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Connecting orbits for a reversible Hamiltonian system

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Abstract. The existence of heteroclinic and homoclinic solutions which shadow corresponding chains of such solutions is established for a class of reversible Hamiltonian systems. The proof involves elementary minimization arguments.

1. *Introduction*

The goal of this paper is to find heteroclinic and homoclinic solutions for a class of reversible Hamiltonian systems. The systems have the form

$$\ddot{q} + W_q(t, q) = f(t), \tag{HS}$$

where W and f satisfy:

$$W \in C^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \text{ and is 1-periodic in } t \text{ and in the components } q_1, \ldots, q_n \text{ of } q; \tag{W_1}$$

$$f \in C(\mathbb{R}, \mathbb{R}^n) \text{ and is 1-periodic in } t; \tag{f_1}$$

$$[f] \equiv \int_0^1 f(t)\, dt = 0. \tag{f_2}$$

Set $V(t, q) = W(t, q) - f(t) \cdot q$ and assume

$$V(-t, q) = V(t, q), \tag{V_1}$$

i.e. (HS) is a reversible system. Such systems arise as models for the n-pendulum and were treated, in particular, in [**9**]. See also Bolotin [**1, 2**] and [**8**].

To describe the problem studied here, it is necessary to recall what was shown in [**9**]. Let

$$L(q) = \tfrac{1}{2}|\dot{q}|^2 - V(t, q),$$

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the Lagrangian associated with (HS). More generally, the kinetic energy term could be replaced by

$$K(q) = \sum_{i,j=1}^{n} a_{ij}(t,q)\dot{q}_i\dot{q}_j,$$

where $(a_{ij}(t,q))$ is a positive definite matrix with a_{ij} satisfying (W_1) and even in t. Then (HS) would be replaced by the Lagrangian system for $K - V$.

Set

$$I_1(q) = \int_0^1 L(q)\,dt$$

for $q \in W_1^{1,2}$, where

$$W_1^{1,2} = \{q \in W^{1,2}[0,1] \mid q \text{ is 1-periodic}\}.$$

As usual $W^{1,2}[0,1]$ denotes the Hilbert space of functions having square integrable derivatives on $[0,1]$ under

$$\|q\|^2_{W^{1,2}[0,1]} = \int_0^1 (|\dot{q}|^2 + |q|^2)\,dt.$$

Define

$$c_1 = \inf_{q \in W_1^{1,2}} I_1(q).$$

In [**9**] it was shown that

$$\mathcal{M} = \{q \in W_1^{1,2} \mid I_1(q) = c_1\}$$

is non-empty. Note that $q \in \mathcal{M}$ implies $q + k \in \mathcal{M}$ for all $k \in \mathbb{Z}^n$ via (W_1) and (f_2). Assume

$$\mathcal{M} \text{ consists of isolated points.} \tag{$\mathcal{M}$}$$

The main result of [**9**] was the following.

THEOREM 1.1. *If* (W_1), (f_1), (f_2), (V_1) *and* $(\mathcal{M})$ *hold, then for each* $v \in \mathcal{M}$, *there is a* $w \in \mathcal{M}\backslash\{v\}$ *and a solution,* Q, *of* (HS) *such that* $Q(t) - v(t) \to 0$ *as* $t \to -\infty$ *and* $Q(t) - w(t) \to 0$ *as* $t \to \infty$.

For notational simplicity, henceforth we write $Q(-\infty) = v$ and $Q(\infty) = w$ to indicate the asymptotic behavior of Theorem 1.1. The proof of Theorem 1.1 was by a minimization argument employing a renormalized functional. A renormalization was required since the natural functional associated with (HS) is infinite on the class of curves asymptotic to v and $\mathcal{M}\backslash\{v\}$.

Subsequent to [**9**], Maxwell [**6**] extended Theorem 1.1 by proving that for any $v \neq w \in \mathcal{M}$, there exists a heteroclinic chain of solutions of (HS) asymptotic to v and w, i.e. there is an $\ell > 0$ and distinct heteroclinic solutions $Q_1, \dots, Q_\ell$ of (HS) with $Q_1(-\infty) = v \equiv v_0$, $Q_{i+1}(-\infty) = Q_i(\infty) \equiv v_i \in \mathcal{M}$, $1 \le i < \ell$, and $Q_\ell(\infty) = w \equiv v_\ell$. Furthermore, the chain is minimal. By a minimal heteroclinic chain it is meant that if $v_i, v_{i+1} \in \mathcal{M}$,

$$\Gamma(v_i, v_{i+1}) = \{q \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n) \mid q(-\infty) = v_i \text{ and } q(\infty) = v_{i+1}\},$$

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and

$$c(v_i, v_{i+1}) = \inf_{q \in \Gamma(v_i, v_{i+1})} J(q), \tag{1.2}$$

where J is the renormalized functional (whose definition will be recalled in §2), then

$$c(v_0, v_\ell) = \sum_{i=1}^{\ell} c(v_{i-1}, v_i). \tag{1.3}$$

Moreover, there is no heteroclinic chain $P_1, \dots, P_m$ joining v_i to v_{i+1}, $0 \le i \le m-1$, with

$$\sum_1^m J(P_k) = c(v_i, v_{i+1})$$

unless $m = 1$.

Our main result is that given any pair $v \neq w \in \mathcal{M}$ and a minimal heteroclinic chain joining v and w, if certain mild non-degeneracy conditions are satisfied, there exist infinitely many actual solutions of (HS) which are heteroclinic from v to w. These solutions are distinguished by the amount of time they spend near the intermediate periodic solutions $v_1, \dots, v_{\ell-1}$. These solutions do not necessarily shadow the chain $Q_1, \dots, Q_\ell$. However, if the time intervals spent near the periodic states are large enough, then the heteroclinic does shadow some minimal heteroclinic chain joining v and w.

To describe the non-degeneracy condition, set

$$\mathcal{S}_i \equiv \mathcal{S}(v_{i-1}, v_i) \equiv \{q(0) \mid q \in \Gamma(v_{i-1}, v_i) \text{ and } J(q) = c(v_{i-1}, v_i)\}.$$

Let $\mathcal{C}_{v_{i-1}}(v_{i-1}, v_i)$ denote the component of $\overline{\mathcal{S}}_i$ to which $v_{i-1}(0)$ belongs. Similarly, $\mathcal{C}_{v_i}(v_{i-1}, v_i)$ is the component of $\overline{\mathcal{S}}_i$ to which $v_i(0)$ belongs. The non-degeneracy assumption is

$$\mathcal{C}_{v_{i-1}}(v_{i-1}, v_i) = \{v_{i-1}(0)\}, \quad 1 \le i \le \ell. \tag{$*$}$$

As will be seen in §2, if $(*)$ fails, (HS) has a continuum of heteroclinics from v_{i-1} to v_i for some i, $1 \le i \le \ell$.

To describe our next result, observe that by the reversibility of (HS), whenever $Q(t)$ is a solution of (HS), heteroclinic to v and w, then $Q(-t)$ is a solution heteroclinic to w and v. Given a minimal heteroclinic chain, $\mathcal{Q} = (Q_1, \dots, Q_\ell)$, such as Maxwell found, one can form a larger chain by gluing further admissible heteroclinics from $\{Q_i(t), Q_j(-t) \mid 1 \le i, j \le \ell\}$ to $\mathcal{Q}$. For example, the possible $\ell + 1$ chains are $\{Q_1(-t), Q_1, \dots, Q_\ell\}$ and $\{Q_1, \dots, Q_\ell, Q_\ell(-t)\}$. Similarly, $\ell + 2$ chains can be constructed by gluing a pair of admissible heteroclinics to the ends of $\mathcal{Q}$ or a pair $Q_i(-t), Q_i$ or $Q_{i+1}, Q_{i+1}(-t)$ between Q_i and Q_{i+1}.

