Proceedings of the Royal Society of Edinburgh, **149**, 655–689, *2019*
DOI:10.1017/prm.2018.43

Lagrangian submanifolds of the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$ from minimal surfaces in $\mathbb{S}^3$

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(MS received 28 November 2016; accepted 2 March 2017)

We study non-totally geodesic Lagrangian submanifolds of the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$ for which the projection on the first component is nowhere of maximal rank. We show that this property can be expressed in terms of the so-called angle functions and that such Lagrangian submanifolds are closely related to minimal surfaces in $\mathbb{S}^3$. Indeed, starting from an arbitrary minimal surface, we can construct locally a large family of such Lagrangian immersions, including one exceptional example. We also show that locally all such Lagrangian submanifolds can be obtained in this way.

Keywords: Lagrangian submanifolds; Nearly Kaehler $\mathbb{S}^3 \times \mathbb{S}^3$; minimal surfaces, $\mathbb{S}^3$.

2010 *Mathematics subject classification:* 53B25; 53C42; 53D12

1. Introduction

The nearly Kähler manifolds are almost Hermitian manifolds with almost complex structure J for which the tensor field $\tilde{\nabla}J$ is skew-symmetric, where $\tilde{\nabla}$ is the Levi Civita connection. They have been studied intensively in the 1970s by Gray ([**12**]).

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https://doi.org/10.1017/prm.2018.43 Published online by Cambridge University Press

Nagy ([**17**, **18**]) made a further contribution to the classification of nearly Kähler manifolds and more recently, it has been shown by Butruille ([**4**]) that the only homogeneous 6-dimensional nearly Kähler manifolds are the nearly Kähler 6-sphere $\mathbb{S}^6$, $\mathbb{S}^3 \times \mathbb{S}^3$, the projective space $\mathbb{C}P^3$ and the flag manifold $SU(3)/U(1) \times U(1)$, where the last three are not endowed with the standard metric. All these spaces are compact 3-symmetric spaces. Note that in 2014, V. Cortés and J. J. Vásquez have discovered the first non homogeneous (but locally homogeneous) nearly Kähler structures in [**5**], while more recently, the first complete non homogeneous nearly Kähler structures were discovered on $\mathbb{S}^6$ and $\mathbb{S}^3 \times \mathbb{S}^3$ in [**11**].

A natural question for the above mentioned four homogeneous nearly Kähler manifolds is to study their submanifolds. There are two natural types of submanifolds of nearly Kähler (or more generally, almost Hermitian) manifolds, namely almost complex and totally real submanifolds. Almost complex submanifolds are submanifolds whose tangent spaces are invariant under J. For a totally real submanifold, a tangent vector is mapped by the almost complex structure J into a normal vector. In this case, if additionally, the dimension of the submanifold is half the dimension of the ambient manifold, then the submanifold is Lagrangian.

Note that the Lagrangian submanifolds of nearly Kähler manifolds are especially interesting as they are always minimal and orientable (see [**9**] for $\mathbb{S}^6$ or [**13**, **20**] for the general case). Lagrangian submanifolds of $\mathbb{S}^6$ have been studied by many authors (see, amongst others, [**6**, **7**, **9**, **10**, **15**, **19**, **22**, **23**]), whereas the study of Lagrangian submanifolds of $\mathbb{S}^3 \times \mathbb{S}^3$ only started recently. The first examples of those were given in [**16**, **20**]. Moreover, in [**8**, **24**], the authors obtained a classification of the Lagrangian submanifolds, which are either totally geodesic or have constant sectional curvature. An important tool in the study in [**8**, **24**] is the use of an almost product structure P on $\mathbb{S}^3 \times \mathbb{S}^3$, which was introduced in [**2**]. Its definition is recalled in §2. The decomposition of P into a tangential part and a normal part along a Lagrangian submanifold allows us to introduce three principal directions, E_1, E_2, E_3, with corresponding angle functions $\theta_1, \theta_2, \theta_3$.

In this paper, we are interested in studying non-totally geodesic Lagrangian submanifolds $f : M \to \mathbb{S}^3 \times \mathbb{S}^3 : x \mapsto f(x) = (p(x), q(x))$, for which the first component has nowhere maximal rank. Basic properties of the structure and its Lagrangian submanifolds are given in §2. In §3, we show that in this case $\theta_1 = \pi/3$ (theorem 1) and $p(M)$ has to be a (branched) minimal surface in $\mathbb{S}^3$ (theorem 3). Conversely, for a non-totally geodesic minimal surface in $\mathbb{S}^3$ which locally corresponds to a solution of the Sinh–Gordon equation, $\Delta\omega = -8\sinh\omega$, and for an additional arbitrary solution of the Liouville equation, $\Delta\mu = -e^\mu$, we can construct locally a Lagrangian immersion of $\mathbb{S}^3 \times \mathbb{S}^3$. Thus, we obtain a large class of examples of Lagrangian immersions. We also obtain that a similar class of Lagrangian immersions can be associated with a totally geodesic surface in $\mathbb{S}^3$. This last case contains, in particular, the constant curvature sphere obtained in [**8**]. Additionally, for each non-totally geodesic minimal surface, we obtain also one exceptional example. In case of the Clifford torus in $\mathbb{S}^3$, this additional example is the flat Lagrangian torus in $\mathbb{S}^3 \times \mathbb{S}^3$ discovered in [**8**]. We also show that any non-totally geodesic Lagrangian immersion for which the first component has nowhere maximal rank is obtained by applying one of the three previously mentioned constructions. The main results are summarized in §4.

2. Preliminaries

In this section, we recall the homogeneous nearly Kähler structure of $\mathbb{S}^3 \times \mathbb{S}^3$ and we mention some of the known results from [**8, 24**].

By the natural identification $T_{(p,q)}(\mathbb{S}^3 \times \mathbb{S}^3) \cong T_p\mathbb{S}^3 \oplus T_q\mathbb{S}^3$, we write a tangent vector at (p,q) as $Z(p,q) = (U(p,q), V(p,q))$ or simply $Z = (U, V)$. We regard the 3-sphere as the set of all unit quaternions in $\mathbb{H}$ and we use the notations i, j, k to denote the imaginary units of $\mathbb{H}$. In computations it is often useful to write a tangent vector $Z(p,q)$ at (p,q) on $\mathbb{S}^3 \times \mathbb{S}^3$ as $(p\alpha, q\beta)$, with α and β imaginary quaternions. This is also possible for $v \in T_p\mathbb{S}^3$ as we know that $\langle v, p \rangle = 0$ and, in addition, for $p \in \mathbb{S}^3$ we can always find $\tilde{v} \in \mathbb{H}$ such that $v = p\tilde{v}$. Moreover, $Re(\tilde{v}) = 0$ as $0 = \langle p, v \rangle = Re(\bar{p}v) = Re(\bar{p}p\tilde{v}) = Re(\tilde{v})$. We define the vector fields

$$
\begin{aligned}
\tilde{E}_1(p,q) &= (pi, 0), & \tilde{F}_1(p,q) &= (0, qi), \\
\tilde{E}_2(p,q) &= (pj, 0), & \tilde{F}_2(p,q) &= (0, qj), \\
\tilde{E}_3(p,q) &= -(pk, 0), & \tilde{F}_3(p,q) &= -(0, qk),
\end{aligned}
\tag{1}
$$

which are mutually orthogonal with respect to the usual Euclidean product metric on $\mathbb{S}^3 \times \mathbb{S}^3$. The Lie brackets are $[\tilde{E}_i, \tilde{E}_j] = -2\varepsilon_{ijk}\tilde{E}_k$, $[\tilde{F}_i, \tilde{F}_j] = -2\varepsilon_{ijk}\tilde{F}_k$ and $[\tilde{E}_i, \tilde{F}_j] = 0$, where

$$
\varepsilon_{ijk} = \begin{cases} 1, & \text{if } (ijk) \text{ is an even permutation of } (123), \\ -1, & \text{if } (ijk) \text{ is an odd permutation of } (123), \\ 0, & \text{otherwise.} \end{cases}
$$

The almost complex structure J on the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$ is defined by

$$
J(U,V)_{(p,q)} = \frac{1}{\sqrt{3}}(2pq^{-1}V - U, -2qp^{-1}U + V),
\tag{2}
$$

for $(U, V) \in T_{(p,q)}(\mathbb{S}^3 \times \mathbb{S}^3)$. The nearly Kähler metric on $\mathbb{S}^3 \times \mathbb{S}^3$ is the Hermitian metric associated with the usual Euclidean product metric on $\mathbb{S}^3 \times \mathbb{S}^3$:

$$
\begin{aligned}
g(Z, Z') &= \frac{1}{2}(\langle Z, Z' \rangle + \langle JZ, JZ' \rangle) \\
&= \frac{4}{3}(\langle U, U' \rangle + \langle V, V' \rangle) - \frac{2}{3}(\langle p^{-1}U, q^{-1}V' \rangle + \langle p^{-1}U', q^{-1}V \rangle),
\end{aligned}
\tag{3}
$$

where $Z = (U, V)$ and $Z' = (U', V')$. In the first line $\langle \cdot, \cdot \rangle$ stands for the usual Euclidean product metric on $\mathbb{S}^3 \times \mathbb{S}^3$ and in the second line $\langle \cdot, \cdot \rangle$ stands for the usual Euclidean metric on $\mathbb{S}^3$. By definition, the almost complex structure is compatible with the metric g.

From [**2**], we have the following lemma.

LEMMA 1. *The Levi-Civita connection $\tilde{\nabla}$ on $\mathbb{S}^3 \times \mathbb{S}^3$ with respect to the metric g is given by*

$$
\begin{aligned}
\tilde{\nabla}_{\tilde{E}_i}\tilde{E}_j &= -\varepsilon_{ijk}\tilde{E}_k & \tilde{\nabla}_{\tilde{E}_i}\tilde{F}_j &= \frac{\varepsilon_{ijk}}{3}(\tilde{E}_k - \tilde{F}_k) \\
\tilde{\nabla}_{\tilde{F}_i}\tilde{E}_j &= \frac{\varepsilon_{ijk}}{3}(\tilde{F}_k - \tilde{E}_k) & \tilde{\nabla}_{\tilde{F}_i}\tilde{F}_j &= -\varepsilon_{ijk}\tilde{F}_k.
\end{aligned}
$$

Then we have

$$(\tilde{\nabla}_{\tilde{E}_i} J)\tilde{E}_j = -\frac{2}{3\sqrt{3}}\varepsilon_{ijk}(\tilde{E}_k + 2\tilde{F}_k), \quad (\tilde{\nabla}_{\tilde{E}_i} J)\tilde{F}_j = -\frac{2}{3\sqrt{3}}\varepsilon_{ijk}(\tilde{E}_k - \tilde{F}_k),$$

$$(\tilde{\nabla}_{\tilde{F}_i} J)\tilde{E}_j = -\frac{2}{3\sqrt{3}}\varepsilon_{ijk}(\tilde{E}_k - \tilde{F}_k), \quad (\tilde{\nabla}_{\tilde{F}_i} J)\tilde{F}_j = -\frac{2}{3\sqrt{3}}\varepsilon_{ijk}(2\tilde{E}_k + \tilde{F}_k). \tag{4}$$

Let $G := \tilde{\nabla}J$. Then G is skew-symmetric and satisfies

$$G(X, JY) = -JG(X, Y), \quad g(G(X, Y), Z) + g(G(X, Z), Y) = 0, \tag{5}$$

for any vectors fields X, Y, Z tangent to $\mathbb{S}^3 \times \mathbb{S}^3$. Therefore, $\mathbb{S}^3 \times \mathbb{S}^3$ equipped with g and J, becomes a nearly Kähler manifold.

The almost product structure P introduced in [2] is defined as

$$PZ = (pq^{-1}V, qp^{-1}U), \quad \forall Z = (U, V) \in T_{(p,q)}(\mathbb{S}^3 \times \mathbb{S}^3) \tag{6}$$

plays an important role in the study of the Lagrangian submanifolds of the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$. It has the following properties:

$$P^2 = Id \quad (P \text{ is involutive}),$$
$$PJ = -JP \quad (P \text{ and } J \text{ anti-commute}),$$
$$g(PZ, PZ') = g(Z, Z') \quad (P \text{ is compatible with } g),$$
$$g(PZ, Z') = g(Z, PZ') \quad (P \text{ is symmetric}).$$

Moreover, the almost product structure P can be expressed in terms of the usual product structure $QZ = Q(U, V) = (-U, V)$ and vice versa:

$$QZ = \frac{1}{\sqrt{3}}(2PJZ - JZ),$$

$$PZ = \frac{1}{2}(Z - \sqrt{3}QJZ).$$

Next, we recall the relation between the Levi-Civita connections $\tilde{\nabla}$ of g and ∇^E of the Euclidean product metric $\langle \cdot, \cdot \rangle$.

LEMMA 2 [8]. *The relation between the nearly Kähler connection $\tilde{\nabla}$ and the Euclidean connection ∇^E is*

$$\nabla^E_X Y = \tilde{\nabla}_X Y + \frac{1}{2}(JG(X, PY) + JG(Y, PX)).$$

We recall here a useful formula, already known in [8].

Let D be the Euclidean connection on $\mathbb{R}^8$. For vector fields $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ on $\mathbb{S}^3 \times \mathbb{S}^3$, we may decompose $D_X Y$ along the tangent and the normal

directions as follows:

$$D_X Y = \nabla_X^E Y + \frac{1}{2}\langle D_X Y, (p,q)\rangle(p,q) + \frac{1}{2}\langle D_X Y, (-p,q)\rangle(-p,q). \qquad (7)$$

Here, notice the factor $1/2$ due to the fact that (p,q) and $(-p,q)$ have length $\sqrt{2}$. Moreover, as $\langle Y, (p,q)\rangle = 0$, (7) is equivalent with

$$D_X Y = \nabla_X^E Y - \frac{1}{2}\langle Y, X\rangle(p,q) - \frac{1}{2}\langle Y, (-X_1, X_2)\rangle(-p,q).$$

In the special case that $Y_2 = 0$, the previous formula reduces to

$$D_X(Y_1, 0) = \nabla_X^E(Y_1, 0) - \langle X_1, Y_1\rangle(p, 0). \qquad (8)$$

We find it appropriate here to prove an additional important formula not explicitly mentioned in [**2**], that allows us to evaluate G for any tangent vector fields.

