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Fixed points of contractive maps on dcpo's

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In this paper we study approach structures on dcpo's. A dcpo $(X, \leqslant)$ will be endowed with several other structures: the Scott topology; an approach structure generated by a collection of weightable quasi metrics on X; and a collection $\mathcal{W}$ of weights corresponding to the quasi metrics. Understanding the interaction between these structures on X will eventually lead to some fixed-point theorems for the morphisms in the category of approach spaces, which are called contractions. Existing fixed-point theorems on both monotone and non-monotone maps are obtained as special cases.

1. Introduction

Domain theory originated with the work of D. Scott (Scott 1972) in an effort to build a mathematical framework for modelling computer algorithms. A given directed complete partial order (dcpo) $(X, \leqslant)$ is endowed with the Scott topology $\sigma(X)$, which is then used as a tool to study convergence phenomena in X and to describe the Scott continuous functions. In Scott's model, the latter represent the computable functions. For a Scott continuous map on a dcpo with a bottom element, this setting provides the 'Scott least fixed-point theorem' where the least fixed point is obtained by iterating the function on the bottom element (Gierz *et al.* 2003). In Scott's model, fixed-point theorems are extremely important since they represent the 'meaning' of the algorithm. As is known from the work of A. Edalat (Edalat and Heckmann 1998), the Scott least fixed-point theorem implies the classical 'Banach fixed-point theorem' for Lipschitz functions with Lipschitz factor strictly smaller than 1 on a complete metric space.

Moreover, in quantitative domain theory, a continuous dcpo (domain) $(X, \leqslant)$ with a countable basis carries a weightable quasi metric inducing the Scott topology – see Matthews (1994), Waszkiewicz (2001), Waszkiewicz (2003) and Schellekens (2003). This refinement was created in order to model quantitative data, and is applied, for instance, in complexity analysis (Schellekens 1995; García *et al.* 2008; Romaguera and Schellekens 1999; Romaguera *et al.* 2011). In complexity analysis, fixed-point theorems are the clue for estimating the complexity of an algorithm. As a generalisation of the classical Banach theorem, S. Oltra and O. Valero proved the existence of fixed points for Lipschitz functions with Lipschitz factor strictly smaller than 1 on a bicomplete quasi metric space (Oltra and Valero 2004). K. Martin considered a measurement on a domain, which can be seen as an alternative for a weightable quasi metric in quantitative domain theory, and he developed fixed-point theorems for non-monotone maps in that setting (Martin 2000b; Martin 2000a).

The construction of a weightable quasi metric inducing the Scott topology on a domain having a countable basis, as proposed in Waszkiewicz (2003) and Schellekens (2003), is intrinsic, but the actual numerical values of the quasi metric depend on certain *ad hoc* choices. Recall, however, that the domain of formal balls of a complete metric space (X, d) provides a distinguished example of a domain where the Scott topology is induced by a weightable quasi metric that can be constructed directly from the given metric (Heckmann 1999).

In Colebunders *et al.* (2011), the current authors proposed the use of an approach structure in the sense of Lowen (1997) rather than a quasi metric. The approach structure on a continuous dcpo $(X, \leqslant)$ is supposed to induce $\sigma(X)$. As shown in Colebunders *et al.* (2011), such an approach structure can be intrinsically defined, regardless of the cardinality conditions on bases. Moreover, as will become clear in the course of this paper, an approach structure deals with convergence of a net or a filter by estimating 'how far' a point is from being a limit point. Useful applications of approach theory often rely on this fact – see, for instance, Berckmoes *et al.* (2011) for applications to probability theory.

In the current paper, we study approach structures on dcpo's. With respect to the morphisms in the category of approach spaces, which are called contractions, we study fixed-point theorems. In the current paper, a dcpo $(X, \leqslant)$ will be endowed with several other structures: the Scott topology; an approach structure generated by a collection of weightable quasi metrics on X; and a collection $\mathcal{W}$ of weights corresponding to the quasi metrics. Understanding the interaction between all these structures on X will eventually lead to some fixed-point theorems. In particular, we obtain existing fixed-point theorems due to D. Scott (Gierz *et al.* 2003) and K. Martin (Martin 2000b; Martin 2000a) for both monotone and non-monotone maps as special cases. Concrete applications of these fixed-point theorems to complexity analysis are currently being worked on in collaboration with M. Schellekens and will appear in a forthcoming paper (Colebunders *et al.* 2012). The investigation of the relation between our present context and the theory of measurements as developed by K. Martin (Martin 2000a; Martin 2000b) is also the subject of ongoing research.

To fix the notation, we consider $[0, \infty]$ as a lattice-ordered semigroup with the usual order and addition denoted by $\leqslant$ and $+$. An extended pre-quasi-pseudo metric on a set X is usually a function $q : X \times X \to [0, \infty]$ that vanishes on the diagonal: if q also satisfies the triangular inequality, it is called an extended quasi-pseudo metric, and if q satisfies both the triangular inequality and symmetry, it is called an extended pseudo metric. In the current paper, all such $q : X \times X \to [0, \infty]$ are allowed to take the value ∞, and both distances between two different points can be zero. From now on, we will simplify the terminology by omitting the words 'extended' and 'pseudo', so in this respect our terminology differs from common usage, but it does conform with the terminology used in Berckmoes *et al.* (2011) and Colebunders *et al.* (2011), and agrees with the practice in more categorically oriented papers on the subject such as Gutierres and Hofmann (2012). We use qMet to denote the construct of all quasi metric spaces with non-expansive maps

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as morphisms, and Met to denote the full subconstruct of all metric spaces. The collection of all (quasi) metrics on a set X is denoted by (q)Met(X). Given a pre-quasi metric q on a set X and $x \in X$ fixed, we write $q(x,.) : X \to [0,\infty]$ to denote the function defined by $q(x,.)(y) = q(x,y)$. Given a collection $\mathcal{Q}$ of pre-quasi metrics on X and $x \in X$ fixed, we use the denotation

$$\mathcal{Q}(x,.) = \{\ q(x,.) \mid q \in \mathcal{Q}\ \}.$$

All powers, such as $[0,\infty]^X$ or $[0,\infty]^{X\times X}$, as well as their subspaces, are ordered pointwise.

The following notion is crucial when describing approach spaces in terms of a gauge of quasi metrics or by approach systems. If $\mathcal{C} \subseteq [0,\infty]^X$ is a collection of functions and $\varphi \in [0,\infty]^X$, we say that φ is *dominated by* $\mathcal{C}$ if

$$\forall \epsilon > 0,\ \forall N < \infty\ \ \exists \varphi_\epsilon^N \in \mathcal{C}\ \text{ such that }\ \varphi \wedge N \leqslant \varphi_\epsilon^N + \epsilon.$$

Approach spaces can be described in many equivalent ways. We will now recall some definitions and results we will need to use frequently later in the paper – for more information, see Lowen (1997).