A new heteroclinic chain $\hat{\mathcal{Q}}$ obtained by such a gluing process will be called an augmented chain. It will be shown that if $(*)$ is satisfied, then corresponding to any augmented chain $\hat{\mathcal{Q}} = (u_1, \dots, u_p)$ of $\mathcal{Q}$, there are infinitely many actual solutions of (HS) heteroclinic to $u_1(-\infty)$ and $u_p(\infty)$ and distinguished by the amount of time they spend near $u_i(\infty)$, $1 \le i \le p-1$. The simplest example of this result is to take the setting of Theorem 1.1, gluing the trivial chain Q to $Q(-t)$ to get a homoclinic chain joining v to

v. Then the augmented chain theorem shows that there are infinitely many actual solutions of (HS) homoclinic to v and distinguished by the amount of time they spend near w.

The results on augmented chains will be obtained from a theorem combining two or more (generally a finite number) minimal heteroclinic chains. The final set of applications are to situations where one has an infinite heteroclinic chain. Here the subtleties of dealing with a limiting situation must be dealt with.

A few results have been obtained that are related to ours. In particular, in an unpublished manuscript [**7**], Maxwell gave a version of our main result. His non-degeneracy condition is stronger than $(*)$; it implies that the functions Q_i are isolated minimizers of J in $\Gamma(v_{i-1}, v_i)$. Maxwell's heteroclinics shadow appropriate translates of $Q_1, \ldots, Q_\ell$.

There is also earlier related work of Strobel, who in his thesis [**12**] studied (HS) for $f \equiv 0$, $V(t, 0) = 0 > V(t, x)$, $x \in \mathbb{R}^n \backslash \mathbb{Z}^n$. Thus Strobel is dealing with heteroclinics to equilibria and no reversibility is required for V. Under his assumptions, Strobel first found a minimal heteroclinic chain joining any pair of equilibria $\alpha \neq \beta \in \mathbb{Z}^n$ for (HS). With a further strong non-degeneracy condition on the basic heteroclinics, Strobel found actual heteroclinics that shadow the heteroclinic chain joining α and β.

Both Strobel and Maxwell use delicate and somewhat technical variational deformation arguments in the spirit of those others have used to find multibump homoclinic solutions of Hamiltonian systems; see, for example, Séré [**11**] or [**4**] for such results. In contrast, the existence theorems here are based on elementary minimization arguments. Indeed, our work was partially motivated by a recent paper of Calanchi and Serra [**3**]. They studied the setting of Theorem 1.1 when $n = 1$. Using a nice minimization argument, they obtained the family of homoclinics corresponding to the simple augmented homoclinic chain described above (for $n = 1$). A comparison argument they use plays an important role in our work. See also the paper [**5**] of Mather which at least in spirit is connected to the current work.

The definition of the renormalized functional will be recalled in §2 and a study of the non-degeneracy condition $(*)$, which is of independent interest, will be made. The main result, extending Maxwell's work, will be carried out in §3. Finally, §4 deals with augmented chains, gluing minimal chains, and infinite chains.

2. *Some preliminaries*

To begin, the technical framework used in [**9**] will be recalled. The reversibility condition (V_1) implies (see [**9**])

$$c_1 = \inf_{q \in W^{1,2}[0,1]} I_1(q). \tag{2.1}$$

Hence, for each $p \in \mathbb{Z}$ and $q \in W^{1,2}[p-1, p]$,

$$a_p(q) \equiv \int_{p-1}^{p} L(q)\,dt - c_1 \geq 0.$$

Let $v \in \mathcal{M}$ and

$$\Gamma = \{q \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n) \mid q(-\infty) = v \text{ and } q(\infty) = w \text{ for some } w \in \mathcal{M}\backslash\{v\}\}$$

Set

$$J(q) = \sum_{p \in \mathbb{Z}} a_p(q)$$

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and

$$c = \inf_{q \in \Gamma} J(q).$$

In [**9**], Theorem 1.1 was proved by showing that there is a $Q \in \Gamma$ such that $J(Q) = c$ and Q satisfies (HS). Likewise, the result of Maxwell [**6**] was proved by minimizing J over $\Gamma(v, w)$ and showing that a minimizing sequence converges to a heteroclinic chain of solutions.

Once one has a basic heteroclinic solution of (HS) or a chain of such solutions, to construct more complicated solutions some sort of non-degeneracy condition is required. Classically, to get a symbolic dynamics of solutions, it is assumed that there is a transversal intersection of stable and unstable manifolds for an appropriate Poincaré map at a heteroclinic point. For variational constructions, milder conditions are generally required.

To formulate such a condition for the current setting, suppose first that $V = V(q)$. Then $\mathcal{M}$ consists of points at which V achieves its maximum on $\mathbb{R}^n$. Thus $v \equiv v(0)$ for $v \in \mathcal{M}$ and Q of Theorem 1.1 is heteroclinic to a pair of equilibrium points of (HS). Let

$$\mathcal{S} = \{q(0) \mid q \in \Gamma \text{ and } J(q) = c\}. \tag{2.2}$$

When V is autonomous, if $q \in \Gamma$, so is

$$\tau_\theta q(t) = q(t - \theta)$$

for all $\theta \in \mathbb{R}$. Hence $\overline{\mathcal{S}} \supset \overline{q(\mathbb{R})}$ and in particular $v, w \in \overline{\mathcal{S}}$. Simple examples when $n = 1$ show no more complicated connecting orbits can be expected for this time independent case.

More generally, when V depends on t and $\mathcal{S}$ is defined by (2.2), $q \in \Gamma$ implies $\tau_k q \in \Gamma$ for $k \in \mathbb{Z}$. Hence $v(0)$ and $w(0)$ belong to $\overline{\mathcal{S}}$. The behavior of $\overline{\mathcal{S}}$ will be studied more closely.

LEMMA 2.3. *$\overline{\mathcal{S}}$ is bounded and therefore compact.*

Proof. It can be assumed that $V(t, q) \le 0$ for $q \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Let r be small compared to

$$\max_{z \ne w \in \mathcal{M}} \|z - w\|_{L^\infty[0,1]}.$$

By Proposition 2.18 of [**9**], there is an $\alpha(r) > 0$ such that if $\varphi \in W^{1,2}[0, 1]$ and $\|\varphi - z\|_{L^\infty[0,1]} \ge r$ for all $z \in \mathcal{M}$, i.e. $\|\varphi - \mathcal{M}\|_{L^\infty[0,1]} \ge r$, then

$$I_1(\varphi) \ge c_1 + \alpha(r). \tag{2.4}$$

Let $q \in \Gamma$ with $J(q) = c$ and ℓ be the number of intervals $[p, p-1]$ (with $p \in \mathbb{Z}$) on which $\|q - \mathcal{M}\|_{L^\infty[p-1,p]} \ge r$. Then by (2.4),

$$\ell\alpha(r) \le c. \tag{2.5}$$

This provides an upper bound on ℓ. For any $q \in \Gamma$ and $s > \sigma$,

$$|q(s) - q(\sigma)| \le \int_\sigma^s |q'(t)|\, dt \le (s - \sigma)^{1/2} \left(\int_\sigma^s |q'(t)|^2\, dt \right)^{1/2}. \tag{2.6}$$

Hence, if $s, \sigma \in [p-1, p]$ and $J(q) = c$,

$$|q(s) - q(\sigma)| \le (a_p(q) + c_1)^{1/2} \le (c + c_1)^{1/2}. \tag{2.7}$$

Now (2.5) and (2.7) yield an $L^\infty(\mathbb{R}, \mathbb{R}^n)$ bound for q and the lemma follows. □

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Let

$$\mathcal{K} = \{(q(0), q'(0)) \mid q \in \Gamma \text{ and } J(q) = c\}.$$

Let P denote the projector of $\mathcal{K}$ to $\mathcal{S}$, i.e. $P(q(0), q'(0)) = q(0)$.

