Flows of measures generated by vector fields

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The scope of the paper is twofold. We show that for a large class of measurable vector fields in the sense of Weaver (i.e. derivations over the algebra of Lipschitz functions), called in the paper laminated, the notion of integral curves may be naturally defined and characterized (when appropriate) by an ordinary differential equation. We further show that for such vector fields the notion of a flow of the given positive Borel measure similar to the classical one generated by a smooth vector field (in a space with smooth structure) may be defined in a reasonable way, so that the measure 'flows along' the appropriately understood integral curves of the given vector field and the classical continuity equation is satisfied in the weak sense.

1. Introduction

Every sufficiently smooth and bounded vector field V in the Euclidean space $E = \mathbb{R}^n$ is well known to generate a canonical flow of a given finite Borel measure μ in E by setting $\mu_t := \varphi_{V\#}^t \mu$, $t \in \mathbb{R}^+$, where $\varphi_V^t(x) := \theta(t)$, θ standing for the unique solution to the differential equation

$$\dot{\theta}(t) = V(\theta(t)) \tag{1.1}$$

satisfying the initial condition $\theta(0) = x$. Such a flow satisfies the continuity equation

$$\frac{\partial \mu_t}{\partial t} + \operatorname{div} v_t \mu_t = 0 \tag{1.2}$$

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in the weak sense in $E \times \mathbb{R}^+$, where $v_t \colon E \to E$ is some velocity field (non-unique, of course, for the given V, but, in particular, in this case this equation is satisfied with $v_t := V$). The goal of this paper is to try to explain to what extent the above assertions can be extended to the case in which the vector field V can be non-smooth, and even strongly discontinuous, for instance when it is just measurable. Namely, we are interested in establishing whether one can say that a measurable vector field V defines a reasonable flow of a given measure along the *integral curves* of V while satisfying the continuity equation (1.2). Of course, the notion of an integral curve of V must be now understood in a weaker sense, namely, as absolutely continuous Carathèodory solutions of (1.1) (i.e. the latter equations must be satisfied almost everywhere in $\mathbb{R}^+$). The following simple example shows, however, that we might not expect this to be possible for all measurable vector fields, because in fact some of them may admit no non-constant integral curves.

EXAMPLE 1.1. Let $K \subset [0,1]$ be a Cantor set of positive Lebesgue measure (or just any closed totally disconnected set of positive Lebesgue measure) and let $V(x) := \mathbf{1}_K(x)$. Then there is no absolutely continuous solution to (1.1) different from the constant curves $\theta(t) \equiv x \notin K$.

Looking at the above example from another point of view, one might guess that in fact to overcome the possible difficulty with the pointwise definition of the integral curve it would be helpful to have the right 'relaxation' of the notion of a smooth vector field. The idea of the appropriate weak concept comes from differential geometry. In fact, a classical smooth vector field can be viewed as a linear operator over the algebra of smooth functions over E satisfying the Leibniz rule, or, in other words, the derivation (in this way one customarily identifies the vector field, i.e. a map $V \colon E \to E$ with the directional derivative along the latter). The correct weak analogue of this notion can be obtained by substituting smooth functions with appropriately less smooth ones. Namely, we will view the measurable vector field as a linear operator that maps Lipschitz functions into measurable ones (with respect to some measure) and satisfies the Leibniz rule and some extra continuity assumption. This notion was first introduced in [22] and adapted to metric measure spaces in [16] (for a detailed treatment of the subject as well as recent results related to it, see also [12,13]). It is well known that such vector fields can in fact be identified with the one-dimensional metric currents introduced in [3]. It is the main goal of this paper to show that for a large class of such measurable vector fields the notion of integral curves still makes sense, and the flow of a given measure along the respective integral curves can be defined in a reasonable way, and, in particular, it satisfies the continuity equation (1.2) with some natural velocity field v_t related to V.

Measurable vector fields and integral curves

The measurable vector fields (or, equivalently, one-dimensional metric currents) admitting the natural notion of 'integral curves', called in this paper *laminated*, are those that can be represented as integrals of vector fields associated with single absolutely continuous curves in some possibly σ-finite measure η over the space of curves $C(\mathbb{R}^+; E)$. As shown in [19, 20], this class includes *normal* vector fields, i.e. those

for which the weak divergence is a signed Radon measure (in the Euclidean space this is a consequence of representation results for normal de Rham one-dimensional currents due to Smirnov [21]), in which case there is a finite measure η representing such a vector field. However, the class of laminated vector fields is strictly larger, although it does not include all measurable vector fields (in particular, the one from example 1.1; see remark A.12). Note that in view of proposition A.4 a laminated vector field must have rather particular structure: namely (at least in a Euclidean space), it corresponds to a vector measure with the field of directions tangential to its mass measure in the sense of Bouchittè *et al.* [9], which is not the case for generic measurable vector fields by remark A.5. Our first main result is theorem 5.1, which says in particular that in a Banach space E with the Radon–Nikodym property (for example, in a finite-dimensional space), the following hold.