If $\mathcal{Q} \subseteq [0,\infty]^{X\times X}$ is a collection of quasi metrics, we say that $\mathcal{G}$ is *the saturation of* $\mathcal{Q}$ if $\mathcal{G}$ consists of exactly those quasi metrics e such that for every $x \in X$, the function $e(x,.)$ is dominated by $\mathcal{Q}(x,.)$.

An *approach gauge* $\mathcal{G}$ of quasi metrics on X is an ideal in the lattice qMet(X) that is *saturated* in the sense that $\mathcal{G}$ equals its saturation. The pair $(X,\mathcal{G})$ is called an *approach space*. Morphisms between approach spaces are called contractions. A map $f : (X,\mathcal{G}_X) \to (Y,\mathcal{G}_Y)$ is a *contraction* if

$$\forall q \in \mathcal{G}_Y\ : q \circ (f \times f) \in \mathcal{G}_X.$$

Approach spaces and contractions constitute a topological construct in the sense of Adámek *et al.* (1990), and are denoted by App. Moreover, initial structures have an easy description. Given a structured source $(f_i : X \to (X_i,\mathcal{G}_{X_i}))_{i\in I}$, the initial gauge on X consists of all quasi metrics e such that for all $x \in X$, the function $e(x,.)$ is dominated by

$$\{q_i \circ f_i \times f_i \mid i \in I, q_i \in \mathcal{G}_i\}^\vee(x),$$

where

$$\{q_i \circ f_i \times f_i \mid i \in I, q_i \in \mathcal{G}_i\}^\vee$$

consists of all finite suprema of quasi metrics taken from

$$\{q_i \circ f_i \times f_i \mid i \in I, q_i \in \mathcal{G}_i\}.$$

Another concept we will need is a localisation of the previous one. So, a collection $\mathcal{A} = (\mathcal{A}(x))_{x\in X}$, where each $\mathcal{A}(x) \subseteq [0,\infty]^X$, is said to satisfy the *mixed triangular inequality* if

$$\forall \varphi \in \mathcal{A}(x),\ \forall \epsilon > 0,\ \forall N < \infty$$
$$\exists (\varphi_z)_{z\in X} \in \prod_{z\in X} \mathcal{A}(z)\ \text{ such that }\ \forall z,y \in X\ \varphi(y) \wedge N \leqslant \varphi_x(z) + \varphi_z(y) + \epsilon.$$

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An *approach system* on X is a collection $\mathcal{A} = (\mathcal{A}(x))_{x\in X}$, where each $\mathcal{A}(x) \subseteq [0,\infty]^X$ satisfies the following axioms (Lowen 1997):

(1) Each $\mathcal{A}(x)$ is an ideal in the lattice $[0,\infty]^X$.
(2) $\forall \varphi \in \mathcal{A}(x) \quad \varphi(x) = 0$.
(3) $\mathcal{A}(x)$ is *saturated* in the sense that every function φ *dominated by* $\mathcal{A}(x)$ already belongs to $\mathcal{A}(x)$.
(4) $\mathcal{A}$ satisfies the *mixed triangular inequality*.

The foregoing gauge and approach system concepts are equivalent in the sense that each uniquely determines the other. The formulas for going from one structure to another can be found in Lowen (1997) and will be recalled later whenever needed.

On a given set X, we will often denote a given approach space simply by X, and then we will use its gauge $\mathcal{G}_X$ or its approach system $\mathcal{A}_X$ whenever appropriate. Contractions can be equivalently described in terms of approach systems by

$$\forall x \in X, \forall \varphi' \in \mathcal{A}_Y(f(x)) \quad \varphi' \circ f \in \mathcal{A}_X(x),$$

and the categories of sets with approach gauges on the one hand and sets with approach systems on the other are concretely isomorphic. In terms of the approach systems, the initial lift $(X, \mathcal{A}_X)$ in $x \in X$ of $(f_i : X \to (X_i, \mathcal{A}_{X_i}))_{i\in I}$ is described as the class of all functions dominated by a finite sup of functions taken from

$$\{\xi_i \circ f_i \mid \xi_i \in \mathcal{A}_i(f_i(x)), i \in I\}.$$

Convergence in an approach space X is described by means of a limit operator. For a given filter $\mathcal{F}$ and a point $x \in X$, the value $\lambda\mathcal{F}(x)$ is interpreted as the distance that the point is away from being a limit point of the filter. If $\mathbf{F}(X)$ is the set of all filters on X, the limit operator is a function

$$\lambda : \mathbf{F}(X) \to [0,\infty]^X.$$

For an approach space with gauge $\mathcal{G}$ or approach system $\mathcal{A}$, for $\mathcal{F} \in \mathbf{F}(X)$ and $x \in X$, the limit operator is calculated by

$$\lambda\mathcal{F}(x) = \sup_{q\in\mathcal{G}} \inf_{F\in\mathcal{F}} \sup_{y\in F} q(x,y)$$

or

$$\lambda\mathcal{F}(x) = \sup_{\varphi\in\mathcal{A}(x)} \inf_{F\in\mathcal{F}} \sup_{y\in F} \varphi(y).$$

Using the axioms for the limit operator as described in Lowen (1997), we get yet another equivalent description of approach spaces. Using the following characterisation for a map $f : (X, \lambda_X) \to (Y, \lambda_Y)$ to be a contraction

$$\lambda_Y(\mathrm{stack} f(\mathcal{F})) \circ f \leqslant \lambda_X \mathcal{F}$$

for every $\mathcal{F} \in \mathbf{F}(X)$, the category App can be isomorphically described in terms of the limit operator.

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The construct App constitutes a framework in which other important constructs can be fully embedded. Top is embedded as a full concretely reflective and concretely coreflective subconstruct, and qMet is embedded as a concretely coreflective subconstruct.

A way to embed a topological space $(X, \mathcal{T})$ is by describing the gauge, the approach system or the limit operator associated with it. The gauge is

$$\mathcal{G}_{\mathcal{T}} = \{q \in \mathsf{qMet(X)} \mid \mathcal{T}_q \subseteq \mathcal{T}\}.$$

Using the following indicator function on X

$$\theta_A(z) := \begin{cases} 0 & z \in A \\ \infty & z \notin A, \end{cases}$$

the approach system $\mathcal{A}_{\mathcal{T}}(x)$ in x consists of all functions φ that are dominated by the collection

$$\mathcal{B}_{\mathcal{T}}(x) = \{\theta_V \mid V \in \mathcal{V}_{\mathcal{T}}(x)\}$$

with $\mathcal{V}_{\mathcal{T}}(x)$ the neighbourhood filter of the topology in x, and the limit operator on a filter $\mathcal{F}$ is defined by

$$\lambda_{\mathcal{T}}\mathcal{F} = \theta_{\lim \mathcal{F}}$$

where $\lim\mathcal{F}$ is the set of all topological convergence points of $\mathcal{F}$.