LEMMA 2.8. *P is a homeomorphism of $\mathcal{K}$ to $\mathcal{S}$ (and $\overline{\mathcal{K}}$ to $\overline{\mathcal{S}}$).*

Proof. It suffices to prove P is one-to-one. Suppose to the contrary that there exist $p \neq q$ with $p(0) = q(0)$. Note that both p and q are asymptotic to v as $t \to -\infty$ but may have different asymptotes as $t \to \infty$. Since $J(p) = J(q) = c$,

$$\sum_{i \le 0} a_i(p) = \sum_{i \le 0} a_i(q) \tag{2.9}$$

since otherwise, say,

$$\sum_{i \le 0} a_i(p) < \sum_{i \le 0} a_i(q). \tag{2.10}$$

Then

$$\sum_{i > 0} a_i(p) > \sum_{i > 0} a_i(q) \tag{2.11}$$

and gluing $p|_{-\infty}^{0}$ to $q|_0^{\infty}$ produces a new function $z \in \Gamma$ with $J(z) < c$, contrary to the definition of c. Therefore there is equality in (2.9) and in (2.11). Hence, with z as just defined, $J(z) = c$. The arguments of [**9**] then imply that z is a solution of (HS). But z coincides with p and with q on open intervals. Thus $z \equiv p \equiv q$ and P is a homeomorphism of $\mathcal{K}$ to $\mathcal{S}$.

To complete the proof, it suffices to show that $\overline{\mathcal{K}} \backslash \mathcal{K}$ consists of $(v(0), v'(0))$ together with $(w(0), w'(0))$ for all $w \in \mathcal{M} \backslash \{v\}$ for which there is a $Q \in \Gamma$ having $Q(\infty) = w$ and $J(Q) = c$. Indeed, if $(a, b) \in \overline{\mathcal{K}} \backslash \mathcal{K}$ and is not in this latter set,

$$(a, b) = \lim_{m \to \infty} (q_m(0), q_m'(0))$$

with q_m in Γ and $J(q_m) = c$. The form of I implies (q_m) is bounded in $W^{1,2}_{\text{loc}}(\mathbb{R}, \mathbb{R}^n)$. By Lemma 2.3, (q_m) is bounded in $L^\infty(\mathbb{R}, \mathbb{R}^n)$ and (HS) then implies (q_m) is bounded in $C^2(\mathbb{R}, \mathbb{R}^n)$. Hence, along a subsequence, (q_m) converges weakly in $W^{1,2}_{\text{loc}}$ and strongly in L^∞_{loc} to $q \in W^{1,2}_{\text{loc}}$ with q a solution of (HS) and $J(q) \le c$. Standard arguments show that q is part of a heteroclinic j-chain of solutions of (HS) joining v and w. But then the definition of c implies $j = 1$, $q \in \Gamma$ and $J(q) = c$. Hence $(a, b) = (q(0), q'(0)) \in \mathcal{K}$. □

Let $w(0) \in \overline{\mathcal{S}}$ for some $w \in \mathcal{M} \backslash \{v\}$ and let $\mathcal{C}$ denote the component of $\overline{\mathcal{S}}$ containing $w(0)$. The next 'all or nothing' lemma describes the possibilities for $\mathcal{C}$.

LEMMA 2.12. *Either:*

(i) $v(0) \in \mathcal{C}$ *or*

(ii) $\mathcal{C} = \{w(0)\}$.

Proof. Otherwise $\mathcal{C}$ is a subcontinuum of $\overline{\mathcal{S}}$ containing $w(0)$ and not meeting $v(0)$. For each $q \in \overline{\mathcal{S}}$ and $k \in \mathbb{N}$, set $f_k(q(0)) = q(-k)$. By Lemma 2.8, $f_k \colon \overline{\mathcal{S}} \to \overline{\mathcal{S}}$ is continuous

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and by Lemma 2.3, $\mathcal{C}$ is compact. Hence $f_k(\mathcal{C})$ is compact, connected, and $w(0) \in f_k(\mathcal{C})$. Consequently, by the definition of $\mathcal{C}$,

$$f_k(\mathcal{C}) \subset \mathcal{C} \tag{2.13}$$

for all $k \in \mathbb{N}$. But for each $q(0) \in \mathcal{S}$, $q(-k) \to v(0)$ as $k \to \infty$, contrary to (2.13). □

Remark 2.14. (a) Of course alternative (i) occurs in the autonomous case.

(b) The combined 'curve shortening' argument of Lemma 2.8 and dynamical system argument of Lemma 2.12 can be applied in other variational settings where minimization is used such as [**10**]. Whether Lemma 2.12 is true for other variational non-minimization settings is an interesting open question.

Remark 2.15. Suppose $v \neq w \in \mathcal{M}$ and there is a $Q \in \Gamma(v, w)$ which is a solution of (HS) heteroclinic to v and w satisfying $J(Q) = c(v, w)$. Thus

$$\mathcal{S}(v, w) \equiv \{q(0) \in \mathbb{R}^n \mid q \in \Gamma(v, w) \text{ and } J(q) = c(v, w)\} \neq \varnothing.$$

Let $\mathcal{C}_v$ and $\mathcal{C}_w$ denote the components of $\overline{\mathcal{S}}(v, w)$ containing $v(0)$ and $w(0)$ respectively. Then the arguments of Lemmas 2.8 and 2.12 show that either there is a subcontinuum of $\overline{\mathcal{S}}(v, w)$ containing $\mathcal{C}_v$ and $\mathcal{C}_w$ or $\mathcal{C}_v = \{v(0)\}$ and $\mathcal{C}_w = \{w(0)\}$.

Remark 2.16. If v, w_1, w_2 are distinct elements of $\mathcal{M}$, the argument of Lemma 2.8 shows $\mathcal{S}(v, w_1) \cap \mathcal{S}(v, w_2) = \varnothing$.

3. *The generalization of Maxwell's result*

Let $v \neq w \in \mathcal{M}$ and let $Q_1, \ldots, Q_\ell$ be a minimal heteroclinic chain of solutions of (HS) joining v and w as obtained by Maxwell and described in the introduction. Again, set $v_0 = v$, $v_i = Q_i(\infty)$, $1 \leq i \leq \ell - 1$, and $v_\ell = w$.

THEOREM 3.1. *Let* (W_1), (f_1), (f_2), (V_1) *and* $(\mathcal{M})$ *be satisfied and let* $(*)$ *hold. Let* $v \neq w \in \mathcal{M}$. *Then* (HS) *has infinitely many heteroclinic solutions from* v *to* w *characterized by the amount of time they spend near* v_i, $1 \leq i \leq \ell - 1$.

Remark 3.2. Note that if $(*)$ does not hold, (HS) has a continuum of solutions joining v_{i-1} to v_i for some i, $1 \leq i \leq \ell$. This does not provide more information than is already known for the autonomous case, but it is significant when V depends explicitly on t.

