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On critical and supercritical pseudo-relativistic nonlinear Schrödinger equations

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In this paper, we investigate existence and non-existence of a nontrivial solution to the pseudo-relativistic nonlinear Schrödinger equation

$$\left(\sqrt{-c^2\Delta + m^2c^4} - mc^2\right)u + \mu u = |u|^{p-1}u \quad \text{in } \mathbb{R}^n \ (n \geqslant 2)$$

involving an $H^{1/2}$-critical/supercritical power-type nonlinearity, that is, $p \geqslant ((n+1)/(n-1))$. We prove that in the non-relativistic regime, there exists a nontrivial solution provided that the nonlinearity is $H^{1/2}$-critical/supercritical but it is H^1-subcritical. On the other hand, we also show that there is no nontrivial bounded solution either (i) if the nonlinearity is $H^{1/2}$-critical/supercritical in the ultra-relativistic regime or (ii) if the nonlinearity is H^1-critical/supercritical in all cases.

Keywords: pseudo-relativistic NLS; supercriticality; existence; non-existence

2010 *Mathematics subject classification:* Primary: 35J10; 35J61; 35Q51

1. Introduction

We consider the pseudo-relativistic nonlinear Schrödinger equation (NLS)

$$i\partial_t\psi = \left(\sqrt{-c^2\Delta + m^2c^4} - mc^2\right)\psi - |\psi|^{p-1}\psi, \tag{1.1}$$

where $\psi = \psi(t,x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$ denotes a wave function, $c > 0$ is the speed of light and $m > 0$ is the particle mass. The non-local operator $\sqrt{-c^2\Delta + m^2c^4} - mc^2$ is the pseudo-differential operator defined by the symbol $\sqrt{c^2|\xi|^2 + m^2c^4} - mc^2$ which arises from Einstein's energy-momentum relation $E^2 = |p|^2c^2 + m^2c^4$. The equation (1.1) formally converges to the non-relativistic NLS

$$i\partial_t\psi = -\frac{1}{2m}\Delta\psi - |\psi|^{p-1}\psi$$

as $c \to \infty$, because $\sqrt{c^2|\xi|^2 + m^2c^4} - mc^2 \to ((1)/(2m))|\xi|^2$ as $c \to \infty$. On the other hand, it converges to the nonlinear half-wave equation

$$i\partial_t\psi = c\sqrt{-\Delta}\psi - |\psi|^{p-1}\psi$$

as $m \to 0$. In this sense, the pseudo-relativistic NLS describes the intermediate dynamics between the classical and the relativistic models. The equation (1.1) may have completely different characters in the non-relativistic regime ($c^2|\xi|^2 \ll m^2c^4$), where the relativity is taken into account only weakly, from the ultra-relativistic regime ($c^2|\xi|^2 \gg m^2c^4$), where the relativity is dominant.

The goal of this paper is to find criteria for existence and non-existence of a nontrivial standing wave of the form $\psi(t, x) = e^{i\mu t}u(x)$ to the pseudo-relativistic NLS (1.1). To this end, we shall focus on the stationary pseudo-relativistic NLS

$$(\sqrt{-c^2\Delta + m^2c^4} - mc^2)u + \mu u = |u|^{p-1}u, \tag{1.2}$$

where $u = u(x) : \mathbb{R}^n \to \mathbb{C}$.

When the nonlinearity is $H^{1/2}$-subcritical, that is, $1 < p < ((n + 1)/(n - 1))$, by a standard variational argument, it is shown that the pseudo-relativistic NLS (1.2) admits a nontrivial solution for all $m, c, \mu > 0$ (see [**2**, **4**, **5**, **7**, **8**, **15**, **18**] for the related variational results). However, to the best knowledge of the authors, nothing is known in the $H^{1/2}$-critical/supercritical case, that is, $p \geqslant ((n + 1)/(n - 1))$, because the standard variational approach does not work well in the supercritical setting.

Nevertheless, there is still a hope to construct a nontrivial solution in the critical/supercritical case. To see this, we recall that as $c \to \infty$, the pseudo-relativistic equation (1.2) approaches to the non-relativistic equation, that is, the stationary non-relativistic NLS

$$-\frac{1}{2m}\Delta u + \mu u = |u|^{p-1}u. \tag{1.3}$$

As for existence of a nontrivial solution to (1.3), there is a dichotomy divided at the H^1-criticality [**1**, **17**]. Precisely, a positive radially symmetric bounded solution exists in the H^1-subcritical case, that is, $1 < p < ((n + 2)/(n - 2))$. Moreover, such a solution is known to be unique [**13**]. However, by Pohozaev's identities, no nontrivial bounded solution exists in the H^1-critical/supercritcal case $p \geqslant ((n + 2)/(n - 2))$. On the other hand, as $m \to 0$, the equation (1.2) approaches to the ultra-relativistic equation, namely the stationary nonlinear half-wave equation

$$c\sqrt{-\Delta}u + \mu u = |u|^{p-1}u. \tag{1.4}$$

Similarly but differently, for (1.4), the dichotomy arises at the $H^{1/2}$-criticality. Indeed, a positive radial solution exists in the $H^{1/2}$-subcritical case, that is, $1 <$

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Table 1. *Existence and non-existence of a non-trivial solution to* (1.3)/(1.4)

	$1 < p < \frac{n+1}{n-1}$	$\frac{n+1}{n-1} \leqslant p < \frac{n+2}{n-2}$	$\frac{n+2}{n-2} \leqslant p < \infty$
non-relativistic NLS (1.3)	existence	existence	non-existence
half wave equation (1.4)	existence	non-existence	non-existence

$p < ((n+1)/(n-1))$, and its uniqueness is proved by Frank-Lenzmann for $n = 1$ [**9**] and Frank-Lenzmann-Silvestre for $n \geqslant 2$ [**10**], provided that it is also a ground state. However, by Pohozaev's identities again, a bounded nontrivial solution does not exist in the $H^{1/2}$-critical/supercritical case $p \geqslant ((n+1)/(n-1))$.

The above observation suggests a possibility that existence of a nontrivial solution to (1.2) can be determined by the criticality of the equation as well as by the parameters $m, c, \mu > 0$. More specifically, from the results in Table 1 and the connections among the three equations via the formal limits, it is natural to guess that when $((n+1)/(n-1)) \leqslant p < ((n+2)/(n-2))$, a nontrivial solution exists in the non-relativistic regime $c \gg 1$, but it does not in the ultra-relativistic regime $m \ll 1$. No existence is expected when $p \geqslant ((n+2)/(n-2))$.

The first theorem of this paper proves non-existence of a nontrivial solution to (1.2), which fits into Table 1.

Theorem 1.1 Non-existence. *Let $n \geqslant 2$. Suppose that*

$$p \geqslant \frac{n+1}{n-1} \quad \textit{and} \quad mc^2 \leqslant \mu$$

or that

$$p \geqslant \frac{n+2}{n-2}.$$

Then, there is no nontrivial solution to (1.2) *in* $H^{1/2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

We show theorem 1.1 by exploiting the Pohozaev-type identities on the extension problem of (1.2) to the upper half-plane. We note that if $((n+1)/(n-1)) \leqslant p < ((n+2)/(n-2))$, this approach does not work when $mc^2 > \mu$.

The next theorem, which is the main contribution of this paper, provides an affirmative answer for the existence part.

Theorem 1.2 Existence. *Let $n \geqslant 2$. Suppose that*

$$\frac{n+1}{n-1} \leqslant p < \frac{n+2}{n-2}.$$

Then, there exists $\kappa_0 \geqslant 1$ such that if

$$mc^2 \geqslant \kappa_0 \mu,$$

then the pseudo-relativistic NLS (1.2) has a nontrivial solution $u_c \in H^1_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$*.*

Even though a lot of works have been devoted to the pseudo-relativistic NLS (1.2), to the best knowledge of the authors, theorem 1.2 is the first result in the literature, proving existence of its non-trivial solution in the supercritical setting. Another important remark is that only the quantity $((mc^2)/(\mu))$, not all three independent parameters $m, c, \mu > 0$, determines the regime of the equation concerning existence and non-existence of a nontrivial solution.

Due to supercriticality of the problem, it is difficult to apply the standard variational method. Indeed, the action functional (see (5.1)) is indefinite in this case. Cutting off the nonlinearity is also not suitable in this setting. Even in the $H^{1/2}$-critical case, due to the lack of L^2-convergence of Palais-Smale sequences, it looks impossible to apply a well-known method that characterizes energy sub-levels such that the Palais-Smale condition holds and subsequently tests a family of extremal functions of the Sobolev inequality.