PROPOSITION 1. *Let $X = (p\alpha, q\beta), Y = (p\gamma, q\delta) \in T_{(p,q)}\mathbb{S}^3 \times \mathbb{S}^3$. Then*

$$G(X, Y) = \frac{2}{3\sqrt{3}}(p(\beta \times \gamma + \alpha \times \delta + \alpha \times \gamma - 2\beta \times \delta),$$
$$\times q(-\alpha \times \delta - \beta \times \gamma + 2\alpha \times \gamma - \beta \times \delta)). \qquad (9)$$

Proof. As α is an imaginary unit quaternion, we may write $\alpha = \alpha_1 \cdot i + \alpha_2 \cdot j + \alpha_3 \cdot k$ and similarly for β, γ, δ. Then, using (1), we write for more convenience in computations $X = U_\alpha + V_\beta$, where $U_\alpha = \alpha_1 \tilde{E}_1 + \alpha_2 \tilde{E}_2 - \alpha_3 \tilde{E}_3$ and $V_\beta = \beta_1 \tilde{F}_1 + \beta_2 \tilde{F}_2 - \beta_3 \tilde{F}_3$. Similarly, $Y = U_\gamma + V_\delta$. We now use the relations in (4) and compute

$$G(U_\alpha, V_\beta) = \frac{2}{3\sqrt{3}}(U_{\alpha \times \beta} - V_{\alpha \times \beta}), \quad G(U_\alpha, U_\beta) = \frac{2}{3\sqrt{3}}(U_{\alpha \times \beta} + 2V_{\alpha \times \beta}).$$

As $PU_\alpha = V_\alpha$, we obtain

$$G(V_\alpha, V_\beta) = -\frac{2}{3\sqrt{3}}(V_{\alpha \times \beta} + 2U_{\alpha \times \beta}).$$

Finally, by linearity, we get the relation in (9). □

From now on we will restrict ourselves to 3-dimensional Lagrangian submanifolds M of $\mathbb{S}^3 \times \mathbb{S}^3$. It is known from [**8**, **24**] that, as the pull-back of $T(\mathbb{S}^3 \times \mathbb{S}^3)$ to M splits into $TM \oplus JTM$, there are two endomorphisms $A, B : TM \to TM$ such that the restriction $P|_{TM}$ of P to the submanifold equals $A + JB$, that is $PX = AX + JBX$, for all $X \in TM$. Note that the previous formula, together with the fact that P and J anti-commute, also determines P on the normal space by $PJX = -JPX = BX - JAX$. In addition, from the properties of J and P it follows that A and B are symmetric operators which commute and satisfy moreover, $A^2 + B^2 = Id$ (see [**8**]). Hence A and B can be diagonalized simultaneously at a point p in M and there is an orthonormal basis $e_1, e_2, e_3 \in T_p M$ such that

$$Pe_i = \cos(2\theta_i)e_i + \sin(2\theta_i)Je_i. \qquad (10)$$

The functions θ_i are called the angle functions of the immersion. Next, for a point p belonging to an open dense subset of M on which the multiplicities of the eigenvalues of A and B are constant (see [**21**]), we may extend the orthonormal basis

e_1, e_2, e_3 to a frame on a neighbourhood in the Lagrangian submanifold. Finally, taking into account the properties of G, we know that there exists a local orthonormal frame $\{E_1, E_2, E_3\}$ on an open subset of M such that

$$AE_i = \cos(2\theta_i)E_i, \quad BE_i = \sin(2\theta_i)E_i \tag{11}$$

and

$$JG(E_i, E_j) = \frac{1}{\sqrt{3}}\varepsilon_{ijk}E_k. \tag{12}$$

The following result is known ([**8**]):

PROPOSITION 2. *The sum of the angles* $\theta_1 + \theta_2 + \theta_3$ *is zero modulo* π.

For the Levi-Civita connection ∇ on M, we introduce (see [**8**]) the functions ω_{ij}^k satisfying

$$\nabla_{E_i}E_j = \sum_{k=1}^3 \omega_{ij}^k E_k \quad \text{and} \quad \omega_{ij}^k = -\omega_{ik}^j.$$

As usual, we write:

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$
$$\tilde{\nabla}_X JY = -S_{JY}X + \nabla_X^\perp JY,$$

where h is the second fundamental form on M and S_{JY} is the shape operator in the direction of JY. As for the Lagrangian manifolds of a strict 6-dimensional nearly Kähler manifold, we have $G(X, Y)$ that is normal (see [**13, 20**]) and follows

$$\nabla_X^\perp JY = J\nabla_X Y + G(X, Y),$$
$$Jh(X, Y) = -S_{JY}X.$$

The latter equation implies, in particular, that the cubic form $g(h(X, Y), JZ)$ is totally symmetric. We denote by h_{ij}^k the components of this cubic form on M:

$$h_{ij}^k = g(h(E_i, E_j), JE_k). \tag{13}$$

3. Results

3.1. Elementary properties of orientable minimal surfaces in $\mathbb{S}^3$

We recall some elementary properties of minimal surfaces. Let $p : S \to \mathbb{S}^3 \subset \mathbb{R}^4$ be an oriented minimal surface. We are going to check that the immersion either admits local isothermal coordinates for which the conformal factor satisfies the Sinh–Gordon equation or is totally geodesic. First, we take isothermal coordinates u, v such that $\partial u, \partial v$ is positively oriented, $\langle \partial u, \partial u \rangle = \langle \partial v, \partial v \rangle = 2e^\omega$ and $\langle \partial u, \partial v \rangle = 0$ in a neighbourhood of a point of S. As it is often more useful to

use complex notation, we write $z = u + Iv$ and consider $\partial z = 1/2(\partial u - I\partial v)$ and $\partial \bar{z} = 1/2(\partial u + I\partial v)$. Note that we use I here in order to distinguish between the i, j, k introduced in the quaternions. We also extend everything in a linear way in I. This means that $\langle \partial z, \partial z \rangle = \langle \partial \bar{z}, \partial \bar{z} \rangle = 0$ and $\langle \partial z, \partial \bar{z} \rangle = e^{\omega}$. If we write $\partial u = p\alpha$ and $\partial v = p\beta$, the unit normal is given by $N = p((\alpha \times \beta)/(2e^{\omega}))$. It is elementary to check that this is independent of the choice of complex coordinate and that the matrix $(p((\partial u)/(|\partial u|))((\partial v)/(|\partial v|)) N)$ belongs to $SO(4)$. We denote by σ the component of the second fundamental form in the direction of N. Remark that with this choice, the minimality of the surface implies $\sigma(\partial z, \partial \bar{z}) = 0$ and we may determine the components of the connection ∇ on the surface:

$$\nabla_{\partial z} \partial z = \omega_z \partial z, \quad \nabla_{\partial z} \partial \bar{z} = \nabla_{\partial \bar{z}} \partial z = 0 \quad \text{and} \quad \nabla_{\partial \bar{z}} = \omega_{\bar{z}} \partial \bar{z}. \tag{14}$$

The Codazzi equation of a surface in $\mathbb{S}^3$ states that

$$\nabla \sigma(\partial z, \partial \bar{z}, \partial z) = \nabla \sigma(\partial \bar{z}, \partial z, \partial z).$$

So it follows that $\partial \bar{z}(\sigma(\partial z, \partial z)) = 0$. Hence $\sigma(\partial z, \partial z)$ is a holomorphic function. Then we have two cases:

Case 1. If $\sigma(\partial z, \partial z) = 0$ on an open set, then by conjugation $\sigma(\partial \bar{z}, \partial \bar{z}) = 0$ and therefore, using the analyticity of a minimal surface, $\sigma = 0$ everywhere.

Case 2. If $\sigma(\partial z, \partial z) \neq 0$, then there exists a function $g(z)$ such that $\sigma(\partial z, \partial z) = g(z)$. Away from isolated points, we can always make a change of coordinates if necessary such that $\sigma(\partial z, \partial z) = -1$. Notice that by conjugation, we also get $\sigma(\partial \bar{z}, \partial \bar{z}) = -1$. Such a change of coordinates is unique up to translations and replacing z by $-z$.

Next, given the immersions $p : S \to \mathbb{S}^3(1) \overset{i}{\hookrightarrow} \mathbb{R}^4$, from the Gauss formula, we obtain:

$$\begin{aligned} p_{zz} &= \omega_z p_z - N, \\ p_{z\bar{z}} &= -e^{\omega} p, \\ p_{\bar{z}\bar{z}} &= \omega_{\bar{z}} p_{\bar{z}} - N, \end{aligned} \tag{15}$$

where N is the normal on $\mathbb{S}^3$ and $N_z = e^{-\omega} p_{\bar{z}}$, $N_{\bar{z}} = e^{-\omega} p_z$. Therefore,

$$p_{zz\bar{z}} = (\omega_{z\bar{z}} - e^{-\omega})p_z - \omega_z e^{\omega} p, \quad p_{z\bar{z}\bar{z}} = -e^{\omega} \omega_z p - e^{\omega} p_z,$$

which shows that ω satisfies

$$\begin{aligned} \omega_{z\bar{z}} &= -2\sinh\omega \quad \Longleftrightarrow \\ \Delta\omega &= -8\sinh\omega \quad \text{(Sinh–Gordon equation)}. \end{aligned} \tag{16}$$

Notice that by $\Delta\omega$, we denote the Euclidean Laplacian of ω in $\mathbb{R}^2 = \mathbb{C}$.

Let $\mathcal{P}$ be the lift of the minimal immersion to the immersion of the frame bundle in $SO(4)$, that is,

$$\mathcal{P} : US \to SO(4) : w \mapsto (p \ w \ \tilde{J}w \ N),$$

where US denotes the unit tangent bundle of S and $\tilde{J}$ denotes the natural complex structure on an orientable surface. In terms of our chosen isothermal coordinate this map can be parametrized by

$$\mathcal{P}(u,v,t) = \left(p(u,v), \cos t \frac{p_u}{\mid p_u \mid} + \sin t \frac{p_v}{\mid p_v \mid}, -\sin t \frac{p_u}{\mid p_u \mid} + \cos t \frac{p_v}{\mid p_v \mid}, N(u,v) \right),$$

for some real parameter t. Note that we have the frame equations which state

$$d\mathcal{P} = \mathcal{P}\Omega^t = -\mathcal{P}\Omega,$$

where in terms of the coordinates u, v and t the matrix Ω is given by

$$\begin{pmatrix} 0 & \sqrt{2}e^{\omega/2}(\cos(t)du + \sin(t)dv) \\ -\sqrt{2}e^{\omega/2}(\cos(t)du + \sin(t)dv) & 0 \\ -\sqrt{2}e^{\omega/2}(\cos(t)dv - \sin(t)du) & -\frac{1}{2}(\omega_u dv - \omega_v du) - dt \\ 0 & \sqrt{2}e^{-\omega/2}(\cos(t)du - \sin(t)dv) \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{2}e^{\omega/2}(\cos(t)dv - \sin(t)du) & 0 \\ \frac{1}{2}(\omega_u dv - \omega_v du) + dt & -\sqrt{2}e^{-\omega/2}(\cos(t)du - \sin(t)dv) \\ 0 & \sqrt{2}e^{-\omega/2}(\sin(t)du + \cos(t)dv) \\ -\sqrt{2}e^{-\omega/2}(\sin(t)du + \cos(t)dv) & 0 \end{pmatrix}.$$

3.2. From the Lagrangian immersion to the minimal surface

Now we will consider Lagrangian submanifolds in the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$. We write the Lagrangian submanifold M as

$$f : M \to \mathbb{S}^3 \times \mathbb{S}^3$$
$$x \mapsto f(x) = (p(x), q(x)),$$

and we assume that the first component has nowhere maximal rank. We have the following:

THEOREM 1. *Let*

$$f : M \to \mathbb{S}^3 \times \mathbb{S}^3$$
$$x \mapsto f(x) = (p(x), q(x)),$$

be a Lagrangian immersion such that $p : M \to \mathbb{S}^3$ has nowhere maximal rank. Then $\pi/3$ is an angle function up to a multiple of π. The converse is also true.

Proof. It is clear that p has nowhere maximal rank if and only if there exists a non zero vector field X such that $dp(X) = 0$. As usual, we identify $df(X)$ with X, so we have that $X = df(X) = (dp(X), dq(X))$ and $QX = (-dp(X), dq(X))$. Therefore p

has nowhere maximal rank if and only if

$$
\begin{aligned}
X = QX &\\
&= \frac{1}{\sqrt{3}}(2PJX - JX) \\
&= \frac{1}{\sqrt{3}}(2BX - 2JAX - JX).
\end{aligned}
$$

Comparing tangent and normal components, we see that this is the case if and only if

$$
AX = -\frac{1}{2}X \quad BX = \frac{\sqrt{3}}{2}X.
$$

So we see that X is an eigenvector of both A and B and that the corresponding angle function is $\pi/3$ (up to a multiple of π). $\qquad\square$

For the remainder of the paper, we will consider Lagrangian immersions for which the map p has nowhere maximal rank. In view of the previous lemma, this means that one of the angle functions is constant, namely $\theta_1 = \pi/3$. Then using that the angles are only determined up to a multiple of π and given that $2\theta_1 + 2\theta_2 + 2\theta_3$ is a multiple of 2π, we may write

$$
\begin{aligned}
2\theta_1 &= \frac{2\pi}{3}, \\
2\theta_2 &= 2\Lambda + \frac{2\pi}{3}, \\
2\theta_3 &= -2\Lambda + \frac{2\pi}{3},
\end{aligned}
\tag{17}
$$

for Λ is an arbitrary function which takes values in $[-\pi/2, \pi/2]$. If necessary by interchanging E_2, E_3 with $-E_3, E_2$, we may assume that $\Lambda \geqslant 0$ and, therefore, Λ takes values only in $[0, \pi/2]$. Similarly, if necessary interchanging E_1, E_3 by $-E_1, -E_3$, we may also assume that $h_{13}^3 \leqslant 0$ (see equation (13)).