Theorem 3.1 follows from a more precise result that will be formulated next. Let $0 < \rho < r < 1$ with r small compared to 1 and ρ small compared to r, and ρ, r otherwise free for now. Let $\mathcal{O}_{i,1}, \mathcal{O}_{i,2} \subset \mathbb{R}^n$ be neighborhoods of $v_{i-1}(0)$, $v_i(0)$ respectively, $1 \leq i \leq \ell$, with the diameter of $\mathcal{O}_{i,1}, \mathcal{O}_{i,2} \leq \rho$ and such that for $1 \leq i \leq \ell$ and $j = 1, 2$,

$$\partial \mathcal{O}_{i,j} \cap \mathcal{S}_i = \varnothing, \tag{3.3}$$

where $\mathcal{S}_i = \mathcal{S}(v_{i-1}, v_i)$ as in §2. The existence of $\mathcal{O}_{i,1}$ and $\mathcal{O}_{i,2}$ follows from $(*)$. Let $m \in \mathbb{Z}^{2\ell}$ with $m_{j+1} > m_j$, $1 \leq j \leq 2\ell - 1$. Define

$$X_m = \{q \in W^{1,2}_{\text{loc}}(\mathbb{R}, \mathbb{R}^n) \mid q(-\infty) = v_0, q(m_{2k-1}) \in \overline{\mathcal{O}}_{k,1}, \\ q(m_{2k}) \in \overline{\mathcal{O}}_{k,2}, 1 \leq k \leq \ell, \text{ and } q(\infty) = v_\ell\}.$$

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Thus X_m consists of curves of the type sought as heteroclinic solutions of (HS). Adapting an idea from [**3**], a constrained minimization problem will be used to find these solutions. Set

$$b_m = \inf_{q \in X_m} J(q). \tag{3.4}$$

Remark 3.5. For $j \in \mathbb{Z}$ and $j^* = (j, \ldots, j) \in \mathbb{Z}^{2\ell}$,

$$b_m = b_{m-j^*}, \tag{3.6}$$

i.e. it is only the difference in the m_i's that is significant.

Theorem 3.1 now follows from the following.

THEOREM 3.7. *For $m_{i+1} - m_i$ sufficiently large, $1 \le i \le 2\ell - 1$, there is a $\mathcal{Q}_m \in X_m$ such that $J(\mathcal{Q}_m) = b_m$. Moreover, $\mathcal{Q}_m$ is a solution of* (HS) *heteroclinic from v to w and*

$$\begin{cases} |\mathcal{Q}_m(t) - v(t)| \le r, & t \in (-\infty, m_1], \\ |\mathcal{Q}_m(t) - v_j(t)| \le r, & t \in [m_{2j}, m_{2j+1}], 1 \le j \le \ell - 1, \\ |\mathcal{Q}_m(t) - w(t)| \le r, & t \in [m_{2\ell}, \infty). \end{cases} \tag{3.8}$$

The proof of Theorem 3.7 requires several preliminaries.

LEMMA 3.9. *Let Ω_i be a neighborhood of $v_i(0)$ of diameter $\le r$, $0 \le i \le \ell$. Then for each $\xi \in \overline{\Omega}_i$, there is a function $\varphi_i(\xi, t)$ continuous for $t \in [0, 1]$ and a positive constant a independent of r, ξ, and i such that $\varphi_i(\xi, 0) = \xi$, $\varphi_i(\xi, 1) = v_i(1)$, and*

$$\int_0^1 L(\varphi_i)\, dt - c_1 \le ar. \tag{3.10}$$

Proof. Take

$$\varphi_i(\xi, t) = v_i(t) + (\xi - v_i(0))(1 - t).$$

Then φ_i satisfies the boundary conditions and (3.10) follows from

$$\int_0^1 L(\varphi_i)\, dt - c_1 = |\xi - v_i(0)|^2 - \int_0^1 (V(t, \varphi_i) - V(t, v_i))\, dt. \qquad \square$$

Remark 3.11. Similarly, for each $\xi \in \overline{\Omega}_i$, there is a continuous $\psi_i(\xi, t)$ and a positive constant which can also be taken (independently of ξ, r, i) such that $\psi_i(\xi, 0) = v_i(0)$, $\psi_i(\xi, 1) = \xi$, and (3.10) holds with φ_i replaced by ψ_i.

To continue, assume r is small compared to D, where

$$D = \inf_{u \ne z \in \mathcal{M}} c(u, z) \tag{3.12}$$

and ρ satisfies

$$2a\rho < \alpha(r) \tag{3.13}$$

with α as in (2.4) and a as in (3.10) and Remark 3.11.

Remark 3.14. (i) If (q_k) is a minimizing sequence for (3.4), it can be assumed that

$$\sum_{-\infty}^{m_1} a_i(q_k) \leq a\rho. \tag{3.15}$$

Indeed if (3.15) holds for q_k, set $\widehat{q}_k = q_k$; if not, replace $q_k|_{-\infty}^{m_1}$ by the curve $\widehat{q}_k \in X_m$ obtained by gluing $v_0|_{-\infty}^{m_1-1}$ to $\psi_0(q_k(m_1), t)|_0^1$ to $q_k|_{m_1}^{\infty}$ and (3.15) holds for this new function. Moreover, $J(\widehat{q}_k) \leq J(q_k)$ so the sequence $(\widehat{q}_k)$ is also minimizing for (3.4).

(ii) It can also be assumed that

$$\|q_k - v_0\|_{L^\infty[-\infty, m_1]} \leq r. \tag{3.16}$$

Otherwise, if

$$\|q_k - \mathcal{M}\|_{L^\infty[i-1,i]} > r$$

holds for some $i \in \mathbb{Z}$, $i \leq m_1$, as in (2.4),

$$a_i(q_k) \geq \alpha(r). \tag{3.17}$$

But (3.15) and (3.17) are contrary to (3.13). Hence

$$\|q_k - \mathcal{M}\|_{L^\infty[i-1,i]} \leq r \tag{3.18}$$

for all $i \in \mathbb{Z}$. By the choice of r and ρ, (3.16) follows.

(iii) Similarly,

$$\|q_k - v_\ell\|_{L^\infty[m_{2\ell}, \infty)} \leq r \tag{3.19}$$

and

$$\sum_{m_{2\ell}+1}^{\infty} a_i(q_k) \leq a\rho. \tag{3.20}$$

(iv) As in (i) of this remark,

$$\sum_{m_{2j}+1}^{m_{2j+1}} a_i(q_k) \leq 2a\rho \tag{3.21}$$

and as in (ii), this implies

$$\|q_k - v_j\|_{L^\infty[m_{2j}, m_{2j+1}]} \leq r. \tag{3.22}$$

(v) The estimates (3.16), (3.19) and (3.22) are valid independently of the choice of m provided that $m_{i+1} > m_i + 1$, $i = 1, 2$.

PROPOSITION 3.23. *There exists a $\mathcal{Q}_m \in X_m$ such that $J(\mathcal{Q}_m) = b_m$. Moreover, (3.8) holds for $\mathcal{Q}_m$.*

Proof. Let (q_k) be a minimizing sequence for (3.4). The form of J implies (q_k) is bounded in $W^{1,2}_{\text{loc}}(\mathbb{R}, \mathbb{R}^n)$ and therefore q_k converges along a subsequence both weakly in $W^{1,2}_{\text{loc}}$ and strongly in L^∞_{loc} to $\mathcal{Q}_m \in W^{1,2}_{\text{loc}}$. As in [**9**],

$$J(\mathcal{Q}_m) \leq b_m. \tag{3.24}$$

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By (3.15) and (3.16),

$$\sum_{-\infty}^{m_1} a_i(\mathcal{Q}_m) \le a\rho \tag{3.25}$$

and

$$\|\mathcal{Q}_m - v_0\|_{(-\infty,m_1]} \le r. \tag{3.26}$$

As in [**9**], (3.25) and (3.26) imply $\mathcal{Q}_m(-\infty) = v_0$.

Similarly, $\mathcal{Q}_m(\infty) = v_1$ via (3.19) and (3.20). These facts and the L^∞_{loc} convergence of (q_k) imply $\mathcal{Q}_m \in X_m$. Hence by (3.24), $J(\mathcal{Q}_m) = b_m$. □

Remark 3.27. As in [**9**], $\mathcal{Q}_m$ satisfies (HS) except possibly at the components of m. Furthermore, (HS) is satisfied at m_i unless $\mathcal{Q}_m(m_{2i-1}) \in \partial\mathcal{O}_{i,1}$ or $\mathcal{Q}_m(m_{2i}) \in \partial\mathcal{O}_{i,2}$, $1 \le i \le \ell$. Thus it remains to show that $\mathcal{Q}_m$ at the constraint points is interior to the appropriate set, $\mathcal{O}_{i,j}$. This is where the fact that $m_{i+1} - m_i$ is large is required.