Every approach space X has two natural topological spaces associated with it: the topological coreflection (which has to be thought of as the underlying topology in the same way as for metric spaces) and the topological reflection. In the current paper, we will mainly deal with the coreflection. When the approach space is defined through its gauge $\mathcal{G}$, the topology of the topological coreflection $(X, \mathcal{T}_X)$ can be calculated as the supremum in the lattice of all topologies on X of the collection

$$\{\mathcal{T}_q \mid q \in \mathcal{G}\}.$$

When the approach space is defined by means of the approach system $\mathcal{A} = (\mathcal{A}(x))_{x \in X}$, the topological coreflection is given by the neighbourhood filters, where for x

$$\mathcal{V}(x) = \{V \subseteq X \mid \exists \epsilon > 0,\ \exists \varphi \in \mathcal{A}(x)\ \{\varphi < \epsilon\} \subseteq V\}.$$

When X is given through its limit operator λ, the convergence in $\mathcal{T}_X$ is given by

$$\mathcal{F} \to x \text{ if and only if } \lambda\mathcal{F}(x) = 0.$$

The embedding of quasi metric spaces in App is described as follows. Given (X, q), the gauge of the associated approach space is

$$\mathcal{G}_q = \{e \in \mathsf{qMet(X)} \mid e \leqslant q\},$$

the approach system of the associated approach space in x is

$$\mathcal{A}_q(x) = \{\varphi \in [0, \infty]^X \mid \varphi \leqslant q(x, .)\}$$

and the limit operator is calculated by

$$\lambda_q\mathcal{F}(x) = \inf_{F \in \mathcal{F}} \sup_{y \in F} q(x, y).$$

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The quasi metric coreflection of an approach space X with gauge $\mathcal{G}$ is determined by

$$q_X(x, y) = \sup_{q \in \mathcal{G}} q(x, y).$$

In the current paper, we will consider approach structures on a given directed complete partial order (dcpo) – see Gierz *et al.* (2003) for terminology and basic results on dcpo's. To fix the notation and terminology, recall that for a partially ordered set (poset) $(X, \leqslant)$, a subset $D \subseteq X$ is *directed* if it is non-empty and any pair of elements of D has an upperbound in D. A poset in which every directed subset D has a supremum ($\bigvee D$) is called a *directed complete poset (dcpo)*.

In a dcpo $(X, \leqslant)$, a net $(x_j)_{j \in J}$ converges to $x \in X$ if and only if $x \leqslant \bigvee D$ for some directed set D consisting of eventual lower bounds of the net. The net convergence defines a convergence structure on $(X, \leqslant)$, which is called the *liminf-convergence*. The topological reflection of this convergence space is denoted by $(X, \sigma(X))$, and $\sigma(X)$ is called the *Scott Topology*. The opens are the upsets U such that for every directed D, if the supremun $\bigvee D$ belongs to U, then U intersects D. It is clear that $\sigma(X)$ is a T_0 topology that, in non-trivial cases, never fulfills higher separation properties. The specialisation order of $\sigma(X)$ coincides with the original order.

A function $f : (X, \leqslant) \to (X', \leqslant')$ between dcpo's is said to be *Scott continuous* if and only if $f : (X, \sigma(X)) \to (X', \sigma(X'))$ is continuous. This can be characterised as f is monotone and preserves directed sup's.

Other intrinsic topologies we will encounter on $(X, \leqslant)$ are the *lower topology* $\downarrow\mathcal{T}$ generated by the basis

$$\{\downarrow x \mid x \in X\},$$

and the *Martin topology* (Waszkiewicz 2003) (the μ topology in Martin (2000a)),

$$M(X) = \sigma(X) \vee \downarrow\mathcal{T},$$

where the supremum is taken in the lattice of topologies on X. Note that the Martin topology coincides with the b-topology of $\sigma(X)$ since the downsets $\downarrow x$ coincide with the singleton closures $\mathrm{cl}\{x\}$ in $\sigma(X)$. See Salbany (1984) and Dikranjan and Tholen (1995) for further references on the b-topology and b-closure.

2. Weightable pre-quasi metrics

We follow Matthews (1994) and Künzi (2001) and adapt their definition of a weightable quasi metric to our setting, where the functions $q : X \times X \to [0, \infty]$ are allowed to take the value ∞, and where both distances between two different points can be zero.

Definition 2.1.

(1) A pre-quasi metric space (X, q) is *weightable* if there exists a function $w : X \to [0, \infty]$ (called a weight) that is not identically ∞ and satisfies

$$q(x, y) + w(x) = q(y, x) + w(y)$$

whenever $x, y \in X$.

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(2) We say that a weight w is *forcing* for q if $x \in X$ and $w(x) = \infty$ imply that the function $q(x, .)$ is identically zero on X.
(3) For a weightable pre-quasi metric q, we write $\mathcal{W}_q$ to denote the collection of all its weights.

Matthews (1994) showed that weightable quasi metric spaces (taking only finite values and with $d(x, y) = d(y, x) = 0$ if and only if $x = y$) are in one-to-one correspondence with partial metric spaces. In the same setting, Künzi and Vajner (1994) studied topological spaces that can be induced by a weightable quasi metric and formulated both necessary and sufficient conditions on the topology to ensure quasi metrisability by some weightable quasi metric.

The following example is well known, and will appear to be crucial in the current paper.

Example 2.2. Consider $[0, \infty]$ endowed with the following quasi metric:

$$q_\sigma(x, y) = (y - x) \vee 0 \text{ for } x \text{ and } y \text{ not both equal to } \infty; \quad q_\sigma(\infty, \infty) = 0.$$

The function $w_{q_\sigma} : [0, \infty] \to [0, \infty]$ defined as $w_{q_\sigma}(x) = x$ is a weight for q_σ. Note that it is forcing since the only point in which the weight is infinite is ∞ and $q_\sigma(\infty, y) = 0$ for all y. The underlying topology $\mathcal{T}_{q_\sigma}$ on $[0, \infty]$ is

$$\{[0, b[\mid b \leqslant \infty\} \cup \{[0, \infty]\} \cup \{\varnothing\}.$$

When we endow $[0, \infty]$ with the opposite order $x \preceq y \Leftrightarrow y \leqslant x$, the Scott topology $\sigma([0, \infty]^{op})$ associated with the dcpo $[0, \infty]^{op} = ([0, \infty], \preceq)$ is exactly the topology $\mathcal{T}_{q_\sigma}$.

The importance of Example 2.2 is illustrated by the following results.

Proposition 2.3 (Colebunders *et al.* 2011). The object $([0, \infty], q_\sigma)$ is initially dense in App, meaning that for every approach space X, the total source

$$(g : X \to ([0, \infty], q_\sigma))_{g \text{ contraction}}$$

in App is initial in App.

