leftrightarrow (Uz \vee \exists y(y \in h \wedge (z \in y \vee z \subseteq y)))$.
2. We shall prove that acc $\{w{:}w \in v\} = \{z{:}(Uz \vee \exists y(y \in \{w{:}w \in v\} \wedge (z \in y \vee z \subseteq y)))\}$. With a view to using Axiom 2(iii), we first prove (a) $mz(Uz \vee \exists y(y \in \{w{:}w \in v\} \wedge (z \in y \vee z \subseteq y)))$ and (b) $\exists u \forall z((Uz \vee \exists y(y \in \{w{:}w \in v\} \wedge (z \in y \vee z \subseteq y))) \rightarrow z \in u)$.
3. For (a), by hypothesis $E!v$. Hence Mv by Corollary (i) of Theorem 1, whence $\exists x Mx$. Hence $mzUz$ by Axiom 2(ii), a fortiori $mz(Uz \vee \exists y(y \in \{w{:}w \in v\} \wedge (z \in y \vee z \subseteq y)))$.
4. For (b), consider an arbitrary z such that $Uz \vee \exists y(y \in \{w{:}w \in v\} \wedge (z \in y \vee z \subseteq y))$. We consider the three possibilities for z and

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deduce $z \in v$ in each case. First, suppose Uz. Then by Corollary (ii) of Theorem 1, $z \in v$. Second, suppose $z \in y$ for some $y \in \{w{:}w \in v\}$. Then $y \in v$, whence $z \in v$ by Theorem 7. Third, suppose $z \subseteq y$ for some $y \in \{w{:}w \in v\}$. Then $y \in v$, whence $z \in v$ by Theorem 8. Since z was arbitrary, we can generalize to get $\forall z((Uz \vee \exists y(y \in \{w{:}w \in v\} \wedge (z \in y \ \vee z \subseteq y))) \rightarrow z \in v)$, whence $\exists u \forall z((Uz \vee \exists y(y \in \{w{:}w \in v\} \wedge (z \in y \ \vee z \subseteq y))) \rightarrow z \in u)$.

5. From (a) and (b) it follows that $M\{z{:}(Uz \vee \exists y(y \in \{w{:}w \in v\} \wedge (z \in y \vee z \subseteq y)))\}$ by Axiom 2(iii), whence by the definition of acc and Lemma 3, acc $\{w{:}w \in v\} = \{z{:}(Uz \vee \exists y(y \in \{w{:}w \in v\} \wedge (z \in y \vee z \subseteq y)))\}$.
6. By Lemma 4(i), M(acc $\{w{:}w \in v\}$) and $z \in$ acc $\{w{:}w \in v\} \leftrightarrow (Uz \vee \exists y(y \in \{w{:}w \in v\} \wedge (z \in y \vee z \subseteq y)))$. We go onto prove $z \in v \leftrightarrow z \in$ acc $\{w{:}w \in v\}$.
7. For the $\rightarrow$ half, suppose $z \in v$. Then $Uz \vee \exists y(y \in h \wedge (z \in y \vee z \subseteq y))$. By Theorems 2(i) and (ii), and Theorem 5, $y \in h \rightarrow (y \in v \wedge Vy)$. Hence $y \in h \rightarrow y \in \{w{:}w \in v\}$, whence $Uz \vee \exists y(y \in \{w{:}w \in v\} \wedge (z \in y \vee z \subseteq y))$. Hence $z \in$ acc $\{w{:}w \in v\}$.
8. For the $\leftarrow$ half, suppose $z \in$ acc $\{w{:}w \in v\}$. It follows that $Uz \vee \exists y(y \in \{w{:}w \in v\} \wedge (z \in y \vee z \subseteq y))$. By the reasoning in step 4, it follows that $z \in v$ for each of the three possibilities for z.
9. Since Mv and M(acc $\{w{:}w \in v\}$) and $z \in v \leftrightarrow z \in$ acc $\{w{:}w \in v\}$, then by Axiom 1(ii), $v =$ acc $\{w{:}w \in v\}$, whence $v =$ acc $[w{:}w \in v]$.

THEOREM 11. *Lower levels III* $\quad H[w{:}w \in v]$

Theorems 10 and 11 together ensure that any level v has $[w{:}w \in v]$ as a history.

PROOF.

By Corollary (i) of Theorem 1, Mv, whence $E!v$ by Lemma 3. It also follows from Mv by Lemma 7(iii) that $V_1 = \{z{:}Uz\}$, whence MV_1 by Lemma 4(i). We tackle three cases separately: (i) $v = V_1$, (ii) $v = V_2$, and (iii) $v \neq V_1$ and $v \neq V_2$.

CASE (i) $v = V_1$

By Theorem 9(i), $[w{:}w \in v] \equiv O$. Since $\neg E!O$, it follows by Lemma 3 and Axiom 1(iv) that $\neg \exists x\ x \in O$. Hence $H[w{:}w \in v]$ by the definition of H.

CASE (ii) $v = V_2$

By Theorem 9(ii), $[w{:}w \in v] = V_1$, which together with MV_1 entails $H[w{:}w \in v]$ by the definition of H.

CASE (iii) $v \neq V_1$ and $v \neq V_2$

1. By Theorem 9(iii), $mw\ w \in v$ and $[w{:}w \in v] = \{w{:}w \in v\}$, whence by Lemma 4(i), $M[w{:}w \in v]$ and $mw_1\ w_1 \in [w{:}w \in v]$. Consider an arbitrary level $u \in [w{:}w \in v]$. We prove that (a) $u = V_1 \vee u = V_2 \rightarrow u =$ accum $[w{:}w \in v] \cap u$, and (b) $u \neq V_1 \wedge u \neq V_2 \rightarrow u =$ acc $[w{:}w \in v] \cap u$.

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2. For (a), we prove: (ai) $u = V_1 \rightarrow u = \text{accum}\,[w{:}w \in v] \cap u$, and (aii) $u = V_2 \rightarrow u = \text{accum}\,[w{:}w \in v] \cap u$.
3. For (ai), suppose $u = V_1$. Then $u = \text{accum}\ O$ by the definition of V_1. By Lemma 4(i), Corollary (i) of Theorem 1 and the definition of U, $w_1 \in \{w{:}w \in v\} \rightarrow \neg Uw_1$. Also by Lemma 4(i), $w_1 \in u \rightarrow Uw_1$. Hence $\neg\exists w_1(w_1 \in \{w{:}w \in v\} \wedge w_1 \in u)$, whence by Lemma 2(i), $[w_1{:}\ w_1 \in \{w{:}w \in v\} \wedge w_1 \in u] \equiv O$. Hence $\{w{:}w \in v\} \cap u \equiv O$ by the definition of $\cap$. So $u = \text{accum}\ \{w{:}w \in v\} \cap u$, whence $u = \text{accum}\,[w{:}w \in v] \cap u$.
4. For (aii): suppose $u = V_2$. Then $u = \text{accum}\ V_1$ by the definition of V_2. Since $E!v$ and $v \neq V_1$ and $v \neq V_2$, it follows by the definition of V that $v = \text{acc}\ h$ for some history h. By Theorem 2(ii) $Mh \wedge h \neq V_1$, whence $my\ y \in h$ by the definition of M. Hence $w_1 \in h$ for some level w_1 by Theorem 5. By Corollary (ii) of Theorem 1, $Uz \rightarrow z \in w_1$, whence by Lemma 4(i), $z \in V_1 \rightarrow z \in w_1$. Since MV_1, it follows that $V_1 \subseteq w_1$ by the definition of $\subseteq$. By the definition of acc and Lemma 4(i), $z \in v \leftrightarrow (Uz \vee \exists y(y \in h \wedge (z \in y \vee z \subseteq y)))$. Hence $V_1 \in v$, whence $V_1 \in \{w{:}w \in v\}$ by Lemma 4(i) and the definition of V. Since by hypothesis $u = V_2$, it follows by the strength of identity that $E!V_2$. Hence $V_1 = \iota w(w \in u)$ by Theorem 9(ii). Since $V_1 \in \{w{:}w \in v\}$, it follows that $V_1 = \iota w_1(w_1 \in \{w{:}w \in v\} \wedge w_1 \in u)$. Hence by the strength of identity, $E!\iota w_1(w_1 \in \{w{:}w \in v\} \wedge w_1 \in u)$, whence $\exists_1 w_1(w_1 \in \{w{:}w \in v\} \wedge w_1 \in u)$. Hence $\{w{:}w \in v\} \cap u = V_1$ by Lemma 2(ii) and the definition of $\cap$. Hence $u = \text{accum}\ \{w{:}w \in v\} \cap u$, whence $u = \text{accum}\,[w{:}w \in v] \cap u$.
5. By (ai) and (aii), $u = V_1 \vee u = V_2 \rightarrow u = \text{accum}\,[w{:}w \in v] \cap u$.
6. For (b), suppose $u \neq V_1 \wedge u \neq V_2$. Since $u \in \{w{:}w \in v\}$, it follows by Lemma 4(i) that $u \in v$. Hence by Theorem 7, $w_1 \in u \rightarrow w_1 \in v$, whence $(w_1 \in v \wedge w_1 \in u) \leftrightarrow w_1 \in u$. Hence by Lemma 4(i), $(w_1 \in \{w{:}w \in v\} \wedge w_1 \in u) \leftrightarrow w_1 \in u$, whence by Lemma 1(ii), $[w_1{:}\ w_1 \in \{w{:}w \in v\} \wedge w_1 \in u] \equiv [w_1{:}w_1 \in u]$. Since $E!u$ and $u \neq V_1$ and $u \neq V_2$, it follows by Theorem 9(iii) that $[w_1{:}w_1 \in u] = \{w_1{:}w_1 \in u\}$, whence $E![w_1{:}w_1 \in u]$ by the strength of identity. Hence $[w_1{:}\ w_1 \in \{w{:}w \in v\} \wedge w_1 \in u] = [w_1{:}w_1 \in u]$, whence $\{w{:}w \in v\} \cap u = [w_1{:}w_1 \in u]$ by the definition of $\cap$. By Theorem 10(ii), $u = \text{acc}\,[w_1{:}w_1 \in u]$. So $u = \text{acc}\{w{:}w \in v\} \cap u$, whence $u = \text{acc}\,[w{:}w \in v] \cap u$.
7. (a) and (b) hold for any $u \in [w{:}w \in v]$. Since $[w{:}w \in v]=\{w{:}w \in v\}$, it follows by Lemma 4(i) that $y \in \{w{:}w \in v\} \rightarrow Vy$. So we can generalize to get $\forall y(y \in [w{:}w \in v] \rightarrow (y = V_1 \vee y = V_2 \rightarrow y = \text{accum}\,[w{:}w \in v] \cap y) \wedge (y \neq V_1 \wedge y \neq V_2 \rightarrow y = \text{acc}\,[w{:}w \in v] \cap y))$, which together with $M[w{:}w \in v]$ entails $H[w{:}w \in v]$ by the definition of H.