To overcome the aforementioned difficulty, we employ the following non-variational approach, combined with the uniform L^q-estimates for the pseudo-differential operator $\sqrt{-c^2\Delta + m^2c^4} - mc^2$. First, by some algebraic manipulation for notational simplicity, we reduce to the case $m = 1/2$ and $\mu = 1$, and consider

$$P_c(D)u = |u|^{p-1}u, \tag{1.5}$$

where

$$P_c(D) = \left(\sqrt{-c^2\Delta + \frac{c^4}{4}} - \frac{c^2}{2}\right) + 1$$

(see § 3.1). Next, we choose a ground state u_∞ to the non-relativistic limit equation

$$-\Delta u + u = |u|^{p-1}u. \tag{1.6}$$

Considering the $H^{1/2}$-supercritical pseudo-relativistic equation (1.5) as a perturbation of the H^1-subcritical non-relativistic equation (1.6), we formulate an equation for the perturbation w from the ground state u_∞ (see (3.3)). Then, we establish existence of a solution w via the contraction mapping principle (see theorem 3.7). The main advantage of this approach is that we may take the full advantage of extra properties of the ground state u_∞, including its smoothness and decay. In particular, the non-degeneracy of the linearized operator $\mathcal{L}_\infty$ about the ground state u_∞ (see (3.5)) plays a crucial role in this procedure. This kind of perturbation argument has been employed previously in the literature for other problems. For example, we refer to [**6**, **12**, **16**] for the nonlinear Dirac equation and to [**14**] for the nonlinear Schrödinger equation with slightly supercritical nonlinearity.

Another new ingredient of our analysis is to use the L^q-estimates for the pseudo-differential operator $P_c(D)$ based on the symbolic analysis. Indeed, for our contraction mapping argument, it is important to find a uniform (in $c \geqslant 2$) boundedness of the inverse operator $P_c(D)^{-1} : L^q \to W^{1,q}$. In the special case $q = 2$, such an estimate immediately follows from a simple pointwise bound on the symbol

[**3**, lemma 4.3]. In this paper, we obtain the following extended inequalities covering all exponents $1 < q < \infty$.

THEOREM 1.3 Norm comparability. *For* $1 < q < \infty$*, there exists a constant* $C_{q,n} > 0$ *such that for* $c \geqslant 2$*,*

$$C_{q,n}^{-1}\|f\|_{W^{1,q}} \leqslant \|P_c(D)f\|_{L^q} \leqslant C_{q,n}\|f\|_{W^{2,q}}$$

REMARK 1.4. Since $P_c(D)$ is a first-order elliptic operator, it is obvious that

$$C_{q,n,c}^{-1}\|f\|_{W^{1,q}} \leqslant \|P_c(D)f\|_{L^q} \leqslant C_{q,n,c}\|f\|_{W^{1,q}},$$

with $C_{q,n,c} > 0$ depending on $c \geqslant 2$. Contrary to these trivial inequalities, theorem 1.3 provides upper and lower bounds uniformly in $c \geqslant 2$. Note that the upper bound in theorem 1.3 is optimal, because $\|P_c(D)f\|_{L^q} \to \|(-\Delta+1)f\|_{L^q}$ as $c \to \infty$ for $f \in C_c^\infty$.

We prove theorem 1.3 by the Hörmander-Mikhlin theorem. For this aim, we carefully estimate the derivatives of the associated symbols. We also prove here that for sufficiently large $c \geqslant 1$, the inverse of the pseudo-relativistic operator is close to that of the non-relativistic operator as operators acting on L^q (see theorem 2.1). Theorems 1.3 and 2.1 are employed to obtain existence in the full H^1-subcritical range. Indeed, without these extended inequalities, only a narrow range of nonlinearities, $1 < p < ((n)/(n-2))$, is covered (see remark 3.6).

The rest of this paper is organized as follows. In § 2, we establish several mapping properties for the pseudo-differential operator $P_c(D)$. Given those properties, in § 3, we prove the existence result (theorem 1.2). § 4 is devoted to establish non-existence (theorem 1.1). Finally, in § 5, we discuss some properties of the solution constructed previously and propose an open question related to these properties.

2. Symbol calculus for the pseudo-relativistic Schrödinger operator

Given a symbol $\mathfrak{m} : \mathbb{R}^d \to \mathbb{R}$, the associated Fourier multiplier operator $\mathfrak{m}(D)$ is defined by

$$\widehat{\mathfrak{m}(D)f}(\xi) = \mathfrak{m}(\xi)\hat{f}(\xi).$$

We introduce the pseudo-differential operator $P_c(D)$ (or $P_\infty(D)$, respectively) as the Fourier multiplier with the symbol

$$P_c(\xi) := \left(\sqrt{c^2|\xi|^2 + \frac{c^4}{4}} - \frac{c^2}{2}\right) + 1 \quad \left(P_\infty(\xi) := |\xi|^2 + 1, \text{respectively}\right). \tag{2.1}$$

The purpose of this section is to provide the connection between these two operators. Precisely, we show that as inverse operators, $P_c(D)$ converges to $P_\infty(D)$ as $c \to \infty$ (theorem 2.1 below). Here, we also prove the norm comparability (theorem 1.3).

THEOREM 2.1 Difference between the inverses of two operators.

(1) *For* $1 < q < \infty$, *there exists a constant* $C_{q,n} > 0$ *such that*

$$\left\| \left(\frac{1}{P_\infty(D)} - \frac{1}{P_c(D)} \right) f \right\|_{L^q} \leqslant \frac{C_{q,n}}{c^2} \|f\|_{L^q} \quad \forall\, c \in [2, \infty). \tag{2.2}$$

(2) *For* $1 < q < \infty$, *there exists a constant* $C_{q,n} > 0$ *such that*

$$\left\| \left(\frac{1}{P_\infty(D)} - \frac{1}{P_c(D)} \right) f \right\|_{L^q} \leqslant \frac{C_{q,n}}{c} \left\| \frac{1}{P_\infty(D)^{1/2}} f \right\|_{L^q} \quad \forall\, c \in [2, \infty). \tag{2.3}$$

For the proof, we recall the Hörmander-Mikhlin multiplier theorem (see [11] for instance).

THEOREM 2.2 Hörmander-Mikhlin. *Suppose that* $\mathfrak{m} : \mathbb{R}^n \to \mathbb{R}$ *satisfies*

$$|\nabla^\alpha \mathfrak{m}(\xi)| \leqslant \frac{B_\alpha}{|\xi|^{|\alpha|}} \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}$$

for all multi-indices $\alpha \in (\mathbb{Z}_{\geqslant 0})^n$ *such that* $0 \leqslant |\alpha| \leqslant n/2 + 1$, *where* $\mathbb{Z}_{\geqslant 0} = \{0, 1, 2, 3 \ldots\}$. *Then for any* $1 < q < \infty$, *there exists a constant* $C_{q,n} > 0$ *such that*

$$\|\mathfrak{m}(D) f\|_{L^q} \leqslant C_{q,n} \left(\sup_{0 \leqslant |\alpha| \leqslant \frac{n}{2}+1} B_\alpha \right) \|f\|_{L^q}.$$

By the Hörmander-Mikhlin multiplier theorem, the proofs of theorems 2.1 and 1.3 are reduced to those of the following bounds on the derivatives of the symbols.

PROPOSITION 2.3.

(1) *For any multi-index* $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}_{\geqslant 0})^n$, *there is a constant* $C_\alpha > 0$ *such that for all* $c \in [2, \infty)$,

$$\left| \nabla_\xi^\alpha \left(\frac{1}{P_\infty(\xi)} - \frac{1}{P_c(\xi)} \right) \right| \leqslant \frac{C_\alpha}{|\xi|^{|\alpha|}} \min \left\{ \frac{1}{c^2}, \frac{1}{c(|\xi|^2 + 1)^{1/2}} \right\}. \tag{2.4}$$

(2) *For any multi-index* $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}_{\geqslant 0})^n$, *there is a constant* $C_\alpha > 0$ *such that for all* $c \in [2, \infty)$,

$$\left| \nabla_\xi^\alpha \left(\frac{P_c(\xi)}{P_\infty(\xi)} \right) \right| \leqslant \frac{C_\alpha}{|\xi|^{|\alpha|}}. \tag{2.5}$$

Proof of theorem 2.1 assuming proposition 2.3. For any multi-index $\alpha \in (\mathbb{Z}_{\geqslant 0})^n$, it follows from estimate (2.4) that

$$\left|\nabla_\xi^\alpha \left(\frac{1}{P_\infty(\xi)} - \frac{1}{P_c(\xi)}\right)\right| \leqslant \frac{C_\alpha}{c^2|\xi|^{|\alpha|}}$$

and

$$\begin{aligned}\left|\nabla_\xi^\alpha \left(\left(\frac{1}{P_\infty(\xi)} - \frac{1}{P_c(\xi)}\right) P_\infty(\xi)^{1/2}\right)\right| &\leqslant \sum_{\alpha_1+\alpha_2=\alpha} \left|\nabla_\xi^{\alpha_1}\left(\frac{1}{P_\infty(\xi)} - \frac{1}{P_c(\xi)}\right)\right| \\ &\quad \left|\nabla_\xi^{\alpha_2}\left(P_\infty(\xi)^{1/2}\right)\right| \\ &\leqslant \sum_{\alpha_1+\alpha_2=\alpha} \frac{C_{\alpha_1}}{c|\xi|^{|\alpha_1|+1}} \frac{C_{\alpha_2}}{|\xi|^{|\alpha_2|-1}} \\ &= \sum_{\alpha_1+\alpha_2=\alpha} \frac{C_{\alpha_1}C_{\alpha_2}}{c|\xi|^{|\alpha|}} \\ &\leqslant \frac{C_\alpha}{c|\xi|^{|\alpha|}}.\end{aligned}$$