Corollary 2.4. For an arbitrary approach space X, the gauge is the saturation of the collection

$$\{q_\sigma \circ g \times g \mid g : X \to ([0, \infty], q_\sigma) \text{ contraction }\}^\vee,$$

where, as before, $\vee$ indicates that finite sups are added to the collection.

Corollary 2.5. For an arbitrary dcpo $(X, \leqslant)$ and $\sigma(X)$ the Scott topology, the total source

$$(g : (X, \sigma(X)) \to [0, \infty]^{op})_{g \text{ Scott continuous}}$$

in Top is initial in Top.

Proof. We embed the topological space $(X, \sigma(X))$ in App and let $\mathcal{G}$ be the corresponding gauge. Applying Proposition 2.3, we get that the total source

$$(g : (X, \mathcal{G}) \to ([0, \infty], q_\sigma))_{g \text{ contraction}}$$

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is initial in App. Applying the coreflector to Top, we get that

$$(g : (X, \sigma(X)) \to ([0,\infty], \mathcal{T}_{q_\sigma}))_{g \text{ contraction}}$$

is initial in Top. Hence, the total source

$$(g : (X, \sigma(X)) \to [0,\infty]^{op})_{g \text{ Scott continuous}}$$

is initial too. □

Definition 2.6. Given a pre-quasi metric space (X, q), we define a reflexive relation, the so-called *specialisation relation*, on X as follows:

$$x \leqslant_q y \Leftrightarrow q(x,y) = 0.$$

Clearly, when q is a quasi metric, the relation $\leqslant_q$ becomes a preorder.

Proposition 2.7. For a weightable pre-quasi metric q on X with weight w, we have the following inequalities:

(1) $q_\sigma \circ w \times w \leqslant q$ and hence $w : (X, q) \to ([0,\infty], q_\sigma)$ is non-expansive.
(2) $w : (X, \leqslant_q) \to [0,\infty]^{op}$ is monotone.
(3) If w is forcing for q, then $q \leqslant q^{-1} + q_\sigma \circ w \times w$.
(4) If w is forcing for q, then for $x \in X$ we have

$$q(x,.) \leqslant q_\sigma \circ w \times w(x,.) \vee \theta_{\{y \leqslant_q x\}}.$$

Proof.

(1) Let $x, y \in X$. The only non-trivial case to be considered is with $w(x) < \infty, w(x) < w(y)$ and $q(x,y) < \infty$. It follows that $q(y,x) < \infty$ and $w(y) < \infty$. Hence $(w(y) - w(x)) \vee 0 \leqslant q(x,y)$.

(2) Let $x \leqslant_q y$. Then, by assumption, $q(x,y) = 0$, and applying (1), we have

$$q_\sigma(w(x), w(y)) = 0.$$

Hence, $w(x) \preceq w(y)$.

(3) Since w is assumed to be forcing for q, when $w(x) = \infty$, the inequality is trivially fulfilled. When $w(x) < \infty$, we have

$$\begin{aligned} q(x,y) = q(y,x) + w(y) - w(x) &\leqslant q(y,x) + (w(y) - w(x)) \vee 0 \\ &= q^{-1}(x,y) + q_\sigma \circ w \times w(x,y). \end{aligned}$$

(4) When evaluating both sides in $y \in X$, the only non-trivial case is with $q(x,y) \neq 0$ and $\theta_{\{y \leqslant_q x\}} \neq \infty$. So we may assume $w(x) < \infty$, $y \leqslant_q x$ and $q(y,x) = 0$. Then

$$q(x,y) = w(y) - w(x) \leqslant (w(y) - w(x)) \vee 0 = q_\sigma \circ w \times w(x,y) \vee \theta_{\{y \leqslant_q x\}}(y).$$ □

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3. Approach structures on dcpo's

Instead of working with a single quasi metric on a dcpo, as Waszkiewicz (2003; 2001) and Schellekens (2003) did, we will consider collections of quasi metrics generating an approach space.

Definition 3.1. A collection $\mathcal{Q} \subseteq [0,\infty]^{X\times X}$ is said to be *locally directed* if

$$\forall x\ \forall d, e \in \mathcal{Q}\ \exists q \in \mathcal{Q}\quad d \vee e(x,.) \leqslant q(x,.).$$

Proposition 3.2. If $\mathcal{Q} \subseteq [0,\infty]^{X\times X}$ consists of quasi metrics and is locally directed, then the saturation $\mathcal{G}$ is the gauge of an approach space on X, which is called the approach gauge generated by $\mathcal{Q}$.

Proof. Since $\mathcal{Q}$ is locally directed, the collection of quasi metrics e such that for every $x \in X$ the function $e(x,.)$ is dominated by $\mathcal{Q}(x,.)$ is an ideal in $[0,\infty]^{X\times X}$. The rest follows from the fact that taking the saturation is an idempotent operation. □

Note that when $\mathcal{Q} \subseteq [0,\infty]^{X\times X}$ consists of quasi metrics and is locally directed and $\mathcal{G}$ is its saturation, the collection $\mathcal{D}$ of all finite sups of quasi metrics in $\mathcal{Q}$ forms an approach (gauge) basis for $\mathcal{G}$ in the sense of Lowen (1997).

Next we start with a given collection of pre-quasi metrics rather than with a collection of quasi metrics. We show how an approach space can be generated from it using approach systems rather than gauges. The following result follows immediately from Lowen (1997).

Proposition 3.3.

(1) If $\mathcal{Q} \subseteq [0,\infty]^{X\times X}$ consists of pre-quasi metrics and is locally directed, and if $(\mathcal{Q}(x,.))_{x\in X}$ satisfies the mixed triangular inequality, then the collection $(\mathcal{Q}(x,.))_x$ defines an approach (system) basis in the sense that:

 (a) $\mathcal{Q}(x,.)$ is an ideal basis in the lattice $[0,\infty]^X$.

 (b) $\forall \varphi \in \mathcal{Q}(x,.)\quad \varphi(x) = 0$.

 (c) $(\mathcal{Q}(x,.))_{x\in X}$ satisfies the mixed triangular inequality.

(2) The collection $\mathcal{A}_{\mathcal{Q}} = (\mathcal{A}(x))_{x\in X}$, with $\mathcal{A}(x)$ consisting of all functions in $[0,\infty]^X$ dominated by $\mathcal{Q}(x,.)$ for $x \in X$ defines an approach system on X with approach basis $(\mathcal{Q}(x,.))_x$. The gauge $\mathcal{G}_{\mathcal{Q}}$ associated with the approach system is

$$\mathcal{G}_{\mathcal{Q}} = \{d \mid \text{quasi metric},\ d(x,.) \in \mathcal{A}(x),\ \forall x\}.$$

Note that every locally directed collection $\mathcal{Q} \subseteq [0,\infty]^{X\times X}$ consisting of quasi metrics automatically satisfies the mixed triangular condition. When the construction given in Proposition 3.3 is applied to such a collection $\mathcal{Q} \subseteq [0,\infty]^{X\times X}$, the approach gauge $\mathcal{G}_{\mathcal{Q}}$ coincides with the gauge $\mathcal{G}$ obtained directly as in Proposition 3.2.

