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On the finite-element approximation of ∞-harmonic functions

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In this paper we show that conforming Galerkin approximations for p-harmonic functions tend to ∞-harmonic functions in the limit $p \to \infty$ and $h \to 0$, where h denotes the Galerkin discretization parameter.

Keywords: Galerkin methods; ∞-Laplacian; viscosity solutions

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1. Introduction and the ∞-Laplacian

Let $\Omega \subset \mathbb{R}^d$ be an open and bounded set. For a given function $u\colon \Omega \to \mathbb{R}$ we denote the gradient of u by $\mathrm{D}u\colon \Omega \to \mathbb{R}^d$ and its Hessian $\mathrm{D}^2u\colon \Omega \to \mathbb{R}^{d\times d}$. The ∞-Laplacian is the partial differential equation (PDE)

$$\Delta_\infty u := (\mathrm{D}u \otimes \mathrm{D}u) : \mathrm{D}^2 u = \sum_{i,j=1}^{d} \partial_i u \partial_j u \partial_{ij}^2 u = 0, \tag{1.1}$$

where '$\otimes$' is the tensor product between d-vectors and ':' is the Frobenius inner product between matrices.

This problem is the prototypical example of a PDE from calculus of variations in L^∞, arising as the analogue of the *Euler–Lagrange* equation of the functional [2]

$$\mathscr{J}[u;\infty] := \|\mathrm{D}u\|_{\mathrm{L}^\infty(\Omega)} \tag{1.2}$$

and as the (weighted) formal limit of the variational p-Laplacian

$$\Delta_p u := \operatorname{div}\left(|\mathrm{D}u|^{p-2}\mathrm{D}u\right) = 0. \tag{1.3}$$

The p-Laplacian is a divergence-form problem and appropriate *weak* solutions to this problem are defined in terms of duality, or integration by parts. In passing to the limit $p \to \infty$ the problem loses its divergence structure. In the non-divergence setting we do not have access to the same stability framework as in the variational case and a different class of 'weak' solution must be sought. The correct concept to use is that of viscosity solutions (see [10, 20]). The main idea behind this solution concept is to pass derivatives to test functions through the maximum principle, that is, *without* using duality.

The design of numerical schemes to approximate this solution concept is limited, particularly in the finite element context, where the only provably convergent scheme is given in [18] (although it is inapplicable to the problem at hand). In the finite-difference setting some techniques have been developed [27, 28] and applied to this problem and also the associated eigenvalue problem [6]. In fact, both in the finite difference and finite element setting the methods of convergence are based on the discrete monotonicity arguments of [4], which is an extremely versatile framework. Other methods exist for the problem. For example, in [15] Feng and Neilan proposed a biharmonic regularization, which yields convergence for the case in which (1.1) admits a strong solution. In [25] Lakkis and Pryer proposed an h-adaptive finite-element scheme based on a residual-type error indicator. The underlying scheme was based on the method derived in [24] for fully nonlinear PDEs.

In this paper we examine a different route. We will review and use the known theory used in the derivation of the ∞-Laplacian [3, 17, 20], where a *p-limiting* process is employed to derive (1.1). We study how well Galerkin approximations of (1.3) approximate the solutions of (1.1) and show that by forming an appropriate limit we are able to select candidates for numerical approximation along a 'good' sequence of solutions. This is due to the equivalence of weak and viscosity solutions to (1.3) [19]. To be very clear about where the novelty lies in this work, the techniques we use are not new. We are summarizing existing tools from two fields, one set from PDE theory and the other from numerical analysis. While both sets of results are relatively standard in their own field, to the author's knowledge, they have yet to be combined in this fashion.

We use this exposition to conduct some numerical experiments that demonstrate the rate of convergence both in terms of p-approximation[1] and h-approximation. These results illustrate that for practical purposes, as one would expect, the approximation of p-harmonic functions for large p gives good resolution of ∞-harmonic functions. The numerical approximation of p-harmonic functions is by now quite standard in finite element literature; see, for example, [9, § 5.3]. There has been a lot of activity in the area since then, however. In particular, the quasi-norm introduced in [5] gave significant insight into the numerical analysis of this problem and spawned much subsequent research for which [8, 12, 26] form a non-exhaustive list.

While it is not the focus of this work, we are interested in this approach as it allows us to extend quite simply and reliably to the study of coupled systems of ∞-Laplacian type. When moving from scalar to vectorial calculus of variations in L^∞ the solution concept of viscosity solutions is no longer applicable. One notion of solution currently being investigated is $\mathcal{D}$-solutions [22], which is based on concepts of Young measures. The ultimate goal of this line of research is the construction of reliable numerical schemes that allow for various conjectures to be made as to the nature of solutions and even what the correct solution concept is when studying systems of nonlinear PDEs without a divergence structure [23].

[1] The terminology 'p-approximation' we use here should not be confused with 'p-adaptivity', which is local polynomial enrichment of the underlying discrete function space.

The rest of the paper is set out as follows. In § 2 we formalize notation and begin exploring some of the properties of the p-Laplacian. In particular, we recall that the notion of weak and viscosity solutions to this problem coincide, allowing the passage to the limit $p \to \infty$. In § 3 we describe a conforming discretization of the p-Laplacian and its properties. We show that the method converges to the weak solution for fixed p. Numerical experiments are given in § 4 illustrating the behaviour of numerical approximations to this problem.

2. Approximation via the p-Laplacian

In this section we describe how ∞-harmonic functions can be approximated using p-harmonic functions. We give a brief introduction to the p-Laplacian problem, beginning by introducing the Sobolev spaces [9, 13]

$$\left.\begin{aligned} \mathrm{L}^p(\Omega) &= \left\{\phi\colon \int_\Omega |\phi|^p < \infty\right\} \quad \text{for } p \in [1,\infty) \text{ and} \\ \mathrm{L}^\infty(\Omega) &= \left\{\phi\colon \operatorname*{ess\,sup}_\Omega |\phi| < \infty\right\}, \\ \mathrm{W}^{l,p}(\Omega) &= \{\phi \in \mathrm{L}^p(\Omega)\colon \mathrm{D}^{\boldsymbol{\alpha}}\phi \in \mathrm{L}^p(\Omega) \text{ for } |\boldsymbol{\alpha}| \leqslant l\}, \\ \mathrm{H}^l(\Omega) &= \mathrm{W}^{l,2}(\Omega), \end{aligned}\right\} \tag{2.1}$$