PROPOSITION 3.28. *Let $\epsilon > 0$. Then for $m_{2i} - m_{2i-1}$ sufficiently large, $1 \le i \le \ell$,*

$$b_m \le \sum_{i=1}^{\ell} c(v_{i-1}, v_i) + \epsilon = c(v_0, v_\ell) + \epsilon. \tag{3.29}$$

Proof. It suffices to produce a $q^* \in X_m$ such that

$$J(q^*) \le c(v_0, v_\ell) + \epsilon. \tag{3.30}$$

The function q^* will be obtained by appropriately modifying the chain $\{Q_1, \ldots, Q_\ell\}$. Recall that $Q_i \in \Gamma(v_{i-1}, v_i)$ is a solution of (HS) heteroclinic from v_{i-1} to v_i with $J(Q_i) = c(v_{i-1}, v_i)$, $1 \le i \le \ell$. The minimizer Q_i is not unique in $\Gamma(v_{i-1}, v_i)$. In particular, for each $k \in \mathbb{Z}$, $\tau_k Q_i \in \Gamma(v_{i-1}, v_i)$ and $J(\tau_k Q_i) = J(Q_i)$. For $x \in \mathbb{R}^n$, let $B_\sigma(x)$ denote the open ball about x of radius σ. Let $\sigma > 0$ be such that $\overline{B}_\sigma(v_0(0)) \subset \mathcal{O}_{1,1}$, $\overline{B}_\sigma(v_\ell(0)) \subset \mathcal{O}_{\ell,2}$, $\overline{B}_\sigma(v_i(0)) \subset \mathcal{O}_{i,2} \cap \mathcal{O}_{i+1,1}$, $1 \le i \le \ell - 1$, and

$$2(\ell+2)\sigma < \epsilon. \tag{3.31}$$

Given m_1, from $\{\tau_k Q_1 \mid k \in \mathbb{Z}\}$, choose k_1 so that $\tau_{k_1} Q_1(t) \in \overline{B}_\sigma(v_0(t))$ for all $t \le m_1$ and $\tau_{k_1} Q_1(t) \notin \overline{B}_\sigma(v_0(t))$ for some $t \in (m_1, m_1+1]$. Since $\tau_{k_1} Q_1(\infty) = v_1$, for large t, $\tau_{k_1} Q_1(t) \in \overline{B}_\sigma(v_1(t))$. Choose m_2 so that $\tau_{k_1} Q_1(t) \in \overline{B}_\sigma(v_1(t))$ for $t \ge m_1 - 1$. Let $m_3 \ge m_2 + 2$. As above, choose $k_2 \in \mathbb{Z}$ so that $\tau_{k_2} Q_2(t) \in \overline{B}_\sigma(v_1(t))$ for $t \le m_3$ and $\tau_{k_2} Q_2(t) \notin \overline{B}_\sigma(v_1(t))$ for some $t \in (m_3, m_3+1]$. Since $\tau_{k_2} Q_2(\infty) = v_2$, for large t, $\tau_{k_2} Q_2(t) \in \overline{B}_\sigma(v_2(t))$. Choose m_4 so large that $\tau_{k_2} Q_2(t) \in \overline{B}_\sigma(v_2(t))$ for $t \ge m_4 - 1$. Then choose $m_5 \ge m_4 + 2$. Continuing in this fashion determines $k_1, \ldots, k_\ell$ and $m_2, \ldots, m_{2\ell}$. Now define the function q^* as follows. Glue

$$\begin{aligned} &\tau_{k_1} Q_1|_{-\infty}^{m_2-1} \quad \text{to} \quad \varphi_1(\tau_{k_1} Q_1(m_2-1)), \cdot)|_0^1 \quad \text{to} \quad v_1|_{m_2}^{m_3-1} \\ &\text{to} \quad \psi_1(\tau_{k_2} Q_2(m_3), \cdot)|_0^1 \quad \text{to} \quad \tau_{k_2} Q_2|_{m_3}^{m_4-1} \quad \text{to} \quad \ldots, \end{aligned}$$

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where $\Omega_1 = B_\sigma(v_1(0))$, etc. in Lemma 3.9. Then by Lemma 3.9, Remark 3.11, and (3.31),

$$J(q^*) \le \sum_1^\ell c(v_{i-1}, v_i) + 2\ell a\sigma < c(v_0, v_\ell) + \epsilon \tag{3.32}$$

as claimed. □

PROPOSITION 3.33. *Suppose* $\mathcal{Q}_m(m_{2i-1}) \in \partial\mathcal{O}_{i,1}$ *or* $\mathcal{Q}_m(m_{2i}) \in \partial\mathcal{O}_{i,2}$ *for some* i, $1 \le i \le \ell$. *Then for* $m_{2p+1} - m_{2p}$ *sufficiently large,* $1 \le p \le \ell - 1$, *there is a* $d > 0$ *(independently of large* $m_{2p+1} - m_{2p}$*) such that*

$$J(\mathcal{Q}_m) \ge c(v_0, v_\ell) + d. \tag{3.34}$$

Assume Proposition 3.33 for the moment.

Completion of the proof of Theorem 3.1. In Proposition 3.28, choose

$$\epsilon < d/2. \tag{3.35}$$

Then by (3.29), (3.34) and (3.35),

$$c(v_0, v_\ell) + d \le b_m \le c(v_0, v_\ell) + d/2, \tag{3.36}$$

a contradiction. Hence $\mathcal{Q}_m(m_{2i-1}) \in \mathcal{O}_{i,1}$ and $\mathcal{Q}_m(m_{2i}) \in \mathcal{O}_{i,2}$, $1 \le i \le \ell$ and Theorem 3.1 is proved. □

Proof of Proposition 3.33. An auxiliary variational problem will be introduced. For $1 \le i \le \ell$, define

$$\Lambda(v_{i-1}, v_i) = \{q \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n) \mid q(-\infty) = v_{i-1},\\ q(\infty) = v_i \text{ and } q(0) \in \partial(\mathcal{O}_{i,1} \cup \mathcal{O}_{i,2})\}.$$

Define

$$c^*(v_{i-1}, v_i) = \inf_{q \in \Lambda(v_{i-1}, v_i)} J(q).$$

It is straightforward, via arguments from [**9**], to show that there is a $P_i \in \Lambda(v_{i-1}, v_i)$ such that $J(P_i) = c^*(v_{i-1}, v_i)$. Moreover,

$$c^*(v_{i-1}, v_i) > c(v_{i-1}, v_i). \tag{3.37}$$

Indeed, if there were equality in (3.37), as in [**9**], P_i would be a solution of (HS) heteroclinic to v_{i-1} and v_i and

$$P_i(0) \in \partial(\mathcal{O}_{i,1} \cup \mathcal{O}_{i,2}). \tag{3.38}$$

But by the choices of the sets $\mathcal{O}_{i,j}$, (3.38) is incompatiable with P_i being a solution of (HS) heteroclinic to v_{i-1} and v_i. Therefore

$$2d \equiv \inf_{1 \le i \le \ell} c^*(v_{i-1}, v_i) - c(v_{i-1}, v_i) > 0. \tag{3.39}$$

By hypothesis, $\mathcal{Q}_m(m_{2i-1}) \in \partial\mathcal{O}_{i,1}$ or $\mathcal{Q}_m(m_{2i}) \in \partial\mathcal{O}_{i,2}$ for some i. The arguments are slightly different depending on whether $i = 1$ or ℓ (the simpler cases) or $i \ne i, \ell$. Choosing

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the more difficult case, suppose that $1 \le i \le \ell - 1$ and, for example, $Q_m(m_{2i-1}) \in \partial\mathcal{O}_{i,1}$. By Remark 3.5, it can be assumed that $m_{2i-1} = 0$. By (3.21),

$$\sum_{m_{2i-2}+1}^{m_{2i-1}} a_i(Q_m), \quad \sum_{m_{2i}+1}^{m_{2i+1}} a_i(Q_m) \le 2a\rho. \tag{3.40}$$

Let γ be small compared to ρ. Then for the differences $m_{2j+1} - m_{2j}$ sufficiently large, $1 \le j \le \ell - 1$, there is at least one $s_{i-1} \in [m_{2i-2} + 1, m_{2i-1} - 1] \cap \mathbb{Z}$ and $s_i \in [m_{2i} + 1, m_{2i+1} - 1] \cap \mathbb{Z}$ such that

$$a_s(Q_m) + a_{s+1}(Q_m) \le \gamma \tag{3.41}$$

with $s = s_{i-1}$ and s_i. By the proof of Proposition 2.18 of [**9**], this implies

$$\|Q_m - v_s\|_{L^\infty[s,s+2]} \le \eta(\gamma) \tag{3.42}$$

with $s = s_{i-1}, s_i$, and $\eta(\gamma) \to 0$ as $\gamma \to 0$.