Thus, theorem 2.1 follows from theorem 2.2. □

Proof of theorem 1.3 assuming proposition 2.3. By the triangle inequality and (2.3), we prove that

$$\begin{aligned}\left\|\frac{1}{P_c(D)}f\right\|_{L^q} &\leqslant \left\|\frac{1}{P_\infty(D)}f\right\|_{L^q} + \left\|\left(\frac{1}{P_\infty(D)} - \frac{1}{P_c(D)}\right)f\right\|_{L^q} \\ &\leqslant \left\|\frac{1}{P_\infty(D)}f\right\|_{L^q} + \frac{C_q}{c}\left\|\frac{1}{P_\infty(D)^{1/2}}f\right\|_{L^q} \\ &\leqslant C_q \|f\|_{W^{-1,q}}.\end{aligned}$$

By inserting $f \to P_c(D)\sqrt{P_\infty(D)}f$, we prove the first inequality. The second inequality follows from (2.5) and theorem 2.2. □

In the rest of this section, we prove proposition 2.3. For this aim, we first observe that the pseudo-relativistic symbol $P_c(\xi)$ is comparable with the non-relativistic symbol $P_\infty(\xi)$ inside a large ball, while it is like the half-wave symbol $c|\xi|+1$ outside.

Lemma 2.4 Pointwise comparability on the pseudo-relativistic symbol.

$$\begin{cases} \frac{|\xi|^2+1}{2} \leqslant P_c(\xi) \leqslant |\xi|^2+1 & \text{if } |\xi| \leqslant \frac{\sqrt{3}c}{2}, \\ \frac{c|\xi|+1}{2} \leqslant P_c(\xi) \leqslant c|\xi|+1 & \text{if } |\xi| \geqslant \frac{\sqrt{3}c}{2}. \end{cases}$$

Proof. Suppose that $|\xi| \leqslant ((\sqrt{3}c)/(2))$. Then, $P_c(\xi) = ((c^2)/(2))f(((4|\xi|^2)/(c^2))) + 1$, where $f(t) = \sqrt{1+t} - 1$. Since $f(0) = 0$ and $1/4 \leqslant f'(t) = ((1)/(2\sqrt{1+t})) \leqslant 1/2$

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on $[0,3]$, by Taylor's theorem, we have

$$\frac{|\xi|^2}{2} = \frac{c^2}{2}\cdot\frac{1}{4}\cdot\frac{4|\xi|^2}{c^2} \leqslant P_c(\xi)-1 \leqslant \frac{c^2}{2}\cdot\frac{1}{2}\cdot\frac{4|\xi|^2}{c^2} = |\xi|^2.$$

On the other hand, if $|\xi| \geqslant ((\sqrt{3}c)/(2))$, then

$$P_c(\xi)-1 = c|\xi|\left(\sqrt{1+\frac{c^2}{4|\xi|^2}}-\frac{c}{2|\xi|}\right) = \frac{c|\xi|}{\sqrt{1+(c^2/(4|\xi|^2))}+(c/(2|\xi|))}$$

obeys the desired upper and lower bound, because $((c|\xi|)/(\sqrt{3})) \leqslant P_c(\xi)-1 \leqslant c|\xi|$. □

Remark 2.5. Indeed, the inequality $P_c(\xi) \leqslant |\xi|^2+1$ holds for all ξ, since in the proof of Lemma 2.4, we can use $|f'(t)| \leqslant 1/2$ for all $t \geqslant 0$.

Next, we show that the pseudo-relativistic symbol $P_c(\xi)$ approximates the non-relativistic symbol $P_\infty(\xi)$ near the origin.

Lemma 2.6 Pointwise estimate on the difference between the two symbols.

$$|P_c(\xi)-P_\infty(\xi)| \leqslant \frac{|\xi|^4}{c^2}.$$

Proof. Let $f(t)$ be the function given in the proof of Lemma 2.6. Then, by Taylor's theorem together with $f''(t) = -1/4(1+t)^{-3/2}$, we have

$$|P_c(\xi)-P_\infty(\xi)| = \left|\frac{c^2}{2}\left(f(0)+f'(0)\tfrac{4|\xi|^2}{c^2}+\frac{1}{2}f''(t_*)(\tfrac{4|\xi|^2}{c^2})^2\right)-|\xi|^2\right| \leqslant \frac{|\xi|^4}{c^2}$$

for some $t_* \in (0,((4|\xi|^2)/(c^2)))$. □

Now we are ready to prove proposition 2.3.

Proof of proposition 2.3. We now prove (2.4) and (2.5) separately. □

Proof of (2.4). We denote

$$a(\xi) := \frac{1}{P_\infty(\xi)} - \frac{1}{P_c(\xi)} = \frac{P_c(\xi)-P_\infty(\xi)}{P_c(\xi)P_\infty(\xi)}.$$

First, we find a structure of the derivatives of the symbol a. Precisely, we claim that $\nabla_\xi^\alpha a(\xi)$ is the sum of products of the following factors,

$$a(\xi),\quad \frac{1}{P_c(\xi)^{\ell_1}},\quad \frac{1}{P_\infty(\xi)^{\ell_2}},\ \frac{1}{(c^2|\xi|^2+(c^4/4))^{((\ell_3)/(2))}},$$
$$\text{a polynomial of } c,\xi_1,\ldots,\xi_n, \tag{2.6}$$

where $\ell_1,\ell_2,\ell_3 \in \mathbb{Z}_{\geqslant 0}$. The claim is obviously true when $\alpha = 0$. Suppose that the claim holds for some multi-index α, and consider its derivative. By the induction

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hypothesis, $(\nabla_{\xi_j}\nabla_\xi^\alpha a)(\xi)$ is the sum of the derivatives of the products. By the Leibniz rule, the derivative of each product term is the sum of the derivative of one factor in (2.6) times the product of others in (2.6). Thus, it suffices to show that the derivative of any of (2.6) is again a product of terms in (2.6). Of course, the derivative of a polynomial of $c, \xi_1, \ldots, \xi_n$ is also a polynomial of the same variables. By direct calculations, we observe that

$$\begin{aligned}
\nabla_{\xi_j}\left(P_c(\xi) - P_\infty(\xi)\right) &= -\frac{2\xi_j P_c(\xi)}{(c^2|\xi|^2 + ((c^4)/(4)))^{1/2}} \\
&\quad + \frac{2\xi_j}{(c^2|\xi|^2 + ((c^4)/(4)))^{1/2}}, \\
\nabla_{\xi_j}\left(\frac{1}{P_c(\xi)^{\ell_1}}\right) &= -\frac{\ell_1 c^2 \xi_j}{(c^2|\xi|^2 + ((c^4)/(4)))^{1/2} P_c(\xi)}\left(\frac{1}{P_c(\xi)^{\ell_1}}\right), \\
\nabla_{\xi_j}\left(\frac{1}{P_\infty(\xi)^{\ell_2}}\right) &= -\frac{2\ell_2\xi_j}{P_\infty(\xi)}\left(\frac{1}{P_\infty(\xi)^{\ell_2}}\right), \\
\nabla_{\xi_j}\left(\frac{1}{(c^2|\xi|^2 + ((c^4)/(4)))^{((\ell_3)/(2))}}\right) &= -\frac{\ell_3\xi_j}{|\xi|^2 + ((c^2)/(4))} \\
&\quad \times \left(\frac{1}{(c^2|\xi|^2 + ((c^4)/(4)))^{((\ell_3)/(2))}}\right).
\end{aligned} \tag{2.7}$$

Hence, by the Leibniz rule,

$$\begin{aligned}
(\nabla_{\xi_j} a)(\xi) &= \nabla_{\xi_j}\left(\frac{P_c(\xi) - P_\infty(\xi)}{P_c(\xi)P_\infty(\xi)}\right) \\
&= -\frac{2\xi_j}{P_\infty(\xi)(c^2|\xi|^2 + ((c^4)/(4)))^{1/2}} - \frac{\ell_1 c^2\xi_j}{(c^2|\xi|^2 + ((c^4)/(4)))^{1/2}P_c(\xi)} a(\xi) \\
&\quad - \frac{2\ell_2\xi_j}{P_\infty(\xi)} a(\xi).
\end{aligned} \tag{2.8}$$

From this, we conclude that when a derivative hits a factor in (2.6), it does not make a new type of factors other than (2.6). Thus, the claim is proved.

