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AUTHORS' REJOINDER

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First of all, we express our sincere thanks to all the discussants for their valuable comments and suggestions as well as for their own significant contributions to the area of order statistics in general, and to the topic of stochastic comparison in particular. We shall now provide our response to the comments and suggestions of all the discussants.

Moshe Shaked: We thank Professor Shaked for his careful reading of our paper and for his insightful comments and overall assessment of developments in the topic of stochastic comparison of order statistics from heterogeneous populations.

We do agree with his comment that a stochastic comparison of order statistics from a collection of independent heterogeneous exponential random variables and a stochastic comparison of order statistics from a collection of independent heterogeneous random variables with proportional hazard rates (PHR), are, in some respect, essentially the same comparison. To conclude this, Professor Shaked has described a unified method in detail to this kind of problem using which many comparison results under PHR setup, for instance, Theorems 2.4, 2.6, 2.11, 2.13, 2.15, and 2.16, can all be derived immediately from the corresponding results under the exponential setup.

In his discussion, Professor Shaked has provided some other references pertaining to comparisons of order statistics and also of some statistics closely related to order statistics based on samples of heterogeneous exponential random variables, that forms a nice addition to the present discussion. It is important to mention here that, due to limitation of space, we limit our discussion to sample range and not deal with general spacings in our review paper. In fact, Kochar and Xu [3] have provided a good review of the recent developments on stochastic comparisons of sample spacings when the observations are not necessarily identically distributed.

As aptly mentioned by Professor Shaked in his discussion, another interesting topic that deserves a similar study is the stochastic comparison of convolutions of independent heterogeneous random variables. There are many papers on this topic in the literature and Professor Shaked has mentioned some of them. In this regard, Zhao and Li [8] have provided

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a review on this topic entitled "Ordering properties of convolutions from heterogeneous populations: A review on some recent developments" which will soon appear in *Communications in Statistics – Theory and Methods*.

To conclude, we thank Professor Shaked for pointing out some typographical errors in the earlier version of this paper and we have corrected all of them in the present version. Also, we have replaced the reference of Marshall and Olkin [4] with the newer edition by Marshall, Olkin, and Arnold [5] as suggested.

Maochao Xu: Professor Xu has mainly focused his attention on the skewness of order statistics and order statistics from heterogeneous populations with different shape parameters. Firstly, he has described the significance of studying the skewness of order statistics from two different aspects: detecting the heterogeneity in the population, and unifying and simplifying the study of stochastic comparisons. Next, he has focused on the topic of order statistics from populations with different shapes. There does exist a lot of literature on stochastic comparisons of order statistics from heterogeneous populations for the case when the populations have different scale parameters but common shape parameter. As mentioned by Professor Xu, order statistics from distributions with regular variation tails have close relations with tail indices α's, which are just shape parameters for many distributions and so the research on order statistics from heterogeneous populations with different shape parameters may also be of interest in this connection. We completely agree with Professor Xu's comment that this line of research has not received much attention and more work is needed. As a matter of fact, we are currently working on this problem and hope to report our findings in the near future.

Xiaohu Li and Yinping You: Our sincere thanks go to Professors Xiaohu Li and Yinping You for their insightful comments. We are indeed delighted to see that they have brought up the issue of dependence among individual observations as we believe that this will breathe new life into this line of research. Incorporating their own research work in the areas of network security, reliability, etc., Professors Li and You have provided a good discussion on heterogeneous observations with some dependence structures and the discussion has mainly focused on the [multivariate] usual stochastic order.

Similar to the result of multivariate stochastic order established earlier in Proschan and Sethuraman [7], Professors Li and You have established two new results under the assumption of two interesting structures of dependence that have a close relationship with the well-known Marshall–Olkin structure of dependence; it is evident that both these dependence structures have interesting applications in network design. To the best of our knowledge, only limited results are available in this direction and this interesting topic certainly deserves more attention.