which are equipped with the norms and semi-norms

$$\left.\begin{aligned} \|v\|^p_{\mathrm{L}^p(\Omega)} &= \int_\Omega |v|^p \quad \text{for } p \in [1,\infty), \\ \|v\|_{\mathrm{L}^\infty(\Omega)} &= \operatorname*{ess\,sup}_\Omega |v|, \\ \|v\|^p_{\mathrm{W}^{l,p}(\Omega)} &= \sum_{|\boldsymbol{\alpha}|\leqslant l} \|\mathrm{D}^{\boldsymbol{\alpha}} v\|^p_{\mathrm{L}^p(\Omega)}, \\ |v|^p_{\mathrm{W}^{l,p}(\Omega)} &= \sum_{|\boldsymbol{\alpha}|= l} \|\mathrm{D}^{\boldsymbol{\alpha}} v\|^p_{\mathrm{L}^p(\Omega)}, \end{aligned}\right\} \tag{2.2}$$

where $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_d\}$ is a multi-index, $|\boldsymbol{\alpha}| = \sum_{i=1}^d \alpha_i$ and derivatives $\mathrm{D}^{\boldsymbol{\alpha}}$ are understood in the weak sense. We pay particular attention to the case in which $l = 1$ and

$$\mathrm{W}^{1,p}_g(\Omega) := \{\phi \in \mathrm{W}^{1,p}(\Omega)\colon \phi|_{\partial\Omega} = g\}, \tag{2.3}$$

for a prescribed function $g \in \mathrm{W}^{1,\infty}(\Omega)$. Let $L = L(\boldsymbol{x}, u, \mathrm{D}u)$ be the *Lagrangian*. We will let

$$\left.\begin{aligned} &\mathscr{J}[\cdot\,; p]\colon \mathrm{W}^{1,p}_g(\Omega) \to \mathbb{R} \\ &\qquad \phi \mapsto \mathscr{J}[\phi; p] := \int_\Omega L(\boldsymbol{x}, \phi, \mathrm{D}\phi)\,\mathrm{d}\boldsymbol{x} \end{aligned}\right\} \tag{2.4}$$

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be known as the *action functional*. For the p-Laplacian the action functional is given as[2]

$$\mathscr{J}[u;p] := \int_\Omega L(\boldsymbol{x}, u, \mathrm{D}u) = \int_\Omega |\mathrm{D}u|^p. \tag{2.5}$$

We then look to find a minimizer over the space $\mathrm{W}^{1,p}_g(\Omega)$, that is, to find $u \in \mathrm{W}^{1,p}_g(\Omega)$ such that

$$\mathscr{J}[u;p] = \min_{v\in \mathrm{W}^{1,p}_g(\Omega)} \mathscr{J}[v;p]. \tag{2.6}$$

If we assume temporarily that we have access to a smooth minimizer, i.e. $u \in \mathrm{C}^2(\Omega)$, then, given that the Lagrangian is of first order, we have that the Euler–Lagrange equations are (in general) second order.

The Euler–Lagrange equations for this problem are

$$\mathscr{L}[u;p] := \operatorname{div}(|\mathrm{D}u|^{p-2}\mathrm{D}u) = 0. \tag{2.7}$$

Note that, for $p = 2$, the problem coincides with the Poisson problem $\Delta u = 0$. In general, the p-Laplace problem is to find u such that

$$\left.\begin{aligned} \Delta_p u := \operatorname{div}(|\mathrm{D}u|^{p-2}\mathrm{D}u) = 0 \quad & \text{in } \Omega, \\ u = g \quad & \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.8}$$

Definition 2.1 (weak solution). The problem (2.8) is associated with a weak formulation. Set

$$\mathscr{A}(u,v) = \int_\Omega (|\mathrm{D}u|^{p-2}\mathrm{D}u)\cdot \mathrm{D}v \tag{2.9}$$

to be a semilinear form. Then $u \in \mathrm{W}^{1,p}_g(\Omega)$ is a *weak solution* of (2.8) if it satisfies

$$\mathscr{A}(u,v) = 0 \quad \forall v \in \mathrm{W}^{1,p}_0(\Omega). \tag{2.10}$$

Proposition 2.2 (existence and uniqueness). *There exists a unique weak solution to (2.8).*

Proof. The proof is standard and can be found in [9, theorem 5.3.1], for example. It is based on the strict convexity of $\mathscr{J}[\cdot;p]$, yielding uniqueness, together with appropriate growth conditions for existence. □

Definition 2.3 (viscosity supersolutions and subsolutions). A function $u \in \mathrm{C}^0(\Omega)$ is a viscosity subsolution of a general second-order PDE

$$F(u, \mathrm{D}u, \mathrm{D}^2u) = 0 \tag{2.11}$$

at a point $\boldsymbol{x} \in \Omega$ if for any $\phi \in \mathrm{C}^2(\Omega)$ satisfying $u(\boldsymbol{x}) = \phi(\boldsymbol{x})$, and touching u from above, that is, $u - \phi \leqslant 0$ in a neighbourhood of $\boldsymbol{x}$, we have

$$F(\phi, \mathrm{D}\phi, \mathrm{D}^2\phi) \geqslant 0. \tag{2.12}$$

[2] Typically, $L(\boldsymbol{x}, u, \mathrm{D}u) = (1/p)|\mathrm{D}u|^p$. Note here that the rescaling of L has no effect on the resulting Euler–Lagrange equations as L is independent of u.

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In the case in which F is the p-Laplacian operator, we have

$$\Delta_p \phi \geqslant 0. \tag{2.13}$$

Similarly, a function $u \in \mathrm{C}^0(\Omega)$ is a viscosity supersolution of (2.11) at a point $\boldsymbol{x} \in \Omega$ if for any $\phi \in \mathrm{C}^2(\Omega)$ satisfying $u(\boldsymbol{x}) = \phi(\boldsymbol{x})$ and touching u from below we have

$$F(\phi, \mathrm{D}\phi, \mathrm{D}^2\phi) \leqslant 0, \tag{2.14}$$

or, in particular,

$$\Delta_p \phi \leqslant 0 \tag{2.15}$$

for the p-Laplacian.

Definition 2.4 (viscosity solution). The function u is a viscosity solution of (2.11) in Ω if it is both a viscosity supersolution and a subsolution at any $\boldsymbol{x} \in \Omega$.