We will now extend some notions from Section 2 to approach spaces.

Definition 3.4. If $\mathcal{H} \subseteq [0,\infty]^{X\times X}$ is a collection of weightable pre-quasi metrics, an element $(w_q)_{q\in\mathcal{H}} \in \prod_{q\in\mathcal{H}} \mathcal{W}_q$ is called a *weight* associated with $\mathcal{H}$. We say the weight $(w_q)_{q\in\mathcal{H}}$ is *forcing* for $\mathcal{H}$ if every w_q with $q \in \mathcal{H}$ is forcing for q.

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We now recall the following definition from Colebunders *et al.* (2011).

Definition 3.5. Given an approach space X with gauge $\mathcal{G}$ and approach system $\mathcal{A} = (\mathcal{A}(x))_x$, we define the *specialisation preorder* by

$$x \leqslant_X y \Leftrightarrow (q(x, y) = 0 \text{ whenever } q \in \mathcal{G}).$$

In terms of the approach system, we get

$$x \leqslant_X y \Leftrightarrow (\varphi(y) = 0 \text{ whenever } \varphi \in \mathcal{A}(x)).$$

Note that as the quasi metric coreflection (X, q_X) of an approach space X is given by $q_X(x, y) = \sup_{q\in\mathcal{G}} q(x, y)$, the following expressions are equivalent:

$$(q(x, y) = 0 \text{ whenever } q \in \mathcal{G}) \quad \Leftrightarrow \quad q_X(x, y) = 0.$$

Moreover, since the topology $\mathcal{T}_X$ of the topological coreflection of X is the supremum of the topologies $\{\mathcal{T}_q \mid q \in \mathcal{G}\}$, we also have

$$(q(x, y) = 0 \text{ whenever } q \in \mathcal{G}) \;\Leftrightarrow\; x \in cl_{\mathcal{T}_X}\{y\}.$$

So the specialisation preorder of X defined in Definition 3.5 coincides with both the specialisation preorders determined by the quasi metric or topological coreflections. An approach space is said to be T_0 if its topological coreflection is a T_0 topological space. The following results follow immediately by observing that the specialisation preorder of an approach space is the same as the one derived from the topological coreflection.

Proposition 3.6.

(1) An approach space X is T_0 if and only if the specialisation preorder $\leqslant_X$ is a partial order.

(2) Let X be an approach space. The open sets in the topological coreflection $(X, \mathcal{T}_X)$ are upsets in the preorder $\leqslant_X$.

(3) Every contraction between approach spaces $f : X \to Y$ is monotone as map $f : (X, \leqslant_X) \to (Y, \leqslant_Y)$.

Next we investigate the situation where $(X, \leqslant)$ is a given dcpo and we assume that there is some relation between $\leqslant_X$ and $\leqslant$ or between $\mathcal{T}_X$ and the Scott topology $\sigma(X)$. Note that if $\mathcal{T}_X \leqslant \sigma(X)$, the specialisation preorder $\leqslant_X$ induced by X satisfies $\leqslant \subseteq \leqslant_X$. We say that the approach space X *induces the Scott topology* if its topological coreflection coincides with the Scott topology, that is, $\mathcal{T}_X = \sigma(X)$. Note that in this case the specialisation preorder $\leqslant_X$ coincides with the original dcpo order. In order to formulate the next result on the interaction between an approach space on X, its weight $\mathcal{W}$ and a given dcpo structure $\leqslant$ on X, we first fix some notation and terminology.

Let $(X, \leqslant)$ be a preordered space and $\mathcal{W} \subseteq [0, \infty]^X$.

Definition 3.7.

(1) $\mathcal{W}$ is said to be *monotone for* $\leqslant$ if all functions

$$w : (X, \leqslant) \to [0, \infty]^{op}$$

are monotone.

(2) $\mathcal{W}$ is said to be *strictly monotone for* $\leqslant$ if $\mathcal{W}$ is monotone and

$$(y \leqslant x \text{ and } w(x) = w(y)\ \forall w \in \mathcal{W}) \Rightarrow x = y.$$

The source $(w : X \to ([0,\infty], q_\sigma))_{w \in \mathcal{W}}$ has an initial lift $X^{in}_{\mathcal{W}}$ in App and we write $\mathcal{A}^{in}_{\mathcal{W}}$ to denote its approach system. Similarly, we write $\mathcal{T}^{in}_{\mathcal{W}}$ to denote the initial topology determined by the source $(w : X \to ([0,\infty], \mathcal{T}_{q_\sigma}))_{w \in \mathcal{W}}$ in Top.

We also consider the lower topology $\downarrow\mathcal{T}$ related to the dcpo order $\leqslant$, and when $(X, \downarrow\mathcal{T})$ is embedded in App, we denote the approach space by $X_{\downarrow\mathcal{T}}$ and its approach system by $\mathcal{A}_{\downarrow\mathcal{T}}$.

Proposition 3.8. Let $(X, \leqslant)$ be a dcpo and suppose $\mathcal{H} \subseteq [0,\infty]^{X \times X}$ is a collection of weightable pre-quasi metrics with forcing weight $(w_q)_{q \in \mathcal{H}} \in \prod_{q \in \mathcal{H}} \mathcal{W}_q$ such that with $\mathcal{Q} = \mathcal{H}^\vee$, the collection $(\mathcal{Q}(x,.))_{x \in X}$ is an approach basis. Let $\mathcal{A} = (\mathcal{A}(x))_x$ be the approach system for X as in Proposition 3.3. With $\mathcal{W} = \{w_q | q \in \mathcal{H}\}$, we have:

(1) $w : X \to ([0,\infty], q_\sigma)$ is a contraction, $w : (X, \mathcal{T}_X) \to ([0,\infty]^{op}, \sigma([0,\infty]^{op}))$ is continuous and $w : (X, \leqslant_X) \to [0,\infty]^{op}$ is monotone, for every $w \in \mathcal{W}$.

(2) If the specialisation preorder satisfies $\leqslant\ \subseteq\ \leqslant_X$, we have:

(a) $X \leqslant X^{in}_{\mathcal{W}} \vee X_{\downarrow\mathcal{T}}$ with the supremum taken in App.

(b) $X \vee X_{\downarrow\mathcal{T}} = X^{in}_{\mathcal{W}} \vee X_{\downarrow\mathcal{T}}$.

(c) $\mathcal{T}_X \leqslant \mathcal{T}^{in}_{\mathcal{W}} \vee \downarrow\mathcal{T}$ with the supremum taken in Top.