THEOREM 12. *Foundation for levels*

Let $\exists uA(u)$, *then* $\exists v(A(v) \wedge \neg\exists w(w \in v \wedge A(w)))$.

In English: let some level satisfy a condition; then there is a lowest level satisfying the condition.

https://doi.org/10.1017/bsl.2018.10 Published online by Cambridge University Press

PROOF. By hypothesis $A(u)$ for some level u. We tackle three cases separately: (i) $\neg\exists w(w \in u \wedge A(w))$ then (ii) $\exists_1 w(w \in u \wedge A(w))$ and (iii) $mw\ (w \in u \wedge A(w))$.

CASE (i) $\neg\exists w(w \in u \wedge A(w))$
It follows immediately that $\exists v(A(v) \wedge \neg\exists w(w \in v \wedge A(w)))$.

CASE (ii) $\exists_1 w(w \in u \wedge A(w))$
By hypothesis $w_1 \in u \wedge A(w_1)$ for some unique level w_1. For a reductio suppose that $w_2 \in w_1$ and $A(w_2)$ for some level w_2. Since $w_2 \in w_1$ and $w_1 \in u$, it follows by Theorem 7 that $w_2 \in u$. Since $w_2 \in w_1$, it follows by Axiom 1(iii) that $w_2 \neq w_1$. But $w_1 \in u, A(w_1)$, $w_2 \in u$, $A(w_2)$, and $w_2 \neq w_1$ are together contrary to $\exists_1 w(w \in u \wedge A(w))$. If follows that $A(w_1) \wedge \neg\exists w(w \in w_1 \wedge A(w))$, whence $\exists v(A(v) \wedge \neg\exists w(w \in v \wedge A(w)))$.

CASE (iii) $mw\ (w \in u \wedge A(w))$

1. Since $mw\ (w \in u \wedge A(w))$ and $\forall w((w \in u \wedge A(w)) \rightarrow w \in u)$, it follows by Axiom 2(iii) that $M\{w{:}w \in u \wedge A(w)\}$. Hence by Lemma 3, $E!\{w{:}w \in u \wedge A(w)\}$, whence $\{w{:}w \in u \wedge A(w)\} = \{w{:}w \in u \wedge A(w)\}$. Hence by Lemma 4(i), $w_1 \in \{w{:}w \in u \wedge A(w)\} \leftrightarrow (w_1 \in u \wedge A(w_1))$.
2. Since $mw\ (w \in u \wedge A(w))$, it follows that $mw\ w \in u$, which together with $\forall w(w \in u \rightarrow w \in u)$ entails $M\{w{:}w \in u\}$ by Axiom 2(iii). Hence by Lemma 3, $E!\{w{:}w \in u\}$, whence $\{w{:}w \in u\} = \{w{:}w \in u\}$. Hence by Lemma 4(i), $w_1 \in \{w{:}w \in u\} \leftrightarrow w_1 \in u$.
3. Since $M\{w{:}w \in u \wedge A(w)\}$ and $(w_1 \in u \wedge A(w_1)) \rightarrow w_1 \in u$, it follows that $\{w{:}w \in u \wedge A(w)\} \subseteq \{w{:}w \in u\}$ by the definition of $\subseteq$.
4. Since $mw\ w \in u$, it follows by Theorems 9(i) and (ii) that $u \neq V_1$ and $u \neq V_2$, whence by Theorem 9(iii), $[w{:}w \in u] = \{w{:}w \in u\}$. Hence by Theorem 11, $H\{w{:}w \in u\}$.
5. Since $\{w{:}w \in u \wedge A(w)\} \subseteq \{w{:}w \in u\}$ and $H\{w{:}w \in u\}$, it follows by Theorem 6 that for some level w_2, $w_2 \in \{w{:}\ w \in u \wedge A(w)\} \wedge \neg\exists z(z \in \{w{:}\ w \in u \wedge A(w)\} \wedge z \in w_2)$, whence $w_2 \in u \wedge A(w_2)$.
6. For a reductio suppose that $w_3 \in w_2$ and $A(w_3)$ for some level w_3. Since $w_3 \in w_2$ and $w_2 \in u$, it follows by Theorem 7 that $w_3 \in u$. Since $w_3 \in u$ and $A(w_3)$, it follows that $w_3 \in \{w{:}w \in u \wedge A(w)\}$. But $w_3 \in \{w{:}w \in u \wedge A(w)\}$ and $w_3 \in w_2$ are together contrary to $\neg\exists z(z \in \{w{:}\ w \in u \wedge A(w)\} \wedge z \in w_2)$. It follows that $A(w_2) \wedge \neg\exists w(w \in w_2 \wedge A(w))$, whence $\exists v(A(v) \wedge \neg\exists w(w \in v \wedge A(w)))$.

THEOREM 13. *Comparability of levels* $v \in w \vee v = w \vee w \in v$

PROOF.

1. For a reductio suppose for some v, $\exists w(v \notin w \wedge v \neq w \wedge w \notin v)$. So for some v_1, $\exists w(v_1 \notin w \wedge v_1 \neq w \wedge w \notin v_1) \wedge \neg\exists v_2(v_2 \in v_1 \wedge \exists w(v_2 \notin w \wedge v_2 \neq w \wedge w \notin v_2))$ by Theorem 12. Hence $\forall v_2(v_2 \in v_1 \rightarrow \forall w(v_2 \in w \vee v_2 = w \vee w \in v_2))$.
2. Since for some w, $(v_1 \notin w \wedge v_1 \neq w \wedge w \notin v_1)$, it follows by Theorem 12 that for some w_1, $(v_1 \notin w_1 \wedge v_1 \neq w_1 \wedge w_1 \notin v_1) \wedge \neg\exists w_2(w_2 \in w_1 \wedge (v_1 \notin w_2 \wedge v_1 \neq w_2 \wedge w_2 \notin v_1))$. So $\forall w_2(w_2 \in w_1 \rightarrow (v_1 \in w_2 \vee v_1 = w_2 \vee w_2 \in v_1))$. We shall prove $\forall w_3(w_3 \in v_1 \leftrightarrow w_3 \in w_1)$.

3. For the $\rightarrow$ half, suppose $w_3 \in v_1$. Since $w_1 \notin v_1$, then $w_3 \neq w_1$. For a reductio suppose $w_1 \in w_3$. Then from $w_3 \in v_1$ it follows that $w_1 \in v_1$ by Theorem 7. Contradiction. It follows that $w_1 \notin w_3$. Since $\forall v_2(v_2 \in v_1 \rightarrow \forall w(v_2 \in w \vee v_2 = w \vee w \in v_2))$ and $w_3 \in v_1$ and $w_3 \neq w_1$ and $w_1 \notin w_3$, it follows that $w_3 \in w_1$.
4. For the $\leftarrow$ half, suppose $w_3 \in w_1$. Since $v_1 \notin w_1$, then $w_3 \neq v_1$. For a reductio suppose $v_1 \in w_3$. Then from $w_3 \in w_1$ it follows that $v_1 \in w_1$ by Theorem 7. Contradiction. So $v_1 \notin w_3$. Since $\forall w_2(w_2 \in w_1 \rightarrow (v_1 \in w_2 \vee v_1 = w_2 \vee w_2 \in v_1))$ and $w_3 \in w_1$ and $w_3 \neq v_1$ and $v_1 \notin w_3$, it follows that $w_3 \in v_1$.
5. Since $\forall w_3(w_3 \in v_1 \leftrightarrow w_3 \in w_1)$, then by Lemma 1(ii) $[w{:}w \in v_1] \equiv [w{:}w \in w_1]$. We shall show (a) $v_1 \neq V_1 \wedge w_1 \neq V_1$, and (b) $v_1 \neq V_2 \wedge w_1 \neq V_2$.
6. For (a), for a reductio suppose $v_1 = V_1$. By Theorem 9(i), $[w{:}w \in v_1] \equiv O$, whence $[w{:}w \in w_1] \equiv O$. By Theorems 10(i) and (ii), $w_1 = \text{accum } [w{:}w \in w_1]$ or $w_1 = \text{acc } [w{:}w \in w_1]$, whence $w_1 = \text{accum } O$ or $w_1 = \text{acc } O$. For a reductio suppose $w_1 = \text{accum } O$. Then $w_1 = v_1$ by the definition of V_1. Contradiction. Hence $w_1 = \text{acc } O$, whence $w_1 = v_1$ by Lemma 7(i). Contradiction. Hence $v_1 \neq V_1$. By similar reasoning, $w_1 \neq V_1$.
7. For (b), for a reductio suppose $v_1 = V_2$. Then $E!V_2$ by the strength of identity, whence $V_1 \in v_1$ by Lemma 7(vii), and also $[w{:}w \in v_1] = \imath w(w \in v_1) = V_1$ by Theorem 9(ii). Since $\forall w_3(w_3 \in v_1 \leftrightarrow w_3 \in w_1)$, it follows that $V_1 \in w_1$. Hence by Theorems 10(i) and (ii), $w_1 = \text{accum } [w{:}w \in w_1]$ or $w_1 = \text{acc } [w{:}w \in w_1]$. Since $[w{:}w \in v_1] \equiv [w{:}w \in w_1]$, it follows that $w_1 = \text{accum } V_1$ or $w_1 = \text{acc } V_1$. For a reductio suppose $w_1 = \text{accum } V_1$. Then $w_1 = v_1$ by the definition of V_2. Contradiction. Hence $w_1 = \text{acc } V_1$. Then $w_1 = V_1$ by Lemma 7(i). But $V_1 \in w_1$ and $w_1 = V_1$ are together contrary to Axiom 1(iii). Hence $v_1 \neq V_2$. By similar reasoning, $w_1 \neq V_2$.
8. We can now proceed to the reductio initiated in step 1. From (a) and (b) it follows that $v_1 = \text{acc } [w{:}w \in v_1]$ and $w_1 = \text{acc } [w{:}w \in w_1]$ by Theorem 10(ii). Since $[w{:}w \in v_1] \equiv [w{:}w \in w_1]$, it follows that $v_1 = \text{acc } [w{:}w \in v_1] = \text{acc } [w{:}w \in w_1] = w_1$. Contradiction. Hence $v \in w \vee v = w \vee w \in v$.

THEOREM 14. *The lowest level principle*
Let $\exists u A(u)$, *then* $\exists_1 v(A(v) \wedge \neg\exists w(w \in v \wedge A(w)))$.

In English: let some level satisfy a condition; then there is a unique lowest level satisfying the condition.

