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ON THE DIMENSION OF GROUPS THAT SATISFY CERTAIN CONDITIONS ON THEIR FINITE SUBGROUPS

LUIS JORGE SÁNCHEZ SALDAÑA
Departamento de Matemáticas, Facultad de Ciencias, Universidad Nacional Autónoma de México, Circuito Exterior S/N, Cd. Universitaria, Colonia Copilco el Bajo, Delegación Coyoacán, Mexico City 04510, Mexico
e-mail: luisjorge@ciencias.unam.mx

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Abstract. We say a group G satisfies properties (M) and (NM) if every nontrivial finite subgroup of G is contained in a unique maximal finite subgroup, and every nontrivial finite maximal subgroup is self-normalizing. We prove that the Bredon cohomological dimension and the virtual cohomological dimension coincide for groups that admit a cocompact model for $\underline{E}G$ and satisfy properties (M) and (NM). Among the examples of groups satisfying these hypothesis are cocompact and arithmetic Fuchsian groups, one-relator groups, the Hilbert modular group, and 3-manifold groups.

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1. Introduction. For a group G, consider the following properties:

(M) Every nontrivial finite subgroup of G is contained in a unique maximal finite subgroup of G.

(NM) If M is a nontrivial maximal finite subgroup of G, then $N_G(M) = M$, where $N_G(M)$ denotes the normalizer of M in G.

In this paper, $\mathcal{F}$ will always denote the *family of finite subgroups* of G. A *model for the classifying space* $E_{\mathcal{F}}G$ (usually denoted $\underline{E}$G) is a G-CW-complex X such that every isotropy group is finite and the fixed point set X^H is contractible for every finite subgroup H of G. Equivalently, X is a model for $\underline{E}G$ if for every G-CW-complex Y with finite isotropy groups, there exists a map, unique up to G-homotopy, $Y \to X$. In particular, any two models for $\underline{E}G$ are G-homotopy equivalent. We say G *is of type* $\underline{\mathrm{F}}$ if G admits a cocompact model for $\underline{E}G$.

The *orbit category* $\mathcal{O}_{\mathcal{F}}G$ *of* G *with respect to* $\mathcal{F}$ is the category of homogeneous G-sets G/F, where $F \in \mathcal{F}$, and morphisms are given by G-maps. A Bredon module is a contravariant functor from $\mathcal{O}_{\mathcal{F}}G$ to the category of abelian groups, and a morphism of Bredon modules is a natural transformation. The category of Bredon modules, for G and $\mathcal{F}$ choosen, is abelian with enough projectives. The *Bredon cohomological dimension* (or proper cohomological dimension) $\underline{\mathrm{cd}}(G)$ of G is the length of the shortest projective resolution of the constant module $\mathbb{Z}_{\mathcal{F}}$. If G is torsion-free, then $\underline{\mathrm{cd}}(G)$ is the classical cohomological dimension $\mathrm{cd}(G)$ of G.

On the other hand, provided that G is a virtually torsion-free group, the *virtual cohomological dimension* $\mathrm{vcd}(G)$ of G is, by definition, the cohomological dimension of a finite index torsion-free group. By a well-known theorem due to Serre $\mathrm{vcd}(G)$ does not depend on the finite index subgroup of G that we choose, hence $\mathrm{vcd}(G)$ is a well-defined invariant of G.

For every group G, we have the following inequality $\mathrm{vcd}(G) \leq \underline{\mathrm{cd}}(G)$. This inequality can be strict due to examples constructed in [7, 13, 14]. On the other hand $\mathrm{vcd}(G) = \underline{\mathrm{cd}}(G)$ for the following classes of groups: elementary amenable groups of type FP_∞ [10], $\mathrm{SL}_n(\mathbb{Z})$ [3], $\mathrm{Out}(F_n)$ [19], the mapping class group of any surface with boundary components and punctures [2], any lattice in a classical simple Lie group [1], any lattice in the group of isometries of a symmetric space of non-compact type without Euclidean factors [11], and groups acting chamber transitively on a Euclidean building [6].