Theorem 2.5 (weak solutions of the p-Laplacian are viscosity solutions). *Let $g \in \mathrm{W}^{1,\infty}(\Omega)$ and suppose that $p > d \geqslant 2$ is fixed. Then weak solutions of*

$$\left.\begin{aligned} \Delta_p u &= 0 \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{2.16}$$

are viscosity solutions of

$$\left.\begin{aligned} \Delta_\infty u + \frac{|\mathrm{D}u|^2}{p-2}\Delta u &= 0 \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.17}$$

Proof. Expanding the derivatives of the operator (2.16) we have

$$\left.\begin{aligned} \Delta_p u &= \operatorname{div}(|\mathrm{D}u|^{p-2}\mathrm{D}u) \\ &= |\mathrm{D}u|^{p-2}\Delta u + (p-2)|\mathrm{D}u|^{p-4}\mathrm{D}u \otimes \mathrm{D}u : \mathrm{D}^2 u \\ &= |\mathrm{D}u|^{p-2}\Delta u + (p-2)|\mathrm{D}u|^{p-4}\Delta_\infty u, \end{aligned}\right\} \tag{2.18}$$

which shows that (2.16) is a renormalization of (2.17). The two formulations (2.18) and (2.17) of the p-Laplacian are thus equivalent in the viscosity sense; see, for example, [20, lemma 3, § 8].

It remains to show that weak solutions of (2.16) are viscosity solutions of (2.18). As $g \in \mathrm{W}^{1,\infty}(\Omega)$ we have

$$\|\mathrm{D}g\|^p_{\mathrm{L}^p(\Omega)} = \int_\Omega |\mathrm{D}g|^p = \mathscr{J}[g; p] < \infty. \tag{2.19}$$

Since u solves (2.16) weakly, it minimizes the functional $\mathscr{J}[\cdot; p]$, and hence the minimizer must be of finite energy. In view of the existence and uniqueness of the minimization problem from proposition 2.2 and Morrey's inequality, we may infer that $u \in \mathrm{C}^{0,\alpha}(\Omega)$ and hence $u \in \mathrm{C}^0(\Omega)$.

Now assume by contradiction that u is not a viscosity subsolution of

$$|\mathrm{D}u|^{p-2}\Delta u + (p-2)|\mathrm{D}u|^{p-4}\Delta_\infty u = 0. \tag{2.20}$$

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Then, by definition 2.3, we can find an $\boldsymbol{x} \in \Omega$, a $\psi \in \mathrm{C}^2(\Omega)$ and an $r > 0$ such that $u - \psi < 0$ on $B(\boldsymbol{x}, r)$, $(u - \psi)(\boldsymbol{x}) = 0$ and

$$\Delta_p \psi = |\mathrm{D}\psi|^{p-2} \Delta \psi + (p-2)|\mathrm{D}\psi|^{p-4} \Delta_\infty \psi \leqslant -C < 0 \quad \text{in } B(\boldsymbol{x}, r) \tag{2.21}$$

for some $C > 0$. Now we may define a set

$$\omega(\varepsilon) = \{\boldsymbol{x} \colon u(\boldsymbol{x}) - (\psi(\boldsymbol{x}) - \varepsilon) > 0\}, \tag{2.22}$$

and, since $u - \psi$ has a strict maximum at $\boldsymbol{x}$, we may find an $\varepsilon > 0$ such that

$$\omega(\varepsilon) \subset B(\boldsymbol{x}, r). \tag{2.23}$$

Hence,

$$\begin{aligned} C \int_\omega (u - (\psi - \varepsilon)) &\leqslant \int_\omega -\Delta_p \psi (u - (\psi - \varepsilon)) \\ &= \int_\omega (\mathrm{D}\psi)^{p-2} \mathrm{D}\psi \cdot \mathrm{D}(u - \psi), \end{aligned} \tag{2.24}$$

as $u = \psi - \varepsilon$ on $\partial\omega$ and is extended such that $\psi - \varepsilon = u$ outside ω. Now, by the convexity of the Lagrangian $L(\boldsymbol{x}, u, \mathrm{D}u) = |\mathrm{D}u|^p$ we have

$$|\mathrm{D}v|^{p-2} \mathrm{D}v \cdot \mathrm{D}(w - v) \leqslant |\mathrm{D}w|^p - |\mathrm{D}v|^p, \tag{2.25}$$

and hence

$$0 \leqslant C \int_\omega (u - (\psi - \varepsilon)) \leqslant \int_\Omega |\mathrm{D}u|^p - |\mathrm{D}(\psi - \varepsilon)|^p = \mathscr{J}[u; p] - \mathscr{J}[\psi - \varepsilon; p] \tag{2.26}$$

and we see that

$$\mathscr{J}[u; p] \geqslant \mathscr{J}[\psi - \varepsilon; p]. \tag{2.27}$$

This means that by uniqueness of the minimizer we must have $u \equiv \psi - \varepsilon$, which contradicts the fact that $u(\boldsymbol{x}) = \psi(\boldsymbol{x})$. The complete proof in full generality for convex minimization problems can be found in [20]. See also [21], where Katzourakis extends the arguments of [19] to singular PDEs. □

Remark 2.6 (viscosity solutions of the p-Laplacian are weak solutions). The converse to theorem 2.5 is also true; thus weak and viscosity solutions are equivalent for the p-Laplacian and its evolutionary relative. This has been shown in [19].