PROOF. Since by hypothesis $\exists u A(u)$, it follows by Theorem 12 that $A(v) \wedge \neg\exists w(w \in v \wedge A(w))$ for some level v. For a reductio suppose that $A(v_1) \wedge \neg\exists w(w \in v_1 \wedge A(w))$ for some level $v_1 \neq v$. Then by Theorem 13 it follows that $v \in v_1 \vee v_1 \in v$. But if $v \in v_1$ then $v \in v_1 \wedge A(v)$, contrary to $\neg\exists w(w \in v_1 \wedge A(w))$. Similarly, if $v_1 \in v$ then $v_1 \in v \wedge A(v_1)$, contrary to $\neg\exists w(w \in v \wedge A(w))$. Contradiction. Hence $\exists_1 v(A(v) \wedge \neg\exists w(w \in v \wedge A(w)))$.

THEOREM 15. *Uniqueness of histories*

(i) *Let* $V_1 = \text{accum}\, h$, *then* $h \equiv [w{:}w \in V_1]$.
(ii) *Let* $V_2 = \text{accum}\, h$, *then* $h = [w{:}w \in V_2]$.
(iii) *Let* $v \neq V_1$ *and* $v \neq V_2$ *and* $v = \text{acc}\, h$, *then* $h = [w{:}w \in v]$.

PROOF OF (i).

1. Since by hypothesis $V_1 = \text{accum}\, h$, it follows that $E!V_1$ by the strength of identity, whence $z \in V_1 \rightarrow Uz$ by Lemma 7(ii).
2. For a reductio suppose $h \not\equiv [w{:}w \in V_1]$. Then $h \not\equiv O$ by Theorem 9(i), whence Mh by the definition of H. Hence $h \subseteq h$ by Lemma 6(ii). By the definition of accum, $V_1 = \text{accum}\, h = \{z{:}Uz \vee z \in h \vee z \subseteq h\}$, whence $h \in V_1$ by Lemma 4(i). Hence Uh, whence $\neg\, Mh$ by the definition of U. Contradiction. Hence $h \equiv [w{:}w \in V_1]$.

PROOF OF (ii).

1. Since by hypothesis $V_2 = \text{accum}\, h$, it follows that $E!V_2$ by the strength of identity. By Theorem 9(ii), $[w{:}w \in V_2] = V_1$, whence $E!V_1$ by the strength of identity. Hence MV_1 by Lemma 7(ii).
2. For a reductio suppose $h \equiv O$. Then by the definition of V_1, $V_2 = \text{accum}\, h = \text{accum}\, O = V_1$, contrary to Lemma 7(viii). Hence $h \not\equiv O$, whence Mh by the definition of H. Hence $h \subseteq h$ by Lemma 6(ii). By the definition of accum, $V_2 = \text{accum}\, h = \{z{:}Uz \vee z \in h \vee z \subseteq h\}$, whence $h \in V_2$ by Lemma 4(i). Hence by Lemma 7(v), $Uh \vee h \subseteq V_1$. Since Mh, it follows that $\neg Uh$ by the definition of U, whence $h \subseteq V_1$. Hence $z \in h \rightarrow Uz$ by the definition of $\subseteq$ and Lemma 7(ii).
3. For a reductio suppose $h \neq [w{:}w \in V_2]$. Then $h \neq V_1$, whence $h \subset V_1$ by the definition of $\subset$. Hence for some z_1, $z_1 \notin h \wedge z_1 \in V_1$ by Lemma 6(iii). By Lemma 7(vii) $V_1 \in V_2$, whence $UV_1 \vee V_1 \in h \vee V_1 \subseteq h$ by Lemma 4(i). Since MV_1, it follows that $\neg UV_1$ by the definition of U. Hence $V_1 \in h \;\vee\; V_1 \subseteq h$. For a subordinate reductio suppose $V_1 \in h$. Then UV_1, whence $\neg MV_1$ by the definition of U. Contradiction. Hence $V_1 \subseteq h$, whence $z \in V_1 \rightarrow z \in h$ by the definition of $\subseteq$. Contradiction. Hence $h = [w{:}w \in V_2]$.

PROOF OF (iii).

1. Since by hypothesis $v \neq V_1$ and $v = \text{acc}\, h$, it follows by Theorem 2(ii) that $Mh \wedge h \neq V_1$, whence $E!h$ by Lemma 3.
2. For a reductio suppose $h \neq \{w{:}w \in v\}$. Then by Theorem 14, there is a unique level v_1 such that for some history h_1, $v_1 \neq V_1 \wedge v_1 \neq V_2 \wedge v_1 = \text{acc}\, h_1 \wedge h_1 \neq \{w{:}w \in v_1\}$, and $\neg\exists w_1(w_1 \in v_1 \wedge \exists x(Hx \wedge w_1 \neq V_1 \wedge w_1 \neq V_2 \wedge\ w_1 = \text{acc}\, x \wedge x \neq \{w{:}w \in w_1\})$. Hence $Mh_1 \wedge h_1 \neq V_1$ by Theorem 2(ii). By the definition of acc, $v_1 = \text{acc}\, h_1 = \{z{:}(Uz \vee \exists y(y \in h_1 \wedge\ (z \in y \vee z \subseteq y)))\}$, whence by Lemma 4(i), $z \in v_1 \leftrightarrow (Uz \vee \exists y(y \in h_1 \wedge (z \in y \vee z \subseteq y)))$. We shall prove that $w_2 \in h_1 \leftrightarrow w_2 \in v_1$.
3. The $\rightarrow$ half is immediate by Theorem 2(i).

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4. For the $\leftarrow$ half, suppose $w_2 \in v_1$. Then $E!w_2$ by Axiom 1(i). For a reductio suppose $w_3 \in h_1 \rightarrow w_3 \in w_2$. We tackle three cases separately, deriving a contradiction for each: (i) $w_2 = V_1$, (ii) $w_2 = V_2$, and (iii) $w_2 \neq V_1$ and $w_2 \neq V_2$.
5. For case (i), since Mh_1, it follows by the definition of M that $x \in h_1$ for some x. Since $h_1 \neq V_1$, it follows by Theorem 5 that Vx. By supposition $w_3 \in h_1 \rightarrow w_3 \in w_2$, whence $x \in V_1$. But by Theorem 9(i) $\neg\,\exists w\; w \in V_1$. Contradiction.
6. For case (ii), since Mh_1, it follows by the definition of M that $x \in h_1$ and $y \in h_1$, for some x, y where $x \neq y$. Since $h_1 \neq V_1$, it follows by Theorem 5 that Vx and Vy. By supposition $w_3 \in h_1 \rightarrow w_3 \in w_2$, and so $x \in V_2$ and $y \in V_2$. But by Theorem 9(ii), $\exists_1 w\; w \in V_2$. Contradiction.
7. For case (iii), by Theorems 9(iii) and 10(ii), $w_2 = \text{acc}\,\{w_4 : w_4 \in w_2\}$, whence $w_2 = \{z{:}(Uz \vee \exists y(y \in \{w_4 : w_4 \in w_2\} \wedge (z \in y \vee z \subseteq y)))\}$ by the definition of acc. So $z \in w_2 \;\leftrightarrow (Uz \vee \exists w(w \in w_2 \wedge (z \in w \vee z \subseteq w)))$ by Lemma 4(i). Since $h_1 \neq V_1$, it follows by Theorem 5 that $y \in h_1 \;\rightarrow Vy$. Since (from step 2) $z \in v_1 \rightarrow (Uz \vee \exists y(y \in h_1 \wedge (z \in y \vee z \subseteq y)))$ and (by supposition) $w_3 \in h_1 \rightarrow w_3 \in w_2$, it follows that $z \in v_1 \rightarrow (Uz \vee \exists w(w \in w_2 \wedge (z \in w \vee z \subseteq w)))$, whence $z \in v_1 \rightarrow z \in w_2$. But $w_2 \in v_1$, so $w_2 \in w_2$, contrary to Axiom 1(iii).
8. Since each case leads to a contradiction, it follows that $w_5 \in h_1 \;\wedge w_5 \notin w_2$ for some w_5, whence by Theorem 13 $w_2 = w_5 \vee w_2 \in w_5$. Suppose $w_2 = w_5$. Since $w_5 \in h_1$, it follows that $w_2 \in h_1$.
9. Taking the other alternative, suppose $w_2 \in w_5$. We tackle three cases separately: (a) $w_5 = V_1$, (b) $w_5 = V_2$ and (c) $w_5 \neq V_1$ and $w_5 \neq V_2$.
10. For case (a), by Theorem 9(i) $\neg\,\exists w\; w \in V_1$, whence $w_2 \notin w_5$. But also $w_2 \in w_5$. By the tautology $A \wedge \neg A \rightarrow B$ it follows that $w_2 \in h_1$.
11. For case (b), since $w_2 \in w_5$, it follows that $w_2 \in V_2$. Hence by Theorem 9(ii), $w_2 = V_1$. Since $w_5 \in h_1$, it follows that $V_2 \in h_1$. Since $h_1 \neq V_1$, it follows by the reasoning in steps 1–5 of case (ii) of Theorem 5 that $V_1 \in h_1$, whence $w_2 \in h_1$.
12. For case (c), since $h_1 \neq V_1$ and $w_5 \in h_1$ and $w_5 \neq V_1$ and $w_5 \neq V_2$, it follows by Theorems 4(i), (ii), (iii), and 5 that $w_5 = \text{acc}\; h_1 \cap w_5$ and $E!(h_1 \cap w_5)$, $mz(z \in h_1 \wedge z \in w_5)$ and $H(h_1 \cap w_5)$. Since $w_5 \in h_1$ and $v_1 = \text{acc}\; h_1$, it follows by Theorem 2(i) that $w_5 \in v_1$, and so $\neg\exists x(Hx \wedge w_5 \neq V_1 \wedge w_5 \neq V_2 \wedge w_5 = \text{acc}\; x \wedge x \neq \{w{:}w \in w_5\})$ by step 2. Since $E!(h_1 \cap w_5)$ and $H(h_1 \cap w_5)$ and $w_5 \neq V_1$ and $w_5 \neq V_2$ and $w_5 = \text{acc}\;(h_1 \cap w_5)$, it follows that $h_1 \cap\; w_5 = \{w{:}w \in w_5\}$. Since $w_2 \in w_5$, it follows by Lemma 4(i) that $w_2 \in (h_1 \cap w_5)$. Since $mz(z \in h_1 \wedge z \in w_5)$ it follows by the definition of $\cap$ and Lemma 2(iii) that $(h_1 \cap w_5) = \{z{:}\; z \in h_1 \wedge z \in w_5\}$. It follows that $w_2 \in \{z{:}\; z \in h_1 \wedge z \in w_5\}$, whence by Lemma 4(i) $w_2 \in h_1$.
13. We can now proceed to the reductio initiated in step 2. Since $E!v_1$ and $v_1 \neq V_1$ and $v_1 \neq V_2$, it follows by Theorem 9(iii) that $[w{:}w \in v_1] = \{w{:}w \in v_1\}$. Hence $M\{w{:}w \in v_1\}$ by Lemma 4(i). Since $w_2 \in h_1 \leftrightarrow$

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$w_2 \in v_1$, it follows that $w_2 \in h_1 \leftrightarrow w_2 \in \{w{:}w \in v_1\}$ by Lemma 4(i). Since $h_1 \neq V_1$, it follows by Theorem 5 that $x \in h_1 \rightarrow Vx$. Also $x \in \{w{:}w \in v_1\} \rightarrow Vx$ by Lemma 4(i). So $x \in h_1 \leftrightarrow x \in \{w{:}w \in v_1\}$. Since Mh_1 and $M\{w{:}w \in v_1\}$ and $x \in h_1 \leftrightarrow x \in \{w{:}w \in v_1\}$, it follows by Axiom 1(ii) that $h_1 = \{w{:}w \in v_1\}$. Contradiction. Hence $h = \{w{:}w \in v\}$. Since $v = \text{acc}\, h$, it follows that $E!v$ by the strength of identity, whence by Theorem 9(iii) that $h = [w{:}w \in v]$.