We also note from (2.7) that when $((1)/(P_c(\xi)^{\ell_1}))$, $((1)/(P_\infty(\xi)^{\ell_2}))$, $((1)/((c^2|\xi|^2 + ((c^4)/(4)))^{((\ell_3)/(2))}))$ are differentiated, extra factors are produced. Moreover, these extra factors are all bounded by $((C)/(|\xi|))$. Indeed, by lemma 2.4,

$$\left|\frac{\ell_1 c^2\xi_j}{(c^2|\xi|^2 + ((c^4)/(4)))^{1/2}P_c(\xi)}\right| \leqslant \begin{cases} \dfrac{\ell_1 c^2|\xi|}{((c^2)/(2))(((|\xi|^2)/(2))} = \dfrac{4\ell_1}{|\xi|} & \text{if } |\xi| \leqslant \frac{\sqrt{3}c}{2}, \\ \dfrac{\ell_1 c^2|\xi|}{c|\xi|((c|\xi|)/(2))} = \dfrac{2\ell_1}{|\xi|} & \text{if } |\xi| \geqslant \frac{\sqrt{3}c}{2}, \end{cases}$$

$$\left|\frac{2\ell_2\xi_j}{P_\infty(\xi)}\right| \leqslant \frac{2\ell_2|\xi|}{|\xi|^2+1} \leqslant \frac{2\ell_2}{|\xi|},$$

$$\left|\frac{\ell_3\xi_j}{|\xi|^2+((c^2)/(4))}\right| \leqslant \frac{\ell_3|\xi|}{|\xi|^2+1} \leqslant \frac{\ell_3}{|\xi|}. \tag{2.9}$$

We now prove the proposition by induction. For the zeroth induction step, that is, $\alpha = 0$, using lemma 2.4, we show that if $|\xi| \leqslant ((\sqrt{3}c)/(2))$, then

$$\begin{aligned}|a(\xi)| &\leqslant \frac{((|\xi|^4)/(c^2))}{((|\xi|^2+1)/(2))(|\xi|^2+1)} \leqslant \min\left\{\frac{2}{c^2}, \frac{2|\xi|}{c^2(|\xi|^2+1)^{1/2}}\right\}\\ &\leqslant 2\min\left\{\frac{1}{c^2}, \frac{1}{c(|\xi|^2+1)^{1/2}}\right\}.\end{aligned}$$

On the other hand, if $|\xi| \geqslant ((\sqrt{3}c)/(2))$, then by lemma 2.4 again,

$$\begin{aligned}|a(\xi)| &\leqslant \frac{1}{P_\infty(\xi)} + \frac{1}{P_c(\xi)} \leqslant \frac{1}{|\xi|^2+1} + \frac{2}{c|\xi|+1}\\ &\leqslant \min\left\{\frac{4}{3c^2} + \frac{4}{\sqrt{3}c^2}, \frac{2}{\sqrt{3}c(|\xi|^2+1)^{1/2}} + \frac{2\sqrt{2}}{c(|\xi|^2+1)^{1/2}}\right\}\\ &\leqslant 4\min\left\{\frac{1}{c^2}, \frac{1}{c(|\xi|^2+1)^{1/2}}\right\}.\end{aligned}$$

For the first induction step, that is, $|\alpha| = 1$, we consider the sum (2.8). By a trivial inequality, the first term in (2.8) is bounded by

$$\frac{4}{|\xi|}\min\left\{\frac{1}{c^2}, \frac{1}{c(|\xi|^2+1)^{1/2}}\right\}.$$

Moreover, it follows from (2.9) and the zeroth induction step that the second and the last terms in (2.8) also obeys the same bound. Collecting all, we complete the proof of the first induction step.

For induction, we assume that each product in the sum for $(\nabla_\xi^\alpha a)(\xi)$ is bounded by

$$\frac{C_\alpha}{|\xi|^{|\alpha|}}\min\left\{\frac{1}{c^2}, \frac{1}{c(|\xi|^2+1)^{1/2}}\right\}.$$

Then, it suffices to show that each term in the sum for $(\nabla_{\xi_j}\nabla_\xi^\alpha a)(\xi)$ satisfies the desired bound. Indeed, all these terms are obtained by differentiating the product terms in the previous step. However, as mentioned previously, when a product is differentiated, the derivative lands on either a polynomial factor or other types of factors in (2.6). When a polynomial is differentiated, its degree is reduced by one. Otherwise, an extra factor is generated (see (2.7)) and such an extra factor is bounded by $((C)/(|\xi|))$ (see (2.9)). Thus, summing up all bounds, we prove the proposition. □

Proof of (2.5). The proof is very similar to that of estimate (2.4), so we only give a sketch of it. First, by remark 2.5, $|((P_c(\xi))/(P_\infty(\xi)))| \leqslant 1$. Next, we prove the first derivative

$$\nabla_{\xi_j}\left(\frac{P_c(\xi)}{P_\infty(\xi)}\right) = \frac{1}{P_\infty(\xi)}\cdot\frac{c^2\xi_j}{(c^2|\xi|^2+((c^4)/(4)))^{1/2}} - \frac{2\xi_j}{P_\infty(\xi)}\frac{P_c(\xi)}{P_\infty(\xi)}$$

is bounded by

$$\frac{2|\xi|}{P_\infty(\xi)} + \frac{2|\xi|}{P_\infty(\xi)}\cdot 1 \leqslant \frac{4}{|\xi|}.$$

By the induction argument in the proof of estimate (2.4), one can show that $\nabla_\xi^\alpha(((P_c(\xi))/(P_\infty(\xi))))$ is the sum of products of $((P_c(\xi))/(P_\infty(\xi)))$, $((1)/(P_\infty(\xi)^{\ell_1}))$, $((1)/((c^2|\xi|^2+((c^4)/(4)))^{((\ell_2)/(2))}))$ and a polynomial of $c, \xi_1, \dots, \xi_n$. For induction, we assume that each product in the sum for the expansion of $\nabla_\xi^\alpha(((P_c(\xi))/(P_\infty(\xi))))$ is bounded by $((C_\alpha)/(|\xi|^{|\alpha|}))$. If we differentiate each product, then differentiation produces an extra factor keeping the same structure. Moreover, all possible extra factors are bounded by $((C)/(|\xi|))$ (see (2.7) and (2.9)). Therefore, the derivative of each product is bounded by $((C'_\alpha)/(|\xi|^{|\alpha|+1}))$. Then, summing all the bounds, we obtain the desired bound for the derivative of $\nabla_\xi^\alpha(((P_c(\xi))/(P_\infty(\xi))))$. □

3. Existence result

This section is devoted to our main existence theorem whose proof will be divided into several steps. First, in § 3.1, by algebraic manipulation, we simplify to the case $m = 1/2$ and $\mu = 1$. In § 3.2, we reformulate the pseudo-relativistic Schrödinger equation (1.2) as an equation for the perturbation from the non-relativistic ground state (see (3.3)). The goal is then to construct a solution to the equation by a standard contraction mapping argument. To that end, we prove several key estimates for contraction in §§ 3.3–3.5. After being prepared, in § 3.6, we establish existence and uniqueness of a solution to the reformulated equation. Finally, in § 3.7, we complete the proof of theorem 1.2.

3.1. Reduction to the simple case

To begin with, we observe that if v_c is a non-trivial solution to the pseudo-relativistic NLS with $m = 1/2$ and $\mu = 1$, that is,

$$P_c(D)u = |u|^{p-1}u, \tag{3.1}$$

where

$$P_c(D) := \left(\sqrt{-c^2\Delta + \frac{c^4}{4}} - \frac{c^2}{2}\right) + 1,$$

then $u_c(x) = \mu^{((1)/(p-1))} v_c(\sqrt{2m\mu}x)$ solves

$$\left(\sqrt{-\tilde{c}^2\Delta + m^2\tilde{c}^4} - m\tilde{c}^2\right)u + \mu u = |u|^{p-1}u,$$

where $\tilde{c} = c\sqrt{((\mu)/(2m))}$. Thus, we may restrict ourselves to the case $m = 1/2$ and $\mu = 1$.

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3.2. Setup for contraction

We aim to find a nontrivial solution to (3.1) by employing a perturbation argument. Throughout this section, we assume that $1 < p < \infty$ if $n = 1, 2$, and that $1 < p < ((n+2)/(n-2))$ if $n \geqslant 3$.