Theorem 2.7 (the limit as $p \to \infty$). *Let $u_p \in \mathrm{W}^{1,p}_g(\Omega)$ denote a sequence of weak/viscosity solutions to the p-Laplacian. Then there exists a subsequence such that as $p \to \infty$ that sequence converges to a candidate ∞-harmonic function $u_\infty \in \mathrm{W}^{1,\infty}(\Omega)$, that is,*

$$u_{p_j} \to u_\infty \quad \text{in } C^0. \tag{2.28}$$

Proof. We denote $u_p \in \mathrm{W}^{1,p}_g(\Omega)$ as the weak solution of (2.8). In view of proposition 2.2 we know that u_p minimizes the energy functional

$$\mathscr{J}[u_p; p] = \int_\Omega |\mathrm{D}u_p|^p. \tag{2.29}$$

Hence, in particular,

$$\mathscr{J}[u_p; p] \leqslant \mathscr{J}[g; p], \tag{2.30}$$

where g is the associated boundary data of (2.8). Using this fact,

$$\|\mathrm{D}u_p\|^p_{\mathrm{L}^p(\Omega)} = \mathscr{J}[u_p; p] \leqslant \mathscr{J}[g; p] = \|\mathrm{D}g\|^p_{\mathrm{L}^p(\Omega)}, \tag{2.31}$$

and we may infer that

$$\|\mathrm{D}u_p\|_{\mathrm{L}^p(\Omega)} \leqslant \|\mathrm{D}g\|_{\mathrm{L}^p(\Omega)}. \tag{2.32}$$

Now fix a $k > d$ and take $p \geqslant k$. Then, using Hölder's inequality,

$$\|\mathrm{D}u_p\|^k_{\mathrm{L}^k(\Omega)} = \int_\Omega |\mathrm{D}u_p|^k \leqslant \left(\int_\Omega 1^q\right)^{1/q} \left(\int_\Omega |\mathrm{D}u_p|^p\right)^{1/r}, \tag{2.33}$$

with $r = p/k$ and $q = (r-1)/r$ such that $1/r + 1/q = 1$. Hence,

$$\|\mathrm{D}u_p\|^k_{\mathrm{L}^k(\Omega)} \leqslant |\Omega|^{r/(r-1)} \|\mathrm{D}u_p\|^k_{\mathrm{L}^p(\Omega)} = |\Omega|^{1-k/p} \|\mathrm{D}u_p\|^k_{\mathrm{L}^p(\Omega)} \tag{2.34}$$

and we see

$$\|\mathrm{D}u_p\|_{\mathrm{L}^k(\Omega)} \leqslant |\Omega|^{1/k-1/p} \|\mathrm{D}u_p\|_{\mathrm{L}^p(\Omega)}. \tag{2.35}$$

Using the triangle inequality,

$$\begin{aligned} \|u_p\|_{\mathrm{L}^k(\Omega)} &\leqslant \|u_p - g\|_{\mathrm{L}^k(\Omega)} + \|g\|_{\mathrm{L}^k(\Omega)} \\ &\leqslant C(\|\mathrm{D}u_p - \mathrm{D}g\|_{\mathrm{L}^k(\Omega)} + \|g\|_{\mathrm{L}^k(\Omega)}) \end{aligned} \tag{2.36}$$

in view of the Poincaré inequality. Using the triangle inequality again we have

$$\begin{aligned} \|u_p\|_{\mathrm{L}^k(\Omega)} &\leqslant C(\|\mathrm{D}u_p\|_{\mathrm{L}^k(\Omega)} + \|g\|_{\mathrm{W}^{1,k}(\Omega)}) \\ &\leqslant C(\|\mathrm{D}u_p\|_{\mathrm{L}^k(\Omega)} + \|g\|_{\mathrm{W}^{1,k}(\Omega)}), \end{aligned} \tag{2.37}$$

by (2.35). Hence, using (2.32),

$$\|u_p\|_{\mathrm{W}^{1,k}(\Omega)} \leqslant C\|g\|_{\mathrm{W}^{1,k}(\Omega)}. \tag{2.38}$$

This means that for any $k > d$ we have uniformly that

$$\sup_{p>k} \|u_p\|_{\mathrm{W}^{1,k}(\Omega)} \leqslant C. \tag{2.39}$$

Hence, in view of weak compactness, we may extract a subsequence $\{u_{p_j}\}_{j=1}^\infty \subset \{u_p\}_{p=1}^\infty$ and a function $u_\infty \in \mathrm{W}^{1,k}(\Omega)$ such that, for any $k > n$,

$$u_{p_j} \rightharpoonup u_\infty \quad \text{weakly in } \mathrm{W}^{1,k}(\Omega) \tag{2.40}$$

and

$$\begin{aligned} \|u_\infty\|_{\mathrm{W}^{1,k}(\Omega)} &\leqslant \liminf_{j\to\infty} \|u_{p_j}\|_{\mathrm{W}^{1,k}(\Omega)} \\ &\leqslant \liminf_{j\to\infty} C\|g\|_{\mathrm{W}^{1,k}(\Omega)}. \end{aligned} \tag{2.41}$$

Taking the limit $k \to \infty$, we have

$$\|u_\infty\|_{\mathrm{W}^{1,\infty}(\Omega)} \leqslant C\|g\|_{\mathrm{W}^{1,\infty}(\Omega)}, \tag{2.42}$$

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and thus $u_\infty \in \mathrm{W}^{1,\infty}(\Omega)$. The result follows from Morrey's inequality, concluding the proof. □

Remark 2.8 (an alternative to the p-Dirichlet functional). We note that an alternative sequence of solutions is given in [14], where rather than studying the limit of the p-Dirichlet functional, Evans and Smart proposed

$$\widetilde{\mathscr{J}}[u;p] = \int_\Omega \exp(p|\mathrm{D}u|^2). \tag{2.43}$$

This functional may have some merit over the p-Dirichlet functional since the Euler–Lagrange equations

$$\begin{aligned} 0 &= \operatorname{div}(\exp(p|\mathrm{D}u|^2)\mathrm{D}u) \\ &= \exp(p|\mathrm{D}u|^2)\Delta u + p\exp(p|\mathrm{D}u|^2)\Delta_\infty u \end{aligned} \tag{2.44}$$

yield a clearer relation between Δu and $\Delta_\infty u$. We will not explore this issue further in this work.

The following theorem concerns existence and uniqueness of viscosity solutions to the ∞-Laplacian [17].

Theorem 2.9. *The 'candidate' ∞-harmonic function from theorem 2.7, u_∞, is the unique viscosity solution to the ∞-Laplacian (1.1).*

Proof. The proof is detailed in [17]. Roughly, existence of u_∞ has been shown in theorem 2.7. For uniqueness one must prove and make use of the maximum principle for (1.1). Note also the result of [1], where Armstrong and Smart used difference equations to prove the same result in a simpler fashion. □

Corollary 2.10. *Notice that since the candidate ∞-harmonic function is unique, theorem 2.7 can be improved and the entire sequence u_p must converge to the unique viscosity solution u_∞.*

3. Discretization of the p-Laplacian

In this section we describe a conforming finite-element discretization of the p-Laplacian. Let $\mathscr{T}$ be a conforming triangulation of Ω, namely, $\mathscr{T}$ is a finite family of sets such that

(1) $K \in \mathscr{T}$ implies K is an open simplex (segment for $d = 1$, triangle for $d = 2$, tetrahedron for $d = 3$);

(2) for any $K, J \in \mathscr{T}$ we have that $\overline{K} \cap \overline{J}$ is a full lower-dimensional simplex (i.e. it is either $\emptyset$, a vertex, an edge, a face, or the whole of $\overline{K}$ and $\overline{J}$) of both $\overline{K}$ and $\overline{J}$; and

(3) $\bigcup_{K \in \mathscr{T}} \overline{K} = \overline{\Omega}$.