We define $V^*(a)$ to be the lowest level v such that $a \subseteq v$ (for short, *the level of* a). In symbols, $V^*(a) =_{df} \iota v(a \subseteq v \wedge \neg \exists w(w \in v \wedge a \subseteq w))$.

Theorem 16. *Sets and levels I* *Let* Mx*. Then* (i) $\exists u\; x \subseteq u$
(ii) $E!V^*(x)$.

Proof of (i). By Lemma 5(ii), $M\{z{:}z \in x\}$. It follows by Axiom 2(iii) that $\exists u \forall z(z \in x \rightarrow z \in u)$, whence $\exists u\; x \subseteq u$ by the definition of $\subseteq$.

Proof of (ii). By Theorem 16(i), $\exists u\;\; x \subseteq u$. Hence by Theorem 14, $\exists_1 v(x \subseteq v \wedge \neg \exists w(w \in v \wedge x \subseteq w))$, whence $E!\iota v(x \subseteq v \wedge \neg \exists w(w \in v \wedge x \subseteq w))$. Hence $E!V^*(x)$ by the definition of $V^*(x)$.

Theorem 17. *Foundation for sets* *Let* Mx*, then* $\exists y(y \in x \wedge x \cap y \equiv O)$.
This is our version of the familiar foundation or regularity axiom, but for us it goes without saying that the set x is nonempty.

Proof.
1. Suppose $Uy \wedge y \in x$ for some y. Then $\neg My$ by the definition of U. So by Axiom 1(iv) $\neg \exists z\; z \in y$. Hence $\neg \exists z(z \in x \wedge z \in y)$, whence $x \cap y \equiv O$ by Lemma 2(i) and the definition of $\cap$. Hence $\exists y(y \in x \wedge x \cap y \equiv O)$.
2. Suppose instead that $z \in x \rightarrow \neg Uz$. It follows by the definition of U and Axiom 1(i) that $z \in x \rightarrow Mz$. Since Mx then *my* $y \in x$ by the definition of M, whence $\exists y(My \wedge y \in x)$. Hence by Theorem 16(ii), $\exists y(E!V^*(y) \wedge y \in x)$, whence $\exists u \exists y(u = V^*(y) \wedge y \in x)$ by the definition of $V^*(y)$. Hence by Theorem 14, there is a unique level v_1 such that for some y, $v_1 = V^*(y) \wedge y \in x$, and $\neg \exists w(w \in v_1 \wedge \exists z(w = V^*(z) \wedge z \in x))$.
3. For a reductio suppose that $z_1 \in x \wedge z_1 \in y$ for some z_1. Since $z \in x \rightarrow Mz$, it follows that Mz_1, whence $E!V^*(z_1)$ by Theorem 16(ii). Since $y \subseteq V^*(y)$ by the definition of $V^*(y)$, it follows by the definition of $\subseteq$ that $z_1 \in V^*(y)$. We tackle three cases separately—(i) $V^*(y) = V_1$, (ii) $V^*(y) = V_2$ and (iii) $V^*(y) \neq V_1$ and $V^*(y) \neq V_2$—proving in each case that $V^*(z_1) \in V^*(y)$.
4. In case (i), $V^*(y) = V_1$. Then $E!V_1$ by the strength of identity, whence Uz_1 by Lemma 7(ii). Hence by the definition of U, $\neg Mz_1$. But also Mz_1. Hence $V^*(z_1) \in V^*(y)$ by the tautology $A \wedge \neg A \rightarrow B$.
5. In case (ii), $V^*(y) = V_2$. Hence $E!V_2$ by the strength of identity, whence $E!V_1$ by Lemma 7(vi). Since $z_1 \in V^*(y)$, it follows by Lemma 7(v) that $Uz_1 \vee z_1 \subseteq V_1$. Since Mz_1, it follows by the definition of U that $z_1 \subseteq V_1$. By the definitions of V and $V^*(z_1)$, $V(V_1)$ and $V(V^*(z_1))$,

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whence by Theorem 13, $V_1 \in V^*(z_1) \vee V_1 = V^*(z_1) \vee V^*(z_1) \in V_1$. By the definition of $V^*(z_1)$, $\neg\exists w(w \in V^*(z_1) \wedge z_1 \subseteq w)$. Since $V(V_1)$ and $z_1 \subseteq V_1$, it follows that $V_1 \notin V^*(z_1)$, whence $V_1 = V^*(z_1) \vee V^*(z_1) \in V_1$. For a reductio suppose $V^*(z_1) \in V_1$. Then by Lemma 7(ii) and the definition of U, $\neg M(V^*(z_1))$. But since $V(V^*(z_1))$, it follows by Corollary (i) of Theorem 1 that $M(V^*(z_1))$. Contradiction. Hence $V_1 = V^*(z_1)$, whence $V^*(z_1) \in V^*(y)$ by Lemma 7(vii).

6. In case (iii), $V^*(y) \neq V_1$ and $V^*(y) \neq V_2$. Hence $[w{:}w \in V^*(y)] = \{w{:}w \in V^*(y)\}$ by Theorem 9(iii). Since $z_1 \in V^*(y)$, it follows that $z_1 \in \text{acc}\,\{w{:}w \in V^*(y)\}$ by Theorem 10(ii). By the definition of acc, $\text{acc}\,\{w{:}w \in V^*(y)\} = \{x{:}(Ux \vee \exists y_1(y_1 \in \{w{:}w \in V^*(y)\} \wedge (x \in y_1 \vee x \subseteq y_1)))\}$, whence $Uz_1 \vee \exists y_1(y_1 \in \{w{:}w \in V^*(y)\} \wedge (z_1 \in y_1 \vee z_1 \subseteq y_1))$ by Lemma 4(i). Since Mz_1, it follows that $\neg Uz_1$ by the definition of U. Hence by Lemma 4(i), $z_1 \in v \vee z_1 \subseteq v$ for some level $v \in V^*(y)$, whence $z_1 \subseteq v$ by the Corollary of Theorem 7. By Theorem 13, $V^*(z_1) \in v \vee V^*(z_1) = v \vee v \in V^*(z_1)$. By the definition of $V^*(z_1)$, $\neg\exists w(w \in V^*(z_1) \wedge z_1 \subseteq w)$. Since $z_1 \subseteq v$, it follows that $v \notin V^*(z_1)$, whence $V^*(z_1) \in v \vee V^*(z_1) = v$. Suppose $V^*(z_1) \in v$. Then from $v \in V^*(y)$, it follows that $V^*(z_1) \in V^*(y)$ by Theorem 7. Suppose $V^*(z_1) = v$. Then from $v \in V^*(y)$, it again follows that $V^*(z_1) \in V^*(y)$.
7. We can now go on to the reductio begun in step 3. From $V^*(z_1) \in V^*(y)$ and $V^*(y) = v_1$ it follows that $V^*(z_1) \in v_1$. However, $V^*(z_1) \in v_1$ and $z_1 \in x$ are together contrary to $\neg\exists w(w \in v_1 \wedge \exists z(w = V^*(z) \wedge z \in x))$ by the definition of $V^*(z)$. Hence $\neg\, \exists z(z \in x \wedge z \in y)$, whence $x \cap y \equiv O$ by Lemma 2(i) and the definition of $\cap$. Hence $\exists y(y \in x \wedge x \cap y \equiv O)$.

We now turn to various operations, starting with a separation scheme, which follows Cantor's requirement that the separated members are many.

THEOREM 18. *Cantorian Separation*
Let Mx and $my(y \in x \wedge A(y))$, then $M\{y{:}y \in x \wedge A(y)\}$.

PROOF. By Theorem 16(i), $\exists u\; x \subseteq u$, whence $\exists u \forall y(y \in x \rightarrow y \in u)$ by the definition of $\subseteq$. Hence $\exists u \forall y((y \in x \wedge A(y)) \rightarrow y \in u)$, which together with $my(y \in x \wedge A(y))$ entails $M\{y{:}y \in x \wedge A(y)\}$ by Axiom 2(iii).

THEOREM 19. *Intersection* *Let $mz(z \in x \wedge z \in y)$, then $M(x \cap y)$.*

PROOF. Since $mz(z \in x \wedge z \in y)$, a fortiori $mz\; z \in x$, whence Mx by the definition of M. Hence by Theorem 18, $M\{z{:}\, z \in x \wedge z \in y\}$, whence by Lemma 3, $E!\{z{:}\, z \in x \wedge z \in y\}$. Hence by Lemma 2(iii), $[z{:}\, z \in x \wedge z \in y] = \{z{:}\, z \in x \wedge z \in y\}$, whence $x \cap y = \{z{:}\, z \in x \wedge z \in y\}$ by the definition of $\cap$. Hence $M(x \cap y)$ by Lemma 4(i).

We define the intersection of a set of sets a to be the thing that is either the sole common member of each member of a or the set of the common members of the members of a. In symbols, where Ma and $\forall y(y \in a \rightarrow My)$, $\cap a =_{df} [z{:}\forall y(y \in a \rightarrow z \in y)]$.

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THEOREM 20. *Generalized Intersection*

Let Mx *and* $\forall y(y \in x \rightarrow My)$ *and* $mz(\forall y(y \in x \rightarrow z \in y))$, *then* $M(\cap x)$.