Let $u_\infty \in H^1_r$ be a ground state to the non-relativistic Schrödinger equation

$$P_\infty(D)u = |u|^{p-1}u,$$

which is known to be positive and unique. Hoping to find a radially symmetric real-valued solution u_c to the pseudo-relativistic equation (3.1) close to the non-relativistic ground state u_∞, we write the equation for the difference

$$w := u_c - u_\infty : \mathbb{R}^n \to \mathbb{R},$$

that is,

$$\begin{aligned} P_c(D)w &= P_c(D)u_c - P_c(D)u_\infty \\ &= (P_\infty(D) - P_c(D))u_\infty + P_c(D)u_c - P_\infty(D)u_\infty \\ &= (P_\infty(D) - P_c(D))u_\infty + \Big\{|u_\infty + w|^{p-1}(u_\infty + w) - u_\infty^p\Big\}. \end{aligned}$$

Then, subtracting the linear component $pu_\infty^{p-1}w$ from both sides, we get

$$\mathcal{L}_{c;\infty} w = (P_\infty(D) - P_c(D))u_\infty + \mathcal{Q}(w),$$

where

$$\mathcal{L}_{c;\infty} := P_c(D) - pu_\infty^{p-1}$$

and

$$\mathcal{Q}(w) := |u_\infty + w|^{p-1}(u_\infty + w) - u_\infty^p - pu_\infty^{p-1}w. \tag{3.2}$$

Finally, using that the operator $\mathcal{L}_{c;\infty}$ is invertible (see proposition 3.1 below), which is the key ingredient in our analysis, we derive the equation

$$w = \mathcal{R}_c + (\mathcal{L}_{c;\infty})^{-1}\mathcal{Q}(w), \tag{3.3}$$

where

$$\mathcal{R}_c := (\mathcal{L}_{c;\infty})^{-1}(P_\infty(D) - P_c(D))u_\infty.$$

We now wish to construct a radially symmetric real-valued solution w for the equation (3.3) via the standard contraction mapping argument, assuming that $c \geqslant 1$ is large enough. Precisely, we aim to show that the nonlinear map

$$\Phi_c(w) := \mathcal{R}_c + (\mathcal{L}_{c;\infty})^{-1}\mathcal{Q}(w) \tag{3.4}$$

is contractive on a small ball in the Sobolev space $H^1_r \cap W^{1,q}$ of radially symmetric functions so that there is a unique solution u_c to (3.1) in a small neighbourhood of u_∞. It should be noted that the reformulated equation (3.3) is well-suited for our purpose. Indeed, the first term $\mathcal{R}_c$ is small for large c, because the ground state u_∞ is a regular function and the symbol $|\xi|^2+1-P_c(\xi)$ is asymptotically $O(((|\xi|^4)/(c^2)))$ as $c\to\infty$. Moreover, if w is small, then the super-linear nonlinear term $(\mathcal{L}_{c;\infty})^{-1}\mathcal{Q}(w)$ is even smaller. Therefore, it is natural to expect that Φ_c maps a small ball to itself, and it is contractive on the set. These will be justified rigorously in the next subsections.

3.3. Invertibility of $\mathcal{L}_{c;\infty}$

The following proposition asserts that the differential operator $\mathcal{L}_{c;\infty}$ is invertible, and moreover, its inverse gains one derivative.

PROPOSITION 3.1 Invertibility and smoothing property of $\mathcal{L}_{c;\infty}$. *Let $2\leqslant q<\infty$. Then, there exists $c_0>0$ such that if $c\geqslant c_0$, then $\mathcal{L}_{c;\infty}:H_r^1\cap W^{1,q}\to L_r^2\cap L^q$ is invertible. Moreover, its inverse is uniformly bounded, that is,*

$$\sup_{c\geqslant c_0}\|(\mathcal{L}_{c;\infty})^{-1}\|_{\mathcal{L}(L_r^2\cap L^q;H_r^1\cap W^{1,q})}<\infty,$$

where $\|\cdot\|_{\mathcal{L}(X;Y)}$ is the operator norm from the Banach space X to the Banach space Y.

The proof of the proposition heavily relies on the non-degeneracy of the linearized operator

$$\mathcal{L}_\infty:=-\Delta+1-pu_\infty^{p-1}=P_\infty(D)-pu_\infty^{p-1}:H^2\to L^2$$

about the non-relativistic ground state u_∞ for radially symmetric functions, that is,

$$\operatorname{Ker}(\mathcal{L}_\infty)\cap H_r^2=\{0\}. \tag{3.5}$$

In the first step, by the non-degeneracy, we show invertibility of the operator

$$\mathcal{A}:=\operatorname{Id}-pu_\infty^{p-1}P_\infty(D)^{-1}.$$

LEMMA 3.2. *For each $2\leqslant q<\infty$, the operator $\mathcal{A}:L_r^2\cap L^q\to L_r^2\cap L^q$ is invertible.*

Proof. We claim that $pu_\infty^{p-1}P_\infty(D)^{-1}$ is a compact operator on $L_r^2\cap L^q$. Indeed, compactness follows from the well-known localization property of the ground state u_∞ and the compact embedding $H^2(\Omega)\hookrightarrow L^2(\Omega)$ for any bounded set Ω.

If $v\in\operatorname{Ker}\mathcal{A}$, then $P_\infty(D)^{-1}v\in\operatorname{Ker}(\mathcal{L}_\infty)$. Hence, it follows from the non-degeneracy (3.5) that $P_\infty(D)^{-1}v=0$ and thus $v=0$. Therefore, by the Fredholm alternative, we conclude that $\mathcal{A}$ is invertible. □

By the invertibility of $\mathcal{A}$, we may write

$$\begin{aligned}\mathcal{L}_{c;\infty}&=\Big\{\operatorname{Id}-pu_\infty^{p-1}P_c(D)^{-1}\Big\}P_c(D)\\&=\Big\{\mathcal{A}+pu_\infty^{p-1}\big(P_\infty(D)^{-1}-P_c(D)^{-1}\big)\Big\}P_c(D)\\&=\Big\{\operatorname{Id}+pu_\infty^{p-1}\big(P_\infty(D)^{-1}-P_c(D)^{-1}\big)\mathcal{A}^{-1}\Big\}\mathcal{A}P_c(D).\end{aligned} \tag{3.6}$$

Thus, the following lemma implies invertibility of $\mathcal{L}_{c;\infty}$.

LEMMA 3.3. *Let $2 \leqslant q < \infty$. Suppose that $1 < p < \infty$ if $n = 1, 2$, and that $1 < p < ((n+2)/(n-2))$ if $n \geqslant 3$. Then, there exists $c_0 > 0$ such that for $c \geqslant c_0$,*

$$\|pu_\infty^{p-1}(P_\infty(D)^{-1} - P_c(D)^{-1})\mathcal{A}^{-1}\|_{\mathcal{L}(L_r^2 \cap L^q)} \leqslant \frac{1}{2},$$

where $\mathcal{L}(X) = \mathcal{L}(X; X)$.

Proof. By a trivial inequality, we write

$$\begin{aligned} &\|pu_\infty^{p-1}(P_\infty(D)^{-1} - P_c(D)^{-1})\mathcal{A}^{-1}\|_{\mathcal{L}(L_r^2 \cap L^q)} \\ &\leqslant p\|u_\infty^{p-1}\|_{\mathcal{L}(L_r^2 \cap L^q)}\|P_\infty(D)^{-1} - P_c(D)^{-1}\|_{\mathcal{L}(L_r^2 \cap L^q)}\|\mathcal{A}^{-1}\|_{\mathcal{L}(L_r^2 \cap L^q)}. \end{aligned} \tag{3.7}$$

By Hölder inequality, we have

$$\|u_\infty^{p-1}\|_{\mathcal{L}(L_r^2 \cap L^q)} = \|u_\infty^{p-1}\|_{L^\infty} = \|u_\infty\|_{L^\infty}^{p-1}.$$

By (2.2), $\|P_\infty(D)^{-1} - P_c(D)^{-1}\|_{\mathcal{L}(L_r^2 \cap L^q)} \leqslant ((C)/(c^2))$. Moreover, by lemma 3.2, $\|\mathcal{A}^{-1}\|_{\mathcal{L}(L_r^2 \cap L^q)} < \infty$. By inserting these estimates into (3.7), we prove the lemma for $c \geqslant c_0$ with a suitable choice $c_0 > 0$. □

Proof of proposition 3.1. By the expression (3.6) and the above lemmas, we can invert $\mathcal{L}_{c;\infty}$ for $c \geqslant c_0$,

$$\mathcal{L}_{c;\infty}^{-1} = P_c(D)^{-1}\mathcal{A}^{-1}\Big\{\mathrm{Id} + pu_\infty^{p-1}(P_\infty(D)^{-1} - P_c(D)^{-1})\mathcal{A}^{-1}\Big\}^{-1}.$$

Moreover, by the lower bound in theorem 1.3 and lemmas 3.2 and 3.3, we have the bound,

$$\begin{aligned} \|\mathcal{L}_{c;\infty}^{-1}\|_{\mathcal{L}(L_r^2 \cap L^q; H_r^1 \cap W^{1,q})} &\leqslant \|P_c(D)^{-1}\|_{\mathcal{L}(L_r^2 \cap L^q; H_r^1 \cap W^{1,q})}\|\mathcal{A}^{-1}\|_{\mathcal{L}(L_r^2 \cap L^q)} \\ &\quad \cdot \left\|\Big\{\mathrm{Id} + pu_\infty^{p-1}(P_\infty(D)^{-1} - P_c(D)^{-1})\mathcal{A}^{-1}\Big\}^{-1}\right\|_{\mathcal{L}(L_r^2 \cap L^q)} \\ &\leqslant C, \end{aligned}$$

where the implicit constant C is independent of the choice of $c \geqslant c_0$. □

3.4. First term bound in (3.4)

We now prove that we can make the term $\mathcal{R}_c = (\mathcal{L}_{c;\infty})^{-1}(P_\infty(D) - P_c(D))u_\infty$ arbitrarily small choosing large $c \geqslant 1$.