Note, however, that at the points where Λ is 0 or $\pi/2$ modulo π, we have that two of the angle functions coincide. If this is true on an open set, it follows from [**24**] that the Lagrangian submanifold is totally geodesic and is congruent either with $f \colon \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (1, u)$ or $f \colon \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (uiu^{-1}, u^{-1})$. So by restricting to an open dense subset of M which we denote by M^*, we may actually assume that $\Lambda \in (0, \pi/2)$, in which case the function Λ, as well as the vector fields E_1, E_2, E_3 are differentiable.

Notice that the case when Λ is constant is treated in [**1**], where such Lagrangian submanifolds are determined to be either totally geodesic or of constant sectional curvature. As we consider here $\Lambda \in (0, \pi/2)$, the only possibility is $\Lambda = \pi/3$, in which case the Lagrangian submanifold is not totally geodesic, but of constant sectional curvature.

THEOREM 2. *Let M be a Lagrangian submanifold of constant sectional curvature in the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$. If M is not totally geodesic, then up to an isometry of the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$, M is locally congruent with one of the following immersions:*

(1) $f \colon \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (uiu^{-1}, uju^{-1})$,

(2) $f : \mathbb{R}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : (u, v, w) \mapsto (p(u, w), q(u, v))$, *where p and q are constant mean curvature tori in $\mathbb{S}^3$ is given by*

$$p(u, w) = (\cos u \cos w, \cos u \sin w, \sin u \cos w, \sin u \sin w),$$

$$q(u, v) = \frac{1}{\sqrt{2}} \left(\cos v \left(\sin u + \cos u\right), \sin v \left(\sin u + \cos u\right),\right.$$

$$\cos v \left(\sin u - \cos u\right), \sin v \left(\sin u - \cos u\right)).$$

Note that these are precisely the two Lagrangian immersions with constant sectional curvature obtained in [**8**]. These two examples will appear as special solutions in respectively, Case 2 and Case 3. However, we will mainly focus on the case that Λ is not constant.

In the following, we will identify a tangent vector X in $T_x M$ with its image through df in $T_{(p,q)} \mathbb{S}^3 \times \mathbb{S}^3$, that is, $X \equiv df(X) = (dp(X), dq(X))$, and we can write $QX \equiv Q(df(X)) = (-dp(X), dq(X))$. Therefore, if we see $dp(X)$ projected on the first factor of $\mathbb{S}^3 \times \mathbb{S}^3$, that is $dp(X) \equiv (dp(X), 0)$, we can write

$$dp(X) = \frac{1}{2}(X - QX). \tag{18}$$

We use relations (10) and (17) to compute $PE_1 = -1/2E_1 + \sqrt{3}/2JE_1$. As mentioned before, this is equivalent with stating that $dp(E_1) = 0$ and that p has nowhere maximal rank. By straightforward computations, we obtain

$$(dp(E_2), 0) = \left(\frac{1}{2} - \frac{1}{\sqrt{3}}\sin(2\Lambda + \frac{2\pi}{3})\right) E_2 + \frac{1}{\sqrt{3}}\left(\frac{1}{2} + \cos(2\Lambda + \frac{2\pi}{3})\right) JE_2,$$

$$(dp(E_3), 0) = \left(\frac{1}{2} - \frac{1}{\sqrt{3}}\sin(-2\Lambda + \frac{2\pi}{3})\right) E_3 + \frac{1}{\sqrt{3}}\left(\frac{1}{2} + \cos(-2\Lambda + \frac{2\pi}{3})\right) JE_3 \tag{19}$$

and

$$\langle dp(E_2), dp(E_2) \rangle = \sin^2 \Lambda,$$
$$\langle dp(E_3), dp(E_3) \rangle = \sin^2 \Lambda, \tag{20}$$
$$\langle dp(E_2), dp(E_3) \rangle = 0.$$

We denote

$$v_2 := dp(E_2) \equiv (dp(E_2), 0),$$
$$v_3 := dp(E_3) \equiv (dp(E_3), 0), \tag{21}$$
$$\xi = \frac{1}{\sqrt{3}}E_1 - JE_1$$

and we may easily see that $Q\xi = -\xi$, that is, ξ lies entirely on the first factor of $\mathbb{S}^3 \times \mathbb{S}^3$. Moreover, $\langle v_i, v_j \rangle = \delta_{ij} \sin \Lambda$, $\langle \xi, v_2 \rangle = \langle \xi, v_3 \rangle = 0$ and $\langle \xi, \xi \rangle = 1$. Therefore, $p(M)$ is a surface in $\mathbb{S}^3$ and ξ can be seen as a unit normal to the surface.

As far as the Lagrangian immersion itself is concerned, we also have due to the minimality that

$$
\begin{aligned}
h_{11}^1 + h_{12}^2 + h_{13}^3 &= 0, \\
h_{11}^2 + h_{22}^2 + h_{23}^3 &= 0, \\
h_{11}^3 + h_{22}^3 + h_{33}^3 &= 0.
\end{aligned}
\tag{22}
$$

From [**8**], we know that the covariant derivatives of the endomorphisms A and B are

$$
(\nabla_X A)Y = B S_{JX} Y - Jh(X, BY) + \frac{1}{2}(JG(X, AY) - AJG(X, Y)),
\tag{23}
$$

$$
(\nabla_X B)Y = -A S_{JX} Y + Jh(X, AY) + \frac{1}{2}(JG(X, AY) - AJG(X, Y)).
\tag{24}
$$

We are going to use the definition of ∇A and ∇B in the previous expressions and then evaluate them for different vectors in the basis in order to get information about the functions ω_{ij}^k and h_{ij}^k. For $X = Y = E_1$ in (23), we obtain

$$
\begin{aligned}
h_{12}^2 &= -h_{13}^3, \\
\omega_{11}^2 &= h_{11}^2 \cot \Lambda, \\
\omega_{11}^3 &= -h_{11}^3 \cot \Lambda.
\end{aligned}
\tag{25}
$$

If we take $X = E_1$ and $Y = E_2$ in (23) and (24), we see

$$
E_1(\Lambda) = h_{13}^3,
\tag{26}
$$

$$
\omega_{12}^3 = \frac{\sqrt{3}}{6} - h_{12}^3 \cot 2\Lambda
\tag{27}
$$

and, for $X = E_2$ and $Y = E_1$ in (23), we obtain

$$
h_{11}^2 = 0,
\tag{28}
$$

$$
\omega_{21}^2 = -\cot \Lambda h_{13}^3,
\tag{29}
$$

$$
\omega_{21}^3 = -\frac{\sqrt{3}}{6} - h_{12}^3 \cot \Lambda.
\tag{30}
$$

Then we choose successively $X = E_3, Y = E_1$, $X = E_2, Y = E_3$ and $X = E_3, Y = E_2$ in relations (23) and (24) and obtain

$$
h_{11}^3 = 0,
\tag{31}
$$

$$
\omega_{31}^2 = \frac{\sqrt{3}}{6} + \cot \Lambda h_{12}^3,
\tag{32}
$$

$$\omega_{31}^3 = -\cot\Lambda h_{13}^3, \tag{33}$$

$$\omega_{22}^3 = -\cot 2\Lambda h_{22}^3, \tag{34}$$

$$\omega_{32}^3 = -\cot 2\Lambda h_{23}^3, \tag{35}$$

$$E_2(\Lambda) = h_{23}^3, \tag{36}$$

$$E_3(\Lambda) = -h_{22}^3. \tag{37}$$

We can easily see from (25), (28) and (31) that

$$\omega_{11}^2 = 0 \quad \text{and} \quad \omega_{11}^3 = 0$$

and, if we consider as well the relations in (22), we have

$$h_{33}^3 = -h_{22}^3, \quad h_{11}^1 = 0 \quad \text{and} \quad h_{22}^2 = -h_{23}^3.$$

Later on, we will also need to study the Codazzi equations for M. From [8] we know their general form:

$$\nabla h(X,Y,Z) - \nabla h(Y,X,Z) = \frac{1}{3}(g(AY,Z)JBX - g(AX,Z)JBY$$
$$- g(BY,Z)JAX + g(BX,Z)JAY). \tag{38}$$

We are going to use the definition for ∇h in the previous relation and take different values for the vectors X, Y and Z. Thus, we evaluate it successively for E_1, E_2, E_1; E_1, E_2, E_2; E_1, E_3, E_3; E_1, E_3, E_2 and E_2, E_3, E_3 and we obtain the following relations, respectively:

$$E_1(h_{13}^3) = \frac{1}{3}(-\sqrt{3}h_{12}^3 + 6(h_{13}^3)^2\cot\Lambda - 6(h_{12}^3)^2\csc(2\Lambda) + \sin(2\Lambda)),$$

$$E_1(h_{12}^3) = \frac{1}{3}h_{13}^3(\sqrt{3} + 9h_{12}^3\cot\Lambda + 3h_{12}^3\tan\Lambda), \tag{39}$$

$$E_2(h_{13}^3) - E_1(h_{23}^3) = \frac{1}{\sqrt{3}}h_{22}^3 + h_{12}^3 h_{22}^3\cot\Lambda - h_{13}^3 h_{23}^3\cot\Lambda - h_{12}^3 h_{22}^3\cot(2\Lambda),$$

$$E_1(h_{22}^3) - E_2(h_{12}^3) = h_{13}^3 h_{22}^3(2\cot\Lambda - \tan\Lambda)$$
$$+ \frac{1}{6}h_{23}^3(2\sqrt{3} - 3h_{12}^3\cot\Lambda + 9h_{12}^3\tan\Lambda),$$

$$E_3(h_{12}^3) - E_1(h_{23}^3) = \frac{1}{\sqrt{3}}h_{22}^3 + (h_{12}^3 h_{22}^3 - h_{13}^3 h_{23}^3)\cot\Lambda$$
$$- (3h_{12}^3 h_{22}^3 + 2h_{13}^3 h_{23}^3)\cot(2\Lambda),$$

$$E_3(h^3_{13}) + E_1(h^3_{22}) = \frac{1}{\sqrt{3}}h^3_{23} + h^3_{13}h^3_{22}\cot\Lambda + h^3_{12}h^3_{23}\cot\Lambda - h^3_{12}h^3_{23}\cot(2\Lambda), \quad (40)$$

$$E_2(h^3_{13}) - E_3(h^3_{12}) = 2(h^3_{12}h^3_{22} + h^3_{13}h^3_{23})\cot(2\Lambda),$$

$$E_3(h^3_{22}) - E_2(h^3_{23}) = -\frac{1}{2}(8(h^3_{12})^2 + 4(h^3_{13})^2 + 3((h^3_{22})^2 + (h^3_{23})^2))\cot\Lambda$$

$$- \frac{1}{3}(\sqrt{3}h^3_{12} + \sin 4\Lambda) + \frac{3}{2}((h^3_{22})^2 + (h^3_{23})^2)\tan\Lambda,$$

$$E_2(h^3_{22}) + E_3(h^3_{23}) = -\frac{1}{3}h^3_{13}(\sqrt{3} + 6h^3_{12}\cot\Lambda).$$

THEOREM 3. *Let*

$$f : M \to \mathbb{S}^3 \times \mathbb{S}^3$$
$$x \mapsto f(x) = (p(x), q(x)),$$

be a Lagrangian immersion such that $p : M \to \mathbb{S}^3$ *has nowhere maximal rank. Assume that* M *is not totally geodesic. Then* $p(M)$ *is a (branched) minimal surface in* $\mathbb{S}^3$. *Moreover,*

$$\tilde{P} : M^* \to SO(4) : \ x \mapsto \left(p(x) \quad \frac{v_2}{\sin\Lambda} \quad \frac{v_3}{\sin\Lambda} \quad \xi\right),$$

where v_2, v_3 *and* ξ *are defined by* (21), *is a map which is contained into the frame bundle over the minimal surface* p.

Proof. Recall that $dp(E_1) = 0$, hence $p(M)$ is a surface. Denoting the second fundamental form of the surface in the direction of ξ by σ, a straightforward computation yields

$$\sigma(E_2, E_2) = h^3_{13},$$
$$\sigma(E_2, E_3) = \sigma(E_3, E_2) = \frac{1}{\sqrt{3}}\cos\Lambda\sin\Lambda - h^3_{12}, \quad (41)$$
$$\sigma(E_3, E_3) = -h^3_{13}.$$

As $dp(E_2)$ and $dp(E_3)$ are orthogonal and have the same length, the above formulas indeed imply that the surface is minimal.