PROOF.
1. Since by hypothesis Mx, it follows that $mz\, z \in x$ by the definition of M, whence $z_1 \in x$ for some z_1. Since by hypothesis $\forall y(y \in x \rightarrow My)$, it follows that Mz_1. Since $\forall y(y \in x \rightarrow z \in y) \rightarrow z \in z_1$ and $mz(\forall y(y \in x \rightarrow z \in y))$, it follows that $mz(z \in z_1 \wedge \forall y(y \in x \rightarrow z \in y))$. Hence by Theorem 18, $M\{z{:}z \in z_1 \wedge \forall y(y \in x \rightarrow z \in y)\}$, whence by Lemma 3, $E!\{z{:}z \in z_1 \wedge \forall y(y \in x \rightarrow z \in y)\}$.
2. Since $(z \in z_1 \wedge \forall y(y \in x \rightarrow z \in y)) \leftrightarrow \forall y(y \in x \rightarrow z \in y)$, it follows that $M\{z{:}\forall y(y \in x \rightarrow z \in y)\}$ and $E!\{z{:}\forall y(y \in x \rightarrow z \in y)\}$ by Lemma 1(i). Hence by Lemma 2(iii), $[z{:}\forall y(y \in x \rightarrow z \in y)] = \{z{:}\forall y(y \in x \rightarrow z \in y)\}$, whence $\cap x = \{z{:}\forall y(y \in x \rightarrow z \in y)\}$ by the definition of $\cap$. Hence $M(\cap x)$ by Lemma 4(i).

We define the union of sets a and b to be the set of those things that are each members of a or of b. In symbols, where Ma and Mb, $a \cup b =_{df} \{z{:}\ z \in a \vee z \in b\}$.

THEOREM 21. *Union* *Let* Mx *and* My, *then* $M(x \cup y)$.

PROOF.
1. With a view to using Axiom 2(iii) we shall prove (i) $mz(z \in x \vee z \in y)$ and (ii) $\exists u \forall z((z \in x \vee z \in y) \rightarrow z \in u)$.
2. For (i), since Mx, it follows that $mz\, z \in x$ by the definition of M; a fortiori $mz(z \in x \vee z \in y)$.
3. For (ii), since Mx and My, it follows by Theorem 16(i) that $x \subseteq v$ and $y \subseteq w$ for some levels v and w. By Theorem 13, $v \in w \vee v = w \vee w \in v$. Suppose $v \in$ w, then from $x \subseteq v$ it follows that $x \in w$ by Theorem 8, whence $x \subseteq w$ by the Corollary of Theorem 7. Since $y \subseteq w$ too, it follows that $\forall z((z \in x \vee z \in y) \rightarrow z \in w)$ by the definition of $\subseteq$, whence $\exists u \forall z((z \in x \vee z \in y) \rightarrow z \in u))$. Suppose $w \in v$, then by similar reasoning $x \subseteq v$ and $y \subseteq v$, whence $\exists u \forall z((z \in x \vee z \in y) \rightarrow z \in u))$. Suppose $v = w$, then again both $x \subseteq v$ and $y \subseteq v$, whence $\exists u \forall z((z \in x \vee z \in y) \rightarrow z \in u))$ by similar reasoning. In each case, then, $\exists u \forall z((z \in x \vee z \in y) \rightarrow z \in u))$.
4. From (i) and (ii), it follows that $M\{z{:}\ z \in x \vee z \in y\}$ by Axiom 2(iii), whence $M(x \cup y)$ by the definition of $x \cup y$.

Where a is a set of sets, we define the union of a to be the set of those things that are each members of some member of a. In symbols, where Ma and $\forall y(y \in a \rightarrow My)$, $\cup a =_{df} \{z{:}\ \exists y(y \in a \wedge z \in y)\}$.

THEOREM 22. *Generalized union* *Let* Mx *and* $\forall y(y \in x \rightarrow My)$, *then* $M(\cup x)$.

PROOF.
1. With a view to using Axiom 2(iii) we prove (i) $mz(\exists y(y \in x \wedge z \in y))$ and (ii) $\exists u \forall z(\exists y(y \in x \wedge z \in y) \rightarrow z \in u)$.
2. For (i), since Mx it follows that $mz\, z \in x$ by the definition of M, whence $z_1 \in x$ for some z_1. From the hypothesis $\forall y(y \in x \rightarrow My)$ it

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follows that Mz_1, so $mz\ z \in z_1$ by the definition of M. Since $\forall z(z \in z_1 \rightarrow \exists y(y \in x \wedge z \in y))$, it follows that $mz(\exists y(y \in x \wedge z \in y))$.

3. For (ii), since Mx it follows by Theorem 16(i) that $x \subseteq v$ for some level v. Hence $\forall z(\exists y(y \in x \wedge z \in y) \rightarrow \exists y(y \in v \wedge z \in y))$ by the definition of $\subseteq$, whence $\forall z(\exists y(y \in x \wedge z \in y) \rightarrow z \in v)$ by Theorem 7. Hence $\exists u \forall z(\exists y(y \in x \wedge z \in y) \rightarrow z \in u)$.
4. From (i) and (ii) it follows that $M\{z\colon \exists y(y \in x \wedge z \in y)\}$ by Axiom 2(iii), whence $M(\cup x)$ by the definition of $\cup x$.

Theorem 23. *Sets and levels II* *Let* Mx, *then* $\exists_1 v(x \in v \wedge \neg \exists w(w \in v \wedge x \in w))$.

This means that for any set there is a unique lowest level of which it is a member.

Proof. By Theorem 16(i) $x \subseteq u_1$ for some level u_1. By Axiom 2(iv) $u_1 \in u_2$ for some level u_2. Since $x \subseteq u_1$ and $u_1 \in u_2$, it follows that $x \in u_2$ by Theorem 8, whence $\exists u\ x \in u$. Hence $\exists_1 v(x \in v \wedge \neg \exists w(w \in v \wedge x \in w))$ by Theorem 14.

The putative pair set $\{a, b\}$ is defined as $\{z\colon z = a \vee z = b\}$.

Theorem 24. *Pairing* *Let* $E!x$ *and* $E!y$ *and* $x \neq y$, *then* $M\{x, y\}$.

Our pairs are *proper* pairs—they have two members. Hence the condition $x \neq y$.

Proof.

1. Since $E!x$ and $E!y$ and $x \neq y$, it follows that $mz(z = x \vee z = y)$. By Lemma 3, it follows from $E!x$ and $E!y$ that either (i) $Ux \wedge Uy$ or (ii) $Mx \wedge My$ or (iii) $Mx \wedge Uy$ or (iv) $Ux \wedge My$. With a view to using Axiom 2(iii) we prove $\exists u \forall z(z = x \vee z = y) \rightarrow z \in u)$ for each case.
2. For case (i), from $Ux \wedge Uy \wedge x \neq y$ it follows that $mz_1 Uz_1$. Hence by Axiom 2(i) $M\{z_1{:}Uz_1\}$, whence $V_1 = \{z_1{:}Uz_1\}$ by Lemma 7(iii). Since $(z = x \vee z = y) \rightarrow Uz$, it follows that $(z = x \vee z = y) \rightarrow z \in V_1$ by Lemma 4(i). So $\exists u \forall z(z = x \vee z = y) \rightarrow z \in u)$ by the definition of V.
3. For case (ii), from $Mx \wedge My$ it follows by Theorem 23 that $x \in v$ and $y \in w$ for some levels v, w. By Theorem 13, $v \in w \vee v = w \vee w \in v$. Suppose $v \in w$, then by Theorem 7 $x \in w$. Also $y \in w$, so $\exists u \forall z((z = x \vee z = y) \rightarrow z \in u)$. Suppose $v = w$, then $x \in v$ and $y \in v$, whence $\exists u \forall z((z = x \vee z = y) \rightarrow z \in u)$. Suppose $w \in v$, then by Theorem 7, $y \in v$. Also $x \in v$, so $\exists u \forall z((z = x \vee z = y) \rightarrow z \in u)$. From $v \in w \vee v = w \vee w \in v$, then, it follows that $\exists u \forall z((z = x \vee z = y) \rightarrow z \in u)$.
4. For case (iii), from Mx, it follows by Theorem 23 that $x \in v$ for some level v. By Corollary (ii) of Theorem 1 it follows from Uy that $y \in v$. Hence $\exists u \forall z((z = x \vee z = y) \rightarrow z \in u)$.
5. For case (iv), $\exists u \forall z(z = x \vee z = y) \rightarrow x \in u)$ is proved by the same reasoning as in step 4.

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6. Since $mz(z = x \vee z = y)$ and $\exists u \forall z(z = x \vee z = y) \rightarrow z \in u)$, it follows by Axiom 2(iii) that $M\{z\colon z = x \vee z = y\}$, whence $M\{x, y\}$ by the definition of $\{x, y\}$.

We define the power set of a as the set, if any, of the subsets of a. In symbols, $P(a) =_{df} \{y{:}y \subseteq a\}$.

THEOREM 25. *Power set* *Let my $y \subseteq x$, then $M(P(x))$.*

The hypothesis is necessary, since a pair set has only itself as a subset, and therefore has no power set.

PROOF. From the hypothesis $my\ y \subseteq x$ it follows that $z \subseteq x$ for some z, whence Mx by Lemma 6(i). Hence by Theorem 23, $x \in v$ for some level v, whence $y \subseteq x \rightarrow y \in v$ by Theorem 8. Hence $\exists u(\forall y(y \subseteq x \rightarrow y \in u)$. From $my\ y \subseteq x$ and $\exists u(\forall y(y \subseteq x \rightarrow y \in u)$, it follows that $M\{y{:}y \subseteq x\}$ by Axiom 2(iii), whence $M(P(x))$ by the definition of $P(x)$.

We define the power-plus set of a to be the set, if any, of the members and subsets of a. In symbols, $P{+}(a) =_{df} \{y\colon y \in a \vee y \subseteq a\}$.

THEOREM 26. *Power-plus set* *Let Mx, then $M(P{+}(x))$.*

PROOF. From the hypothesis Mx it follows that $my\ y \in x$ by the definition of M; a fortiori $my(y \in x \vee y \subseteq x)$. Also from Mx it follows that $x \in v$ for some level v by Theorem 23. So $y \in x \rightarrow y \in v$ by Theorem 7, and $y \subseteq x \rightarrow y \in v$ by Theorem 8. Hence $\exists u(\forall y((y \in x \vee y \subseteq x) \rightarrow y \in u))$. From $my(y \in x \vee y \subseteq x)$ and $\exists u(\forall y(y \in x \vee y \subseteq x) \rightarrow y \in u))$, it follows that $M\{y{:}y \in x \vee y \subseteq x\}$ by Axiom 2(iii), whence $M(P{+}(x))$ by the definition of $P{+}(x)$.

We now turn to further results about the structure and composition of levels.

THEOREM 27. *Membership and proper subset among levels*: $v \in w \leftrightarrow v \subset w$.

Since membership well-orders levels, and this theorem says that membership and proper subset are equivalent among levels, so proper subset well-orders levels too.