In this note, we prove $\mathrm{vcd}(G) = \underline{\mathrm{cd}}(G)$ for groups of type $\underline{\mathrm{F}}$ that satisfy properties (M) and (NM). This implies, due to the main theorem of [12], the existence of a cocompact model for $\underline{E}G$ of dimension $\max\{3, \mathrm{vcd}(G)\}$. Among the examples of groups that satisfy these properties, we have cocompact Fuchsian groups and arithmetic Fuchsian groups, one-relator groups, the Hilbert modular group, and 3-manifold groups. It is worth noticing that the class of groups for which our main theorem applies is closed under taking free products, this is a particular case of Theorem 3.1.

The proof is very short and relies on [1, Corollary 3.4] (Theorem 2.1 below), which reduces the proof of the main theorem (Theorem 2.5) to computing the dimension of certain fixed point sets of a cocompact X model for $\underline{E}G$. For groups satisfying properties (M) and (NM), we prove that X can be chosen in such a way that the relevant fixed point sets are one-point spaces (Lemma 2.4), hence they have dimension 0.

2. Preliminaries and main theorem. We have the following criterion that we will use to prove $\mathrm{vcd}(G) = \underline{\mathrm{cd}}(G)$ under the hypothesis of our main theorem.

If G acts on a space X, and A is an element of G, we denote X^A the set consisting of all elements of X fixed by A. In a similar way, we define X^H for a subgroup H of G we define.

THEOREM 2.1 ([1, Corollary 3.4]). *Let G be a virtually torsion-free group of type $\underline{\mathrm{F}}$ with a cocompact model X for $\underline{E}G$. Let K be the kernel of the G-action on X. If* $\dim(X^A) < \mathrm{vcd}(G)$ *for every finite order element A of $G \setminus K$, then* $\mathrm{vcd}(G) = \underline{\mathrm{cd}}(G)$.

REMARK 2.2. Note that if G is as in Theorem 2.1, then $\mathrm{vcd}(G)$ is finite. Let H be a finite index torsion-free subgroup of G. Then, a cocompact model X for $\underline{E}G$ is, by restriction, a cocompact model for $\underline{E}H = EH$. Therefore, $\mathrm{vcd}(G) = \mathrm{cd}(H) \leq \dim(X/H) < \infty$.

Our next lemma characterizes properties (M) and (NM) in terms of the existence of a model for $\underline{E}G$ with small fix point sets. Recall that $\mathcal{F}$ is the family of finite subgroups of G.

LEMMA 2.3. *Let G be a group. Then the following to conditions are equivalent:*

(1) There exists a model X for $\underline{E}G$ with the property that X^H consists of exactly one point for every nontrivial finite subgroup H of G.
(2) Properties (M) and (NM) are true for G.

Proof. Assume X is a model for $\underline{E}G$ such that $X^H = \{x_H\}$ for every nontrivial finite subgroup H of G. Let K be a nontrivial finite subgroup of G. Then K is contained in the stabilizer S of x_H for a unique $H \in \mathcal{F}$. Note that S is a maximal finite subgroup of G. In fact, if S is contained in $F \in \mathcal{F}$, then F must fix a unique point y of X since X^F consists of a point. In particular, S fixes y. Hence by uniqueness, F and S fix the same point of X. Therefore, $F \leq S$. This proves that the stabilizer of any x_H is a finite maximal subgroup of G, for all $H \in \mathcal{F}$. Hence, G satisfies (M). The normalizer $N_G(S)$ acts on X^H, hence $N_G(S) \leq S$. Therefore, G satisfies (NM).

Assume that G satisfies (M) and (NM). Let Y by any model for EG. Let I be the set of finite maximal subgroups of G. Note that G acts on I by conjugation. Moreover, the

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stabilizer of $M \in I$ is $N_G(M) = M$, since G satisfies (NM). Then, the join $X = Y * I$ is a model for $\underline{E}G$, where the G-action on X is the diagonal action. In fact, X is contractible since it can be seen as the union of copies cones of Y glued all together by their common base. Let $H \in \mathcal{F}$, then X^H consists of the conic point represented by the unique finite maximal subgroup that contains H. □

Our next lemma tells us that we can collapse down to a point the fixed point sets of any cocompact model for $\underline{E}G$.

LEMMA 2.4. *Let G be a group of type* $\underline{\mathrm{F}}$ *that satisfies properties (M) and (NM). Then G admits a cocompact model X for $\underline{E}G$ such that X^H consists of exactly one point for every nontrivial finite subgroup H of G.*

Proof. Let Y be any cocompact model for $\underline{E}G$. Given a point $y \in Y$, we denote by Gy the G-orbit of y. Denote by Y_{sing} the subspace of Y consisting of points with nontrivial isotropy. Note that Y_{sing} is a G-CW-subcomplex of Y.