Define a function Q as follows:

$$\begin{aligned} Q(t) &= v_{i-1}(t), \quad t \le s_{i-1} \\ &= (t - s_{i-1})Q_m(s_{i-1} + 1) + (s_{i-1} + 1 - t)v_{i-1}(0), \quad s_{i-1} \le t \le s_{i-1} + 1 \\ &= Q_m(t), \quad s_{i-1} + 1 \le t \le s_i \\ &= (t - s_i)v_i(0) + (s_i + 1 - t)Q_m(s_i), \quad s_i \le t \le s_i + 1 \\ &= v_i(t), \quad t \ge s_i + 1. \end{aligned} \tag{3.43}$$

Then $Q \in \Lambda(v_{i-1}, v_i)$ so

$$J(Q) = \sum_{s_{i-1}+1}^{s_i+1} a_j(Q) \ge c(v_{i-1}, v_i) + 2d. \tag{3.44}$$

Now define a function $\mathcal{P}$ as follows:

$$\begin{aligned} \mathcal{P}(t) &= Q_m(t), \quad t \le s_{i-1} \\ &= (t - s_{i-1})v_{i-1}(0) + (s_{i-1} + 1 - t)Q_m(s_{i-1}), \quad s_{i-1} \le t \le s_{i-1} + 1 \\ &= (t - s_{i-1} - 1)\tau_{k_i} Q_i(s_{i-1} + 2) + (s_{i-1} + 2 - t)v_{i-1}(0), \\ &\qquad s_{i-1} + 1 \le t \le s_{i-1} + 2 \\ &= \tau_{k_i} Q_i(t), \quad s_{i-1} + 2 \le t \le s_i \\ &= (t - s_i)v_i(0) + (s_i + 1 - t)\tau_{k_i} Q_i(s_i + 1), \quad s_i \le t \le s_i + 1 \\ &= (t - s_i - 1)Q_m(s_i + 2) + (s_i + 2 - t)v_i(0), \quad s_i + 1 \le t \le s_i + 2 \\ &= Q_m(t), \quad t \ge s_i + 2. \end{aligned} \tag{3.45}$$

Then $\mathcal{P} \in X_m$ and

$$J(Q_m) - J(\mathcal{P}) = \sum_{s_{i-1}+1}^{s_i+2} (a_j(Q_m) - a_j(\mathcal{P})). \tag{3.46}$$

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By (3.43), (3.44) and Lemma 3.9,

$$\sum_{s_{i-1}+1}^{s_i+2} a_j(\mathcal{Q}_m) \geq \sum_{s_{i-1}+2}^{s_i} a_j(\mathcal{Q}_m) = J(\mathcal{Q}) - a_{s_{i-1}+1}(\mathcal{Q}) - a_{s_i+1}(\mathcal{Q})$$
$$\geq c(v_{i-1}, v_i) + 2d - 2a\eta(\gamma), \tag{3.47}$$

while by (3.41) and the argument of Lemma 3.9,

$$\sum_{s_{i-1}+1}^{s_i+2} a_j(\mathcal{P}) \leq a_{s_{i-1}+1}(\mathcal{P}) + a_{s_{i-1}+2}(\mathcal{P}) + \sum_{s_{i-1}+3}^{s_i} a_j(\tau_{k_i}\mathcal{Q}_i) + a_{s_i+1}(\mathcal{P}) + a_{s_i+2}(\mathcal{P})$$
$$\leq c(v_{i-1}, v_i) + 2a\eta(\gamma) + 2a\sigma. \tag{3.48}$$

Combining (3.46)–(3.48) yields

$$J(\mathcal{Q}_m) - J(\mathcal{P}) \geq 2d - 4a\eta(\gamma) - 2a\sigma. \tag{3.49}$$

By (3.31) and (3.35), this becomes

$$J(\mathcal{Q}_m) - J(\mathcal{P}) \geq \tfrac{3}{2}d - 4a\eta(\gamma). \tag{3.50}$$

Choosing γ so small that

$$8a\eta(\gamma) < d \tag{3.51}$$

then shows

$$J(\mathcal{Q}_m) - J(\mathcal{P}) \geq d, \tag{3.52}$$

contrary to the minimality of $\mathcal{Q}_m$ in X_m. Thus Proposition 3.33 and Theorem 3.1 are proved. □

Remark 3.53. Now that the proof of Theorem 3.1 has been completed, a more precise statement about the lower bounds need for the differences $m_j - m_{j-1}$ can be made. Namely, $m_{2i+1} - m_{2i}$, $1 \leq i \leq \ell - 1$, depends on d via (3.51) and d depends on ℓ in its definition. The remaining differences $m_{2i} - m_{2i-1}$, $1 \leq i \leq \ell$, depend on ϵ (see (3.31)) and ϵ depends on d via (3.35).

Remark 3.54. The proof of Proposition 3.33 shows that if $m_{2i+1} - m_{2i} \to \infty$, $1 \leq i \leq \ell - 1$, $\mathcal{Q}_m$ approaches a chain $\{\hat{\mathcal{Q}}_1, \ldots, \hat{\mathcal{Q}}_\ell\}$ with $\hat{\mathcal{Q}}_i \in \Gamma(v_{i-1}, v_i)$, and $J(\hat{\mathcal{Q}}_i) = c(v_{i-1}, v_i)$ so $\hat{\mathcal{Q}}_i$ is a solution of (HS) heteroclinic to v_{i-1} and v_i. The function $\hat{\mathcal{Q}}_i$ may not equal $\tau_p Q_i$ for some $p \in \mathbb{Z}$ unless there is a unique minimizer of J in $\Gamma(v_{i-1}, v_i)$ (modulo τ_p). In any event, for all $m_{i+1} - m_i$ sufficiently large, $\mathcal{Q}_m$ shadows a heteroclinic chain of solutions of (HS).

Remark 3.55. $\mathcal{Q}_m$ is a minimal solution of (HS) in the following sense. For any $s < t \in \mathbb{R}$, consider the set of q in X_m such that $q(s) = \mathcal{Q}_m(s)$ and $q(t) = \mathcal{Q}_m(t)$. Then $\mathcal{Q}_m$ minimizes

$$\int_s^t L(q)\, dt \tag{3.56}$$

over this class. Indeed, otherwise $\mathcal{Q}_m\big|_s^t$ could be replaced by the minimizer of (3.56) producing $\widehat{\mathcal{Q}} \in X_m$ such that $J(\widehat{\mathcal{Q}}) < b_m$, contrary to the definition of b_m.

4. *Some generalizations*

In this section, some generalizations will be given of Theorem 3.1. In particular, the analogues of Theorem 3.1 when minimal heteroclinic chains are glued together will be proved. An important special case is that of an augmented chain. Finally, infinite chains will be studied. In all cases, the ideas that go into the proof of Theorem 3.1 play a major role.