PROOF.
1. For the $\rightarrow$ half, suppose $v \in w$. By Corollary (i) of Theorem 1 Mv, whence $v \subseteq w$ by the Corollary of Theorem 7. By Axiom 1(iii) $v \notin v$, hence $v \neq w$. So $v \subset w$ by the definition of $\subset$.
2. For the $\leftarrow$ half, suppose $v \subset w$, then $v \subseteq w$ and $v \neq w$ by the definition of $\subset$, whence by Theorem 13 $v \in w \vee w \in v$. For a reductio suppose $w \in v$. Since $v \subseteq w$, it follows by the definition of $\subseteq$ that $w \in w$, contrary to Axiom 1(iii). Hence $w \notin v$, whence $v \in w$.

THEOREM 28. *The lowest level* (i) $mzUz \leftrightarrow V_1 = \imath v(\neg\exists w\ w \in v)$
(ii) $\exists x Vx \leftrightarrow E!V_1$

PROOF OF (i).
1. For the $\rightarrow$ half, suppose $mzUz$. Then by Axiom 2(i) $M\{z{:}Uz\}$. Hence $V_1 = \{z{:}Uz\}$ by Lemma 7(iii), whence $E!V_1$ by the strength of identity. Hence $\exists u\ u = u$ by the definition of V, whence $\exists_1 v(v = v \wedge \neg\exists w(w \in v$

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$\wedge\, w = w))$ by Theorem 14. Hence $E!\iota v(\neg\exists w\, w \in v)$. By Theorem 9(i), $\neg\exists w\, w \in V_1$. Hence $V_1 = \iota v(\neg \exists w\, w \in v)$.

2. For the $\leftarrow$ half, suppose $V_1 = \iota v(\neg \exists w\, w \in v)$. Then $E!V_1$ by the strength of identity, So by Lemma 7(ii), $\exists x Mx$. So by Axiom 2(ii) $mzUz$.

Proof of (ii).

1. For the $\rightarrow$ half, suppose $\exists x Vx$. Then $\exists x Mx$ by Corollary (i) of Theorem 1, whence $mzUz$ by Axiom 2(ii). Hence $E!V_1$ by Theorem 28(i) and the strength of identity.
2. For the $\leftarrow$ half, suppose $E!V_1$. Then $\exists x Vx$ by the definition of V.

Theorem 29. *Levels next above I* $\qquad E!u \leftrightarrow E!u'$

Proof.

1. For the $\rightarrow$ half, suppose $E!u$. By Axiom 2(iv) $\exists u_1\, u \in u_1$. Hence by Theorem 14 $\exists_1 v(u \in v \wedge \neg\exists w(w \in v \wedge u \in w))$, whence $E!\iota v(u \in v \wedge \neg\exists w(w \in v \wedge u \in w))$. Hence $E!u'$ by the definition of u'.
2. For the $\leftarrow$ half, suppose $E!u'$, then $u \in u'$ by the definition of u'. Hence $E!u$ by Axiom 1(i).

Theorem 30. *Levels next above II* $\quad$ (i) $u' = \{x{:}Ux \vee x \subseteq u\}$
(ii) $u' = P{+}(u)$

Proof of (i). By Corollary (i) of Theorem 1, Mu. Hence $E!u$ by Lemma 3, whence $E!u'$ by Theorem 29. By the definition of u', $u \in u'$ and Vu'. So Mu' by Corollary (i) of Theorem 1. We tackle three cases separately: (i) $u' = V_1$, (ii) $u' = V_2$ and (ii) $u' \neq V_1$ and $u' \neq V_2$.

Case (i) $u' = V_1$

Since $u \in u'$, it follows that $u \in V_1$. But $u \notin V_1$ by Theorem 9(i). Hence $u' = \{x{:}Ux \vee x \subseteq u\}$ by the tautology $A \wedge \neg A \rightarrow B$.

Case (ii) $u' = V_2$

By the strength of identity $E!V_2$. Hence $\iota w(w \in V_2) = V_1$ by Theorem 9(ii). Since $u \in V_2$, it follows that $u = V_1$, whence $(Ux \vee x \subseteq V_1) \leftrightarrow (Ux \vee x \subseteq u)$. By Lemma 7(iv) $u' = V_2 = \{x{:}Ux \vee x \subseteq V_1\}$, whence by Lemma 1(i) $u' = \{x{:}Ux \vee x \subseteq u\}$.

Case (iii) $u' \neq V_1$ and $u' \neq V_2$

1. We first prove $w \subseteq u \leftrightarrow w \in u'$. For the $\rightarrow$ half, suppose $w \subseteq u$. From $w \subseteq u$ and $u \in u'$ it follows by Theorem 8 that $w \in u'$. For the $\leftarrow$ half, suppose $w \in u'$, then $u \notin w$ by the definition of u'. Hence $w \in u \vee w = u$ by Theorem 13. By Corollary (i) of Theorem 1, Mw. Suppose $w \in u$, then $w \subseteq u$ by the Corollary of Theorem 7. Suppose $w = u$, then $w \subseteq u$ by Lemma 6(ii).
2. Since $E!u'$ and $u' \neq V_1$ and $u' \neq V_2$, it follows by Theorem 10(ii) that $u' = \text{acc}\,[w{:}w \in u']$. whence $u' = \text{acc}\,\{w{:}w \in u'\}$ by Theorem 9(iii). By the definition of acc, $\text{acc}\,\{w{:}w \in u'\} = \{x{:}(Ux \vee \exists y(y \in \{w{:}w \in u'\} \wedge (x \in y \vee x \subseteq y)))\}$. Hence $x \in u' \leftrightarrow (Ux \vee \exists w_1(w_1 \in u' \wedge (x \in w_1 \vee x \subseteq w_1)))$ by Lemma 4(i). We next prove $x \in u' \leftrightarrow (Ux \vee x \subseteq u)$.

3. For the $\rightarrow$ half, suppose $x \in u'$, then $E!x$ by Axiom 1(i), whence $Ux \vee Mx$ by Lemma 3. Suppose Ux, a fortiori $Ux \vee \exists w_1(w_1 \in u' \wedge x \subseteq w_1)$. Suppose instead Mx. Since $x \in u'$, it follows that $Ux \vee \exists w_1(w_1 \in u' \wedge (x \in w_1 \vee x \subseteq w_1))$, whence $Ux \vee \exists w_1(w_1 \in u' \wedge x \subseteq w_1)$ by the Corollary of Theorem 7. Since $w \subseteq u \leftrightarrow w \in u'$, it follows that $Ux \vee \exists w_1(w_1 \subseteq u \wedge x \subseteq w_1)$. So $Ux \vee x \subseteq u$ by the definition of $\subseteq$.
4. For the $\leftarrow$ half, suppose Ux, then $x \in u'$. Suppose instead that $x \subseteq u$, then $u \subseteq u$ by Lemma 6(ii), whence $Ux \vee \exists w_1(w_1 \subseteq u \wedge x \subseteq w_1)$. Since $w \subseteq u \leftrightarrow w \in u'$, it follows that $Ux \vee \exists w_1(w_1 \in u' \wedge x \subseteq w_1)$, whence $x \in u'$.
5. Since $x \in u' \leftrightarrow (Ux \vee x \subseteq u)$, it follows by Lemma 1(i) that $\{x{:}x \in u'\} \equiv \{x{:}Ux \vee x \subseteq u\}$. Since Mu', it follows by Lemma 5(ii) that $u' = \{x{:}x \in u'\}$, whence $u' = \{x{:}Ux \vee x \subseteq u\}$.

Proof of (ii).

1. By Corollary (i) of Theorem 1, Mu. Hence by Theorem 26 $M(P{+}(u))$, whence $E!P{+}(u)$ by Lemma 3. So $P{+}(u) = \{x{:}x \in u \vee x \subseteq u\}$ by the definition of $P{+}(u)$, whence $x \in P{+}(u) \leftrightarrow (x \in u \vee x \subseteq u)$ by Lemma 4(i). We shall prove $x \in P{+}(u) \leftrightarrow (Ux \vee x \subseteq u)$.
2. For the $\rightarrow$ half, suppose $x \in P{+}(u)$, then $x \in u \vee x \subseteq u$. Suppose $x \in u$, then by Axiom 1(i) $E!x$, whence by Lemma 3 $Ux \vee Mx$. Suppose Ux, a fortiori $Ux \vee x \subseteq u$. Suppose Mx, then by the Corollary of Theorem 7 $x \subseteq u$, a fortiori $Ux \vee x \subseteq u$. Suppose instead that $x \subseteq u$, then again $Ux \vee x \subseteq u$.
3. For the $\leftarrow$ half, suppose Ux, then by Corollary (ii) of Theorem 1, $x \in u$, a fortiori $x \in u \vee x \subseteq u$. Hence $x \in P{+}(u)$. Suppose instead that $x \subseteq u$, a fortiori $x \in u \vee x \subseteq u$, whence $x \in P{+}(u)$.
4. Since $x \in P{+}(u) \leftrightarrow (Ux \vee x \subseteq u)$, it follows by Lemma 1(i) that $\{x{:}x \in P{+}(u)\} \equiv \{x{:}Ux \vee x \subseteq u\}$. Since $M(P{+}(u))$, it follows by Lemma 5(ii) that $P{+}(u) = \{x{:}x \in P{+}(u)\}$, whence $P{+}(u) = \{x{:}Ux \vee x \subseteq u\}$, and so $u' = P{+}(u)$ by Theorem 30(i).

Theorem 31. *Sets and levels III*
Let Mx, then $(V^*(x))' = \iota v(x \in v \wedge \neg\exists w(w \in v \wedge x \in w))$.

This means that the level next above the level of a set is the lowest level of which the set is a member.

Proof.

1. Since by hypothesis Mx, it follows that $\neg Ux$ by the definition of U. By Theorem 16(ii) $E!V^*(x)$. Also $V(V^*(x))$ by the definition of $V^*(x)$, whence by Theorem 29 $E!(V^*(x))'$. By Theorem 30(i) $(V^*(x))' = \{x{:}Ux \vee x \subseteq V^*(x)\}$, so by Lemma 4(i) $x \subseteq V^*(x) \rightarrow x \in (V^*(x))'$. Since $x \subseteq V^*(x)$ by the definition of $V^*(x)$, then $x \in (V^*(x))'$.
2. For a reductio suppose that $x \in w$ for some $w \in (V^*(x))'$. By the definition of $(V^*(x))'$ we have that $\neg\exists w_1(w_1 \in (V^*(x))' \wedge V^*(x) \in w_1)$. Hence $V^*(x) \notin w$, whence by Theorem 13 $V^*(x) = w \vee w \in V^*(x)$. Suppose $V^*(x) = w$, then $x \in V^*(x)$. Now suppose instead that $w \in V^*(x)$, then $x \in V^*(x)$ by Theorem 7. We tackle three cases

separately, deducing a contradiction in each case: (i) $V^*(x) = V_1$, (ii) $V^*(x) = V_2$, and (iii) $V^*(x) \neq V_1$ and $V^*(x) \neq V_2$.