We claim that there exist a finite number of points $y_1, \ldots, y_m$ of Y such that the disjoint union $Gy_1 \sqcup \cdots \sqcup Gy_m$ is a G-deformation retract of Y_{sing}.

By Lemma 2.3, there is a model Z for $\underline{E}G$ such that Z^H consists of exactly one point for every nontrivial finite subgroup H of G. On the other hand, we have unique (up to G-homotopy) G-maps $f\colon Y \to Z$ and $g\colon Z \to Y$ such that $f \circ g$ and $g \circ f$ are G-homotopic to the corresponding identity functions. These functions induce G-maps $f'\colon Y_{\text{sing}} \to Z_{\text{sing}}$ and $g'\colon Z_{\text{sing}} \to Y_{\text{sing}}$, and also, by restriction $f' \circ g'$ and $g' \circ f'$ are G-homotopic to the corresponding identity functions. By construction of Z, Z_{sing} is of the form $\bigsqcup_{M \in \mathcal{M}} Gz_M$, where $\mathcal{M}$ is the set of representatives of conjugacy classes of maximal finite subgroups of G and the isotropy of z_M is M. Since G admits a cocompact model for $\underline{E}G$, by [**15**, Theorem 4.2] G has a finite number of conjugacy classes of finite subgroups. Therefore, $Z_{\text{sing}} = Gz_1 \sqcup \cdots \sqcup Gz_m$ for certain points $z_1, \ldots, z_m$ of Z. Define $y_i = f'(z_i)$ for $i = 1, \ldots, m$. We can conclude that $Gy_1 \sqcup \cdots \sqcup Gy_m$ is a G-deformation retract of Y_{sing}. Moreover, if $r\colon Y_{\text{sing}} \to Gy_1 \sqcup \cdots \sqcup Gy_m$ is the mentioned retraction, then $r^{-1}(gy_i)$ is contractible and consists of all points x of Y_{sing} such that $G_x \leq G_{gy_i} = gG_{y_i}g^{-1}$. Hence, the setwise stabilizer of $r^{-1}(gy_i)$ is $N_G(G_{gy_i}) = G_{gy_i}$.

Define X to be the G-CW-complex defined by $Z/\sim$, where $\sim$ is the relation generated by $x \sim y$ if and only if $r(x) = r(y)$. Hence, X is G-homotopically equivalent to Z. Therefore, X is a model for $\underline{E}G$. Clearly, X is cocompact and by construction X^H consists of exactly one point if H is a nontrivial finite subgroup of X. □

Now we are ready to prove our main theorem.

THEOREM 2.5. *Let G be a virtually torsion-free group of type* $\underline{\mathrm{F}}$ *that satisfies properties (M) and (NM). Then,* $\mathrm{vcd}(G) = \underline{\mathrm{cd}}(G)$.

Proof. If G is finite, then there is nothing to prove. From now on, we assume G is infinite.

By Lemma 2.4, there exists a cocompact model X for $\underline{E}G$ satisfying that X^H consists of one point for every nontrivial finite subgroup H of G.

On the other hand, since G is infinite and virtually torsion-free, we conclude that $\mathrm{vcd}(G) > 0$. Hence, we have $\dim(X^F) = 0 < \mathrm{vcd}(G)$ for every nontrivial finite subgroup F of G. Therefore, by Theorem 2.1, we have $\mathrm{vcd}(G) = \underline{\mathrm{cd}}(G)$. □

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3. Examples. Next, we will describe some examples of groups satisfying the hypothesis of Theorem 2.5.

3.1. Groups in the literature.

(1) Extensions $1 \to \mathbb{Z}^n \to G \to F \to 1$ such that F is finite and the conjugation action of F on $\mathbb{Z}^n$ is free outside $0 \in \mathbb{Z}^n$, and G is of type $\underline{\mathrm{F}}$. Properties (M) and (NM) for this groups are established in [5].

It is worth noticing that, in the more general context of virtually poly-cyclic groups, it is already known that the Bredon cohomological dimension and the virtual cohomological dimension are equal, see for instance [16, Example 5.26].