(3) If the topological coreflection satisfies $\mathcal{T}_X \leqslant \sigma(X)$, we have:

$w : (X, \leqslant) \to [0,\infty]^{op}$ is Scott continuous for every $w \in \mathcal{W}$.

(4) If the approach space is T_0 (in particular when $\leqslant_X = \leqslant$), we have:

$\mathcal{W}$ is strictly monotone for $\leqslant_X$.

(5) If $\mathcal{T}_X = \sigma(X)$, we have:

(a) $\sigma(X) \leqslant \mathcal{T}^{in}_{\mathcal{W}} \vee \downarrow\mathcal{T}$ with the supremum taken in Top.

(b) $M(X) = \sigma(X) \vee \downarrow\mathcal{T} = \mathcal{T}^{in}_{\mathcal{W}} \vee \downarrow\mathcal{T}$.

Proof.

(1) From Proposition 2.7, we know that $q_\sigma \circ w \times w \leqslant q$, so for $x \in X$, we have

$$q_\sigma(w(x), .) \circ w = q_\sigma \circ w \times w(x, .) \leqslant q(x, .)$$

for every $q \in \mathcal{H}$. Since $q(x,.)$ belongs to $\mathcal{A}(x)$, the map w is a contraction. The rest follows by application of Example 2.2 and Proposition 3.6.

(2) (a) Under the extra assumption, we have $\leqslant\ \subseteq\ \leqslant_q$ for every $q \in \mathcal{H}$. By application of Proposition 2.7 for $x \in X$, we have

$$q(x, .) \leqslant q_\sigma(w(x), w(.)) \vee \theta_{\{y \leqslant_q x\}} \leqslant q_\sigma(w(x), w(.)) \vee \theta_{\downarrow x}.$$

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It is clear that $q_\sigma(w(x), w(.))$ belongs to $\mathcal{A}^{in}_{\mathcal{W}}(x)$ and $\theta_{\downarrow x}$ to $\mathcal{A}_{\downarrow\mathcal{T}}(x)$. Hence, $q(x, .)$ is dominated by a finite sup of functions belonging to $(\mathcal{A}^{in}_{\mathcal{W}} \cup \mathcal{A}_{\downarrow\mathcal{T}})(x)$ for every $q \in \mathcal{H}$. From this we can conclude that $X \leqslant X^{in}_{\mathcal{W}} \vee X_{\downarrow\mathcal{T}}$.

(b) In view of part (1), the weights w are contractions $w : X \to ([0, \infty], q_\sigma)$, so $X^{in}_{\mathcal{W}} \leqslant X$. From part (2a), we now have

$$X^{in}_{\mathcal{W}} \vee X_{\downarrow\mathcal{T}} \leqslant X \vee X_{\downarrow\mathcal{T}} \leqslant X^{in}_{\mathcal{W}} \vee X_{\downarrow\mathcal{T}},$$

from which the equality follows.

(c) We apply the coreflector from App to Top to the inequality in part (2b). Since the coreflector preserves initial sources, we immediately get the result, where this time the suprema are taken in Top.

(3) This follows immediately from part (1).

(4) First observe that by part (1), $\mathcal{W}$ is monotone for $\leqslant_X$. By Proposition 2.7 (3) under the assumptions $y \leqslant_X x$ and $w(x) = w(y)$, whenever $w \in \mathcal{W}$, we have

$$q(x, y) \leqslant q_\sigma \circ w \times w(x, y) = 0$$

for every $q \in \mathcal{H}$. It follows that $\varphi(y) = 0$ for every $\varphi \in \mathcal{A}(x)$. Hence, $x \leqslant_X y$, and in view of the T_0 property, we have $x = y$.

(5) (a) This follows immediately from part (2c).

(b) This follows by applying the coreflector from App to Top to the equality in part (2b). □

4. Fixed points for contractive functions

In the first part of this section, we consider a function $f : X \to X$, and for an arbitrary point $a \in X$, we will estimate 'how far' a is from being a fixed point. This will be done by comparing $f(a)$ to a. An estimate of the 'distance' between $f(a)$ and a will be obtained by structuring X as an approach space and using its limit operator.

As before, given an arbitrary $\mathcal{W} \subseteq [0, \infty]^X$, we consider the initial lift $X^{in}_{\mathcal{W}}$ in App of the source $(w : X \to ([0, \infty]^{op}, q_\sigma))_{w \in \mathcal{W}}$ with q_σ as in Example 2.2. Note that although q_σ induces the Scott topology on $[0, \infty]^{op}$, that is, $[0, \infty]$ endowed with the opposite order, our uses of $\leqslant$, sup, inf and lim sup on $[0, \infty]$ all refer to the usual order.

Proposition 4.1. Let X be an approach space with limit operator λ_X and let $\mathcal{W} \subseteq [0, \infty]^X$ be arbitrary. For a contraction

$$f : X \to X^{in}_{\mathcal{W}},$$

we fix $x \in X$ and let

$$a_n = f^n(x)$$

be the values obtained by iterating f on x. For $w \in \mathcal{W}$, let

$$l_w = \limsup_{n \to \infty} w(a_n).$$

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For $a \in X$ arbitrary, we have the estimate

$$\sup_{w \in \mathcal{W}} q_\sigma(w(f(a)), w(a)) \leqslant \lambda_X((a_n)_n, a) + \sup_{w \in \mathcal{W}} q_\sigma(l_w, w(a)).$$

Proof. For $w \in \mathcal{W}$, we have

$$\begin{aligned} q_\sigma(w(f(a)), w(a)) &\leqslant q_\sigma(w(f(a)), l_w) + q_\sigma(l_w, w(a)) \\ &= (\limsup_{n\to\infty} w(a_n) - w(f(a))) \vee 0 + q_\sigma(l_w, w(a)) \\ &= \limsup_{n\to\infty}(q_\sigma(w(f(a)), w(a_n)) + q_\sigma(l_w, w(a)) \\ &= \lambda_{q_\sigma}(w(a_n))_n(w(f(a))) + q_\sigma(l_w, w(a)). \end{aligned}$$

By taking the supremum on both sides, it follows that

$$\begin{aligned} \sup_{w \in \mathcal{W}} q_\sigma(w(f(a)), w(a)) &\leqslant \sup_{w \in \mathcal{W}} \lambda_{q_\sigma}(w(a_n))_n(w(f(a))) + \sup_{w \in \mathcal{W}} q_\sigma(l_w, w(a)) \\ &= \lambda_{X^{in}_{\mathcal{W}}}(a_n)_n(f(a)) + \sup_{w \in \mathcal{W}} q_\sigma(l_w, w(a)) \\ &\leqslant \lambda_X(a_n)_n(a) + \sup_{w \in \mathcal{W}} q_\sigma(l_w, w(a)), \end{aligned}$$

where, in the last equality, we use the fact that f is a contraction. □

We will now concentrate on each of the terms on the right-hand side of the final inequality. For the first term, we know that $\lambda_X(a_n)_n(a) = 0$ if and only if $(a_n)_n$ converges to a in the topological coreflection $\mathcal{T}_X$. In order to discuss the second term, we will make some extra assumptions.