3. For case (i), by Lemma 7(ii), $x \in V^*(x) \leftrightarrow Ux$. Since $x \in V^*(x)$, it follows that Ux. Contradiction.
4. For case (ii), by Lemma 7(vi) and the definition of V, $V(V_1)$. Also by Lemma 7(v), $x \in V^*(x) \leftrightarrow (Ux \vee x \subseteq V_1)$. Since $x \in V^*(x)$ and $\neg\, Ux$, it follows that $x \subseteq V_1$. Since by Lemma 7(vii) $V_1 \in V_2$, it also follows that $V_1 \in V^*(x)$. But by the definition of $V^*(x)$, $\neg\, \exists w_1(w_1 \in V^*(x) \wedge x \subseteq w_1)$. Contradiction.
5. For case (iii), by Theorem 9(iii), $[w{:}w \in V^*(x)] = \{w{:}w \in V^*(x)\}$, whence by Theorem 10(ii), $V^*(x) = \text{acc}\ \{w{:}w \in V^*(x)\}$. By its definition, $\text{acc}\ \{w{:}w \in V^*(x)\} = \{z{:}(Uz \vee \exists y(y \in \{w{:}w \in V^*(x)\} \wedge (z \in y \vee z \subseteq y)))\}$. Since $x \in V^*(x)$, it now follows that $Ux \vee \exists y(y \in \{w{:}w \in V^*(x)\} \wedge (x \in y \vee x \subseteq y))$ by Lemma 4(i). Since $\neg Ux$, it now follows that $\exists y(y \in \{w{:}w \in V^*(x)\} \wedge (x \in y \vee x \subseteq y))$, whence $\exists w_1(w_1 \in V^*(x) \wedge (x \in w_1 \vee x \subseteq w_1))$ by Lemma 4(i). Since Mx, it follows by the Corollary of Theorem 7 that $\exists w_1(w_1 \in V^*(x) \wedge x \subseteq w_1)$. But by the definition of $V^*(x)$, $\neg\exists w_1(w_1 \in V^*(x) \wedge x \subseteq w_1)$. Contradiction.
6. Since each case is contradictory, it follows that $\neg\exists w\ (w \in (V^*(x))' \wedge x \in w)$. Then by Theorem 23 $\exists_1 v(x \in v \wedge \neg\exists w(w \in v \wedge x \in w))$, whence $E!\iota v(x \in v \wedge \neg\exists w(w \in v \wedge x \in w))$. Since $x \in (V^*(x))'$ and $\neg\, \exists w(w \in (V^*(x))' \wedge x \in w)$, it follows finally that $(V^*(x))' = \iota v(x \in v \wedge \neg\exists w(w \in v \wedge x \in w))$.

Theorem 32. *Levels next above III* *Let $mx\ x \subseteq u$, then $P{+}(u) = V_1 \cup P(u)$.*

Theorems 30(ii) and 32 jointly entail that $u' = V_1 \cup P(u)$, provided u has many subsets. The condition is necessary, since when there are exactly two ur-elements, V_1 has itself as its only subset, and hence its power set does not exist.

Proof.

1. Since $mx\ x \subseteq u$, it follows that $M(P(u))$ by Theorem 25, whence by Lemma 3 $E!P(u)$. Hence $P(u) = \{y{:}y \subseteq u\}$ by the definition of $P(u)$, whence by Lemma 4(i) $z \in P(u) \leftrightarrow z \subseteq u$. By Corollary (i) of Theorem 1 Mu, and so by Theorem 26, $M(P{+}(u))$. Hence by Lemma 7(iii) $V_1 = \{z{:}Uz\}$, whence MV_1 and $z \in V_1 \leftrightarrow Uz$ by Lemma 4(i). By Theorems 30(i) and (ii) and Lemma 4(i), $z \in P{+}(u) \leftrightarrow (Uz \vee z \subseteq u)$, whence $z \in P{+}(u) \leftrightarrow (z \in V_1 \vee z \in P(u))$.
2. Since MV_1 and $M(P(u))$, it follows by Theorem 21 that $M(V_1 \cup P(u))$, whence $E!(V_1 \cup P(u))$ by Lemma 3. By the definition of $\cup$, $V_1 \cup P(u)) = \{z{:}\, z \in V_1 \vee z \in P(u)\}$. Hence $z \in (V_1 \cup P(u)) \leftrightarrow (z \in V_1 \vee z \in P(u))$ by Lemma 4(i). Since $z \in P{+}(u) \leftrightarrow (z \in V_1 \vee z \in P(u))$, it follows that $z \in P{+}(u) \leftrightarrow z \in (V_1 \cup P(u))$.
3. Since $M(P{+}(u))$ and $M(V_1 \cup P(u))$ and $z \in P{+}(u) \leftrightarrow z \in (V_1 \cup P(u))$, it follows by Axiom 1(ii) that $P{+}(u) = V_1 \cup P(u)$.

Theorem 33. *Limit levels I* $\exists uLu \leftrightarrow mx\, x = x$

Proof.

1. For the $\rightarrow$ half, suppose $\exists uLu$. Then $\exists xMx$ by Corollary (i) of Theorem 1. Hence $mxUx$ by Axiom 2(ii), whence $mx\, x = x$.
2. For the $\leftarrow$ half, suppose $mx\, x = x$. If $\neg\exists xMx$ then $\forall xUx$ by Lemma 3, whence $mxUx$. If $\exists xMx$ then $mxUx$ by Axiom 2(ii). Hence either way $mxUx$, whence by Axiom 2(i) $M\{x{:}Ux\}$. Hence $E!V_1$ by Lemma 7(iii) and the strength of identity. Hence $\exists xVx$ by the definition of V, whence $\exists uLu$ by Axiom 2(v) and the definition of L.

Theorem 34. *Limit levels II* *Let* Lu, *then* $u = \cup\{w{:}w \in u\}$.

A limit level is the union of its history.

Proof.

1. By Corollary (i) of Theorem 1, Mu, whence $E!u$ by Lemma 3. Since Lu, it follows that $u \neq V_1$ by the definition of L. For a reductio suppose $u = V_2$. Then $E!V_2$ by the strength of identity, whence $V_2 = \{z{:}Uz \vee z \subseteq V_1\}$ and $E!V_1$ by Lemmas 7(iv) and (vi). By Theorem 30(i), $V_1' = \{z{:}Uz \vee z \subseteq V_1\}$, whence $V_2 = V_1'$. Hence by the definition of V, $\exists u_1 V_2 = u_1'$, whence $\neg LV_2$ by the definition of L. Contradiction. Hence $u \neq V_2$. Since $E!u$ and $u \neq V_1$ and $u \neq V_2$, it follows by Theorem 9(iii) that $mw\ w \in u$ and $[w{:}w \in u] = \{w{:}w \in u\}$. We shall prove $z \in u \leftrightarrow \exists w(w \in u \wedge z \in w)$.
2. For the $\rightarrow$ half, suppose $z \in u$. By Axiom 1(i) and Lemma 3 $Uz \vee Mz$. Suppose Uz. Then $\forall v\ z \in v$ by Corollary (ii) of Theorem 1. Since $\exists w\ w \in u$, it follows that $\exists w(w \in u \wedge z \in w)$. Suppose instead that Mz. By Theorem 16(ii) $E!V^*(z)$, whence $V(V^*(z))$ by the definition of $V^*(z)$. Hence $E!(V^*(z))'$ by Theorem 29, and then $\exists u_1\,(V^*(z))' = u_1'$ by the definition of $(V^*(z))'$. So $u \neq (V^*(z))'$ by the definition of L, whence by Theorem 13 $u \in (V^*(z))' \vee (V^*(z))' \in u$. For a reductio suppose $u \in (V^*(z))'$. Since $z \in u$, it follows that $\exists w(w \in (V^*(z))' \wedge z \in w))$. But $(V^*(z))' = \imath v(z \in v \wedge \neg\,\exists w(w \in v \wedge z \in w))$ by Theorem 31. Contradiction. Hence $(V^*(z))' \in u$, whence $\exists w(w \in u \wedge z \in w)$.
3. For the $\leftarrow$ half, suppose $\exists w(w \in u \wedge z \in w)$. Then $z \in u$ by Theorem 7.
4. Since $[w{:}w \in u] = \{w{:}w \in u\}$, it follows that $M\{w{:}w \in u\}$ by Lemma 4(i). Also by Lemma 4(i), $\forall y(y \in\{w{:}w \in u\} \rightarrow Vy)$, whence $\forall y(y \in\{w{:}w \in u\} \rightarrow My)$ by Corollary (i) of Theorem 1. Hence $M(\cup\{w{:}w \in u\})$ by Theorem 22. Hence $E!\cup\{w{:}w \in u\}$ by Lemma 3, whence $\cup\{w{:}w \in u\} = \{z{:}\ \exists y(y \in \{w{:}w \in u\} \wedge z \in y)\}$ by the definition of $\cup\{w{:}w \in u\}$. Hence $z \in \cup\{w{:}w \in u\} \leftrightarrow \exists y(y \in\{w{:}w \in u\} \wedge z \in y)$ by Lemma 4(i), so $z \in \cup\{w{:}w \in u\} \leftrightarrow \exists w(w \in u \wedge z \in w)$ by Lemma 4(i) again. Since $z \in u \leftrightarrow \exists w(w \in u \wedge z \in w)$, it follows that $z \in u \leftrightarrow z \in \cup\{w{:}w \in u\}$. Since Mu and $M(\cup\{w{:}w \in u\})$ and $z \in u \leftrightarrow z \in \cup\{w{:}w \in u\}$, it follows by Axiom 1(ii) that $u = \cup\{w{:}w \in u\}$.

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We define V_ω to be the lowest limit level, or in symbols, $V_\omega =_{df} \imath v(Lv \wedge \neg\exists w(w \in v \wedge Lw))$.

THEOREM 35. *The lowest limit level* $E!V_\omega \leftrightarrow \exists uLu$

PROOF.

1. For the $\rightarrow$ half, suppose $E!V_\omega$. Then $\exists uLu$ by the definition of V_ω.
2. For the $\leftarrow$ half, suppose $\exists uLu$. Then $\exists_1 v(Lv \wedge \neg\exists w(w \in v \wedge Lw))$ by Theorem 14. Hence $E!\imath v(Lv \wedge \neg\exists w(w \in v \wedge Lw))$, whence $E!V_\omega$ by the definition of V_ω.

We define a to be *inductive* if (i) V_1 belongs to a and (ii) the level next above any level that belongs to a also belongs to a. In symbols, $Ia =_{df} V_1 \in a \wedge \forall v(v \in a \rightarrow v' \in a)$.

THEOREM 36. *V_ω is inductive* *Let $E!V_\omega$, then $I(V_\omega)$.*

PROOF.