Moreover, we also see that the surface is totally geodesic if and only if $h^3_{13} = 0$ and $h^3_{12} = 1/\sqrt{3}\cos\Lambda\sin\Lambda$. Note also that if we write $(dp(E_2), 0) = (p\alpha, 0)$ and $(dp(E_3), 0) = (p\gamma, 0)$, we have

$$G((dp(E_2), 0), (dp(E_3), 0)) = G((p\alpha, 0), (p\gamma, 0))$$

$$= \frac{2}{3\sqrt{3}}(p(\alpha \times \gamma), 2q(\alpha \times \gamma)).$$

Therefore,

$$(p(\alpha \times \gamma), 0) = \frac{3\sqrt{3}}{4}\left(G((dp(E_2), 0), (dp(E_3), 0)) - Q(G((dp(E_2), 0), (dp(E_3), 0)))\right).$$

A straightforward computation, using (19) and (12), shows that this gives

$$(p(\alpha \times \gamma), 0) = (\sin \Lambda)^2 \xi.$$

Therefore, ξ corresponds with the normal N on the surface.   □

3.3. The reverse construction

In the following, we will separate the study of the submanifold into three cases, according to whether the surface is totally geodesic or not and whether the map to the frame bundle is an immersion or not.

3.3.1. *Case 1. $p(M)$ is not a totally geodesic surface and the map $\tilde{\mathcal{P}}$ is an immersion.* In that case, we can identify M with the frame bundle on the minimal surface induced earlier. Recall that

$$\tilde{\mathcal{P}} : x \in M^* \mapsto \left(p \; \frac{v_2}{\sin \Lambda} \; \frac{v_3}{\sin \Lambda} \; \xi \right).$$

Writing again $d\tilde{\mathcal{P}} = -\tilde{\mathcal{P}}\tilde{\Omega}$, we can express the matrix $\tilde{\Omega}$ in terms of $\{E_1, E_2, E_3\}$ by

$$
\begin{pmatrix}
0 & \sin(\Lambda)\omega_2 \\
-\sin(\Lambda)\omega_2 & 0 \\
-\sin(\Lambda)\omega_3 & -\left(\frac{1}{\sqrt{3}} + h^3_{12}\csc(2\Lambda)\right)\omega_1 - h^3_{22}\csc(2\Lambda)\omega_2 - h^3_{23}\csc(2\Lambda)\omega_3 \\
0 & \left(-\frac{\cos\Lambda}{\sqrt{3}} + h^3_{12}\csc\Lambda\right)\omega_3 - h^3_{13}\csc(\Lambda)\omega_2
\end{pmatrix}
$$

$$
\begin{pmatrix}
\sin(\Lambda)\omega_3 & 0 \\
\left(\frac{1}{\sqrt{3}} + h^3_{12}\csc(2\Lambda)\right)\omega_1 & h^3_{13}\csc(\Lambda)\omega_2 \\
+ h^3_{22}\csc(2\Lambda)\omega_2 + h^3_{23}\csc(2\Lambda)\omega_3 & + \left(\frac{\cos\Lambda}{\sqrt{3}} - h^3_{12}\csc\Lambda\right)\omega_3 \\
0 & \left(\frac{\cos\Lambda}{\sqrt{3}} - h^3_{12}\csc(\Lambda)\right)\omega_2 \\
& -h^3_{13}\csc(\Lambda)\omega_3 \\
\left(-\frac{\cos\Lambda}{\sqrt{3}} - h^3_{12}\csc(\Lambda)\right)\omega_2 - h^3_{13}\csc(\Lambda)\omega_3 & 0
\end{pmatrix},
$$

where $\omega_i(E_j) = \delta_{ij}$. The above matrix implies that the map $\tilde{\mathcal{P}}$ into $SO(4) \subset \mathbb{R}^{16}$ is an immersion if and only if

$$\frac{1}{\sqrt{3}} + h^3_{12}\csc(2\Lambda) \neq 0.$$

As it is an immersion, in view of the dimensions, its image is an open part of the frame bundle and we can identify M with an open part of the frame bundle on the

minimal surface. Moreover, we can write

$$\frac{v_2}{\sin \Lambda} = \cos(t + \gamma(t, u, v)) \frac{p_u}{\mid p_u \mid} + \sin(t + \gamma(t, u, v)) \frac{p_v}{\mid p_v \mid},$$

where γ is some function. As $\tilde{\mathcal{P}}$ is an immersion, we have $t + \gamma(t, u, v)$ that depends on t and can be taken as the new variable t on the frame bundle. Doing so, we have $\tilde{\mathcal{P}} = \mathcal{P}$ and $\tilde{\Omega} = \Omega$ (for $\mathcal{P}, \Omega$ as in §3.1). Comparing both expressions for the matrix Ω, we deduce

$$
\begin{aligned}
\omega_1 = {} & \frac{1}{1/\sqrt{3} + h_{12}^3 \csc 2\Lambda} \left(- \left(\sqrt{2} \frac{\csc 2\Lambda}{\sin \Lambda} e^{\omega/2} (h_{22}^3 \cos t - h_{23}^3 \sin t) + \frac{1}{2} \omega_v \right) du \right. \\
& \left. - \left(\sqrt{2} \frac{\csc 2\Lambda}{\sin \Lambda} e^{\omega/2} (h_{22}^3 \sin t + h_{23}^3 \cos t) - \frac{1}{2} \omega_u \right) dv + dt \right), \\
\omega_2 = {} & \frac{1}{\sin \Lambda} \sqrt{2} e^{\omega/2} (\cos(t) du + \sin(t) dv), \\
\omega_3 = {} & \frac{1}{\sin \Lambda} \sqrt{2} e^{\omega/2} (\cos(t) dv - \sin(t) du),
\end{aligned}
$$

as well as

$$
\begin{cases}
e^{-\omega} \cos(2t) + h_{13}^3 \dfrac{1}{\sin^2 \Lambda} = 0, \\[2mm]
e^{-\omega} \sin(2t) + \left(h_{12}^3 \csc \Lambda - \dfrac{\cos \Lambda}{\sqrt{3}} \right) \dfrac{1}{\sin \Lambda} = 0,
\end{cases}
\tag{42}
$$

which implies

$$
\begin{cases}
h_{13}^3 = -e^{-\omega} \cos(2t) \sin^2 \Lambda, \\[2mm]
h_{12}^3 = \left(-e^{-\omega} \sin(2t) \sin \Lambda + \dfrac{\cos \Lambda}{\sqrt{3}} \right) \sin \Lambda.
\end{cases}
\tag{43}
$$

We may express E_1, E_2, E_3 with respect to the basis $\{\partial t, \partial u, \partial v\}$ as follows. For $E_i = a_i \partial t + b_i \partial u + c_i \partial v$, we use the previously obtained expressions of ω_j in $\omega_j(E_i) = \delta_{ij}$ and by straightforward computations we get

$$
\begin{aligned}
E_1 = {} & \left(\frac{1}{\sqrt{3}} + h_{12}^3 \csc(2\Lambda) \right) \partial t, \\
E_2 = {} & \left(\csc(2\Lambda) h_{22}^3 + \frac{1}{2\sqrt{2}} \sin \Lambda e^{-\omega/2} (\cos(t) \omega_v - \sin(t) \omega_u) \right) \partial t \\
& + \frac{e^{-\omega/2} \cos t \sin \Lambda}{\sqrt{2}} \partial u + \frac{e^{-\omega/2} \sin t \sin \Lambda}{\sqrt{2}} \partial v, \\
E_3 = {} & \left(\csc(2\Lambda) h_{23}^3 - \frac{1}{2\sqrt{2}} \sin \Lambda \ e^{-\omega/2} (\cos(t) \omega_u + \sin(t) \omega_v) \right) \partial t \\
& - \frac{e^{-\omega/2} \sin t \sin \Lambda}{\sqrt{2}} \partial u + \frac{e^{-\omega/2} \cos t \sin \Lambda}{\sqrt{2}} \partial v.
\end{aligned}
\tag{44}
$$

In order to be able to proceed with the reverse construction, that is, in order to be able to construct a Lagrangian immersion starting from the minimal surface, we

need to express Λ, h_{22}^3 and h_{23}^3 in terms of the variables t, u, v. Remark that, as $E_1(\Lambda) = h_{13}^3$, we may use (43) and the expression of E_1 in (44) to determine how Λ depends on the variable t. We get

$$\Lambda_t = -\frac{2\cos(2t)\sin^2\Lambda}{\sqrt{3}e^\omega - 2\cos t \sin t \tan \Lambda}. \tag{45}$$

In order to solve the above differential equation, we use (45) to compute the derivative of the expression $((\sqrt{3}e^\omega)/(\tan\Lambda)) - \sin(2t)$:

$$\partial t\left(\frac{\sqrt{3}e^\omega}{\tan\Lambda} - \sin(2t)\right)^2 = 2\sin(4t),$$

which, by integration, gives

$$\left(\frac{\sqrt{3}e^\omega}{\tan\Lambda} - \sin(2t)\right)^2 = -\frac{1}{2}\cos(4t) + \frac{c_1}{4},$$

where c_1 does not depend on t. Notice that this implies

$$\tan\Lambda = \frac{2\sqrt{3}e^\omega}{\varepsilon_1\sqrt{c_1 - 2\cos(4t) + 2\sin(2t)}}, \tag{46}$$

where $\varepsilon_1 = \pm 1$ and, at the same time, the surface is defined on an open set where $c_1 - 2\cos(4t) \geqslant 0$. Note that as the above expression contains a square root which would complicate simplifications later on, we will avoid its use as much as possible. For later use, remark that we can write

$$\left(\frac{2\sqrt{3}e^\omega}{\tan\Lambda} - 2\sin(2t)\right)^2 = c_1 - 2\cos(4t). \tag{47}$$

As we can rewrite the above equation as

$$\left(\frac{2\sqrt{3}e^\omega}{\tan\Lambda} - 2\sin(2t)\right)^2 + 2\cos(4t) + 2 = c_1 + 2,$$

we see that $c_1 \geqslant -2$ and equality can hold if $t \in \{\pm\pi/4, \pm 5\pi/4\}$ and $((2\sqrt{3}e^\omega)/(\tan\Lambda)) \pm 2 = 0$. So on an open dense subset we can write

$$c_1 = e^{\omega+\mu} - 2.$$

Combining this with the previous expression of c_1 and taking the derivative with respect to u and v, we can compute

$$\Lambda_u = -\frac{\sin^2\Lambda\left(\mu_u + e^\omega\cot\Lambda\left(3e^\omega\cot\Lambda(\mu_u - \omega_u) - 2\sqrt{3}\mu_u\sin(2t)\right) + \omega_u\right)}{6e^{2\omega}\cot\Lambda - 2\sqrt{3}e^\omega\sin(2t)}$$

$$\Lambda_v = -\frac{\sin^2\Lambda\left(\mu_v + e^\omega\cot\Lambda\left(3e^\omega\cot\Lambda(\mu_v - \omega_v) - 2\sqrt{3}\mu_v\sin(2t)\right) + \omega_v\right)}{6e^{2\omega}\cot\Lambda - 2\sqrt{3}e^\omega\sin(2t)}.$$

Using this, together with (44), we can solve in (36) and (37), for h_{22}^3 and h_{23}^3. This gives us

$$
\begin{aligned}
h_{22}^3 &= \frac{e^{-3\omega/2}\sin^2\Lambda}{6\sqrt{2}}\big(3e^\omega\cos\Lambda((\omega_u - \mu_u)\sin t + (\mu_v - \omega_v)\cos t)\\
&\quad - \sqrt{3}\sin\Lambda((\mu_u + \omega_u)\cos(3t) + (\mu_v + \omega_v)\sin(3t))\big),\\
h_{23}^3 &= \frac{e^{-3\omega/2}\sin^2\alpha}{6\sqrt{2}}\big(\sqrt{3}\sin\Lambda((\mu_u + \omega_u)\sin(3t) + (-\mu_v - \omega_v)\cos(3t))\\
&\quad - 3e^\omega\cos\Lambda(\mu_u - \omega_u)\cos t - 3e^\omega\cos\Lambda(\mu_v - \omega_v)\sin t\big).
\end{aligned}
$$

In order to determine a differential equation for the function μ, we now apply the previously obtained Codazzi equations for M. By (44), it turns out (39) and the first five equations of (40) are trivially satisfied. Recall from (16) that $\Delta\omega = -8\sinh\omega$. The seventh equation of (40) reduces to

$$
\Delta\mu = -4e^\omega(\cos(2\Lambda) + 2)\csc^2\Lambda + 8\sqrt{3}\cot\Lambda\sin(2t) + 8\sinh\omega. \tag{48}
$$

A straightforward computation, using the definition of μ and (47) shows that this reduces to

$$
\Delta\mu = -e^\mu. \tag{49}
$$

Further on, with these new notations, we may see by straightforward computations that the sixth equation of (40) is now trivially satisfied.