- Every laminated vector field X has a pointwise representation as a pair consisting of a 'total variation' measure m_X (or, equivalently, the mass measure of the respective metric current) and a Borel 'field of directions', i.e. a Borel function $V\colon E \to E$ with $\|V(y)\| = 1$ for m_X-almost every (a.e.) $y \in E$ ($\|\cdot\|$ standing for the norm in E) such that

$$(X\pi)(y) = \langle V(y), \mathrm{d}\pi(y)\rangle \quad \text{for } m_X\text{-a.e. } y \in E,$$

for every $\pi \in \mathrm{Lip}(E)$ a quasi-differentiable function with bounded derivative $\mathrm{d}\pi$ satisfying some rather weak continuity property (the respective class of functions being denoted by $Q^1(E)$), $\langle\cdot,\cdot\rangle$ standing for the usual duality pairing between E and its continuous dual; the map V is unique up to m_X-almost everywhere equality.

- An η-a.e. curve $\theta \in C(\mathbb{R}^+; E)$ parametrized by arc length is an absolutely continuous integral curve of X in the sense that when E is strictly convex, θ is a Carathéodory solution of (1.1) with V the above 'field of directions' of X and $t \in (0, \ell(\theta))$, where $\ell(\theta)$ stands for the length of θ; if E is not strictly convex, then (1.1) has to be substituted by a differential inclusion.

Transportable measures and their flows

Let us describe in very rough and heuristic terms the notion of the flow of a given measure μ by a given laminated vector field X that we propose in this paper. The flow is seen as a movement of an ensemble of particles, each one moving along some integral curve of X. The movement of this ensemble is assumed to be given by a measure σ over $C(\mathbb{R}^+; E) \times \mathbb{R}^+$ so that, intuitively, $\sigma(\theta, s)$ stands for the number of particles that move along the curve $\theta \in C(\mathbb{R}^+; E)$ at time $s \in \mathbb{R}^+$. Suppose that we set our timer at $t := 0$ and observe the distribution of the particles. Each particle at this instant is moving along some curve $\theta \in C(\mathbb{R}^+; E)$ for some time $s \in \mathbb{R}^+$ and hence is observed at the position $\beta_0(\theta, s) := \theta(s) \in E$, so that we observe the distribution of particles given by the measure $\mu := \beta_{0\#}\sigma$. After a time $t \in \mathbb{R}^+$ has elapsed (i.e. when our timer shows t), we observe the measure $\mu_t := \beta_{t\#}\sigma$, where $\beta_t(\theta, s) := \theta(s + t) \in E$. The family $\{\mu_t\}_{t\in\mathbb{R}^+}$ may be viewed thus as a flow of the measure $\mu = \mu_0$. In order that each particle in the flow of the measure μ be flowing along an integral curve of X, one has to request that the projection

of the measure σ to the space of curves $C(\mathbb{R}^+; E)$ be absolutely continuous with respect to η. It is worth remarking that our notion of the measure σ producing the flow is very similar to that of a *randomized stopping time* introduced in [18] and recently successfully employed in [6]. A curious feature of this definition of the flow is its sensibility to equivalent choices of distances in E, which is due to the same sensibility of the notion of 'integral curves' to a measurable vector field; in fact, even for a finite-dimensional normed space, different choices of the norm produce different representations of the same measurable vector field and hence different flows.

Clearly, there is a very strong non-uniqueness of the constructed flow due to both non-uniqueness of a measure η representing the given vector field X and also that of the measure σ producing the flow. Part (but not all) of this non-uniqueness may be avoided by providing natural constructions of flows for particular subclasses of vector fields. We do this in § 7 for so-called *cyclic*, or divergence-free, and *acyclic* vector fields separately (it is worth keeping in mind that every vector field can be represented, though again not uniquely, as a sum of a cyclic one and an acyclic one).