To begin, suppose $u \neq v$ and $v \neq w$ belong to $\mathcal{M}$. By Maxwell's result, there exist minimal heteroclinic chains of solutions $\{P_1, \dots, P_k\}$ and $\{Q_1, \dots, Q_\ell\}$ of (HS) with the P chain joining u and v and the Q chain joining v and w. Under the further assumption that $(*)$ is satisfied by the u, v and v, w problems, Theorem 3.1 provides actual heteroclinics from u to v and from v to w. What about heteroclinics from u to w? Then there are infinitely many such solutions of (HS). Indeed, let $R = (P_1, \dots, P_k, Q_1, \dots, Q_\ell) \equiv (R_1, \dots, R_{k+\ell})$, the chain obtained by concatenating the P and Q chains, and let $z_0 = R_1(-\infty)$, $z_i = R_i(\infty)$, $1 \leq i \leq k + \ell$. Let $\mathcal{O}_{i,j}$ be the neighborhoods of $z_{i-1}(0)$, $z_i(0)$ as in §2 and $m \in \mathbb{Z}^{2(k+\ell)}$. Define X_m as earlier and likewise b_m. Then under the above hypotheses on the P and Q chains, we have the following.

THEOREM 4.1. *If $m_{i+1} - m_i$ is sufficiently large, $1 \leq i \leq 2(\ell + k) - 1$, there is a solution $\mathcal{R}_m$ of* (HS) *heteroclinic from u to w with $J(\mathcal{R}_m) = b_m$ and*

$$\begin{aligned} &|\mathcal{R}_m(t) - z_0(t)| \leq r, \quad t \in (-\infty, m_1] \\ &|\mathcal{R}_m(t) - z_i(t)| \leq r, \quad t \in [m_{2i}, m_{2i+1}], \quad 1 \leq i \leq 2(\ell + k) - 1 \\ &|\mathcal{R}_m(t) - z_{k+\ell}(t)| \leq r, \quad t \in [m_{2(k+\ell)}, \infty). \end{aligned} \tag{4.2}$$

Proof. The chain R may not be a minimal heteroclinic chain. However, the only role minimality played in the proof of Theorem 3.1 was via Lemma 2.12 and the construction of the sets $\mathcal{O}_{i,j}$. Here these sets are provided by the minimality of the P and Q chains. An examination of the proof of Theorem 3.1 now shows that if (3.31), (3.35), (3.39) and (3.51) hold with ℓ replaced by $\ell + k$, the proof carries over unchanged for the current setting. □

What was just done for two pairs u, v and v, w of course extends to any finite number of such pairs. Observe that if $\{Q_1, \dots, Q_\ell\}$ is a minimal heteroclinic chain in the sense of §1, so is any subchain $\{Q_i, Q_{i+1}, \dots, Q_j\}$ of it. Since an augmented chain as defined in §1 is just obtained by gluing together a finite number of such minimal chains formed from $\{Q_1, \dots, Q_\ell\}$ and $\{Q_1(-t), \dots, Q_\ell(-t)\}$, an immediate consequence of Theorem 4.1 is the following.

COROLLARY 4.3. *Under the hypotheses of Theorem 4.1, for any augmented chain $\{u_1, \dots, u_j\}$ constructed from $\{Q_1, \dots, Q_\ell\}$ and $\{Q_1(t), \dots, Q_\ell(-t)\}$, there exist infinitely many solutions of (HS) heteroclinic to $u_1(-\infty)$ and $u_j(\infty)$ and characterized by the amount of time they spend near the intermediate periodic states $u_1(\infty), \dots, u_{j-1}(\infty)$.*

Our final topic is the question of infinite chains. Observe that if several minimal heteroclinic chains are glued together and if N is the total number of heteroclinics in the chains, the restrictions on σ and γ needed to apply the argument of Theorem 3.1 become

$$(2N + 2)a\sigma < \epsilon < d/2, \tag{4.4}$$

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where

$$2d = \min_{1\le i\le N}(c^*(u_{i-1}(\infty), u_i(\infty)) - c(u_{i-1}(\infty), u_i(\infty))) \tag{4.5}$$

(with $u_0(\infty) \equiv u_1(-\infty)$) and

$$8a\eta(\gamma) < d. \tag{4.6}$$

Thus as $N \to \infty$, $\sigma \to 0$, and $m_{2i} - m_{2i-1} \to \infty$. Moreover, as $N \to \infty$, $d \to 0$ is a possibility and if so, $\gamma \to 0$ and $m_{2i+1} - m_{2i} \to 0$. Consequently some care must be taken in dealing with infinite chains.

The case of an infinite augmented chain is simpler to study so it will be treated first. As earlier, let $Q^+ = \{Q_1, \ldots, Q_\ell\}$ be a minimal heteroclinic chain of solutions of (HS) with $Q_1(-\infty) = v_0$ and $Q_i(\infty) = v_i$, $1 \le i \le \ell$. Assume $(*)$ holds for this setting so we have neighborhoods, $\mathcal{O}_{i,j}$, as in §3 with diam $\mathcal{O}_{i,j} \le \rho$. By Theorem 3.1 there is an $m_* > 0$, $m_* = m_*(\rho)$ such that if $m \in \mathbb{Z}^{2\ell}$ with $m_{i+1} - m_i \ge m_*$, (HS) has a solution $\mathcal{Q}_m$ in X_m with $J(\mathcal{Q}_m) = b_m$. Now let $U = \{u_j \mid j \in \mathbb{Z}\}$ be an augmented chain constructed from the Q^+ chain and its time reversal chain, Q^-. Associated with each u_i are its periodic asymptotic states $u_i(\pm\infty) \in \{v_0, \ldots, v_\ell\}$ and corresponding $\Omega_{i,1} \in \{\mathcal{O}_{k,1} \mid i \le k \le \ell\}$, $\Omega_{i,2} \in \{\mathcal{O}_{k,2} \mid 1 \le k \le \ell\}$. Let $M = (M_j)_{j\in\mathbb{Z}} \in \mathbb{Z}^\infty$ with $M_{j+1} - M_j \ge m_*$. Define

$$X_M = \{q \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n) \mid q(M_{2i-1}) \in \Omega_{i,1}, q(M_{2i}) \in \Omega_{i,2}, i \in \mathbb{Z}\}.$$

Then we have the following.

THEOREM 4.7. *Under the above hypotheses, there is a solution of* (HS), $\mathcal{Q}_M \in X_M$ *with*

$$\|\mathcal{Q}_M - u_i(\infty)\|_{L^\infty[m_{2i}, m_{2i+1}]} \le r, \quad i \in \mathbb{Z}. \tag{4.8}$$

Proof. Let $m(j) = (M_{-2(j-1)}, \ldots, M_{2j}) \in Z^{4j}$. By Corollary 4.3, there is a solution of (HS), $\mathcal{Q}_{m(j)} \in X_{m(j)}$ such that $J(\mathcal{Q}_{m(j)}) = b_{m(j)}$ provided that $M_{i+1} - M_i$ is large enough, $-(2j-1) \le i \le 2j-1$. Thus it must be verified that m_* is an appropriate lower bound for the differences $M_{i+1} - M_i$, independently of j. Observe that in (4.5), although N may be large, only ℓ different terms occur on the right-hand side for an augmented chain. Hence here d and γ in (4.6) can be chosen independently of j and $M_{2i+1} - M_{2i} \ge m_*$ suffices. To show that this is also the case for $M_{2i} - M_{2i-1}$, an improvement of Proposition 3.28 is needed for an augmented chain. Namely, it will be shown that for $m_* = m_*(\ell)$ sufficiently large and an augmented chain $R = \{R_1, \ldots, R_N\}$ constructed from Q^+ and Q^-,