The shape regularity constant of $\mathscr{T}$ is defined as the number

$$\mu(\mathscr{T}) := \inf_{K \in \mathscr{T}} \frac{\rho_K}{h_K}, \tag{3.1}$$

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where ρ_K is the radius of the largest ball contained inside K and h_K is the diameter of K. An indexed family of triangulations $\{\mathscr{T}^n\}_n$ is called *shape regular* if

$$\mu := \inf_n \mu(\mathscr{T}^n) > 0. \tag{3.2}$$

Furthermore, we define $h\colon \Omega \to \mathbb{R}$ to be the piecewise constant *mesh-size function* of $\mathscr{T}$ given by

$$h(\boldsymbol{x}) := \max_{\overline{K} \ni \boldsymbol{x}} h_K. \tag{3.3}$$

A mesh is called quasi-uniform when there exists a positive constant C such that $\max_{x\in\Omega} h \leqslant C \min_{x\in\Omega} h$. In what follows we shall assume that all triangulations are shape regular and quasi-uniform, although the results may be extendable even to the non-quasi-uniform case using techniques developed in [12].

We let $\mathscr{E}$ be the skeleton (set of common interfaces) of the triangulation $\mathscr{T}$ and say that $e \in \mathscr{E}$ if e is on the interior of Ω and $e \in \partial\Omega$ if e lies on the boundary $\partial\Omega$; we set h_e to be the diameter of e.

We let $\mathbb{P}^k(\mathscr{T})$ denote the space of piecewise polynomials of degree k over the triangulation $\mathscr{T}$, i.e.

$$\mathbb{P}^k(\mathscr{T}) = \{\phi \text{ such that } \phi|_K \in \mathbb{P}^k(K)\}, \tag{3.4}$$

and introduce the *finite-element space*

$$\mathbb{V}_h := \mathbb{P}^k(\mathscr{T}) \cap \mathrm{C}^0(\Omega) \tag{3.5}$$

to be the usual space of continuous piecewise polynomial functions of degree k over the triangulation.

Definition 3.1 (finite-element sequence). A finite-element sequence $(v_h, \mathbb{V}_h)$ is a sequence of discrete objects indexed by the mesh parameter h and individually represented on a particular finite-element space $\mathbb{V}_h$, with discretization parameter h.

Definition 3.2 (the $\mathrm{L}_2(\Omega)$ projection operator). The $\mathrm{L}_2(\Omega)$ projection operator $P_h\colon \mathrm{L}_2(\Omega) \to \mathbb{V}_h$ is defined for $v \in \mathrm{L}_2(\Omega)$ such that

$$\int_\Omega P_h v \Phi = \int_\Omega v \Phi \quad \forall \phi \in \mathbb{V}_h. \tag{3.6}$$

It is well known that this operator satisfies the following approximation properties for $v \in \mathrm{W}^{1,p}(\Omega)$:

$$\lim_{h\to 0} \|v - P_h v\|_{\mathrm{L}^p(\Omega)} = 0, \tag{3.7}$$

$$\lim_{h\to 0} \|\mathrm{D}v - \mathrm{D}(P_h v)\|_{\mathrm{L}^p(\Omega)} = 0. \tag{3.8}$$

3.1. Galerkin discretization

We consider the Galerkin discretization of (2.8) to find $u_h \in \mathbb{V}_h$ with $u_h|_{\partial\Omega} = P_h g$ such that

$$\mathscr{A}(u_h, \Phi) = 0 \quad \forall \Phi \in \mathbb{V}_h. \tag{3.9}$$

Proposition 3.3. *There exists a unique solution of (3.9).*

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Proof. The proof is standard and, in fact, equivalent to that of the smooth case; see [9, theorem 5.3.1]. □

THEOREM 3.4 (convergence of the discrete scheme to weak solutions). *Let the finite-element sequence generated by solving (3.9) be $(u_{h,p}, \mathbb{V}_h)$ and let u_p be the weak solution of (2.16). Then for fixed p we have that*

$$u_{h,p} \to u_p \quad \text{in } \mathrm{C}^0(\Omega). \tag{3.10}$$

Proof. We begin by noting that the discrete weak formulation (3.9) is equivalent to the following minimization problem: find $u_{h,p} \in \mathbb{V}_h$ such that

$$\mathscr{J}[u_{h,p}; p] = \min_{V \in \mathbb{V}_h} \mathscr{J}[v_h; p]. \tag{3.11}$$

Using this, we immediately have

$$\|\mathrm{D}u_{h,p}\|^p_{\mathrm{L}^p(\Omega)} \leqslant \mathscr{J}[u_{h,p}; p] \leqslant \mathscr{J}[P_h g; p] \leqslant \|\mathrm{D}(P_h g)\|^p_{\mathrm{L}^p(\Omega)}. \tag{3.12}$$

In view of the stability of the L_2 projection in $\mathrm{W}^{1,p}(\Omega)$ [11], we have

$$\|\mathrm{D}u_{h,p}\|_{\mathrm{L}^p(\Omega)} \leqslant C \tag{3.13}$$

uniformly in h. Hence, by weak compactness there exists a (weak) limit to the finite-element sequence that we will call u^*. Due to the weak semi-continuity of $\mathscr{J}[\cdot; p]$ we have

$$\mathscr{J}[u^*; p] \leqslant \mathscr{J}[u_{h,p}; p]. \tag{3.14}$$

In addition, in view of the approximation properties of P_h given in definition 3.2 we have for any $v \in \mathrm{C}^\infty$ that

$$\mathscr{J}[v; p] = \liminf_{h \to 0} \mathscr{J}[P_h v; p]. \tag{3.15}$$

Using the fact that $u_{h,p}$ is a discrete minimizer of (3.11), we have

$$\mathscr{J}[u^*; p] \leqslant \mathscr{J}[u_{h,p}; p] \leqslant \mathscr{J}[P_h v; p], \tag{3.16}$$

whence, sending $h \to 0$, we see that

$$\mathscr{J}[u^*; p] \leqslant \mathscr{J}[v; p]. \tag{3.17}$$

Now, as v was generic we may use density arguments and that u_p was the unique minimizer to conclude that $u^* = u_p$, thus concluding the proof. □