More importantly, it is not guaranteed that for the given μ and X (and η representing X) the flow (i.e. the measure σ producing the latter) in fact exists. When it exists, we say that μ is *transportable* by X. One might anyhow easily assume that each μ is transportable by every laminated vector field X by admitting possibly very unnatural flows (in particular, such that every measure be not moving at all, that is, $\mu_t = \mu$ for all $t \in \mathbb{R}^+$). In order to avoid this we must not consider the measures η over curves that charge constant curves (those concentrated over singletons). This produces physically reasonable flows but the transportability of all measures is sacrificed. In particular, for the measure μ to be transportable by X it has to be concentrated over the support of X. We show that, given a laminated vector field X, a lot of measures are still transportable by X, in particular, the negative part of its weak divergence (if the latter is a finite measure) and its mass measure; and, moreover, every $\nu \ll \mu$ is transportable by X if μ is transportable as well (see proposition 4.2).

Continuity equations

The principal result of this paper regarding flows of transportable measures is theorem 6.1, which shows in particular that in a Banach space with the Radon–Nikodym property (for example, in a finite-dimensional normed space) every flow μ_t corresponding to X satisfies the continuity equation (1.2) in the appropriately weak sense. Moreover, if the norm in E is strictly convex, then the velocity field v_t is collinear to V, the Borel map pointwise representing X (in fact, $v_t = V(1 - \varphi_t)$, where $\varphi_t \colon E \to [0,1]$ is some bounded function that may be non-zero due to the fact that some mass is stopped during the movement).

Note that the results established in this paper may be considered complementary to superposition principles obtained in [5, §§ 7 and 8] and in [2, theorem 12], although they look in the opposite direction: there one considers families of measures satisfying the continuity equation (1.2) and shows that under appropriate conditions they have to 'flow' along the trajectories of some ordinary differential equation.

The basic notation as well as some useful auxiliary results on metric currents are reported in appendix A. Some remarks on the representation of currents by curves are provided in appendix A.3. Appendix B contains more or less folkloric technical results that are only provided for the reader's convenience and to explain some notation.

2. Notation and preliminaries

2.1. Framework and basic notation

The metric spaces we deal with in what follows are always assumed to be complete. Unless explicitly stated otherwise, E will stand for a complete metric space (our main results, theorems 5.1 and 6.1, are among notable exceptions: there E stands for a Banach space). For a $D \subset E$, we will denote by $\bar{D}$ its closure, by ∂D its topological boundary, by $\mathbf{1}_D$ its characteristic function, by $\mathcal{H}^k$ the k-dimensional Hausdorff measure and let $D^c := E \setminus D$. The notation $B_r(x) \subset E$ stands for the open ball of E centred at $x \in E$ with radius $r > 0$. For real numbers α and β we employ the usual brief notation $\alpha \vee \beta := \max\{\alpha, \beta\}$ and $\alpha \wedge \beta := \min\{\alpha, \beta\}$. By $\bar{e}_i$, $i = 1, \ldots, n$, we denote the unit vectors along the axes x_i in $\mathbb{R}^n$. The notation $\mathcal{L}^n$ stands for the Lebesgue measure in $\mathbb{R}^n$.

If E is a Banach space with norm $\| \cdot \|$, its dual will be denoted by E' and assumed to be equipped with the dual norm $\| \cdot \|'$, the duality being denoted by $\langle \cdot, \cdot \rangle \colon E \times E' \to \mathbb{R}$. The space E will also be silently identified, if necessary, with its natural isometric embedding in E''. A Banach space E is said to be strictly convex if its unit ball is strictly convex.

2.2. Measures

The Borel σ-algebra of a metric space E will be denoted by $\mathscr{B}(E)$. In what follows, all the measures over a metric space E we will consider are Borel. We say that a positive measure μ over E is concentrated over $D \in \mathscr{B}(E)$ if $\mu(D^c) = 0$. Furthermore, all finite positive Borel measures we will be working with are silently assumed to be tight (i.e. concentrated over a σ-compact set), and hence, in fact, Radon, since E is assumed to be complete. This is true for all finite positive Borel measures if E is Polish (i.e. complete separable) or, more generally, if the density character (i.e. the minimum cardinality of a dense subset) of E is an Ulam number (see, for example, [8, proposition 7.2.10]). Note that it is consistent with the Zermelo–Fraenkel set theory to assume that the latter holds for every metric space. Anyhow, if μ is a tight positive finite Borel measure over E, then it is concentrated over a closed separable subspace $S \subset E$ (for S one can take, for instance, the support $\operatorname{supp} \mu$ of μ, or the closed linear span of $\operatorname{supp} \mu$ if E is a Banach space and one wants S to be linear), and hence, in all such situations for the purposes of this paper, up to substituting E by its separable subspace S we may suppose, without loss of generality, that E is separable itself. For two measures μ and ν over the same measurable space we use the standard notation $\nu \ll \mu$ to say that ν is absolutely continuous with respect to μ, and $\nu \perp \mu$ to say that these measures are mutually singular. We say that μ and ν are equivalent if $\mu \ll \nu \ll \mu$.