(2) Cocompact Fuchsian groups and arithmetic Fuchsian groups. Let G be a Fuchsian group, that is, G acts properly discontinuously and by orientation-preserving isometries on the hyperbolic plane $\mathbb{H}$. A subgroup of G is finite and nontrivial if and only if it fixes a unique point in $\mathbb{H}$. Also any element of infinite order does not fix any point of $\mathbb{H}$ because the action is proper. This implies that $\mathbb{H}$ is a model for $\underline{E}G$ such that the point set $\mathbb{H}^H$ consists of one point for every nontrivial finite subgroup H of G. Thus, by Lemma 2.3, we have that G satisfies properties (M) and (NM). If additionally G acts cocompactly on $\mathbb{H}$, then clearly satisfies the hypothesis of Theorem 2.5. If G is an arithmetic Fuchsian group, then the Borel–Serre bordification of $\mathbb{H}$ is a cocompact model for $\underline{E}G$ with X^H is a one-point space for H finite nontrivial. For more information about Fuchsian groups, see [8].

(3) One-relator groups admiting a cocompact model for $\underline{E}G$. Properties (M) and (NM) are verified in [5].

A few comments on the existence of a model for $\underline{E}G$ for one-relator groups. For one-relator groups without torsion, the Cayley 2-complex is an $\underline{E}G = EG$. For one-relator groups with torsion, if the relator has the form w^n, then the torsion is all conjugate into the cyclic subgroup generated by w, which has order n. If you take the version of the Cayley 2-complex which has just one 2-cell with label w^n bounding each such word in the Cayley graph (not the universal cover of the presentation 2-complex which would have n different 2-cells bounding each such word), this becomes a cocompact model for $\underline{E}G$ provided that the 2-cells (whose stabilizer is cyclic of order n) are subdivided to ensure that it becomes a G-CW-complex.

(4) The Hilbert modular group. A totally real number field k is an algebraic extension of $\mathbb{Q}$ such that all its embeddings $\sigma_i : k \to \mathbb{C}$ have image contained in $\mathbb{R}$. Let k denotes a totally real number field of degree n and $\mathcal{O}_k$ its ring of integers. The Hilbert modular group is by definition $PSL_2(\mathcal{O}_k)$. If $K = \mathbb{Q}$, we recover the classical modular group $PSL_2(\mathbb{Z})$. Properties (M) and (NM) are verified for the Hilbert modular group in [4, Lemma 4.3]. Since the Hilbert modular group is a lattice in $PSL_2(\mathbb{R}) \times \cdots \times PSL_2(\mathbb{R})$, then it is an arithmetic group acting diagonally in the symmetric space $\mathbb{H} \times \cdots \times \mathbb{H}$. Hence, the Borel–Serre bordification again provides a cocompact model for $\underline{E}G$. See [8] for more information of the Hilbert modular group.

3.2. Groups acting on trees and properties (M) and (NM). Let us quickly recall the notation of graph of groups from [18]. A graph of groups $\mathbf{Y}$ consists of a graph Y (in the sense of Serre), one group Y_y for every edge y of Y, one group Y_P for each vertex P of Y, and injective homomorphism $Y_e \to Y_P$ of P is a vertex of the edge y. Recall that associated with $\mathbf{Y}$, we have the fundamental group $\pi_1(\mathbf{Y})$ and the Bass–Serre tree T, in such a way that $\pi_1(\mathbf{Y})$ acts on T by simplicial automorphism and the quotient graph is isomorphic to Y. Denote by $f: T \to Y$, the quotient projection.

We will be able to construct more examples using the following theorem.

THEOREM 3.1. *Let* $\mathbf{Y}$ *be a graph of groups in the sense of Serre with compact underlying graph. Assume that the vertex groups are of type* $\underline{\mathrm{F}}$ *and satisfy properties (M) and (NM) and assume that the edge groups are torsion-free. Then, the fundamental group* $\pi_1(\mathbf{Y})$ *of* Y *is of type* $\underline{\mathrm{F}}$ *and satisfies properties (M) and (NM).*