COROLLARY 3.5 (convergence of the discrete scheme to viscosity solutions). *In view of theorem 2.5 the discrete scheme converges to viscosity solutions of the p-Laplacian.*

LEMMA 3.6 (convergence in the limit $p \to \infty$). *Let $u_{h,p}$ solve the discrete problem (3.9). Then, for fixed h, along a subsequence we have $u_{h,p} \to u_{h,\infty}$.*

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Proof. The proof follows similarly to theorem 2.7. Since $u_{h,p}$ is the Galerkin solution to (3.9), it minimizes $\mathscr{J}[\cdot,p]$ over $\mathbb{V}_h$. Hence we know that

$$\|\mathrm{D}u_{h,p}\|_{\mathrm{L}^p(\Omega)} \leqslant \|\mathrm{D}(P_h g)\|_{\mathrm{L}^p(\Omega)} \leqslant C\|\mathrm{D}g\|_{\mathrm{L}^p(\Omega)}, \tag{3.18}$$

in view of the stability of P_h in $\mathrm{W}^{1,p}(\Omega)$. In addition, analogously to (2.33)–(2.38) we may find a constant such that

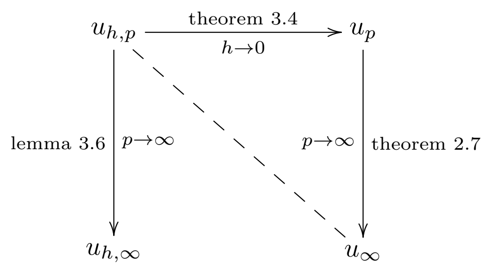
$$\|u_{h,p}\|_{\mathrm{W}^{1,k}} \leqslant C, \tag{3.19}$$

allowing the extraction of a subsequence $(u_{h,p_j})_{j=1}^{\infty}$ and a limit $u_{h,\infty}$ such that, for $k > d$,

$$u_{h,p_j} \rightharpoonup u_{h,\infty} \quad \text{weakly in } \mathrm{W}^{1,k}(\Omega). \tag{3.20}$$

The rest of the proof runs parallel to that of theorem 2.7. □

Remark 3.7 (summarizing the results thus far). Up to this point we have shown that the solid arrows on the following diagram hold:



We would like to select a route for which we can pass the limits together, that is, we want to select an appropriate route for which the dashed line is true.

Theorem 3.8 (convergence). *Let $u_{h,p}$ be the Galerkin solution of (3.9) and let u_∞ be the unique viscosity solution of (1.1). Then,*

$$u_{h,p} \to u_\infty \ \textit{in}\ \mathrm{C}^0 \quad \textit{as}\ p\to\infty\ \textit{and}\ h\to 0. \tag{3.21}$$

Proof. The proof is a consequence of theorems 2.7 and 3.4, noting that along the same subsequence used in theorem 2.7 and corollary 2.10 we have that

$$\|u_{h,p}-u_\infty\|_{\mathrm{C}^0(\Omega)} \leqslant \|u_{h,p}-u_p\|_{\mathrm{C}^0(\Omega)} + \|u_p-u_\infty\|_{\mathrm{C}^0(\Omega)}, \tag{3.22}$$

and hence

$$\|u_{h,p}-u_\infty\|_{\mathrm{C}^0(\Omega)} \to 0 \quad \text{as } p\to\infty \text{ and } h\to 0.$$ □

Remark 3.9 (consequences of theorem 3.8). An immediate consequence of theorem 3.8 and the previous arguments are that for $H\colon \Omega\times\mathbb{R}\times\mathbb{R}^d$ with appropriate conditions (convexity, for example), finite-element approximations to the p-functional

$$\mathscr{J}[u;p] = \|H(\cdot,u,\mathrm{D}u)\|_{\mathrm{L}^p(\Omega)} \tag{3.23}$$

can be used as approximations to

$$\mathscr{J}[u;\infty] = \|H(\cdot,u,\mathrm{D}u)\|_{\mathrm{L}^\infty(\Omega)}. \tag{3.24}$$

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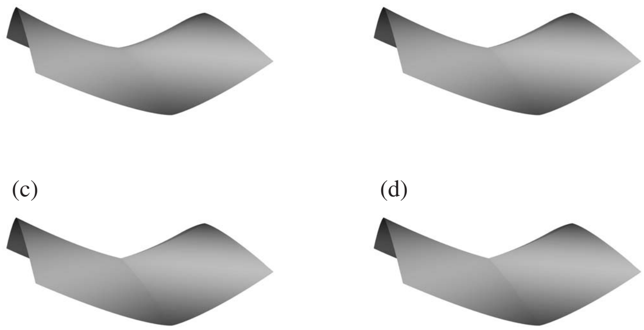

Figure 1. Finite-element approximations to the ∞-harmonic Aronsson function (4.1) using p-harmonic functions for various p. The finite-element approximations to the 5-Laplacian, 15-Laplacian, 50-Laplacian and 100-Laplacian are given by (a), (b), (c) and (d), respectively. Notice that as p increases the approximation better catches the singularity on the coordinate axis.

REMARK 3.10 (discontinuous Galerkin approximations). All the above results can be extended into the discontinuous Galerkin framework, where continuity is not globally enforced in the polynomial space. This is based on the discrete action functional

$$\mathscr{J}_h[u_h; p] := \int_\Omega |G(u_h)|^p + \int_{\mathscr{E}} h_e^{1-p} |[\![u_h]\!]|^p, \tag{3.25}$$

where

$$\int_\Omega G(u_h)\phi = \sum_{K \in \mathscr{T}} \int_K \mathrm{D}u_h \phi - \int_{\mathscr{E}} [\![u_h]\!] \{\!\{\phi\}\!\} \quad \forall \phi \in \mathbb{P}^k(\mathscr{T}). \tag{3.26}$$

Here, $[\![u_h]\!] = u_h|_{K^+} - u_h|_{K^-}$ denotes the *jump* over an edge e shared by neighbouring elements K^+ and K^-, and

$$\{\!\{\phi\}\!\} = \tfrac{1}{2}(\phi|_{K^+} + \phi|_{K^-})$$

is the *average* of a quantity over an edge. Using the results of [7], discrete minimizers to (3.25) satisfy the equivalent weak convergence results as the conforming finite elements.