If a Borel measure μ over a metric space E can be represented (we customarily use the term *disintegrated*) as

$$\mu(B) = \int_X \mu_x(B)\,\mathrm{d}\nu \quad \text{for all } B \in \mathscr{B}(E),$$

where ν is a Borel measure over another metric space X and each μ_x is a Borel measure over E, the map $x \in E \mapsto \mu_x(B)$ being ν-measurable for all $B \in \mathscr{B}(E)$, then we will write $\mu = \nu \otimes \mu_x$ in order to emphasize that the above disintegration formula may be seen as a generalization of the Fubini theorem for product measures.

2.3. Curves

Curves in a metric space E will always be assumed to be parametrized over $\mathbb{R}^+$ and continuous. The space of such curves $C(\mathbb{R}^+; E)$ is equipped with uniform convergence over bounded intervals. The latter convergence is provided, for instance, by the distance

$$d(u, v) := \sum_{k=1}^{+\infty} \frac{1}{2^k} \frac{d_k(u \llcorner [0, k], v \llcorner [0, k])}{1 + d_k(u \llcorner [0, k], v \llcorner [0, k])},$$

where d_k stands for the usual uniform distance in $C([0, k]; E)$ and $u \llcorner [a, b]$ stands for the restriction of the curve $u \in C(\mathbb{R}^+; E)$ to the interval $[a, b]$. In what follows we find it suitable to think of the space $C([a, b]; E)$ of continuous functions over $[a, b]$ with values in E (equipped with the usual uniform norm $\| \cdot \|_\infty$) as continuously embedded in $C(\mathbb{R}^+; E)$ (equipped with the above distance) by the embedding map $\imath \colon C([a, b]; E) \to C(\mathbb{R}^+; E)$ defined by $\imath(\theta)(t) := \theta(a \wedge t \vee b)$. Note that once E is separable, $C(\mathbb{R}^+; E)$ equipped with the latter distance is separable too (in fact, once V_k is a dense subset of $C([0, k]; E)$, $\bigcup_k \imath(V_k)$ is dense in $C(\mathbb{R}+; E)$).

For an absolutely continuous curve $\theta \in C(\mathbb{R}^+; E)$ we let

$$\ell(\theta) = \int_{\mathbb{R}^+} |\dot{\theta}|(t)\,\mathrm{d}t,$$

where $|\dot{\theta}|(t)$ stands for the metric derivative of θ at $t \in \mathbb{R}^+$, which exists almost everywhere in $\mathbb{R}^+$ and is everywhere in what follows silently assumed to be locally integrable as part of the definition of absolutely continuous curves. Note that if E is a Banach space with the Radon–Nikodym property, then in fact the derivative $\dot{\theta}(t)$ is well defined for a.e. $t \in \mathbb{R}^+$ (with the limit in the definition of the derivative intended in the sense of the norm), and hence $\|\dot{\theta}(t)\| = |\dot{\theta}|(t)$ for a.e. $t \in \mathbb{R}^+$. The value $\ell(\theta)$ is usually referred to as the *parametric length* of θ. An absolutely continuous curve $\theta \in C(\mathbb{R}^+; E)$ will be called rectifiable if $\ell(\theta) < +\infty$. For every rectifiable curve θ the sequence $\{\theta(t_k)\}$ for $t_k \to +\infty$ is Cauchy, since for $t_k \leqslant t_j$ one has

$$d(\theta(t_k), \theta(t_j)) \leqslant \ell(\theta \llcorner [t_k, t_j]) = \ell(\theta \llcorner [0, t_k]) - \ell(\theta \llcorner [0, t_j]) \to 0$$

as $k, j \to +\infty$. Thus, the limit

$$\mathrm{end}(\theta) := \lim_{t \to +\infty} \theta(t)$$

is well defined (we will refer to it as the *ending* point of θ). The starting point start(θ) is just start(θ) := $\theta(0)$. The function $e_t \colon C(\mathbb{R}^+; E) \to E$ is defined by $e_t(\theta) := \theta(t)$, $t \in \mathbb{R}^+$. We will say that a rectifiable curve θ is parametrized by arc length if $|\dot{\theta}|(t) = 1$ for a.e. $t \in [0, \ell(\theta)]$ and $|\dot{\theta}|(t) = 0$ for a.e. $t \geqslant \ell(\theta)$ (so that, in particular, start(θ) = $\theta(0)$ and end(θ) = $\theta(t)$ for all $t \geqslant \ell(\theta)$).