Reverse construction
We denote by $p : S \to \mathbb{S}^3 \subset \mathbb{R}^4$ a given minimal surface S which is not totally geodesic, on which we take suitable isothermal coordinates as introduced before. Hence we have a solution ω of $\Delta\omega = -8\sinh\omega$. Additionally, we take a solution of

$$
\Delta\mu = -e^\mu \tag{50}
$$

and we take the open part of the frame bundle such that

$$
\left(\frac{2\sqrt{3}e^\omega}{\tan\Lambda} - 2\sin(2t)\right)^2 = e^{\omega+\mu} - 2 - 2\cos(4t) \tag{51}
$$

has a solution for the function Λ on an open domain. We define

$$
\begin{aligned}
h_{13}^3 &= -e^\omega\cos(2t)\sin^2\Lambda,\\
h_{12}^3 &= \left(-e^{-\omega}\sin(2t)\sin\Lambda + \frac{\cos\Lambda}{\sqrt{3}}\right)\sin\Lambda,
\end{aligned}
$$

$$h_{22}^3 = \frac{e^{-3\omega/2}\sin^2\Lambda}{6\sqrt{2}}\big(3e^\omega\cos\Lambda((\omega_u - \mu_u)\sin t + (\mu_v - \omega_v)\cos t)$$
$$-\sqrt{3}\sin\Lambda((\mu_u + \omega_u)\cos(3t) + (\mu_v + \omega_v)\sin(3t))\big),$$

$$h_{23}^3 = \frac{e^{-3\omega/2}\sin^2\Lambda}{6\sqrt{2}}\big(\sqrt{3}\sin\Lambda((\mu_u + \omega_u)\sin(3t) + (-\mu_v - \omega_v)\cos(3t))$$
$$-3e^\omega\cos\Lambda(\mu_u - \omega_u)\cos t - 3e^\omega\cos\Lambda(\mu_v - \omega_v)\sin t)$$

and we define as well a metric on the open part of the frame bundle by assuming that the vectors

$$E_1 = \frac{1}{2}\left(\sqrt{3} - 2e^{-\omega}\tan\Lambda\sin t\cos t\right)\partial t,$$

$$E_2 = -\frac{e^{-3\omega/2}\sin\Lambda}{12\sqrt{2}}\Big(\sqrt{3}\tan\Lambda((\mu_u + \omega_u)\cos(3t) + (\mu_v + \omega_v)\sin(3t))$$
$$+ 3e^\omega((\mu_u + \omega_u)\sin t + (-\mu_v - \omega_v)\cos t)\Big)\partial t$$
$$+ \frac{e^{-\omega/2}\cos t\sin\Lambda}{\sqrt{2}}\partial u + \frac{e^{-\omega/2}\sin t\sin\Lambda}{\sqrt{2}}\partial v, \tag{52}$$

$$E_3 = \frac{e^{-3\omega/2}\sin\Lambda}{12\sqrt{2}}\Big(\sqrt{3}\tan\Lambda((\mu_u + \omega_u)\sin(3t) + (-\mu_v - \omega_v)\cos(3t))$$
$$- 3e^\omega((\mu_u + \omega_u)\cos t + (\mu_v + \omega_v)\sin t)\Big)\partial t$$
$$- \frac{e^{-\omega/2}\sin t\sin\Lambda}{\sqrt{2}}\partial u + \frac{e^{-\omega/2}\cos t\sin\Lambda}{\sqrt{2}}\partial v$$

form an orthonormal basis.

We now want to determine the Lagrangian immersion

$$f : S \times I \to \mathbb{S}^3 \times \mathbb{S}^3$$
$$(u, v, t) \mapsto f(u, v, t) = (p(u, v, t), q(u, v, t)).$$

We already know that the first component is the given minimal surface p. We write for both bases

$$\frac{\partial}{\partial t}(q) = q\beta_1, \quad \frac{\partial}{\partial t}(p) = p\alpha_1, \qquad\qquad E_1(q) = q\tilde{\beta}_1, \quad E_1(p) = p\tilde{\alpha}_1,$$
$$\frac{\partial}{\partial u}(q) = q\beta_2, \quad \frac{\partial}{\partial u}(p) = p\alpha_2, \quad \text{and} \quad E_2(q) = q\tilde{\beta}_2, \quad E_2(p) = p\tilde{\alpha}_2,$$
$$\frac{\partial}{\partial v}(q) = q\beta_3, \quad \frac{\partial}{\partial v}(p) = p\alpha_3, \qquad\qquad E_3(q) = q\tilde{\beta}_3, \quad E_3(p) = p\tilde{\alpha}_3.$$

Note that $\alpha_1 = 0$ and α_2 and α_3 are determined by the minimal surface. In particular, α_2 and α_3 are mutually orthogonal imaginary quaternions with length squared

$2e^\omega$. From (52), we then get

$$\tilde{\alpha}_1 = 0,$$

$$\tilde{\alpha}_2 = \frac{e^{-\omega/2}\cos t \sin \Lambda}{\sqrt{2}}\alpha_2 + \frac{e^{-\omega/2}\sin t \sin \Lambda}{\sqrt{2}}\alpha_3,$$

$$\tilde{\alpha}_3 = -\frac{e^{-\omega/2}\sin t \sin \Lambda}{\sqrt{2}}\alpha_2 + \frac{e^{-\omega/2}\cos t \sin \Lambda}{\sqrt{2}}\alpha_3$$

and from the properties of the minimal surface, we obtain

$$\frac{\partial \alpha_2}{\partial u} = -\frac{\partial \alpha_3}{\partial v} = \frac{1}{2}\omega_u\alpha_2 - \frac{1}{2}\omega_v\alpha_3 - e^\omega\alpha_2 \times \alpha_3,$$

$$\frac{\partial \alpha_2}{\partial v} = \frac{1}{2}\omega_v\alpha_2 + \frac{1}{2}\omega_u\alpha_3 + \alpha_2 \times \alpha_3,$$

$$\frac{\partial \alpha_3}{\partial u} = \frac{1}{2}\omega_v\alpha_2 + \frac{1}{2}\omega_u\alpha_3 - \alpha_2 \times \alpha_3.$$

Using the properties of the vector cross product, this also implies

$$\frac{\partial \alpha_2 \times \alpha_3}{\partial u} = 2\alpha_2 + 2e^\omega\alpha_3 + \omega_u\alpha_2 \times \alpha_3,$$

$$\frac{\partial \alpha_2 \times \alpha_3}{\partial v} = -2e^\omega\alpha_2 - 2\alpha_3 + \omega_v\alpha_2 \times \alpha_3.$$

Now, in order to find $\tilde{\beta}_i$, we remark that the vectors E_1, E_2 and E_3 need to correspond with eigenvectors of the operators A and B with suitable eigenfunctions. We have

$$E_1 = (0, q\tilde{\beta}_1),$$
$$E_2 = (p\tilde{\alpha}_2, q\tilde{\beta}_2), \tag{53}$$
$$E_3 = (p\tilde{\alpha}_3, q\tilde{\beta}_3).$$

The angle functions are $\theta_1 = 2\pi/3, \theta_2 = 2\Lambda + 2\pi/3, \theta_3 = -2\Lambda + 2\pi/3$ and

$$PE_i = \cos(2\theta_i)E_i + \sin(2\theta_i)JE_i, \tag{54}$$

for $i = 1, 2, 3$. At the same time, by the definition of P in (6) and by (53), we have

$$PE_1 = (p\tilde{\beta}_1, 0), \quad PE_2 = (p\tilde{\beta}_2, q\tilde{\alpha}_2), \quad PE_3 = (p\tilde{\beta}_3, q\tilde{\alpha}_3). \tag{55}$$

Now we use the definition of J to write out JE_i:

$$JE_1 = \frac{1}{\sqrt{3}}(2p\tilde{\beta}_1, q\tilde{\beta}_1),$$

$$JE_2 = \frac{1}{\sqrt{3}}(p(2\tilde{\beta}_2 - \tilde{\alpha}_2), q(-2\tilde{\alpha}_2 + \tilde{\beta}_2)), \tag{56}$$

$$JE_3 = \frac{1}{\sqrt{3}}(p(2\tilde{\beta}_3 - \tilde{\alpha}_3), q(-2\tilde{\alpha}_3 + \tilde{\beta}_3)).$$

Then, by using (56), (53) and the values of θ_i in (17), we rewrite equation (54) and, by comparing it with (55), we obtain

$$\tilde{\beta}_2 = \frac{\cos(2\Lambda + 2\pi/3) - 1/\sqrt{3}\sin(2\Lambda + 2\pi/3)}{1 - 2/\sqrt{3}\sin(2\Lambda + 2\pi/3)}\tilde{\alpha}_2 = \frac{1}{2}(1 - \sqrt{3}\cot\Lambda)\tilde{\alpha}_2,$$

$$\tilde{\beta}_3 = \frac{\cos(-2\Lambda + 2\pi/3) - 1/\sqrt{3}\sin(-2\Lambda + 2\pi/3)}{1 - 2/\sqrt{3}\sin(-2\Lambda + 2\pi/3)}\tilde{\alpha}_3 = \frac{1}{2}(1 + \sqrt{3}\cot\Lambda)\tilde{\alpha}_3.$$

Next, we continue the computations in order to determine $\tilde{\beta}_1$. For this, we compute $G(E_2, E_3)$ in two different ways, once using (12) and once using (9). We obtain, respectively,

$$G(E_2, E_3) = -\frac{1}{\sqrt{3}}JE_1 = -\frac{1}{3}(p2\tilde{\beta}_1, q\tilde{\beta}_1),$$

and

$$\begin{aligned}
G(E_2, E_3) &= G((p\tilde{\alpha}_2, q\tilde{\beta}_2), (p\tilde{\alpha}_3, q\tilde{\beta}_3)) \\
&= \frac{2}{3\sqrt{3}}(p(\tilde{\beta}_2 \times \alpha_3 + \tilde{\alpha}_2 \times \tilde{\beta}_3 + \tilde{\alpha}_2 \times \tilde{\alpha}_3 - 2\tilde{\beta}_2 \times \tilde{\beta}_3, \\
&\quad q(-\tilde{\beta}_2 \times \alpha_3 - \tilde{\alpha}_2 \times \tilde{\beta}_3 + 2\tilde{\alpha}_2 \times \tilde{\alpha}_3 - \tilde{\beta}_2 \times \tilde{\beta}_3) \\
&= \frac{2}{3\sqrt{3}}\left(p\left(2 - \frac{1}{2}(1 - 3\cot^2\Lambda)\right)\tilde{\alpha}_2 \times \tilde{\alpha}_3, \right. \\
&\quad \left. q\left(1 - \frac{1}{3}(1 - 3\cot^2\Lambda)\right)\tilde{\alpha}_2 \times \tilde{\alpha}_3\right) \\
&= \frac{1}{2\sqrt{3}}(1 + \cot^2\Lambda)(2p\tilde{\alpha}_2 \times \tilde{\alpha}_3, q\tilde{\alpha}_2 \times \tilde{\alpha}_3).
\end{aligned}$$

Hence, comparing both expressions, we get

$$\tilde{\beta}_1 = -\frac{\sqrt{3}}{2}\csc^2\Lambda \ \tilde{\alpha}_2 \times \tilde{\alpha}_3 = -\frac{\sqrt{3}}{4}\mathrm{e}^{-\omega}\alpha_2 \times \alpha_3.$$

Moreover, we also obtain

$$\tilde{\beta}_2 = \frac{1}{2\sqrt{2}}(1 - \sqrt{3}\cot\Lambda)e^{-\omega/2}\sin\Lambda(\cos t\alpha_2 + \sin t\alpha_3),$$

$$\tilde{\beta}_3 = \frac{1}{2\sqrt{2}}(1 + \sqrt{3}\cot\Lambda)e^{-\omega/2}\sin\Lambda(-\sin t\alpha_2 + \cos t\alpha_3).$$

We then take the inverse of (52) and deduce

$$\beta_1 = -\frac{\sqrt{3}\alpha_2 \times \alpha_3}{2\sqrt{3}e^\omega - 2\sin(2t)\tan(\Lambda)},$$

$$\beta_2 = \frac{1}{8}\left(e^{-\omega}\left(\mu_v + \omega_v - \frac{(\mu_u + \omega_u)\cos(2t)\tan(\Lambda)}{\sqrt{3}e^\omega - \sin(2t)\tan(\Lambda)} \right)\alpha_2 \right.$$

$$\left. \times \alpha_3 - 4(\sqrt{3}\cot(\Lambda)\cos(2t) + 1)\alpha_2 - 4\sqrt{3}\sin(2t)\cot\Lambda\alpha_3 \right),$$

$$\beta_3 = \frac{1}{8}\left(-e^{-\omega}\left(\mu_u + \omega_u + \frac{(\mu_v + \omega_v)\cos(2t)\tan(\Lambda)}{\sqrt{3}e^\omega - \sin(2t)\tan(\Lambda)} \right)\alpha_2 \right.$$

$$\left. \times \alpha_3 - 4\sqrt{3}\cot(\Lambda)\sin(2t)\alpha_2 + 4(1 + \sqrt{3}\cos(2t)\cot\Lambda)\alpha_3 \right).$$

By straightforward computations, it now follows

$$\frac{\partial\beta_1}{\partial u} - \frac{\partial\beta_2}{\partial t} - 2\beta_1 \times \beta_2 = 0,$$

$$\frac{\partial\beta_1}{\partial v} - \frac{\partial\beta_3}{\partial t} - 2\beta_1 \times \beta_3 = 0,$$

$$\frac{\partial\beta_3}{\partial u} - \frac{\partial\beta_2}{\partial v} - 2\beta_3 \times \beta_2 = 0,$$

from which we deduce that the integrability conditions for the immersion q are satisfied.