4. Numerical experiments

In this section we summarize numerical experiments validating the analysis done in previous sections and allowing us to make conjectures on reasonable methods for coupling h and p.

REMARK 4.1 (practical computation of (3.9) for large p). The computation of p-harmonic functions is an extremely challenging problem in its own right. The class

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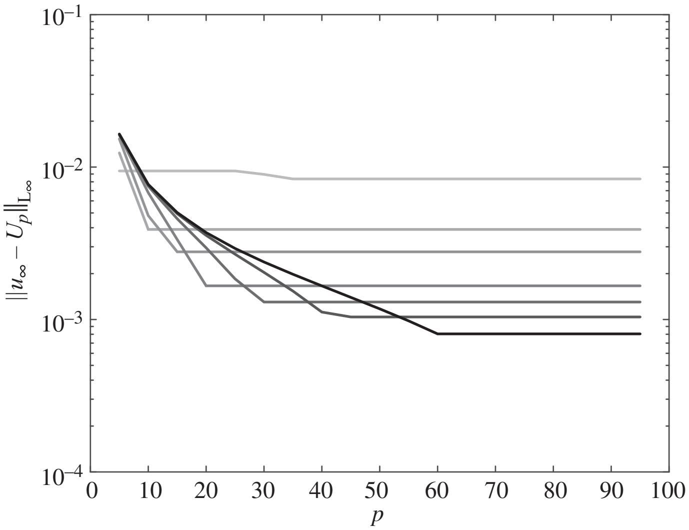

Figure 2. The error of the finite-element approximation to the p-Laplacian compared to the viscosity solution to the ∞-Laplacian for various p. The shades represent different mesh-refinement levels. The darker the shade, the more refined the mesh. In this experiment the mesh-size ranges from $h \sim 0.7$ to $h \sim 0.005$. Notice that as the mesh is refined, a better approximation is achieved for higher and higher p.

Table 1. *The convergence of the finite-element approximation $u_{h,p}$ to u_∞, a viscosity solution of (1.1), as the mesh-size is decreased.*

(We study the L^∞ error of the approximation, the associated convergence rate, and give p^*, the smallest such p for which $\inf_p \|u_\infty - u_{h,p}\|_{L^\infty(\Omega)}$ is attained. Notice that as the mesh is refined, the critical value increases.)

$\dim \mathbb{V}_h$	$\inf_p \|u_\infty - u_{h,p}\|_{L^\infty(\Omega)}$	EOC	p^*
25	0.0162	0.00	5
81	0.00836	0.95	5
289	0.00390	1.10	10
1089	0.00278	0.49	15
4225	0.00166	0.74	20
16641	0.00130	0.35	30
66049	0.00104	0.33	45
263169	0.000805	0.37	60

of nonlinearity in the problem results in the algebraic system, which ultimately yields the finite-element solution, being ill-conditioned. One method to tackle this class of problem is the use preconditioners based on descent algorithms [16]. For extremely large p, say $p \geqslant 1000$, this may be required, however, for our purposes we restrict our attention to $p \sim 100$. This yields sufficient accuracy for the results we want to illustrate.

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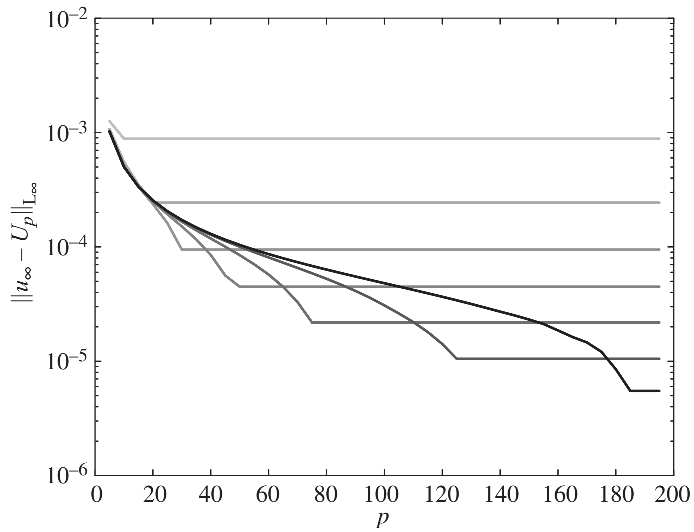

Figure 3. The error of the finite-element approximation to the p-Laplacian compared to a smooth solution of the ∞-Laplacian for various p. The shades represent different mesh-refinement levels. The darker the shade, the more refined the mesh. In this experiment the mesh-size ranges from $h \sim 0.7$ to $h \sim 0.005$. Notice that as the mesh is refined, a better approximation is achieved for higher and higher p.

Table 2. *The convergence of the finite-element approximation $u_{h,p}$ to u_∞, a smooth solution of (1.1), as the mesh-size is decreased.*

(We study the L^∞ error of the approximation, the associated convergence rate, and give p^*, the smallest such p for which $\inf_p \|u_\infty - u_{h,p}\|_{L^\infty(\Omega)}$ is attained. Notice that as the mesh is refined, the critical value increases much quicker than in the non-smooth case of table 1.)

$\dim \mathbb{V}_h$	$\inf_p \|u_\infty - u_{h,p}\|$	EOC	p^*
25	0.00301	0.00	5
81	0.000883	1.77	10
289	0.000244	1.86	20
1089	0.0000946	1.37	30
4225	0.0000448	1.08	50
16641	0.0000218	1.04	75
66049	0.0000105	1.06	125
263169	0.0000052	1.01	185

Even tackling the $p \sim 100$ case is computationally tough. Our numerical approximation is based on a Newton solver. As is well known, Newton solvers require a sufficiently close initial guess to converge. For large p a reasonable initial guess is given by numerically approximating the q-Laplacian for $q < p$ sufficiently close to p. This leads to an iterative process in the generation of the initial guess.