$$b_m < \sum_1^N c(R_{i-1}(\infty), R_i(\infty)). \tag{4.9}$$

To display the idea behind (4.9) in its simplest setting, which comes from [**3**], suppose that we are dealing with the chain $Q_1(t)$, $Q_1(-t)$ from v_0 to v_0 via v_1. Then a suitable choice of q^* in Proposition 3.28 is $Q_1|_{-\infty}^k$ glued to its reversal, i.e. $Q_1(-t)|_{-k}^\infty$, where k is sufficiently large. Then

$$J(q^*) < 2c(v_0, v_1) \tag{4.10}$$

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yielding (4.9) for this simple case. For further reference, note that the left-hand side of (4.10) is less than $2c(v_0, v_1)$ by

$$2\sum_{k}^{\infty} a_i(Q_1). \tag{4.11}$$

More generally, a finite augmented heteroclinic chain R is obtained by gluing a subchain H_1 of $Q^{\pm}$ to a subchain H_2 of $Q^{\mp}$, etc. Say $R = \{H_1, \ldots, H_k\}$. For definiteness suppose H_1 is a Q^+ subchain with its last heteroclinic piece being Q_i. Then the first piece of H_2 is $Q_i(-t)$. Similar pairs of time reversed orbits form the junctures between all H_p, H_{p+1} subchains. Thus to construct a q^* approximating R, as in (4.10), the approximation of $Q_i(t)$, $Q_i(-t)$ will be a piece of $Q_i : Q_i|_{\bullet}^{k}$ glued to $Q_i(-t)|_{\bullet}^{k}$. Suppose $H_1 = \{Q_{i-\beta}, \ldots, Q_i\}$. Choosing q^* exactly as in Proposition 3.28 to approximate $Q_{i-\beta}, \ldots, Q_{i-1}$, and using $Q_i|_{\bullet}^{k}$ for Q_i, (3.32) and (4.11) shows that the contribution of q^* corresponding to H_1 does not exceed

$$\sum_{s=i-\beta+1}^{i} c(Q_{s-1}, Q_s) + 2\beta\sigma - \sum_{k}^{\infty} a_s(Q_i). \tag{4.12}$$

Since $\beta \le \ell$, if σ is small enough (and therefore $m_{2i} - m_{2i-1}$ large enough), the number in (4.12) is < 0.

The same analysis applied to the other subchains of R then yields (4.9). Consequently for m_* sufficiently large, for each choice of j, there is a solution $\mathcal{Q}_{m(j)}$ of (HS) in $X_{m(j)}$. Observe that (4.2) provides $L^\infty[M_{2i}, M_{2i+1}]$ bounds for $\mathcal{Q}_{m(j)}$, and arguing as in the proof of Lemma 2.3 gives bounds for $\mathcal{Q}_{m(j)}$ in $L^\infty[M_{2i-1}, M_{2i}]$. Since $\mathcal{Q}_{m(j)}$ is a solution of (HS), the equations then yield bounds for $\mathcal{Q}_{m(j)}$ in C^2. Standard arguments then imply $\mathcal{Q}_{m(j)}$ converges along a subsequence to $\mathcal{Q}_M$ which is a solution of (HS) in X_M which satisfies (4.8). □

Remark 4.13. The solution, $\mathcal{Q}_m$, of (HS) is minimal in the sense of Remark 3.55 by the same argument.

For the final application of this section, consider a sequence $(z_i)_{i\in\mathbb{Z}}$ with $z_i \in \mathcal{M}$ and $z_{i+1} \neq z_i$. As earlier, each successive pair z_i, z_{i+1} can be joined by a minimal heteroclinic chain $H_i = \{h_{i,1}, \ldots, h_{i,\ell_i}\}$. As earlier, in trying to find a solution of (HS) which spends at least a prescribed amount of time near each z_i (as well as the intermediate periodics $h_{i,j}(\infty)$), one encounters the potential difficulties that were mentioned following (4.4)–(4.6). In particular, for the approximation argument of the proof of Theorem 4.7, $d = d(j)$ may tend to 0 as $j \to \infty$ and even if this is not the case, $\sigma = \sigma(j)$ may go to 0 as $j \to \infty$. The first difficulty disappears if $\{h_{i,j} \mid i \in \mathbb{Z}, 1 \le j \le \ell_i\}$ merely consists of a finite set of basic heteroclinics together with their translates (i.e. $h \to h + k$, $k \in \mathbb{Z}^n$). Then $d(j) \ge d_0 > 0$, independently of j but the second difficulty remains.

It is possible to bypass the problem of $\sigma(j) \to 0$ as $j \to \infty$, but only at a price. To be more explicit, first note that a direct consequence of Maxwell's result [**6**] is that there are a finite number of basic heteroclinics that together with their translates and time reversals can be used to generate a heteroclinic chain between any pair $z \neq w \in \mathcal{M}$. Of course this chain may not be a minimal one. Let $\mathcal{B}$ denote this set of basic heteroclinics, their translates and time reversals. Assume as usual that $(*)$ holds for this family. Now given $(z_i)_{i\in\mathbb{Z}} \subset \mathcal{M}$ with $z_{i+1} \neq z_i$, let h_i denote a heteroclinic chain from z_{i-1} to z_i consisting of numbers of $\mathcal{B}$ and let H be the infinite chain $(h_i)_{i\in\mathbb{Z}}$. H can also be written as $(u_k)_{k\in\mathbb{Z}}$, where $u_k \in \mathcal{B}$. Approximating this chain as in the proof of Theorem 4.7, by the remarks of the previous paragraph, $d(j) \geq d_0 > 0$ independently of j. Now the constructions of Theorems 4.7 and 3.1 will be modified so that as $|i| \to \infty$, $m_{i+1} - m_i \to \infty$. In particular, let $m(j)$ be as in the proof of Theorem 4.7. Suppose at step j, the construction of $q^*(j)$ in Proposition 3.28 result in the estimate

$$2a\ell\sigma(j) < \left(\sum_1^j 2^{-p}\right)\epsilon. \tag{4.14}$$

This is certainly true for $j = 1$. To apply Theorem 3.1 at step $j+1$, suppose that $M_{-(2j+1)} < M_{-2j} < M_{-(2j-1)}$ and $M_{2j-1} < M_{2j} < M_{2j+1}$. Replace the balls $B_\sigma(v_i(0))$ by balls $B_{\sigma_{j+1}}(v_i(0))$, $i = -(j+1)$, $j+1$. Modifying $q^*(j)$ by gluing on pieces of $\tau_{k_{-(j+1)}} u_{-j+1}$ and $\tau_{k_{j+1}} u_{j+1}$ to construct $q^*(j+1)$, gives an additional contribution to the right-hand side of (4.14) of the form $2a\sigma_{j+1}$. Thus for σ_{j+1} sufficiently small (which can be achieved by taking $M_{i+1} - M_i$ sufficiently large, $i = -(2j+1), -2j, 2j-1, 2j$), the analogue of (4.14) for $j+1$ holds.

With this modification, the sequence $\mathcal{Q}_{m(j)}$ of heteroclinics of Theorem 4.7 can be constructed and as earlier this leads to a solution of (HS) satisfying (4.8). Stating what has been shown somewhat informally, we have the following.

THEOREM 4.15. *Let $(z_i)_{i\in\mathbb{Z}} \subset \mathcal{M}$ with $z_{i+1} \neq z_i$. If $(*)$ holds, there is a sequence $(\gamma_i)_{i\in\mathbb{N}}$ with $\gamma_i \to \infty$ as $i \to \infty$ such that whenever $\mathcal{M} \in \mathbb{Z}^\infty$ with $\mathcal{M}_{i+1} - \mathcal{M}_i \geq \gamma_i$, there is a solution, $\mathcal{Q}_\mathcal{M}$, of* (HS) *satisfying*

$$|\mathcal{Q}_\mathcal{M}(t) - z_i(t)| \leq r, \quad t \in [\mathcal{M}_{2i}, \mathcal{M}_{2i+1}],\ i \in \mathbb{Z}. \tag{4.16}$$

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