3.3.2. *Case 2. The minimal surface* $p(M)$ *is totally geodesic, that is,* $\sigma = 0$. As mentioned before, this means that $h_{13}^3 = 0$, $h_{12}^3 = ((\cos\Lambda\sin\Lambda)/(\sqrt{3}))$. The equations following from (23) and (24), just like in the first case, give

$$
\begin{aligned}
&h_{12}^2 = 0, \quad \omega_{11}^2 = 0, \qquad \omega_{21}^3 = -\frac{2 + \cos(2\Lambda)}{2\sqrt{3}}, \\
&h_{11}^2 = 0, \quad \omega_{11}^3 = 0, \qquad \omega_{22}^3 = -h_{22}^3\cot(2\Lambda), \\
&h_{11}^3 = 0, \quad \omega_{12}^3 = \frac{\sin^2\Lambda}{\sqrt{3}}, \quad \omega_{31}^2 = \frac{2 + \cos(2\Lambda)}{2\sqrt{3}}, \\
&\omega_{21}^2 = 0, \quad \omega_{31}^3 = 0, \qquad \omega_{32}^3 = -h_{23}^3\cot(2\Lambda)
\end{aligned}
\tag{57}
$$

and

$$
\begin{aligned}
E_1(\Lambda) &= 0, \\
E_2(\Lambda) &= h_{23}^3, \\
E_3(\Lambda) &= -h_{22}^3.
\end{aligned}
\tag{58}
$$

In this case, the equations of Codazzi become

$$E_1(h_{23}^3) = -\frac{\sqrt{3}}{2}h_{22}^3, \quad E_1(h_{22}^3) = \frac{\sqrt{3}}{2}h_{23}^3, \quad E_2(h_{22}^3) = -E_3(h_{23}^3) \tag{59}$$

and

$$-1 - (1 + 12(h_{22}^3)^2 + 12(h_{23}^3)^2)\cos(2\Lambda) + \cos(4\Lambda) + \cos(6\Lambda)$$
$$+ 4(E_2(h_{23}^3) - E_3(h_{22}^3))\sin(2\Lambda) = 0. \tag{60}$$

In what follows, we are going to introduce new vector fields on M by:

$$X_1 = \frac{4}{\sqrt{3}}E_1,$$
$$X_2 = -\frac{2h_{22}^3\csc^2\Lambda\sec\Lambda}{\sqrt{3}}E_1 + 2\csc\Lambda\ E_2, \tag{61}$$
$$X_3 = -\frac{2h_{23}^3\csc^2\Lambda\sec\Lambda}{\sqrt{3}}E_1 + 2\csc\Lambda\ E_3.$$

We can easily check

$$[X_1, X_2] = 2X_3,$$
$$[X_2, X_3] = 2X_1, \tag{62}$$
$$[X_3, X_1] = 2X_2.$$

Taking a canonical metric on M such that X_1, X_2 and X_3 have unit length and are mutually orthogonal, it follows from the Koszul formula that all connection components are determined. From (4.1), proposition 5.2 and its preceding paragraph in [**8**], it follows that we can locally identify M with $\mathbb{S}^3$ and we can consider X_1, X_2 and X_3 as the standard vector fields on $\mathbb{S}^3$ with

$$X_1(x) = xi,$$
$$X_2(x) = xj, \tag{63}$$
$$X_3(x) = xk.$$

Using the above formulas, the component p of the map can now be determined explicitly. First, we write

$$D_{X_i}p = p\alpha_i, \tag{64}$$

for $i = 1, 2, 3$, where D denotes the Euclidean covariant derivative. Of course, by theorem 1, $D_{X_1}p = 0$. Then, we may compute by (18)

$$(dp(X_2), 0) = \left(\frac{2\cos\Lambda}{\sqrt{3}} + 2\sin\Lambda\right)E_2 + \left(-2\cos\Lambda + \frac{\sin\Lambda}{\sqrt{3}}\right)JE_2,$$
$$(dp(X_3), 0) = \left(-\frac{2\cos\Lambda}{\sqrt{3}} + 2\sin\Lambda\right)E_3 + \left(2\cos\Lambda + \frac{\sin\Lambda}{\sqrt{3}}\right)JE_3$$

and we see that

$$\nabla_{X_1}^E(dp(X_2), 0) = (2dp(X_3), 0), \quad \nabla_{X_2}^E(dp(X_3), 0) = (0, 0),$$
$$\nabla_{X_1}^E(dp(X_3), 0) = (-2dp(X_2), 0), \quad \nabla_{X_3}^E(dp(X_2), 0) = (0, 0), \tag{65}$$
$$\nabla_{X_2}^E(dp(X_2), 0) = (0, 0), \quad \nabla_{X_3}^E(dp(X_3), 0) = (0, 0).$$

Moreover, it is straightforward to get

$$\langle dp(X_2), dp(X_2)\rangle = \langle dp(X_3), dp(X_3)\rangle = 4, \quad \langle dp(X_2), dp(X_3)\rangle = 0. \quad (66)$$

Next, we want to determine a system of differential equations satisfied by α_2 and α_3. For this, we consider $\mathbb{S}^3 \times \mathbb{S}^3 \in \mathbb{R}^4 \times \mathbb{R}^4$. On the one hand, we use (64) together with $D_X(dp(Y), 0) = (D_X dp(Y), 0)$. On the contrary, we use (8) and, therefore, we obtain

$$
\begin{array}{ll}
X_1(\alpha_2) = 2\alpha_3, & X_1(\alpha_3) = -2\alpha_2, \\
X_2(\alpha_2) = 0, & X_2(\alpha_3) = -\alpha_2 \times \alpha_3, \\
X_3(\alpha_2) = -\alpha_3 \times \alpha_2, & X_3(\alpha_3) = 0.
\end{array}
\quad (67)
$$

We choose a unit quaternion h such that at the point $p(x) = 1$ we have

$$\alpha_2(1) = -2hjh^{-1},$$

$$\alpha_3(1) = -2hkh^{-1},$$

$$\alpha_2 \times \alpha_3(1) = 4hih^{-1}.$$

Using (63), we can check that $\alpha_2 = -2hxjx^{-1}h^{-1}$, $\alpha_3 = -2hxkx^{-1}h^{-1}$ and $\alpha_2 \times \alpha_3 = 4hxix^{-1}h^{-1}$ are the unique solutions for the system of differential equations in (67):

$$
\begin{aligned}
X_1(\alpha_2) = X_1(-2hxjx^{-1}h^{-1}) &= -2(hX_1(x)jx^{-1}h^{-1} + hxjX_1(x^{-1})h^{-1}) \\
&= -4hxkx^{-1}h^{-1} \\
&= 2\alpha_3,
\end{aligned}
$$

$$
\begin{aligned}
X_1(\alpha_3) = X_1(-2hxkx^{-1}h^{-1}) &= -2(hX_1(x)kx^{-1}h^{-1} + hxkX_1(g^{-1})h^{-1}) \\
&= 4hxjx^{-1}h^{-1} \\
&= -2\alpha_2,
\end{aligned}
$$

$$
\begin{aligned}
X_2(\alpha_3) = X_2(-2hxkx^{-1}h^{-1}) &= -2(hxjkx^{-1}h^{-1} + hxk(-j)x^{-1}h^{-1}) \\
&= -4hxix^{-1}h^{-1} \\
&= -\alpha_2 \times \alpha_3,
\end{aligned}
$$

$$
\begin{aligned}
X_2(\alpha_2) = X_2(-2hxjx^{-1}h^{-1}) &= -2(hxjjx^{-1}h^{-1} + hxj(-j)x^{-1}h^{-1}) \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
X_3(\alpha_3) = X_3(-2hxkx^{-1}h^{-1}) &= -2(hxkkx^{-1}h^{-1} + hxk(-k)x^{-1}h^{-1}) \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
X_3(\alpha_2) = X_3(-2hxjx^{-1}h^{-1}) &= -2(hxkjx^{-1}h^{-1} + hxj(-k)x^{-1}h^{-1}) \\
&= 4hxix^{-1}h^{-1} \\
&= \alpha_2 \times \alpha_3.
\end{aligned}
$$

This in its turn implies

$$p(x) = -hixix^{-1}h^{-1} \quad (68)$$

is the unique solution of $X_i(p) = p\alpha_i$ with initial conditions $p(1) = 1$. Indeed, we have

$$
\begin{aligned}
X_1(p) &= X_1(-hixix^{-1}h^{-1}) = 0 = p\alpha_1, \\
X_2(p) &= X_2(-hixix^{-1}h^{-1}) = 2hixkx^{-1}h^{-1} \\
&= (-hixix^{-1}h^{-1})(-2hxjx^{-1}h^{-1}) = p\alpha_2, \\
X_3(p) &= X_3(-hixix^{-1}h^{-1}) = -2hixjx^{-1}h^{-1} \\
&= (-hixix^{-1}h^{-1})(-2hxkx^{-1}h^{-1}) = p\alpha_3.
\end{aligned}
$$

Before we can determine the second component q of the Lagrangian immersion, we need to explore the Codazzi equations further. First, we look at the system of differential equations for the function Λ in (59) and (60). Notice that by using the relations in (61), we have

$$
\begin{aligned}
X_1(\Lambda) &= 0, \\
X_2(\Lambda) &= 2h_{23}^3 \csc\Lambda, \\
X_3(\Lambda) &= -2h_{22}^3 \csc\Lambda,
\end{aligned}
\tag{69}
$$

where the last two equations can be seen as the definition for the functions h_{23}^3 and h_{22}^3. The first one is, of course, a condition for the unknown function of Λ. Three out of the four Codazzi equations then can be seen as integrability conditions for the existence of a solution of this system, whereas the last one reduces to

$$
\begin{aligned}
X_2(X_2(\Lambda)) + X_3(X_3(\Lambda)) &= (\cot(\Lambda) - \tan(\Lambda))((X_2(\Lambda))^2 \\
&\quad + (X_3(\Lambda))^2) + 4(1 + 2\cos(2\Lambda))\cot(\Lambda).
\end{aligned}
$$

Under the change of variable $\Lambda = \arctan(e^{2\beta})$, this equation simplifies to

$$
X_2(X_2(\beta)) + X_3(X_3(\beta)) = \frac{2(3 - e^{4\beta})}{e^{4\beta}}.
\tag{70}
$$

Note also that for $\Lambda = \pi/3$, we get the solution corresponding to example (1) in theorem 2, as it follows. From (61) and (69), we see

$$
\begin{aligned}
X_1 &= \frac{4}{\sqrt{3}} E_1, \\
X_2 &= \frac{4}{\sqrt{3}} E_2, \\
X_3 &= \frac{4}{\sqrt{3}} E_3.
\end{aligned}
$$

This implies that M has constant sectional curvature $\sqrt{3}/4$. Hence this corresponds to example (1) in theorem 2.

REMARK 1. Note that there exist at least locally many solutions of the system

$$X_1(\beta) = 0,$$

$$X_2(X_2(\beta)) + X_3(X_3(\beta)) = \frac{2(3 - e^{4\beta})}{e^{4\beta}}.$$

This can be seen by choosing special coordinates on the usual $\mathbb{S}^3$. We take

$$x_1 = \cos v \cos(t + u),$$
$$x_2 = \cos v \sin(t + u),$$
$$x_3 = \sin v \cos(u - t),$$
$$x_4 = \sin v \sin(u - t).$$

As given (63), at the point $x = (x_1, x_2, x_3, x_4)$, the vectors in the basis are

$$X_1(x) = (-x_2, x_1, x_4, -x_3),$$
$$X_2(x) = (-x_3, -x_4, x_1, x_2),$$
$$X_3(x) = (-x_4, x_3, -x_2, x_1),$$

it is straightforward to see

$$\partial t = X_1,$$
$$\partial u = \cos(2v)X_1 + \sin(2t)\sin(2v)X_2 + \cos(2t)\sin(2v)X_3,$$
$$\partial v = \cos(2t)X_2 - \sin(2t)X_3,$$

and conversely,

$$X_1 = \partial t,$$
$$X_2 = \frac{\sin(2t)}{\sin(2v)}\partial u - \sin(2t)\frac{\cos(2v)}{\sin(2v)}\partial t + \cos(2t)\partial v,$$
$$X_3 = \frac{\cos(2t)}{\sin(2v)}\partial u - \cos(2t)\frac{\cos(2v)}{\sin(2v)}\partial t - \sin(2t)\partial v.$$

At last, the equations in (70) become $((\partial)/(\partial t))\beta = 0$ and

$$\csc^2(2\mathrm{v})\frac{\partial^2 \beta}{\partial \mathrm{u}^2} + \frac{\partial^2 \beta}{\partial \mathrm{v}^2} + 2\cot(2\mathrm{v})\frac{\partial \beta}{\partial \mathrm{v}} = 2(3\mathrm{e}^{-4\beta} - 1). \tag{71}$$

The above differential equation is an elliptic quasilinear second order PDE. Hence, we can apply the Cauchy–Kowalevskaya theorem (see [**14**]). Therefore, if we start with an analytic regular curve without self-intersections and analytic Cauchy data along the curve, we locally have a unique (analytic) solution. Given that we can choose arbitrarily both the curve and the Cauchy data along the curve, locally there exist many solutions for the system in (71).