With the same notation as in Proposition 4.1, we further suppose that X carries a dcpo structure $\leqslant$.

Proposition 4.2. If $\mathcal{W}$ is monotone for $(X, \leqslant)$ and there is a subsequence $(a_{k_n})_n$ such that $a_{k_n} \leqslant a \;\; \forall n \in \mathbb{N}$, we have

$$\sup_{w \in \mathcal{W}} q_\sigma(l_w, w(a)) = 0.$$

Proof. Let $w \in \mathcal{W}$ be arbitrary. Applying monotonicity, we have

$$a_{k_n} \leqslant a \Rightarrow w(a_{k_n}) \geqslant w(a),$$

for the given subsequence. Hence, we get

$$\limsup_{n\to\infty} w(a_n) \geqslant w(a),$$

and thus $q_\sigma(l_w, w(a)) = 0$. □

Proposition 4.3. Let $(X, \leqslant)$ be a dcpo endowed with an approach structure X and let $\mathcal{W} \subseteq [0,\infty]^X$ be strictly monotone for $\leqslant$. For a contraction

$$f : X \to X^{in}_{\mathcal{W}},$$

we fix $x \in X$ and let

$$a_n = f^n(x)$$

be the values obtained by iterating f on x. We now make the following assumptions on some $a \in X$:

(1) There is a subsequence $(a_{k_n})_n$ such that $a_{k_n} \leqslant a \quad \forall n \in \mathbb{N}$.
(2) $(a_n)_n$ converges to a in the topological coreflection $\mathcal{T}_X$.
(3) $a \leqslant f(a)$.

Then the point a is a fixed point of f.

Proof. For $w \in \mathcal{W}$ let

$$l_w = \limsup_{n\to\infty} w(a_n).$$

By Proposition 4.2, condition (1) implies that $q_\sigma(l_w, w(a)) = 0$, and by condition (2), we also have $\lambda_X((a_n)_n, a) = 0$. Applying Proposition 4.1, we can conclude that

$$\sup_{w\in\mathcal{W}} q_\sigma(w(f(a)), w(a)) = 0.$$

It follows that $w(a) = w(f(a))$ for every $w \in \mathcal{W}$ and by strict monotonicity we have $a = f(a)$.

□

Next we list some situations implying the conditions in Proposition 4.3. Note that in Propositions 4.4 and 4.5, and their corollaries, the contractive map f need not be monotone.

Proposition 4.4. Let $(X, \leqslant)$ be a dcpo endowed with an approach structure X with $\mathcal{T}_X \leqslant M(X)$ and let $\mathcal{W} \subseteq [0,\infty]^X$ be strictly monotone for $\leqslant$. For a contraction

$$f : X \to X^{in}_{\mathcal{W}},$$

we fix $x \in X$ and let

$$a_n = f^n(x)$$

be the values obtained by iterating f on x. If $(a_n)_n$ is monotone and

$$\{z \in X \mid z \leqslant f(z)\}$$

is closed under directed suprema, then $a = \bigvee_n a_n$ is a fixed point.

Proof. Since $(a_n)_n$ is monotone, the supremum $a = \bigvee_n a_n$ is well defined and $(a_n)_n$ converges to a in the Scott topology. Since $a_n \leqslant a$ for every n, the sequence $(a_n)_n$ also converges to a in the Martin topology $M(X)$. In view of the assumption $\mathcal{T}_X \leqslant M(X)$, the convergence also holds in $\mathcal{T}_X$. Clearly, for all n, the term a_n satisfies

$$a_n \leqslant a_{n+1} = f(a_n).$$

Therefore, the directed supremum a satisfies $a \leqslant f(a)$. By Proposition 4.3, the conclusion then follows. □

Proposition 4.5. Let $(X, \leqslant)$ be a dcpo endowed with an approach structure X with $\mathcal{T}_X \leqslant M(X)$, and let $\mathcal{W} \subseteq [0,\infty]^X$ be strictly monotone for $\leqslant$. For a contraction

$$f : X \to X^{in}_{\mathcal{W}},$$

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we have:

(1) If $I \subset X$ is non-empty and closed under directed suprema and $f : I \to I$ is splitting in the terminology of Martin (2000a) (or inflationary in Gierz *et al.* (2003)) in the sense that $z \leqslant f(z)$ whenever $z \in I$, then f has a fixed point.

(2) If f is splitting on X, then f has a fixed point.

Proof. It is clear that we only have to prove the first assertion. Choose $x \in I$. Then the sequence $(a_n)_n$ obtained by iterating f on x is a monotone sequence in I. So the supremum $a = \bigvee_n a_n$ belongs to I. Since $a_n \leqslant a$ for every n, the sequence $(a_n)_n$ also converges to a in $M(X)$ and hence also in $\mathcal{T}_X$. Proposition 4.3 can again be applied to complete the proof. □

By making the appropriate choices for the approach space X and the collection $\mathcal{W}$, we can recover several fixed-point theorems proved by K. Martin – we will just give two examples.

Corollary 4.6 (Martin 2000a, Proposition 3.3.3). Let $(X, \leqslant)$ be a domain with a measurement $\mu : X \to [0, \infty]^{op}$ such that $\sigma(X) \subseteq \mathcal{T}_\mu^{in} \vee \downarrow\mathcal{T}$. If $f : (X, \leqslant) \to (X, \leqslant)$ is splitting and

$$\mu \circ f : (X, M(X)) \to ([0, \infty]^{op}, \sigma([0, \infty]^{op})$$

is continuous, then for every $x \in X$, the sequence $(a_n)_n$ obtained by iterating f on x has a supremum that is a fixed point for f.

Proof. Let $\mathcal{W} = \{\mu\}$ and observe that, as proved by Martin, $\mathcal{W}$ is strictly monotone. Consider $M(X)$ embedded as an approach space X. Then $\mathcal{T}_X = M(X)$. The condition that $\mu \circ f$ is continuous from $M(X)$ to $[0, \infty]^{op}$ is equivalent to saying that $f : M(X) \to (X, \mathcal{T}_\mathcal{W}^{in})$ is continuous. Since Top is concretely coreflective in App, this in turn is equivalent to $f : X \to X_\mathcal{W}^{in}$ being contractive. The rest follows as in the proof of Proposition 4.5 (2). □

Corollary 4.7 (Martin 2000b, Proposition 3.). Let $(X, \leqslant)$ be a domain with a measurement $\mu : X \to [0, \infty]^{op}$ such that $\sigma(X) \subseteq \mathcal{T}_\mu^{in} \vee \downarrow\mathcal{T}$. If $I \subset X$ is non-empty and closed under directed suprema and $f : I \to I$ is splitting and

$$\mu \circ f : I \to [0, \infty]^{op}$$

is Scott continuous, then, choosing $x \in I$, the sequence $(a_n)_n$ obtained by iterating f on x has a supremum that is a fixed point for f.