1. From the hypothesis $E!V_\omega$ it follows by the definition of V_ω that $V(V_\omega)$ whence MV_ω by Corollary (i) of Theorem 1. Hence $E!V_1$ by Lemma 7(iii) and the strength of identity, whence $V(V_1)$ by the definition of V.
2. By Theorem 13, $V_1 \in V_\omega \vee V_1 = V_\omega \vee V_\omega \in V_1$. But $V_1 \neq V_\omega$ by the definitions of L and V_ω, and $V_\omega \notin V_1$ by Theorem 9(i). Hence $V_1 \in V_\omega$.
3. Consider an arbitrary $u \in V_\omega$. Then $E!u$ by Axiom 1(i). Hence $E!u'$ by Theorem 29, so Vu' by the definition of u'. Hence $u' \in V_\omega \vee u' = V_\omega \vee V_\omega \in u'$ by Theorem 13. Since $\exists x\ u' = x'$ it follows that $u' \neq V_\omega$ by the definitions of L and V_ω. Also $V_\omega \notin u'$ by the definition of u'. Hence $u' \in V_\omega$. Since u was arbitrary we can generalize to get $\forall v(v \in V_\omega \rightarrow v' \in V_\omega)$. Hence $I(V_\omega)$ by the definition of I.

We define $\mathbf{N}^*$ to be the set of the members common to every inductive set. In symbols, $\mathbf{N}^* =_{df} \{x: \forall y(Iy \rightarrow x \in y)\}$. $\mathbf{N}^*$ represents the set of natural numbers.

THEOREM 37. *$\mathbf{N}^*$ is inductive* *Let $mx\ x = x$, then $I(\mathbf{N}^*)$.*

PROOF.

1. Since by hypothesis $mx\ x = x$ it follows that $E!V_\omega$ by Theorems 33 and 35, whence $M(V_\omega)$ by the definition of V_ω and Corollary (i) of Theorem 1, and also $I(V_\omega)$ by Theorem 36. By the definition of I, $Iy \rightarrow (V_1 \in y \wedge V_1' \in y)$, whence $V_1 \in V_\omega \wedge V_1' \in V_\omega$. Hence $E!V_1$ and $E!V_1'$ by Axiom 1(i). By the reasoning in step 1 of the proof of Theorem 34, $V_1' = V_2$. Hence $V_1 \neq V_1'$ by Lemma 7(viii), whence $mx(x \in V_\omega \wedge \forall y(Iy \rightarrow x \in y))$.
2. Since $M(V_\omega)$ and $mx(x \in V_\omega \wedge \forall y(Iy \rightarrow x \in y))$, it follows that $M\{x: x \in V_\omega \wedge \forall y(Iy \rightarrow x \in y)\}$ by Theorem 18. Since $I(V_\omega)$ it follows that $(x \in V_\omega \wedge \forall y(Iy \rightarrow x \in y)) \leftrightarrow \forall y(Iy \rightarrow x \in y)$, whence

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$M\{x{:}\forall y(Iy \to x \in y)\}$ by Lemma 1(i). Hence $E!\mathbf{N}^*$ by Lemma 3 and the definition of $\mathbf{N}^*$, whence $\mathbf{N}^*=\{x{:}\forall y(Iy \to x \in y)\}$.

3. By the definition of I it follows that $\forall y(Iy \to V_1 \in y)$, whence $V_1 \in \mathbf{N}^*$ by Lemma 4(i). Consider an arbitrary $u \in \mathbf{N}^*$. Then $\forall y(Iy \to u \in y)$ by Lemma 4(i). By the definition of I it follows that $\forall y(Iy \to \forall v(v \in y \to v' \in y))$. Hence $\forall y(Iy \to u' \in y)$, whence $u' \in \mathbf{N}^*$ by Lemma 4(i). Since u was arbitrary we can generalize to get $\forall v(v \in \mathbf{N}^* \to v' \in \mathbf{N}^*)$. Hence $I(\mathbf{N}^*)$ by the definition of I.

We define $[a, b]$ as $[z{:}z = a \vee z = b]$ and use this shorthand in the following definition of the ordered pair

$$<a, b> =_{df} \{\{\{[a, V_1], [a, V_2]\}, V_1\}, \{\{[b, V_1], [b, V_2]\}, V_2\}\}$$

Theorem 38. *Ordered pairs*
Let $mx\ x = x$, *then* (*i*) $E!<x, y>$ *and also* (*ii*) $<x, y> = <w, z> \leftrightarrow (x \equiv w \wedge y \equiv z)$.

(i) ensures the existence of ordered pairs, while (ii) says that they have their so-called characteristic property. Here we supply sketches for the interested reader to develop into full-dress proofs.

The proof of (i) is by repeated application of pairing (Theorem 24), having established on each occasion that the members of the next putative pair exist and are distinct. At the start it is shown that $E![x, V_1]$ and $E![x, V_2]$ and $[x, V_1] \neq [x, V_2]$, and similarly for y. Four cases for each of x and y need to be tackled here, which between them exhaust the possibilities: zilch, V_1, V_2, anything else.

By definition an ordered pair $<a, b>$ is of the form $\{a^*, b^*\}$, where a^* codes coordinate a, and b^* codes b. The markers V_1 and V_2 serve to distinguish the two. The proof of the $\to$ half of (ii) proceeds by showing that different coordinates have different codes, i.e. $a^* = b^* \to a \equiv b$. Four cases for $a^* = b^*$ need to be tackled here, which correspond to the four possibilities for a and b: zilch, V_1, V_2, anything else. Supposing $<x, y> = <w, z>$, it follows that $x^* \neq y^*$ and $w^* \neq z^*$ which in turn entail $x^* = w^*$ and $y^* = z^*$. Since $a^* = b^* \to a \equiv b$, it follows that $x \equiv w \wedge y \equiv z$. The proof of the $\leftarrow$ half proceeds by showing that different items code different coordinates, i.e. $a \equiv b \to a^* = b^*$. Supposing $x \equiv w \wedge y \equiv z$, it follows that $\{x^*, y^*\} \equiv \{w^*, z^*\}$, i.e. $<x, y> \equiv <w, z>$, so $<x, y>=<w, z>$ by (i).

Theorem 39. *Nonexistence*
(i) $\neg E!\{x{:}E!x\}$
(ii) $\neg E!\{x{:}Mx\}$
(iii) $\neg E!\{x{:}Vx\}$
(iv) $\neg E!\{x{:}\mathcal{A}x\}$
(v) $\neg E!\{x{:}Hx\}$

This theorem means that there are no sets corresponding to the predicates $E!$, M, V, $\mathcal{A}$ and H. By Corollary (ii) of Theorem 1, every ur-element is a member of every level, and by Theorem 23, every set is a member of some

https://doi.org/10.1017/bsl.2018.10 Published online by Cambridge University Press

level. But it follows from (i) that there can be no all-encompassing level. A level is, in Scott's phrase, never more than a partial universe.

Proof of (i). For a reductio suppose $E!\{x{:}E!x\}$. Then $\{x{:}E!x\} = \{x{:}E!x\}$, whence by Lemma 4(i) $\{x{:}E!x\} \in \{x{:}E!x\}$, contrary to Axiom 1(iii). Hence $\neg E!\{x{:}E!x\}$.

Proof of (ii). For a reductio suppose $E!\{x{:}Mx\}$. Then $\{x{:}Mx\} = \{x{:}Mx\}$, whence $M\{x{:}Mx\}$ by Lemma 4(i). Hence also by Lemma 4(i) $\{x{:}Mx\} \in \{x{:}Mx\}$, contrary to Axiom 1(iii). Hence $\neg E!\{x{:}Mx\}$.

Proof of (iii). For a reductio suppose $E!\{x{:}Vx\}$. Then $\{x{:}Vx\} = \{x{:}Vx\}$, whence $M\{x{:}Vx\}$ by Lemma 4(i), whence by Theorem 16(i) $\{x{:}Vx\} \subseteq v$ for some level v. Hence by the definition of $\subseteq$, $y \in \{x{:}Vx\} \rightarrow y \in v$, whence by Lemma 4(i) $v \in v$, contrary to Axiom 1(iii). Hence $\neg E!\{x{:}Vx\}$.

Proof of (iv).
1. For a reductio suppose $E!\{x{:}\mathcal{A}x\}$. Then $\{x{:}\mathcal{A}x\} = \{x{:}\mathcal{A}x\}$, whence $M\{x{:}\mathcal{A}x\}$ by Lemma 4(i). Hence by Theorem 16(i), $\{x{:}\mathcal{A}x\} \subseteq v$ for some level v. By Axiom 2(iv) it follows that $v \in w$ for some level w, whence by Axiom 1(iii), $v \neq w$. Hence $mxVx$.
2. Since $Vx \rightarrow \mathcal{A}x$ by Theorem 1(iii), it follows that $mx(\mathcal{A}x \wedge Vx)$, whence by Theorem 18, $M\{x{:}\mathcal{A}x \wedge Vx\}$.
3. Since $Vx \rightarrow \mathcal{A}x$, it follows that $\mathcal{A}x \wedge Vx \leftrightarrow Vx$, whence by Lemma 1(i), $M\{x{:}Vx\}$. Hence $E!\{x{:}Vx\}$ by Lemma 3, contrary to Theorem 39(iii). Hence $\neg E!\{x{:}\mathcal{A}x\}$.

Proof of (v).
1. For a reductio suppose $E!\{x{:}Hx\}$. Then $\{x{:}Hx\} = \{x{:}Hx\}$. Hence $M\{x{:}Hx\}$ by Lemma 4(i), whence by Theorem 16(i) $\{x{:}Hx\} \subseteq v$, for some level v. Hence by the definition of $\subseteq$, $y \in \{x{:}Hx\} \rightarrow y \in v$, whence by Lemma 4(i) $Hy \rightarrow y \in v$.
2. Consider an arbitrary level u. By Axiom 2(iv), $u \in v$ and $v \in w$ for some levels v, w. Hence $u \in w$ by Theorem 7. By Axiom 1(iii), $u \neq v$, whence $mw_1(w_1 \in w)$. Hence by Theorems 9(i), (ii), (iii), $[w_1{:}w_1 \in w] = \{w_1{:}w_1 \in w\}$, whence by Lemma 4(i), $u \in [w_1{:}w_1 \in w]$ and by the strength of identity $E![w_1{:}w_1 \in w]$. By Theorem 11, $H[w_1{:}w_1 \in w]$. Hence $\exists y(Hy \wedge u \in y)$. Since u was arbitrary, we can generalize to get $\forall u_1 \exists y(Hy \wedge u_1 \in y)$.
3. Since $Hy \rightarrow y \in v$, it follows that $\forall u_1 \exists y(y \in v \wedge u_1 \in y)$, whence $\forall u_1\ u_1 \in v$ by Theorem 7. Hence $v \in v$, contrary to Axiom 1(iii), whence $\neg E!\{x{:}Hx\}$.

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https://doi.org/10.1017/bsl.2018.10 Published online by Cambridge University Press