In the following part, we are going to determine the second part of the immersion. We start with an arbitrary solution of

$$X_1(\beta) = 0,$$

$$X_2(X_2(\beta)) + X_3(X_3(\beta)) = \frac{2(3 - e^{4\beta})}{e^{4\beta}}$$

and we are going to find a system of differential equations determining the immersion q. We define h_{22}^3 and h_{23}^3 as in (69) and such that $\Lambda = \arctan(e^{2\beta})$. First, we can write for each of the bases that we took, $\{E_i\}$ and $\{X_i\}$, the following:

$$\begin{array}{lll}
X_1(q) = q\beta_1, & X_1(p) = p\alpha_1, & \\
X_2(q) = q\beta_2, & X_2(p) = p\alpha_2, & \text{and} \\
X_3(q) = q\beta_3, & X_3(p) = p\alpha_3, &
\end{array} \qquad \begin{array}{ll}
E_1(q) = q\tilde{\beta}_1, & E_1(p) = p\tilde{\alpha}_1, \\
E_2(q) = q\tilde{\beta}_2, & E_2(p) = p\tilde{\alpha}_2, \\
E_3(q) = q\tilde{\beta}_3, & E_3(p) = p\tilde{\alpha}_3,
\end{array}$$

where $\alpha_1 = 0$ and α_2 and α_3 are as determined previously. Then, we prove as before

$$\tilde{\beta}_1 = -\frac{\sqrt{3}}{2\sin^2\Lambda}\tilde{\alpha}_2 \times \tilde{\alpha}_3,$$

$$\tilde{\beta}_2 = \frac{\cos(2\Lambda + 2\pi/3) - 1/\sqrt{3}\sin(2\Lambda + 2\pi/3)}{1 - 2/\sqrt{3}\sin(2\Lambda + 2\pi/3)}\tilde{\alpha}_2 = \frac{1}{2}(1 - \sqrt{3}\cot\Lambda)\tilde{\alpha}_2, \qquad (72)$$

$$\tilde{\beta}_3 = \frac{\cos(-2\Lambda + 2\pi/3) - 1/\sqrt{3}\sin(-2\Lambda + 2\pi/3)}{1 - 2/\sqrt{3}\sin(-2\Lambda + 2\pi/3)}\tilde{\alpha}_3 = \frac{1}{2}(1 + \sqrt{3}\cot\Lambda)\tilde{\alpha}_3$$

and we continue the computations in order to find the system of differential equations for the immersion q in terms of the basis $\{X_i\}$. As we identify $df(X_1) \equiv X_1$, we have

$$D_{X_1}f = (X_1(p), X_1(q)) = (0, q\beta_1) \equiv X_1 \overset{(61)}{=} \frac{4}{\sqrt{3}}E_1 = \frac{4}{\sqrt{3}}(p\tilde{\alpha}_1, q\tilde{\beta}_1).$$

Therefore, $\beta_1 = 4/\sqrt{3}\tilde{\beta}_1$. We may compute similarly for $D_{X_2}f$ and $D_{X_3}f$ and find

$$\begin{cases} \beta_2 = 2\csc\Lambda\,\tilde{\beta}_2 - \dfrac{2}{\sqrt{3}}h_{22}^3\csc^2\Lambda\sec\Lambda\tilde{\beta}_1, \\ \beta_3 = 2\csc\Lambda\,\tilde{\beta}_3 - \dfrac{2}{\sqrt{3}}h_{23}^3\csc^2\Lambda\sec\Lambda\tilde{\beta}_1, \end{cases} \qquad \begin{cases} \tilde{\alpha}_2 = -\dfrac{1}{\csc\Lambda}hxjx^{-1}h^{-1}, \\ \tilde{\alpha}_3 = -\dfrac{1}{\csc\Lambda}hxkx^{-1}h^{-1} \end{cases}$$

and

$$\tilde{\beta}_1 = -\frac{\sqrt{3}}{2}hxix^{-1}h^{-1}.$$

Using now relations (72), we may express

$$\beta_2 = -(1 - \sqrt{3}\cot\Lambda)\,hxjx^{-1}h^{-1} + h_{22}^3\csc^2\Lambda\sec\Lambda\,hxix^{-1}h^{-1},$$

$$\beta_3 = -(1 + \sqrt{3}\cot\Lambda)\,hxkx^{-1}h^{-1} + h_{23}^3\csc^2\Lambda\sec\Lambda\,hxix^{-1}h^{-1}.$$

Finally, as $X_i(q) = q\beta_i$, we find

$$
\begin{cases}
X_1(q) = -2qhxix^{-1}h^{-1}, \\
X_2(q) = q(h_{22}^3 \csc^2 \Lambda \sec \Lambda \; hxix^{-1}h^{-1} - (1 - \sqrt{3}\cot\Lambda) \; hxjx^{-1}h^{-1}), \\
X_3(q) = q(h_{23}^3 \csc^2 \Lambda \sec \Lambda \; hxix^{-1}h^{-1} - (1 + \sqrt{3}\cot\Lambda) \; hxkx^{-1}h^{-1}),
\end{cases}
$$

which, given (69) and $\Lambda = \arctan(e^{2\beta})$, is equivalent to

$$
\begin{cases}
X_1(q) = -2qhxix^{-1}h^{-1}, \\
X_2(q) = q(-X_3(\beta)hxix^{-1}h^{-1} - (1 - \sqrt{3}e^{-2\beta}) \; hxjx^{-1}h^{-1}), \\
X_3(q) = q(X_2(\beta) \; hxix^{-1}h^{-1} - (1 + \sqrt{3}e^{-2\beta}) \; hxkx^{-1}h^{-1}).
\end{cases} \tag{73}
$$

By straightforward computations, one may see that $X_i(X_j(q)) - X_j(X_i(q)) = [X_i, X_j](q)$ hold for $i, j = 1, 2, 3$. Therefore, the immersion f is completely determined by (68) and (73).

3.3.3. *Case 3. The minimal surface $p(M)$ is not totally geodesic, but the map $\tilde{\mathcal{P}}$ is not an immersion.* As mentioned before, this means

$$
h_{12}^3 = -\frac{\sin(2\Lambda)}{\sqrt{3}}. \tag{74}
$$

Therefore, the equations in §3.2 which follow from (23) and (24) become

$$
h_{12}^2 = -h_{13}^3, \qquad \omega_{12}^3 = \frac{1 + 2\cos(2\Lambda)}{2\sqrt{3}}, \quad \omega_{21}^2 = \omega_{31}^3 = -h_{13}^3 \cot\Lambda,
$$

$$
h_{11}^2 = h_{11}^3 = 0, \quad \omega_{21}^3 = \frac{1 + 2\cos(2\Lambda)}{2\sqrt{3}}, \quad \omega_{31}^2 = -\frac{1 + 2\cos(2\Lambda)}{2\sqrt{3}},
$$

$$
\omega_{11}^2 = \omega_{11}^3 = 0, \quad \omega_{22}^3 = -h_{22}^3 \cot(2\Lambda), \quad \omega_{32}^3 = -h_{23}^3 \cot(2\Lambda)
$$

and

$$
E_1(\Lambda) = h_{13}^3, \quad E_2(\Lambda) = h_{23}^3, \quad E_3(\Lambda) = -h_{22}^3. \tag{75}
$$

Moreover, the equations of Codazzi in (39) yield $h_{13}^3 = 0$ and, therefore, $\omega_{21}^2 = \omega_{31}^3 = 0$. The first two equations in (40) imply

$$
E_1(h_{23}^3) = 0 \quad \text{and} \quad E_1(h_{22}^3) = 0,
$$

while the next three ones vanish identically. The last two equations in (40) become

$$
E_2(h_{22}^3) = -E_3(h_{23}^3) \tag{76}
$$

and

$$
\begin{aligned}
& -1 - [1 + 6(h_{22}^3)^2 + 6(h_{23}^3)^2]\cos 2\Lambda + \cos 4\Lambda + \cos 6\Lambda \\
& + 2[-E_3(h_{22}^3) + E_2(h_{23}^3)]\sin 2\Lambda = 0,
\end{aligned} \tag{77}
$$

respectively. The Lie brackets of the vector fields E_1, E_2, E_3 give

$$[E_1, E_2] = 0,$$
$$[E_1, E_3] = 0,$$
$$[E_2, E_3] = -\frac{1 + 2\cos(2\Lambda)}{\sqrt{3}} E_1 + h_{22}^3 \cot(2\Lambda) E_2 + h_{23}^3 \cot(2\Lambda) E_3.$$

Next, we take new vector fields X_1, X_2, X_3 of the form

$$X_1 = E_1,$$
$$X_2 = \frac{\sqrt{2}(h_{22}^3 - h_{23}^3)}{3^{3/4}(\sin(2\Lambda))^{3/2}} E_1 + \frac{\sqrt{2}}{3^{1/4}\sqrt{\sin(2\Lambda)}} E_2 - \frac{\sqrt{2}}{3^{1/4}\sqrt{\sin(2\Lambda)}} E_3, \qquad (78)$$
$$X_3 = \frac{\sqrt{2}(h_{22}^3 + h_{23}^3)}{3^{3/4}(\sin(2\Lambda))^{3/2}} E_1 + \frac{\sqrt{2}}{3^{1/4}\sqrt{\sin(2\Lambda)}} E_2 + \frac{\sqrt{2}}{3^{1/4}\sqrt{\sin(2\Lambda)}} E_3.$$

We can easily check that $[X_1, X_2] = 0$, $[X_1, X_3] = 0$ and $[X_2, X_3] = 0$, therefore, by the lemma on page 155 in [3], we know that there exist coordinates $\{t, u, v\}$ on M such that

$$X_1 = \partial t,$$
$$X_2 = \partial u,$$
$$X_3 = \partial v.$$

Using (75), we obtain:

$$\Lambda_t = 0,$$
$$\Lambda_u = \frac{h_{22}^3 + h_{23}^3}{3^{1/4}\sqrt{\cos\Lambda\sin\Lambda}},$$
$$\Lambda_v = \frac{-h_{22}^3 + h_{23}^3}{3^{1/4}\sqrt{\cos\Lambda\sin\Lambda}}.$$

Furthermore, we express h_{22}^3 and h_{23}^3 from the previous relations as

$$h_{22}^3 = \frac{1}{2}3^{1/4}(\Lambda_u - \Lambda_v)\sqrt{\cos\Lambda\sin\Lambda},$$
$$h_{23}^3 = \frac{1}{2}3^{1/4}(\Lambda_u + \Lambda_v)\sqrt{\cos\Lambda\sin\Lambda}$$

and therefore, the expression of (78) becomes

$$X_1 = E_1,$$
$$X_2 = -\frac{\Lambda_v\csc(2\Lambda)}{\sqrt{3}} E_1 + \frac{1}{3^{1/4}\sqrt{\cos\Lambda\sin\Lambda}} E_2 - \frac{1}{3^{1/4}\sqrt{\cos\Lambda\sin\Lambda}} E_3, \qquad (79)$$
$$X_3 = \frac{\Lambda_u\csc(2\Lambda)}{\sqrt{3}} E_1 + \frac{1}{3^{1/4}\sqrt{\cos\Lambda\sin\Lambda}} E_2 + \frac{1}{3^{1/4}\sqrt{\cos\Lambda\sin\Lambda}} E_3.$$

Finally, by straightforward computations, one may see that equation (77) becomes

$$-\sqrt{3}(\Lambda_u^2 + \Lambda_v^2)\cos(2\Lambda) + 3^{1/4}(E_2(\Lambda_u) - E_3(\Lambda_u)$$

$$+ E_2(\Lambda_v) + E_3(\Lambda_v))\sqrt{\cos\Lambda\sin\Lambda} - 2(\sin(2\Lambda) + \sin(4\Lambda)) = 0. \tag{80}$$

We compute $dp(\partial u)$ and $dp(\partial v)$:

$$dp(\partial u) = \frac{\sqrt{3} - 2\cos(2\Lambda + \pi/6)}{3^{3/4}\sqrt{2}\sqrt{\sin(2\Lambda)}}E_2 + \frac{2\sin(2\Lambda + \pi/3) - \sqrt{3}}{3^{3/4}\sqrt{2}\sqrt{\sin(2\Lambda)}}E_3$$

$$+ \frac{2\cos(2\Lambda + 2\pi/3) + 1}{3^{3/4}\sqrt{2}\sqrt{\sin(2\Lambda)}}JE_2 + \frac{2\cos(2\Lambda + \pi/3) - 1}{3^{3/4}\sqrt{2}\sqrt{\sin(2\Lambda)}}JE_3,$$

$$dp(\partial v) = \frac{\sqrt{3} - 2\cos(2\Lambda + \pi/6)}{3^{3/4}\sqrt{2}\sqrt{\sin(2\Lambda)}}E_2 - \frac{2\sin(2\Lambda + \pi/3) - \sqrt{3}}{3^{3/4}\sqrt{2}\sqrt{\sin(2\Lambda)}}E_3$$

$$+ \frac{2\cos(2\Lambda + 2\pi/3) + 1}{3^{3/4}\sqrt{2}\sqrt{\sin(2\Lambda)}}JE_2 - \frac{2\cos(2\Lambda + \pi/3) - 1}{3^{3/4}\sqrt{2}\sqrt{\sin(2\Lambda)}}JE_3,$$

and we remark that they are mutually orthogonal and that their length is $((2\tan\Lambda)/(\sqrt{3}))$. So, as u, v are isothermal coordinates on the surface, for which $\langle \partial u, \partial u \rangle = \langle \partial v, \partial v \rangle = 2e^\omega$, we obtain

$$e^\omega = \frac{\tan\Lambda}{\sqrt{3}}. \tag{81}$$

On the one hand, for $z = x + Iy$ as in §3.1, we may compute $dp(\partial z)$:

$$dp(\partial z) = \frac{1}{2}\left[dp(\partial u) - I \cdot dp(\partial v)\right]$$

$$= \frac{1}{2\sqrt{2}\,3^{3/4}\sqrt{\sin(2\Lambda)}}\left[(1 - I)\left(\sqrt{3} - 2\cos\left(2\Lambda + \frac{\pi}{6}\right)\right)E_2\right.$$

$$- (1 + I)\left(\sqrt{3} - 2\sin\left(2\Lambda + \frac{\pi}{3}\right)\right)E_3$$

$$+ (1 - I)\left(2\cos\left(2\Lambda + \frac{2\pi}{3}\right) + 1\right)JE_2$$

$$\left. + (1 + I)\left(2\cos\left(2\Lambda + \frac{\pi}{3}\right) - 1\right)JE_3\right].$$

As

$$\sqrt{3} - 2\cos\left(2\Lambda + \frac{\pi}{6}\right) = 2\sin\Lambda(\sqrt{3}\sin\Lambda + \cos\Lambda),$$

$$2\sin\left(2\Lambda + \frac{\pi}{3}\right) - \sqrt{3} = 2\sin\Lambda(\cos\Lambda - \sqrt{3}\sin\Lambda),$$

$$2\cos\left(2\Lambda + \frac{2\pi}{3}\right) + 1 = 2\sin\Lambda(\sin\Lambda - \sqrt{3}\cos\Lambda),$$

$$2\cos\left(2\Lambda + \frac{\pi}{3}\right) - 1 = -2\sin\Lambda(\sin\Lambda + \sqrt{3}\cos\Lambda),$$