2.4. Spaces

For a metric space E we denote by $\mathrm{Lip}(E)$ (respectively, $\mathrm{Lip}_\mathrm{b}(E)$) the set of all real-valued Lipschitz maps (respectively, all bounded Lipschitz maps), and by $C_\mathrm{b}(E)$ the set of all real-valued bounded continuous maps equipped with the uniform norm. The notation $L^p(E; \mu)$ (respectively, $L^p_\mathrm{loc}(E; \mu)$) will stand for the usual Lebesgue space of (classes of μ-almost everywhere equal) real-valued maps over E integrable (respectively, locally integrable) with power $p \in [1; +\infty)$ with respect to the positive (possibly σ-finite) Borel measure μ (or μ-essentially bounded for $p = +\infty$). If (Ω, Σ, μ) is a measure space with some positive σ-finite measure μ, and E is a Banach space, then $L^1(\Omega, \mu; E)$ stands for the space of Bochner integrable functions $f \colon \Omega \to E$. By $C^1(\mathbb{R}^n)$ (respectively, $C_0^1(\mathbb{R}^n)$) we denote as usual the set of continuously differentiable functions (respectively, continuously differentiable with compact support) in $\mathbb{R}^n$.

For a Banach space E, the function $\varphi \colon E \to \mathbb{R}$ is called quasi-differentiable if for every $x \in E$ there is a functional $\mathrm{d}\varphi(x) \in E'$ such that

$$\varphi(\theta(t + \varepsilon)) = \varphi(\theta(t)) + \langle \dot{\theta}(t), \mathrm{d}\varphi(\theta(t)) \rangle \varepsilon + o(\varepsilon)$$

as $\varepsilon \to 0^+$ whenever $\theta \in C(\mathbb{R}^+; E)$ is differentiable at $t \in \mathbb{R}^+ \setminus \{0\}$ and $\theta(t) = x$. This notion of differentiability is slightly stronger than the Gâteaux one but strictly weaker than the Fréchet one. By $Q^1(E)$ we denote then the set of quasi-differentiable functions $\varphi \colon E \to \mathbb{R}$ such that $\|\mathrm{d}\phi(\cdot)\|'$ is uniformly bounded and the map $(x, y) \mapsto \langle y, \mathrm{d}\varphi(x) \rangle$ is continuous. Note that $Q^1(E) \subset \mathrm{Lip}(E)$, since for $u \in Q^1(E)$ one has

$$\begin{aligned}
|u(y) - u(x)| &= \left| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} u((1-t)x + ty) \, \mathrm{d}t \right| \\
&= \left| \int_0^1 \langle y - x, \mathrm{d}u((1-t)x + ty) \rangle \, \mathrm{d}t \right| \\
&\leqslant \int_0^1 \|y - x\| \|\mathrm{d}u((1-t)x + ty)\|' \, \mathrm{d}t \\
&\leqslant C \|y - x\|,
\end{aligned}$$

where $C > 0$ is such that $\|\mathrm{d}u(z)\|' \leqslant C$ for all $z \in E$.

2.5. Metric currents

For metric currents we use the notation from [19], which is taken from [3], except the notation for the mass measure. In particular, $D^k(E) = \mathrm{Lip}_\mathrm{b}(E) \times (\mathrm{Lip}(E))^k$ stands for the space of metric k-forms, its elements (i.e. k-forms) being denoted by $f \, \mathrm{d}\pi$, where $f \in \mathrm{Lip}_\mathrm{b}(E)$, $\pi \in (\mathrm{Lip}(E))^k$, $\mathcal{M}_k(E)$ stands for the space of

k-dimensional metric currents, $\mathbb{M}(T)$ stands for the mass of a current T, ∂T for its boundary, and m_T stands for the mass measure associated with this current (the latter is denoted by $\|T\|$ in [3] and m_T in [19, 20], but here we introduce a different notation in order to avoid confusion between mass measures of currents, norms, and flows of measures). Since we assume by default that every finite Borel measure is tight, we also presume that the mass measures of the currents are all tight, i.e. concentrated over σ-compact sets (which is automatically true if E is separable, or, more generally, when the density character of E is an Ulam number). In fact, all the statements on metric currents from [3] remain valid under this assumption.