4.1. Test 1: approximation of the Aronsson viscosity solution

We begin by approximating the viscosity solution derived by Aronsson using separation of variables [3]. The function

$$u(x,y) = \tfrac{3}{8}(|x|^{4/3} - |y|^{4/3}) \in \mathrm{C}^{1,1/3}(\Omega) \tag{4.1}$$

is a viscosity solution of the ∞-Laplacian. Notice that this is a weighted version of the Aronsson solution. We have chosen this since $|\mathrm{D}u| \leqslant 1$ on the domains we consider, which helps to overcome the severe restrictions in computing p-harmonic functions with large p. In this test we take $\Omega = [-1.0001, 0.9999]^2$ and triangulate with a criss-cross mesh. This is so that the singularity will not be aligned with the mesh. We approximate the solution of the p-Laplacian with boundary data given by (4.1) for a variety of increasing p. Examples of solutions are given in figure 1. In figure 2 we plot the error against p for various levels of mesh refinement. In table 1 we demonstrate the convergence of the finite-element approximations as $h \to 0$.

4.2. Test 2: approximation of a smooth solution

To test the approximation of a known smooth solution of the ∞-Laplacian (1.1) we consider the Aronnson solution (4.1) away from the coordinate axis. In this test we take $\Omega = [0.5, 1.5]^2$ and triangulate with a criss-cross mesh. As in test 1, we approximate the solution of the p-Laplacian with boundary data given by (4.1) for a variety of increasing p. In figure 3 we plot the error against p for various levels of mesh refinement. In table 2 we demonstrate the convergence of the finite-element approximations as $h \to 0$.

References

1 S. Armstrong and C. Smart. An easy proof of Jensen's theorem on the uniqueness of infinity harmonic functions. *Calc. Var. PDEs* **37** (2010), 381–384.

2 G. Aronsson. Minimization problems for the functional $\sup_x F(x, f(x), f'(x))$. *Ark. Mat.* **6** (1965), 33–53.

3 G. Aronsson. Construction of singular solutions to the p-harmonic equation and its limit equation for $p = \infty$. *Manuscr. Math.* **56** (1986), 135–158.

4 G. Barles and P. E. Souganidis. Convergence of approximation schemes for fully nonlinear second order equations. *Asymp. Analysis* **4** (1991), 271–283.

5 J. W. Barrett and W. B. Liu. Finite element approximation of the parabolic p-Laplacian. *SIAM J. Numer. Analysis* **31** (1994), 413–428.

6 F. Bozorgnia. Numerical investigation of the eigenfunctions of infinity Laplacian operator. Technical report (2015).

7 E. Burman and A. Ern. Discontinuous Galerkin approximation with discrete variational principle for the nonlinear Laplacian. *C. R. Math.* **346** (2008), 1013–1016.

8 C. Carstensen, W. Liu and N. Yan. *A posteriori* FE error control for p-Laplacian by gradient recovery in quasi-norm. *Math. Comp.* **75** (2006), 1599–1616.

9 P. Ciarlet. *The finite element method for elliptic problems*, Studies in Mathematics and Its Applications, vol. 4 (Amsterdam: North-Holland, 1978).

10 M. Crandall, H. Ishii and P. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Am. Math. Soc.* **27** (1992), 1–67.

11 M. Crouzeix and V. Thomée. The stability in L_p and W_p^1 of the L_2-projection onto finite element function spaces. *Math. Comp.* **48** (1987), 521–532.

12 L. Diening and C. Kreuzer. Linear convergence of an adaptive finite element method for the p-Laplacian equation. *SIAM J. Numer. Analysis* **46** (2008), 614–638.

13 L. C. Evans. *Partial differential equations*, Graduate Studies in Mathematics, vol. 19 (Providence, RI: American Mathematical Society, 1998).

14 L. C. Evans and C. Smart. Adjoint methods for the infinity Laplacian partial differential equation. *Arch. Ration. Mech. Analysis* **201** (2011), 87–113.

15 X. Feng and M. Neilan. Vanishing moment method and moment solutions for fully nonlinear second order partial differential equations. *J. Sci. Comput.* **38** (2009), 74–98.

16 Y. Q. Huang, R. Li, and W. Liu. Preconditioned descent algorithms for p-Laplacian. *J. Sci. Comput.* **32** (2007), 343–371.

17 R. Jensen. Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient. *Arch. Ration. Mech. Analysis* **123** (1993), 51–74.

18 M. Jensen and I. Smears. On the convergence of finite element methods for Hamilton–Jacobi–Bellman equations. *SIAM J. Numer. Analysis* **51** (2013), 137–162.

19 P. Juutinen, P. Lindqvist and J. Manfredi. On the equivalence of viscosity solutions and weak solutions for a quasi-linear equation. *SIAM J. Math. Analysis* **33** (2001), 699–717.

20 N. Katzourakis. *An introduction to viscosity solutions for fully nonlinear PDE with applications to calculus of variations in* L^∞, Springer Briefs in Mathematics (Springer, 2015).

21 N. Katzourakis. Nonsmooth convex functionals and feeble viscosity solutions of singular Euler–Lagrange equations. *Calc. Var. PDEs* **54** (2015), 275–298.

22 N. Katzourakis. Generalised solutions for fully nonlinear PDE systems and existence-uniqueness theorems. *J. Diff. Eqns* **263** (2017), 641–686.

23 N. Katzourakis and T. Pryer. On the numerical approximation of ∞-harmonic mappings. *Nonlin. Diff. Eqns Applic.* **23** (2016), article 61.

24 O. Lakkis and T. Pryer. A finite element method for nonlinear elliptic problems. *SIAM J. Sci. Comput.* **35** (2013), A2025–A2045.

25 O. Lakkis and T. Pryer. An adaptive finite element method for the infinity Laplacian. In *Numerical mathematics and advanced applications – ENUMATH 2013*, Lecture Notes in Computational Science and Engineering, vol. 103, pp. 283–291 (Springer, 2013).

26 W. Liu and N. Yan. Quasi-norm local error estimators for p-Laplacian. *SIAM J. Numer. Analysis* **39** (2001), 100–127.

27 A. Oberman. A convergent difference scheme for the infinity Laplacian: construction of absolutely minimizing Lipschitz extensions. *Math. Comp.* **74** (2005), 1217–1230.

28 A. Oberman. Finite difference methods for the infinity Laplace and p-Laplace equations. *J. Computat. Appl. Math.* **254** (2013), 65–80.