LEMMA 3.4 First term bound. *Let $2 \leqslant q < \infty$. Then, we have*

$$\|\mathcal{R}_c\|_{H_r^1 \cap W^{1,q}} = \begin{cases} O\left(\dfrac{1}{c}\right) & \text{if } 1 < p \leqslant 2, \\ O\left(\dfrac{1}{c^2}\right) & \text{if } p > 2. \end{cases}$$

Proof. We recall that the non-relativistic ground state u_∞ is contained in $H_r^{2+\lfloor p\rfloor} \cap W_r^{2+\lfloor p\rfloor,q}$ for all q, where $\lfloor p\rfloor$ is the largest integer less than or equal to p. Thus, if $p > 2$, then by proposition 3.1, (2.2) and the upper bound in theorem 1.3, we get

$$\begin{aligned}\|\mathcal{R}_c\|_{H_r^1\cap W^{1,q}} &\leqslant \|(\mathcal{L}_{c;\infty})^{-1}\|_{\mathcal{L}(L_r^2\cap L^q;H_r^1\cap W^{1,q})} \left\|\frac{P_\infty(D)-P_c(D)}{P_\infty(D)P_c(D)}\right\|_{\mathcal{L}(L_r^2\cap L^q)} \\ &\quad \|P_\infty(D)P_c(D)u_\infty\|_{L_r^2\cap L^q} \\ &\leqslant \frac{C}{c^2}\|P_\infty(D)u_\infty\|_{H_r^2\cap W_r^{2,q}} \leqslant \frac{C}{c^2}\|u_\infty\|_{H_r^4\cap W_r^{4,q}}.\end{aligned}$$

Similarly, if $1 < p < 2$, then by proposition 3.1, (2.3) and the upper bound in theorem 1.3, we obtain

$$\begin{aligned}\|\mathcal{R}_c\|_{H_r^1\cap W^{1,q}} &\leqslant \|(\mathcal{L}_{c;\infty})^{-1}\|_{\mathcal{L}(L_r^2\cap L^q;H_r^1\cap W^{1,q})} \left\|\frac{P_\infty(D)-P_c(D)}{P_\infty(D)^{1/2}P_c(D)}\right\|_{\mathcal{L}(L_r^2\cap L^q)} \\ &\quad \|P_\infty(D)^{1/2}P_c(D)u_\infty\|_{L_r^2\cap L^q} \\ &\leqslant \frac{C}{c}\|P_\infty(D)^{1/2}u_\infty\|_{H_r^2\cap W_r^{2,q}} \leqslant \frac{C}{c}\|u_\infty\|_{H_r^3\cap W_r^{3,q}}.\end{aligned}$$

□

3.5. Nonlinear term bounds in (3.4)

Next, we establish the estimates for the nonlinear term $\mathcal{L}_{c;\infty}^{-1}\mathcal{Q}(w)$ for small w.

PROPOSITION 3.5 Nonlinear estimates. *Fix any $q > n$ and suppose that $0 < \delta \leqslant \|u_\infty\|_{H^1}$. Then for $c \geqslant c_0$, where $c_0 \geqslant 1$ is a large number from proposition 3.1, we have*

$$\|\mathcal{L}_{c;\infty}^{-1}\mathcal{Q}(w)\|_{H_r^1\cap W^{1,q}} \leqslant C\delta^{\min\{p,2\}}, \tag{3.8}$$

$$\|\mathcal{L}_{c;\infty}^{-1}\mathcal{Q}(w) - \mathcal{L}_{c;\infty}^{-1}\mathcal{Q}(\tilde{w})\|_{H_r^1\cap W^{1,q}} \leqslant C\delta^{\min\{p-1,1\}}\|w-\tilde{w}\|_{H_r^1\cap W^{1,q}} \tag{3.9}$$

for any $w, \tilde{w} \in H_r^1 \cap W^{1,q}$ with $\|w\|_{H_r^1\cap W^{1,q}}, \|\tilde{w}\|_{H_r^1\cap W^{1,q}} \leqslant \delta$.

Proof. It suffices to show the second inequality (3.9) in the proposition since the former inequality follows from the latter with $\tilde{w} = 0$.

By the definition (3.2) and the fundamental theorem of calculus, we write

$$\begin{aligned}\mathcal{Q}(w) - \mathcal{Q}(\tilde{w}) &= \Big\{|u_\infty + w|^{p-1}(u_\infty + w) - |u_\infty + \tilde{w}|^{p-1}(u_\infty + \tilde{w})\Big\} \\ &\quad - pu_\infty^{p-1}(w - \tilde{w}) \\ &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\Big[|u_\infty + (1-t)\tilde{w} + tw|^{p-1}(u_\infty + (1-t)\tilde{w} + tw)\Big]\mathrm{d}t \\ &\quad - pu_\infty^{p-1}(w - \tilde{w}) \\ &= p\int_0^1 \big(|u_\infty + (1-t)\tilde{w} + tw|^{p-1} - u_\infty^{p-1}\big)(w - \tilde{w})\mathrm{d}t.\end{aligned}$$

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Suppose that $1 < p \leqslant 2$. Then, by the elementary inequality

$$\big||a|^\ell - |b|^\ell\big| \leqslant \big||a| - |b|\big|^\ell \leqslant |a-b|^\ell \quad \text{if } 0 < \ell < 1,$$

we have

$$|\mathcal{Q}(w) - \mathcal{Q}(\tilde{w})| \leqslant C(|w| + |\tilde{w}|)^{p-1}|w - \tilde{w}|.$$

Thus, by proposition 3.1 and the Hölder inequality and the Sobolev inequality $H_r^1 \cap W^{1,q} \hookrightarrow L_r^2 \cap L_r^\infty$, we prove that

$$\begin{aligned} \|\mathcal{L}_{c;\infty}^{-1}\mathcal{Q}(w) - \mathcal{L}_{c;\infty}^{-1}\mathcal{Q}(\tilde{w})\|_{H_r^1\cap W^{1,q}} &\leqslant C\|\mathcal{Q}(w) - \mathcal{Q}(\tilde{w})\|_{L_r^2\cap L_r^q} \\ &\leqslant C\big(\|w\|_{H_r^1\cap W^{1,q}} + \|\tilde{w}\|_{H_r^1\cap W^{1,q}}\big)^{p-1} \\ &\quad \|w - \tilde{w}\|_{H_r^1\cap W^{1,q}}. \end{aligned} \tag{3.10}$$

If $p > 2$, using the fundamental theorem of calculus again, we find

$$|\mathcal{Q}(w) - \mathcal{Q}(\tilde{w})| \leqslant C(u_\infty + |w| + |\tilde{w}|)^{p-2}(|w| + |\tilde{w}|)|w - \tilde{w}|.$$

From this and estimating as above, we prove that

$$\begin{aligned} &\|\mathcal{L}_{c;\infty}^{-1}\mathcal{Q}(w) - \mathcal{L}_{c;\infty}^{-1}\mathcal{Q}(\tilde{w})\|_{H_r^1\cap W^{1,q}} \leqslant C\|\mathcal{Q}(w) - \mathcal{Q}(\tilde{w})\|_{L_r^2\cap L_r^q} \\ &\leqslant C\|(u_\infty + |w| + |\tilde{w}|)^{p-2}(|w| + |\tilde{w}|)|w - \tilde{w}|\|_{L_r^2\cap L_r^q} \\ &\leqslant C\big(\|u_\infty\|_{H_r^1\cap W^{1,q}} + \|w\|_{H_r^1\cap W^{1,q}} + \|\tilde{w}\|_{H_r^1\cap W^{1,q}}\big)^{p-2} \\ &\quad \cdot \big(\|w\|_{H_r^1\cap W^{1,q}} + \|\tilde{w}\|_{H_r^1\cap W^{1,q}}\big)\|w - \tilde{w}\|_{H_r^1\cap W^{1,q}}. \end{aligned} \tag{3.11}$$

Thus, the proposition is proved. □

REMARK 3.6. As mention in the introduction, if we only use the L^2-boundedness in theorem 2.1 and 1.3 based on the point-wise bounds on the symbols, then we can close the estimates (3.10) and (3.11) with $q = 2$ only when $1 < p \leqslant ((n)/(n-2))$. The symbolic analysis in § 2 allows us to employ the full range of the $W_r^{1,q}$-Sobolev norms in (3.10) and (3.11), thus we can cover the full range of p, that is, $1 < p < ((n+2)/(n-2))$, such that the ground state u_∞ exists.

3.6. Construction of a solution to (3.3)

Now we are ready to construct a solution to the equation (3.3) near the ground state u_∞.