Baha-Eldin Khaledi: We express our sincere thanks to Professor Khaledi for his overall assessment of the developments on this topic and for providing some supplementary references.

As commented by Professor Khaledi, it is surprising that the stronger orders (like the hazard rate order and the likelihood ratio order) than the usual stochastic order between the order statistics from two different heterogeneous samples hold only for the case when the sample size is 2; this is why we limited our discussion to this case in Section 2.1 and then extended our discussion to the outlier models in Section 2.2 which is more applicable. As suggested, we have now cited the reference of Navarro [6] in this version which has discussed this problem when the sample size is any n. Based on Navarro's observation, we conjecture that it may be possible, for the general case when the sample size is any n, that the stronger orders between order statistics from two different heterogeneous samples hold under some stronger restriction on parameters other than majorization. This problem remains open!

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In his discussion, Professor Khaledi has also raised an interesting problem concerning the multiple-outlier models in Section 2.2 whether one can get similar results for the case when the two samples have different sizes and different number of parameters. This problem certainly deserves attention for further work.

We are happy to see that Professor Khaledi is interested in Open Problem 4 and we hope that this problem will be solved soon. As suggested by Professor Khaledi, we have cited the review paper of Boland et al. [1] in the present version as we missed it in the earlier version.

Marco Burkschat: Our final thanks go to Professor Burkschat for his comments and some suggestions. We have taken care of all the minor suggestions and they have certainly improved the presentation of this paper.

Professor Burkschat has made some interesting points and proposals for further work in this direction. We agree with his comment that the situation of discrete underlying distributions has received much less attention than that of continuous distributions. The geometric distribution, for instance, can be regarded as the discrete counterpart of the exponential distribution. Though they have similarities in many respects, we also note that there are some differences between orderings of the order statistics from geometric variables and those arising from exponential variables. New research ideas and techniques are indeed necessary for dealing with the discrete case.

The suggestion of extending the results of order statistics to progressively Type-II censored order statistics is an interesting one. Of course, this will become quite involved due to the complicated distribution theory in this case. The work of Fisher et al. [2] has given the results for the usual (multivariate) stochastic order, and we hope to obtain some new results for some stronger stochastic orderings in this regard.

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References

1. Boland, P.J., Hu, T., Shaked, M. & Shanthikumar, J.G. (2002). Stochastic ordering of order statistics II. In Dror, M., LiEcuyer, P., Szidarovszky, F. (eds.), *Modeling of Stochastic Theory, Methods, and Application*, Boston: Kluwer, Academic Publishers, pp. 607–623.
2. Fischer, T., Balakrishnan, N. & Cramer, E. (2008). Mixture representation for order statistics from INID progressive censoring and its applications. *Journal of Multivariate Analysis* 99: 1999–2015.
3. Kochar, S.C. & Xu, M. (2011). Stochastic comparisons of spacings from heterogeneous samples. In Wells, M.T. and Sengupta, A. (eds.), *Advances in Directional and Linear Statistics,* New York: Springer-Verlag, pp.113–129.
4. Marshall, A.W. & Olkin, I. (1979). *Inequalities: Theory of Majorization and its Applications*. New York: Academic Press.
5. Marshall, A.W., Olkin, I. & Arnold, B.C. (2011). *Inequalities: Theory of Majorization and its Applications*. New York: Springer-Verlag.
6. Navarro, J. (2007). Tail hazard rate ordering properties of order statistics and coherent systems. *Naval Research Logistics* 54: 820–828.
7. Proschan, F. & Sethuraman, J. (1976). Stochastic comparisons of order statistics from heterogeneous populations, with applications in reliability. *Journal of Multivariate Analysis* 6: 608–616.
8. Zhao, P. & Li, X. (2013). Ordering properties of convolutions from heterogeneous populations: A review on some recent developments. *Communications in Statistics-Theory and Methods* (to appear).