Proof. Let T be the Bass–Serre tree of $\mathbf{Y}$. Denote $G = \pi_1(\mathbf{Y})$. Choose cocompact models X_y and X_P for $\underline{E}Y_y$ and $\underline{E}Y_P$, respectively. Then, we can construct a model X for $\underline{E}G$ as follows. Replace each vertex v of T by the corresponding $X_{f(v)}$ and each edge e of T by $X_{p(e)} \times [0, 1]$. Next, if v is a vertex of e, glue one of the $X_e \times 0$ to X_v using map induced by the homomorphism $X_{p(e)} \to X_{p(v)}$ (and $X_{p(e)} \times 1 \to X_{p(v')}$ where v' is the other vertex of e). Hence, X inherits a G-action and we can easily verify that it is a model for $\underline{E}G$ (compare with [**9**, Proposition 4.8]). Moreover, the orbit space X/G can be constructed using a similar construction using instead Y, X_P/Y_P and X_y/Y_y. Therefore, since Y is compact and each X_P/Y_P and X_y/Y_y are compact, we have that X/G is also compact. This proves that G admits a cocompact model for $\underline{E}G$.

Assume now that each X_y and X_P are models satisfying the conclusion of Theorem 2.4. Let H be a nontrivial finite subgroup of G. Then, H cannot fix any edge of T because every edge group of $\mathbf{Y}$ is torsion-free. But, since H is finite, has to fix one vertex of T. Hence, H fixes a unique vertex v of T. Hence, H acts on $X_{p(v)}$, so H fixes a unique point of $X_{p(v)}$. Therefore, every nontrivial finite subgroup of G fixes a unique point of X, and by Theorem 2.3 we conclude that G satisfies properties (M) and (NM). □

Our final example are 3-manifold groups. For more information about 3-manifold groups, JSJ-decomposition, and the geometrization theorem see [**17**].

3.3. 3-manifold groups. Let M be a closed, orientable, connected 3-manifold with fundamental group G. We claim that G is of type $\underline{\mathrm{F}}$ and satisfies properties (M) and (NM). The prime decomposition $M = N_1 \# \cdots \# N_m$ induces a splitting of $G = G_1 * \cdots * G_m$. By Theorem 3.1, it is enough to prove that each G_i is of type $\underline{\mathrm{F}}$ and satisfies properties (M) and (NM).

From now on, assume M is prime. Using the Perelman–Thurston geometrization theorem, we can chop off M along tori to obtain pieces that are either hyperbolic or Sifert fibered. This is the so-called JSJ-decomposition. More explicitly, we can find a collection of tori (possibly empty) $T_1, \ldots, T_r$ embedded in M such that (abusing of notation) $M - \bigsqcup_i T_i$ is a disjoint union of manifolds (with boundary if the collection of tori is not empty) such that each piece is either hyperbolic or Seifert fibered. Hence, G is the fundamental group of a graph of groups $\mathbf{Y}$ with vertex groups the fundamental groups of Seifert fibered manifolds or hyperbolic manifolds, and edge group isomorphic to $\mathbb{Z}^2$. Again, by Theorem 3.1, it is enough to prove that all vertex groups in $\mathbf{Y}$ are of type $\underline{\mathrm{F}}$ and satisfy properties (M) and (NM). If the collection of tori is empty, then M itself is either hyperbolic or Seifert fibered. If M is hyperbolic, then G is torsion-free since M is aspherical, thus G satisfies properties (M) and (NM). Additionally, the universal cover of M is a cocompact model for $\underline{E}G$. If M is Seifert fibered, then M is aspherical unless is covered by the three sphere $\mathbb{S}^3$ or by $\mathbb{S}^2 \times \mathbb{R}$. In the $\mathbb{S}^3$ case G is finite, while in the $\mathbb{S}^2 \times \mathbb{R}$ case G is either isomorphic to $\mathbb{Z}$ or to the infinite dihedral subgroup D_∞. In both cases, G satisfies properties (M) and (NM) and is of type $\underline{\mathrm{F}}$. Finally, we have to deal with the case of

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a nontrivial JSJ-decomposition. In this case, we can verify case by case that every hyperbolic and Seifert fibered manifold in the JSJ-decomposition is an aspherical manifolds, and therefore their fundamental groups are torsion-free. Hence, all vertex and edge groups of $\mathbf{Y}$ are torsion-free, thus $G = \pi_1(\mathbf{Y})$ is torsion-free. Also, we have that M is a cocompact model for G.

We can conclude that the fundamental group of every prime manifold is of type $\underline{\mathrm{F}}$ and satisfies properties (M) and (NM). Therefore, every 3-manifold group is of type $\underline{\mathrm{F}}$ and satisfies properties (M) and (NM).

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