PROPOSITION 3.7 Existence of a fixed point for Φ_c. *Fix any $q > n$. Then, given a sufficiently small $\delta > 0$, there exists $c_0 > 0$ such that if $c \geqslant c_0$, then Φ_c has a unique fixed point in*

$$B_\delta^1 := \left\{w \in H_r^1 \cap W^{1,q} : \|w\|_{H_r^1\cap W^{1,q}} \leqslant \delta\right\}.$$

Proof. First, by lemma 3.4, we choose large $c_0 \geqslant 1$ such that $\|\mathcal{R}_c\|_{H_r^1\cap W^{1,q}} \leqslant \delta/2$ for all $c \geqslant c_0$. Hence, it follows from proposition 3.5 that if $w, \tilde{w} \in$

B^1_δ, then $\|\Phi_c(w)\|_{H^1_r\cap W^{1,q}} \leqslant \delta/2 + C\delta^{\min\{p,2\}} \leqslant \delta$ and $\|\Phi_c(w)-\Phi_c(\tilde w)\|_{H^1_r\cap W^{1,q}} \leqslant C\delta^{\min\{p-1,1\}}\|w-\tilde w\|_{H^1_r\cap W^{1,q}} \leqslant 1/2\|w-\tilde w\|_{H^1_r\cap W^{1,q}}$, provided that $C^{\min\{p-1,1\}}\delta \leqslant 1/2$. Thus, we conclude that Φ_c has a unique fixed point in B^1_δ. □

3.7. Construction of a solution u_c to (3.1)

We prove that $u_c = u_\infty + w$, where w is given in proposition 3.7, is indeed a solution to the pseudo-relativistic Schrödinger equation (3.1).

LEMMA 3.8 Construction of a solution to (3.1). *w is a solution to the equation (3.3), that is, $w = \Phi_c(w)$ in $H^1_r \cap W^{1,q}$ for some $q > n$ if and only if $u_c = u_\infty + w$ solves (3.1).*

Proof. It is proved in proposition 3.7 that $w = \mathcal{R}_c + (\mathcal{L}_{c;\infty})^{-1}\mathcal{Q}(w)$ in $H^1_r \cap W^{1,q}$, and so

$$\begin{aligned}
0 &= \mathcal{L}_{c;\infty} w - \mathcal{L}_{c;\infty}\mathcal{R}_c - \mathcal{Q}(w)\\
&= (P_c(D) - pu_\infty^{p-1})w - (P_\infty(D) - P_c(D))u_\infty - \left(|u_c|^{p-1}u_c - u_\infty^p - pu_\infty^{p-1}w\right)\\
&= P_c(D)w - pu_\infty^{p-1}w - P_\infty(D)u_\infty + P_c(D)u_\infty - |u_c|^{p-1}u_c + u_\infty^p + pu_\infty^{p-1}w\\
&= P_c(D)u_c - |u_c|^{p-1}u_c.
\end{aligned}$$

Thus u_c is a solution to (3.1). □

4. Non-existence result

We prove our non-existence theorem (theorem 1.1). By scaling (see § 3.1), we may take $m = 1/2$ and $\mu = 1$. It suffices to show non-existence assuming that $1 \geqslant ((c^2)/(2))$ and $p \geqslant ((n+1)/(n-1))$, or that $1 < ((c^2)/(2))$ and $p \geqslant ((n+2)/(n-2))$. We employ the standard approach to prove non-existence, involving Pohozaev identities, but we perform it on the extended upper half-plane as in [**2**, **8**]. The extension method is a very convenient tool to detour technical difficulties from the non-locality and the lack of scaling of the pseudo-relativistic operator. A direct proof of non-existence would be an interesting mathematical question, but we are not aware of it at this moment.

Let $u_c \in H^{1/2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ a solution to the pseudo-relativistic NLS (1.2), which will be shown to be zero in the end. Then, it has a unique extension $U(x,t) \in H^1(\mathbb{R}^{n+1}_+)$ to the upper half-plane $\mathbb{R}^{n+1}_+ = \{(x,t) : x \in \mathbb{R}^n \text{ and } t > 0\}$ such that

$$\begin{cases} \left(-c^2\Delta_{(x,t)} + \frac{c^4}{4}\right)U(x,t) = 0 & \text{in}\mathbb{R}^{n+1}_+,\\ U(x,0) = u_c(x) & \text{in}\mathbb{R}^n \end{cases} \tag{4.1}$$

and

$$-c\frac{\partial}{\partial t}U(x,0) = \sqrt{-c^2\Delta_x + \frac{c^4}{4}}\,u_c(x)$$

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in distribution sense, which immediately implies that

$$-c\frac{\partial}{\partial t}U(x,0) = \left(\frac{c^2}{2}-1\right)U(x,0) + |U|^{p-1}U(x,0)$$

because $u_c(x) = U(x,0)$ solves (1.2) (see [**2**, **8**]). Since $u_c \in L^\infty(\mathbb{R}^n)$, by the maximum principle, we have $U \in L^\infty(\mathbb{R}^{n+1}_+)$. Then, it follows from the standard elliptic regularity estimates that $U \in C^\alpha(\overline{\mathbb{R}^{n+1}_+})$ for some $\alpha > 0$. In particular, U is continuous up to the boundary $\partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n$. Moreover, the extension U satisfies the Pohozaev-type identities.

LEMMA 4.1. *Let $U \in H^{1/2}(\mathbb{R}^{n+1}_+)$ be a solution to (4.1). Then we have the following identities.*

$$\begin{aligned}&\int_{\mathbb{R}^{n+1}_+} c^2|\nabla U(x,t)|^2 \mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^{n+1}_+} \frac{c^4}{4}|U(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t \\ &= \left(\frac{c^2}{2}-1\right)\int_{\mathbb{R}^n} c|U(x,0)|^2\,\mathrm{d}x + \int_{\mathbb{R}^n} c|U(x,0)|^{p+1}\,\mathrm{d}x,\end{aligned} \tag{4.2}$$

$$\begin{aligned}&\frac{n-1}{2}\int_{\mathbb{R}^{n+1}_+} c^2|\nabla U(x,t)|^2 \mathrm{d}x\,\mathrm{d}t + \frac{n+1}{2}\int_{\mathbb{R}^{n+1}_+} \frac{c^4}{4}|U(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t \\ &= \left(\frac{c^2}{2}-1\right)\frac{n}{2}\int_{\mathbb{R}^n} c|U(x,0)|^2\,\mathrm{d}x + \frac{n}{p+1}\int_{\mathbb{R}^n} c|U(x,0)|^{p+1}\,\mathrm{d}x,\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}&\frac{n-2}{2}\int_{\mathbb{R}^{n+1}_+} c^2|\nabla_x U(x,t)|^2 \mathrm{d}x\,\mathrm{d}t + \frac{n}{2}\int_{\mathbb{R}^{n+1}_+} c^2|\partial_t U(x,t)|^2 \mathrm{d}x\,\mathrm{d}t \\ &\quad + \frac{n}{2}\int_{\mathbb{R}^{n+1}_+} \frac{c^4}{4}|U(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t \\ &= \left(\frac{c^2}{2}-1\right)\frac{n}{2}\int_{\mathbb{R}^n} c|U(x,0)|^2\,\mathrm{d}x + \frac{n}{p+1}\int_{\mathbb{R}^n} c|U(x,0)|^{p+1}\,\mathrm{d}x.\end{aligned} \tag{4.4}$$

Proof. We multiply (4.1) by the three test functions $U(x,t)$, $(x,t)\cdot\nabla U(x,t)$ and $x\cdot\nabla_x U(x,t)$, and integrate on the $(n+1)$-dimensional upper half ball $B^{n+1}_+(0,R)$ of radius $R > 0$ and centred at 0. Then, by integration by parts, we get the above three integral identities, after taking the limit under a suitable choice of the sequence $R_j \to \infty$. We omit the details of this procedure, because it is quite standard in the literature. Here, we note that continuity of U is required to guarantee that the boundary integral terms, which appear whenever we do integration by parts, are well defined. □

We also recall the following trace inequality.

LEMMA 4.2. *For every $U \in H^1(\mathbb{R}^{n+1}_+)$, we have*

$$\int_{\mathbb{R}^n} |U(x,0)|^2 \, \mathrm{d}x \leqslant 2\|U\|_{L^2(\mathbb{R}^{n+1}_+)} \|\partial_t U\|_{L^2(\mathbb{R}^{n+1}_+)}.$$

Proof. From the density argument, we may assume that $U \in C_c^\infty(\overline{\mathbb{R}^{n+1}_+})$. Observe

$$|U(x,0)|^2 = -\int_0^\infty \partial_t \left(|U(x,t)|^2\right) \, \mathrm{d}t \leqslant 2\int_0^\infty |U(x,t)||\partial_t U(x,t)| \, \mathrm{d}t.$$

Then one has from the Hölder inequality

$$\int_{\mathbb{R}^n} |U(x,0)|^2 \, \mathrm{d}x \leqslant 2\int_{\mathbb{R}^n}\int_0^\infty |U(x,t)||\partial_t U(x,t)| \, \mathrm{d}t \, \mathrm{d}x \leqslant 2\|U\|_{L^2(\mathbb{R}^{n+1}_+)} \|\partial_t U\|_{L^2(\mathbb{R}^{n+1}_+)}.$$

□

Using the Pohozaev-type identities and the trace inequality, we prove non-existence.