Proof. We use a similar argument to that used in the previous corollary. So $\mathcal{W} = \{\mu\}$ again, but this time the approach space X is the embedding in App of the Scott topology on $(X, \leqslant)$. The condition that $\mu \circ f$ is continuous from $\sigma(X)$ to $[0, \infty]^{op}$ is equivalent to $f : X \to X_\mathcal{W}^{in}$ being contractive. □

We now turn to situations where apparently no specific class $\mathcal{W}$ is given.

Proposition 4.8. Let $(X, \leqslant)$ be a dcpo endowed with approach spaces $(X, \mathcal{G})$ and $(X, \mathcal{G}')$ defined by their gauges, with $\mathcal{T}_\mathcal{G} \leqslant M(X)$ and $\leqslant = \leqslant_{\mathcal{G}'}$. For a contraction

$$f : (X, \mathcal{G}) \to (X, \mathcal{G}'),$$

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we fix $x \in X$ and let

$$a_n = f^n(x)$$

be the values obtained by iterating f on x. If $(a_n)_n$ is monotone and

$$\{z \in X \mid z \leqslant f(z)\}$$

is closed under directed suprema, then $a = \bigvee_n a_n$ is a fixed point.

Proof. Applying Corollary 2.4 and the notation introduced there, we consider the collection of quasi metrics

$$\mathcal{H} = \{q_\sigma \circ g \times g \mid g : (X, \mathcal{G}') \to ([0,\infty], q_\sigma) \text{ contraction}\}$$

and $\mathcal{Q} = \mathcal{H}^\vee$ as in Proposition 3.8. It is clear that $(\mathcal{Q}(x,.))_{x\in X}$ is an approach basis for the approach space $(X, \mathcal{G}')$.

Using the notation of Example 2.2, every quasi metric $q_\sigma \circ g \times g$ is weighted by the function $w_{q_\sigma} \circ g = g$. If $g(x) = \infty$, it is clear that $q_\sigma(g(x), g(y)) = 0$, so g is forcing for $q_\sigma \circ g \times g$.

The collection $\mathcal{W}$ of all the weights coincides with the collection of all contractions $g : (X, \mathcal{G}') \to ([0,\infty], q_\sigma)$. Hence, by Proposition 2.3, we have $X^{in}_{\mathcal{W}} = (X, \mathcal{G}')$. Moreover, by Proposition 3.8, the collection $\mathcal{W}$ is strictly monotone for $\leqslant$. The rest follows immediately from Proposition 4.4. □

The following example should be compared with Martin (2000a, Theorem 3.2.1).

Example 4.9. Taking for $\mathcal{G}$ the gauge of the Martin topology $M(X)$ (embedded as an approach space) and for $\mathcal{G}'$ the gauge of the Scott topology on a dcpo $(X, \leqslant)$, we get that both conditions $\mathcal{T}_\mathcal{G} \leqslant M(X)$ and $\leqslant = \leqslant_{\mathcal{G}'}$ are fulfilled. This means that for a continuous map

$$f : (X, M(X)) \to (X, \sigma(X)),$$

assuming that $(a_n)_n$ is monotone and $\{z \in X \mid z \leqslant f(z)\}$ is closed under directed suprema, we have that $a = \bigvee_n a_n$ is a fixed point.

Finally, we will give some applications to monotone functions.

Proposition 4.10. Let $(X, \leqslant)$ be a dcpo endowed with an approach structure on X such that $\mathcal{T}_X \leqslant M(X)$ and $\leqslant = \leqslant_X$. For a contractive map

$$f : X \to X$$

and a fixed $x \in X$ satisfying $x \leqslant f(x)$, the sequence $(a_n)_n$ obtained by iterating f on x has a supremum $a = \bigvee_n a_n$ that is a fixed point for f. If, moreover, $(X, \leqslant)$ has a bottom element and x is taken as $x = \perp$, then a is the least fixed point of f.

Proof. As in Proposition 4.8, applying Corollary 2.4 and the notation introduced there, we consider the collection of quasi metrics

$$\mathcal{H} = \{q_\sigma \circ g \times g \mid g : X \to ([0,\infty], q_\sigma) \text{ contraction}\}.$$

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Using the notation of Example 2.2, every quasi metric $q_\sigma \circ g \times g$ is weighted by the forcing weight $w_{q_\sigma} \circ g = g$. The collection $\mathcal{W}$ of all the weights coincides with the collection of all contractions $g : X \to ([0,\infty], q_\sigma)$, so, by Proposition 2.3, we have $X^{in}_{\mathcal{W}} = X$, and by Proposition 3.8, $\mathcal{W}$ is strictly monotone for $\leqslant$.

Next we consider the given contraction f. By Proposition 3.6, it is monotone as a map $f : (X, \leqslant) \to (X, \leqslant)$. Since the fixed element x satisfies $x \leqslant f(x)$, it follows that the sequence $(a_n)_n$ is monotone and thus the supremum $a = \bigvee_n a_n$ is well defined. Moreover, condition (1) in Proposition 4.3 is trivially fulfilled. Since the sequence converges in $M(X)$, it also converges in $\mathcal{T}_X$, so condition (2) is also satisfied. Finally, in order to prove condition (3), observe that $a_{n-1} \leqslant a$ and thus $a_n \leqslant f(a)$ for every $n \in \mathbb{N}$. Finally, $a \leqslant f(a)$ is also fulfilled. So we can apply Proposition 4.3 to conclude that a is a fixed point for f.

Assuming that $x = \bot$ and applying the monotonicity of f, we get that $a_n \leqslant b$ for every fixed point b. So a is clearly the least fixed point. □

As an application of Proposition 4.10, we can recover Scott's least fixed-point theorem (Gierz *et al.* 2003), which, as was shown by A. Edalat in Edalat and Heckmann (1998), implies the classical Banach fixed-point theorem for Lipschitz functions with Lipschitz factor strictly smaller than 1 on a complete metric space.

Example 4.11. On a dcpo $(X, \leqslant)$, by taking for the approach structure the Scott topology on X (embedded in App), both conditions $\mathcal{T}_X \leqslant M(X)$ and $\leqslant = \leqslant_X$ are fulfilled and we get that:

For a continuous map

$$f : (X, \sigma(X)) \to (X, \sigma(X))$$

and a fixed $x \in X$ satisfying $x \leqslant f(x)$, the sequence $(a_n)_n$ with $a_n = f^n(x)$ obtained by iterating f on x has a supremum $a = \bigvee_n a_n$ thst is a fixed point for f. When $(X, \leqslant)$ has a bottom element $\bot$, and choosing $x = \bot$, the supremum a is the least fixed point of f.

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