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we finally have

$$
dp(\partial z) = \frac{\sin \Lambda}{\sqrt{2}\, 3^{3/4} \sqrt{\sin(2\Lambda)}} \Big[(1 - I) \left(\sqrt{3} \sin \Lambda + \cos \Lambda \right) E_2
$$
$$
+ (1 + I) \left(\cos \Lambda - \sqrt{3} \sin \Lambda \right) E_3 + (1 - I) \left(\sin \Lambda - \sqrt{3} \cos \Lambda \right) J E_2
$$
$$
- (1 + I) \left(\sin \Lambda + \sqrt{3} \cos \Lambda \right) J E_3 \Big].
$$

Moreover, from (81), it follows that $\omega_z = ((1)/(\sin(2\Lambda)))(\Lambda_u - i\Lambda_v)$.
On the contrary, we may compute $\nabla^E_{\partial z} dp(\partial z)$ using the Euclidean connection ∇^E:

$$
\nabla^E_{\partial z} dp(\partial z) = -\frac{1}{\sqrt{3}} E_1 + \frac{e^{-i\pi/4} \sin^2 \Lambda \left(\sqrt{3} \cot \Lambda + 3 \right) (\Lambda_u - i\Lambda_v)}{3 \sqrt[4]{3} \sin^{3/2}(2\Lambda)} E_2
$$
$$
+ \frac{e^{-i\pi/4} \sin \Lambda (\Lambda_v + i\Lambda_u) \left(\sqrt{3} \cos \Lambda - 3 \sin \Lambda \right)}{3 \sqrt[4]{3} \sin^{3/2}(2\Lambda)} E_3 + J E_1
$$
$$
+ \frac{e^{-i\pi/4} \sin \Lambda (\Lambda_u - i\Lambda_v) \left(\sqrt{3} \sin \Lambda - 3 \cos \Lambda \right)}{3 \sqrt[4]{3} \sin^{3/2}(2\Lambda)} J E_2
$$
$$
- \frac{(1/3 + i/3) \sin \Lambda (\Lambda_u - i\Lambda_v) \left(\sqrt{3} \sin \Lambda + 3 \cos \Lambda \right)}{\sqrt{2} \sqrt[4]{3} \sin^{3/2}(2\Lambda)} J E_3.
$$

From the previous computations we see, indeed, that

$$
\nabla^E_{\partial z} dp(\partial z) = -N + \omega_z dp(\partial z),
$$

which corresponds to (15). From here, we remark the component in the direction of the normal $N = \xi$ (see §3.2) and we see that the choice of coordinates $\{t, u, v\}$ following from (78) is the right one, as we have indeed $\sigma(\partial z, \partial z) = -1$, as in §3.1. Using (81), together with the fact that by taking the inverse in (79), we have

$$
E_1 = \partial t,
$$
$$
E_2 = \frac{3^{1/4} \sqrt{\sin(2\Lambda)}}{2\sqrt{2}} \left(\frac{\Lambda_v - \Lambda_u}{\sqrt{3} \sin(2\Lambda)} \partial t + \partial u + \partial v \right),
$$
$$
E_3 = -\frac{3^{1/4} \sqrt{\sin(2\Lambda)}}{2\sqrt{2}} \left(\frac{\Lambda_v + \Lambda_u}{\sqrt{3} \sin(2\Lambda)} \partial t + \partial u - \partial v \right),
$$

we may prove that equation (80) is equivalent to the Sinh–Gordon equation in (16), which characterizes the minimal surface.

Reverse construction

Let S be a minimal surface given by $p : S \to \mathbb{S}^3 \subset \mathbb{R}^4$, on which we take isothermal coordinates u and v as in §3.1. Hence, we have a solution ω of the Sinh–Gordon

equation $\Delta \omega = -8 \sinh \omega$. Next, we define a function $\Lambda \in (0, \pi/2)$ such that

$$e^\omega = \frac{\tan \Lambda}{\sqrt{3}}.$$

REMARK 2. If $\omega = 0$, then $\Lambda = \pi/3$, which corresponds to example (2) in theorem 2.

We then define a metric on an open part of the unit frame bundle of the surface by assuming that the vectors

$$
\begin{aligned}
E_1 &= \partial t, \\
E_2 &= \frac{\sqrt{3}e^{\omega/2}}{2\sqrt{1+3e^{2\omega}}} \left(\frac{\omega_v - \omega_u}{2\sqrt{3}} \partial t + \partial u + \partial v \right), \\
E_3 &= -\frac{\sqrt{3}e^{\omega/2}}{2\sqrt{1+3e^{2\omega}}} \left(\frac{\omega_v + \omega_u}{2\sqrt{3}} \partial t + \partial u - \partial v \right)
\end{aligned}
\tag{82}
$$

form an orthonormal basis. Next, we want to determine the Lagrangian immersion

$$
\begin{aligned}
f &: S \times I \to \mathbb{S}^3 \times \mathbb{S}^3 \\
(u, v, t) &\mapsto f(u, v, t) = (p(u, v, t), q(u, v, t)),
\end{aligned}
$$

for which we already know that the first component is the given minimal surface p. We write for both bases

$$
\begin{array}{lll}
\dfrac{\partial}{\partial t}(q) = q\beta_1, & \dfrac{\partial}{\partial t}(p) = p\alpha_1, & \\[2mm]
\dfrac{\partial}{\partial u}(q) = q\beta_2, & \dfrac{\partial}{\partial u}(p) = p\alpha_2, \quad \text{and} \\[2mm]
\dfrac{\partial}{\partial v}(q) = q\beta_3, & \dfrac{\partial}{\partial v}(p) = p\alpha_3 &
\end{array}
\qquad
\begin{array}{ll}
E_1(q) = q\tilde{\beta}_1, & E_1(p) = p\tilde{\alpha}_1, \\[2mm]
E_2(q) = q\tilde{\beta}_2, & E_2(p) = p\tilde{\alpha}_2, \\[2mm]
E_3(q) = q\tilde{\beta}_3, & E_3(p) = p\tilde{\alpha}_3.
\end{array}
$$

Note that $\alpha_1 = 0$ and α_2 and α_3 are determined by the minimal surface. In particular, α_2 and α_3 are mutually orthogonal imaginary quaternions with length squared $2e^\omega$. From the derivates of p in the latter relations together with (82), we obtain

$$
\begin{aligned}
\tilde{\alpha}_1 &= 0, \\
\tilde{\alpha}_2 &= \frac{\sqrt{3}e^{\omega/2}}{2\sqrt{1+3e^{2\omega}}}(\alpha_2 + \alpha_3), \\
\tilde{\alpha}_2 &= -\frac{\sqrt{3}e^{\omega/2}}{2\sqrt{1+3e^{2\omega}}}(\alpha_2 - \alpha_3).
\end{aligned}
\tag{83}
$$

We then follow the same steps as in *Case 1* and obtain

$$\tilde{\beta}_1 = -\frac{\sqrt{3}e^{-\omega}}{4}\alpha_2 \times \alpha_3,$$

$$\tilde{\beta}_2 = \frac{\sqrt{3}(e^{\omega/2} - e^{-\omega/2})}{4\sqrt{1 + 3e^{2\omega}}}(\alpha_2 + \alpha_3), \tag{84}$$

$$\tilde{\beta}_3 = -\frac{\sqrt{3}(e^{\omega/2} - e^{-\omega/2})}{4\sqrt{1 + 3e^{2\omega}}}(\alpha_2 - \alpha_3).$$

Finally, we take the inverse of the matrix which give $\{E_i\}$ on the basis of $\{\partial t, \partial u, \partial v\}$ in (82) and obtain

$$\beta_1 = -\frac{\sqrt{3}e^{-\omega}}{4}\alpha_2 \times \alpha_3,$$

$$\beta_2 = \frac{e^{-\omega}}{8}(4e^{\omega}\alpha_2 - 4\alpha_3 + \omega_v\alpha_2 \times \alpha_3),$$

$$\beta_3 = -\frac{e^{-\omega}}{8}(4\alpha_2 - 4e^{\omega}\alpha_3 + \omega_u\alpha_2 \times \alpha_3).$$

By straightforward computations, it now follows that

$$\frac{\partial\beta_1}{\partial u} - \frac{\partial\beta_2}{\partial t} - 2\beta_1 \times \beta_2 = 0,$$

$$\frac{\partial\beta_1}{\partial v} - \frac{\partial\beta_3}{\partial t} - 2\beta_1 \times \beta_3 = 0,$$

$$\frac{\partial\beta_3}{\partial u} - \frac{\partial\beta_2}{\partial v} - 2\beta_3 \times \beta_2 = 0,$$

from which we deduce that the integrability conditions for the immersion q are satisfied.

4. Conclusion

The results in §3.3 can now be summarized in the following theorems.

THEOREM 4. *Let ω and μ be solutions of, respectively, the Sinh–Gordon equation $\Delta\omega = -8\sinh\omega$ and the Liouville equation $\Delta\mu = -e^{\mu}$ on an open simply connected domain $U \subseteq \mathbb{C}$ and let $p : U \to \mathbb{S}^3$ be the associated minimal surface with complex coordinate z such that $\sigma(\partial z, \partial z) = -1$.*
Let $V = \{(z, t) \mid z \in U, t \in \mathbb{R}, e^{\omega+\mu} - 2 - 2\cos(4t) > 0\}$ and let Λ be a solution of

$$\left(\frac{2\sqrt{3}e^{\omega}}{\tan\Lambda} - 2\sin(2t)\right) = e^{\omega+\mu} - 2 - 2\cos(4t)$$

on V. *Then, there exists a Lagrangian immersion* $f : V \to \mathbb{S}^3 \times \mathbb{S}^3 : x \mapsto (p(x), q(x))$, *where* q *is determined by*

$$\frac{\partial q}{\partial t} = -\frac{\sqrt{3}}{2\sqrt{3}e^{\omega} - 2\sin(2t)\tan\Lambda} \, q \, \alpha_2 \times \alpha_3,$$

$$\frac{\partial q}{\partial u} = \frac{1}{8}\left(e^{-\omega}\left(\mu_v + \omega_v - \frac{(\mu_u + \omega_u)\cos(2t)\tan\Lambda}{\sqrt{3}e^{\omega} - \sin(2t)\tan\Lambda}\right) q \, \alpha_2 \times \alpha_3 \right.$$
$$\left. - 4(\sqrt{3}\cot\Lambda\cos(2t) + 1) \, q \, \alpha_2 - 4\sqrt{3}\sin(2t)\cot\Lambda \, q \, \alpha_3\right),$$

$$\frac{\partial q}{\partial v} = \frac{1}{8}\left(-e^{-\omega}\left(\mu_u + \omega_u + \frac{(\mu_v + \omega_v)\cos(2t)\tan\Lambda}{\sqrt{3}e^{\omega} - \sin(2t)\tan\Lambda}\right) q \, \alpha_2 \times \alpha_3 \right.$$
$$\left. - 4\sqrt{3}\cot\Lambda\sin(2t) \, q \, \alpha_2 + 4(1 + \sqrt{3}\cos(2t)\cot\Lambda) \, q \, \alpha_3\right),$$

where $\alpha_2 = \bar{p}p_u$ *and* $\alpha_3 = \bar{p}p_v$.

THEOREM 5. *Let* X_1, X_2, X_3 *be the standard vector fields on* $\mathbb{S}^3$. *Let* β *be a solution of the differential equations*

$$X_1(\beta) = 0,$$

$$X_2(X_2(\beta)) + X_3(X_3(\beta)) = \frac{2(3 - e^{4\beta})}{e^{4\beta}},$$

on a connected, simply connected open subset U *of* $\mathbb{S}^3$.
Then there exists a Lagrangian immersion $f : U \to \mathbb{S}^3 \times \mathbb{S}^3 : x \mapsto (p(x), q(x))$, *where* $p(x) = xix^{-1}$ *and* q *is determined by*

$$X_1(q) = -2qhxix^{-1}h^{-1},$$

$$X_2(q) = q\left(-X_3(\beta)hxix^{-1}h^{-1} - (1 - \sqrt{3}e^{-2\beta}) \, hxjx^{-1}h^{-1}\right),$$

$$X_3(q) = q\left(X_2(\beta) \, hxix^{-1}h^{-1} - (1 + \sqrt{3}e^{-2\beta}) \, hxkx^{-1}h^{-1}\right).$$

Note that in the previous theorem the image of p is a totally geodesic surface in $\mathbb{S}^3$.

THEOREM 6. *Let* ω *be a solution of the Sinh–Gordon equation* $\Delta\omega = -8\sinh\omega$ *on an open connected domain of* U *in* $\mathbb{C}$ *and let* $p : U \to \mathbb{S}^3$ *be the associated minimal surface with complex coordinate* z *such that* $\sigma(\partial z, \partial z) = -1$. *Then, there exists a Lagrangian immersion* $f : U \times I \to \mathbb{S}^3 \times \mathbb{S}^3 : x \mapsto (p(x), q(x))$, *where* q *is determined by*

$$\frac{\partial q}{\partial t} = -\frac{\sqrt{3}e^{-\omega}}{4}q \, \alpha_2 \times \alpha_3,$$

$$\frac{\partial q}{\partial u} = \frac{e^{-\omega}}{8}(4e^{\omega}q\alpha_2 - 4q\alpha_3 + \omega_v q \, \alpha_2 \times \alpha_3),$$

$$\frac{\partial q}{\partial v} = -\frac{e^{-\omega}}{8}(4q\alpha_2 - 4e^{\omega}q\alpha_3 + \omega_u q \, \alpha_2 \times \alpha_3).$$

where $\alpha_2 = \bar{p}p_u$ *and* $\alpha_3 = \bar{p}p_v$.

THEOREM 7. *Let $f : M \to \mathbb{S}^3 \times \mathbb{S}^3 : x \mapsto (p(x), q(x))$ be a Lagrangian immersion such that p has nowhere maximal rank. Then every point x of an open dense subset of M has a neighbourhood U such that $f|_U$ is obtained as described in theorems 4, 5 or 6.*

Acknowledgements

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