Proof of theorem 1.1. Suppose that $1 \geqslant ((c^2)/(2))$ and $p \geqslant ((n+1)/(n-1))$. Then, combining (4.2) and (4.3) to cancel out the last term, we obtain that

$$\begin{aligned} &\left(\frac{n-1}{2} - \frac{n}{p+1}\right)\int_{\mathbb{R}^{n+1}_+} c^2|\nabla U(x,t)|^2 \mathrm{d}x \, \mathrm{d}t \\ &\quad + \left(\frac{n+1}{2} - \frac{n}{p+1}\right)\int_{\mathbb{R}^{n+1}_+} \frac{c^4}{4}|U(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t \\ &= \left(\frac{c^2}{2} - 1\right)\left(\frac{n}{2} - \frac{n}{p+1}\right)\int_{\mathbb{R}^n} c|U(x,0)|^2 \, \mathrm{d}x. \end{aligned} \tag{4.5}$$

Thus, it follows that

$$\int_{\mathbb{R}^{n+1}_+} |U(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t = 0,$$

because $((n-1)/(2)) - ((n)/(p+1)) \geqslant 0$, $((n+1)/(2)) - ((n)/(p+1)) > 0$ and $(((c^2)/(2)) - 1)(((n)/(2)) - ((n)/(p+1))) \leqslant 0$. Consequently, $u_c(x) = U(x,0)$ is identically zero by the continuity of U.

Now we assume that $1 < ((c^2)/(2))$ and $p \geqslant ((n+2)/(n-2))$. Then, combining (4.2) and (4.4), we obtain that

$$\begin{aligned} &\left(\frac{n-2}{2} - \frac{n}{p+1}\right)\int_{\mathbb{R}^{n+1}_+} c^2|\nabla_x U(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t \\ &\quad + \left(\frac{n}{2} - \frac{n}{p+1}\right)\int_{\mathbb{R}^{n+1}_+} c^2|\partial_t U(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t \\ &\quad + \left(\frac{n}{2} - \frac{n}{p+1}\right)\int_{\mathbb{R}^{n+1}_+} \frac{c^4}{4}|U(x,t)|^2 \, \mathrm{d}x\mathrm{d}t \\ &= \left(\frac{c^2}{2} - 1\right)\left(\frac{n}{2} - \frac{n}{p+1}\right)\int_{\mathbb{R}^n} c|U(x,0)|^2 \, \mathrm{d}x. \end{aligned} \tag{4.6}$$

We see from lemma 4.2 and Young's inequality that

$$\left(\frac{c^2}{2}-1\right)\int_{\mathbb{R}^n} c|U(x,0)|^2\,\mathrm{d}x \leqslant 2\left(\frac{c^2}{2}-1\right)\|U\|_{L^2(\mathbb{R}^{n+1}_+)}c\|\partial_t U\|_{L^2(\mathbb{R}^{n+1}_+)}$$
$$\leqslant \left(\frac{c^2}{2}-1\right)^2\int_{\mathbb{R}^{n+1}_+}|U(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^{n+1}_+} c^2|\partial_t U(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t.$$

Inserting this to (4.6),

$$\left(\frac{n-2}{2}-\frac{n}{p+1}\right)\int_{\mathbb{R}^{n+1}_+} c^2|\nabla_x U(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+\left(\frac{n}{2}-\frac{n}{p+1}\right)\int_{\mathbb{R}^{n+1}_+}\frac{c^4}{4}|U(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant \left(\frac{n}{2}-\frac{n}{p+1}\right)\left(\frac{c^2}{2}-1\right)^2\int_{\mathbb{R}^{n+1}_+}|U(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t.$$

From this, by the assumption, we finally deduce

$$(c^2-1)\int_{\mathbb{R}^{n+1}_+}|U(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t = 0.$$

This again implies that $u_c(x)$ is identically zero. □

5. Concluding remarks

In this section, we present some properties of the solution u_c to the pseudo-relativistic NLS (1.2), constructed in § 3. Throughout this section, we assume that

$$\begin{cases} 1 < p < \infty & \text{if } n = 1, 2, \\ 1 < p < \frac{n+2}{n-2} & \text{if } n \geqslant 3, \end{cases}$$

and that $mc^2 \geqslant \kappa_0\mu$, where $\kappa_0 = c_0^2$ and $c_0 \geqslant 1$ is a large constant given in in § 3.

First, we prove uniqueness of a solution to (1.2) among radially symmetric functions near the ground state u_∞ to the non-relativistic NLS (1.3).

Proposition 5.1. *There exists some $\delta > 0$ such that a solution to (1.2) is unique in*

$$B_\delta(u_\infty) := \{u \in H^1_r \cap L^\infty :\ \|u-u_\infty\|_{H^1\cap L^\infty} < \delta\}.$$

Proof. Let $u_c \in H^1_r \cap L^\infty$ be a solution to (1.2). Then by the argument in lemma 3.8, $w := u_c - u_\infty$ is a fixed point of the map Φ_c. Then w is a unique fixed point in a small ball $B_\delta(0)$ because Φ_c is a contraction map. Note that two norms $\|\cdot\|_{H^1\cap L^\infty}$ and $\|\cdot\|_{H^1\cap W^{1,q}}$ are equivalent since we assume $q > n$. This shows the uniqueness of u_c in a small ball $B_\delta(u_\infty)$. □

We also obtain the rate of convergence for the non-relativistic limit $u_c \to u_\infty$.

PROPOSITION 5.2 Rate of convergence. *Let u_c be the solution to* (1.2) *constructed in § 3. Then, for any $q > n$, we have*

$$\|u_c - u_\infty\|_{H^1\cap W^{1,q}} = \begin{cases} O\left(\dfrac{1}{c}\right) & \text{if } 1 < p \leqslant 2, \\ O\left(\dfrac{1}{c^2}\right) & \text{if } p > 2. \end{cases}$$

Proof. By lemma 3.4, we may choose $\delta = ((A)/(c^a))$ such that $\|\mathcal{R}_c\|_{H^1_r\cap W^{1,q}} \leqslant \delta$, where $A > 0$ is some large number and $a = 1$ or 2 depending on the rate in lemma 3.4. Then, repeating the proof of proposition 3.7, one can show that Φ_c is contractive on the $((A)/(c^a))$-ball for sufficiently large c. Let $\tilde{w}$ be the fixed point in the $((A)/(c^a))$-ball. Then, by uniqueness, the solution $\tilde{w}$ equals to the solution $w = u_c - u_\infty$ in proposition 3.7. Therefore, we conclude that the difference $u_c - u_\infty = \tilde{w}$ is in the ball of radius $((A)/(c^a))$. □

Combining the above two propositions, we conclude that the solution u_c, in § 3, is the only radially symmetric real-valued solution to the pseudo-relativistic NLS (1.2) converging to the non-relativistic ground state u_∞.

COROLLARY 5.3. *Let $\{u_c\}$ be a sequence of solutions to* (1.2) *in $H^1_r \cap L^\infty$ such that it converges to u_∞ in $H^1_r \cap L^\infty$ as $c \to \infty$. Then for sufficiently large $c \geqslant 1$, u_c is unique, and it converges with the rate given in proposition 5.2.*

In [**2**, **3**], the authors prove that in the $H^{1/2}$-subcritical range $1 < p < ((n+1)/(n-1))$, a positive radial ground state to (1.2) belongs to $H^1 \cap L^\infty$ and converges to u_∞ so, by the uniqueness, our solution u_c, in this case, is the same as the ground state to (1.2) for large c. By a *ground state*, we mean a solution to (1.2) which attains the minimum value of an associated functional I_c among all nontrivial solutions, where

$$I_c(u) = \frac{1}{2}\int_{\mathbb{R}^n} \left(\sqrt{-c^2\Delta + m^2c^4} - mc^2\right) u\overline{u} + \mu|u|^2 \,\mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^n} |u|^{p+1}\,\mathrm{d}x \quad (5.1)$$

Obviously, u solves (1.2) if and only if it is a critical point of the functional I_c, because (1.2) is its Euler-Lagrange equation. Thus, a ground state u_c can be rephrased as a critical point of the functional I_c that minimizes the value of I_c among all nontrivial critical points, that is,

$$I_c(u_c) = \min_{v\in H^{1/2}} \Big\{ I_c(v) \mid v \neq 0,\, I_c'(v) = 0 \Big\}. \quad (5.2)$$

Thus, Corollary 5.3 gives an alternative proof of uniqueness of a radially symmetric non-negative ground state to the $H^{1/2}$-subcritical pseudo-relativistic NLS (1.2).

COROLLARY 5.4. *If $1 < p < ((n+1)/(n-1))$, then a positive radial ground state to* (1.2) *is unique for sufficiently large c.*

We finally remark that a ground state, in the sense of (5.2), is well-defined only when the nonlinearity is $H^{1/2}$-subcritical or critical, that is, $1 < p \leqslant ((n+1)/(n-1))$, since by the Sobolev embedding $H^{1/2} \hookrightarrow L^{p+1}$, the functional I_c is well-defined and continuously differentiable on $H^{1/2}$ in this case. However, if we define a ground state (in a weak sense) as a minimizer of the action functional I_c among all nontrivial solutions $v \in H^1 \cap L^\infty$ to (1.2), then the meaning of a ground state may make sense even in the $H^{1/2}$-supercritical case $((n+1)/(n-1)) < p < ((n+2)/(n-2))$. We strongly speculate that our solution u_c, constructed in theorem 1.2, would be a ground state in this sense. This seems to be an interesting open question worth to be answered in future work.

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