We also use the notion of subcurrents from [19]. Namely, an $S \in \mathcal{M}_k(E)$ is called a subcurrent of a $T \in \mathcal{M}_k(E)$, written $S \leqslant T$, if $\mathbb{M}(S) + \mathbb{M}(T - S) = \mathbb{M}(T)$. Finally, the current $T \in \mathcal{M}_k(E)$ is normal if $\partial T \in \mathcal{M}_{k-1}(E)$, i.e. is still a metric current.

The one-dimensional current associated with a Lipschitz curve $\theta\colon [a, b] \to E$ is denoted by $[\![\theta]\!]$, namely,

$$[\![\theta]\!](f\,\mathrm{d}\pi) := \int_a^b f(\theta(t))\,\mathrm{d}\pi(\theta(t))$$

for every $f\,\mathrm{d}\pi \in D^1(E)$. Recall that $\mathbb{M}([\![\theta]\!]) \leqslant \ell(\theta)$. The sequence of currents $\{T_\nu\} \subset \mathcal{M}_k(E)$ is said to converge weakly to a $T \in \mathcal{M}_k(E)$, written $T_\nu \rightharpoonup T$, if $T_\nu(\omega) \to T(\omega)$ as $\nu \to \infty$ for all $\omega \in D^k(E)$.

3. Vector fields as derivations

In what follows we will be working with the following notion of a measurable vector field introduced in [22]. We use it in the form adapted to metric measure spaces in [16].

DEFINITION 3.1. Let μ be a Borel measure over E. We will call a bounded linear operator $X\colon \mathrm{Lip}_{\mathrm{b}}(E) \to L^\infty(E; \mu)$ a μ-vector field (or measurable vector field or even just vector field when we do not need to mention the underlying measure) if it satisfies the following.

- *Leibniz rule:* $X(fg) = fXg + gXf$ for all $\{f, g\} \subset \mathrm{Lip}_{\mathrm{b}}(X)$.

- *Weak continuity:* $\lim_k Xf_k = Xf$ in the weak* sense of $L^\infty(E; \mu)$, i.e. whenever $\lim_k f_k(x) = f(x)$ for all $x \in E$ and $\sup_k(\|f_k\|_\infty + \mathrm{Lip}\,f_k) < +\infty$.

It is worth mentioning that operators over function algebras satisfying the Leibniz rule are usually called derivations.

As mentioned in appendix A.2, a μ-vector field X can be extended by locality to the whole of $\mathrm{Lip}(E)$ (and we do not change the name for this extension). Furthermore, by proposition A.8 there is a one-to-one correspondence between measurable vector fields and metric currents, namely, every μ-vector field X over E defines a metric current $T \in \mathcal{M}_1(E)$ by the formula

$$T(f\,\mathrm{d}\pi) := \int_E fX\pi\,\mathrm{d}\mu, \tag{3.1}$$

with $m_T \leqslant C\mu$, and, conversely, every $T \in \mathcal{M}_1(E)$ with $m_T \leqslant C\mu$ defines a unique μ-vector field X over E such that (3.1) holds. This gives rise to the possibility of identifying one-dimensional metric currents with measurable vector fields, which we will always do in what follows. In particular, we may attribute the terminology of currents to vector fields, for example, calling the vector field *normal* if the respective current is.

We also introduce the following notion, which will be important in what follows.

DEFINITION 3.2. We will say that a current $T \in \mathcal{M}_1(E)$ is represented by a σ-finite Borel measure η over $C(\mathbb{R}^+; E)$ without cancellation of mass if

$$\left.\begin{aligned} T(\omega) &= \int_{C(\mathbb{R}^+;E)} [\![\theta]\!](\omega)\,\mathrm{d}\eta(\theta) \quad \text{for all } \omega \in D^1(E), \\ \mathbb{M}(T) &= \int_{C(\mathbb{R}^+;E)} \ell(\theta)\,\mathrm{d}\eta(\theta). \end{aligned}\right\} \tag{3.2}$$

A μ-vector field X over E will be said to be represented by a σ-finite Borel measure η over $C(\mathbb{R}^+; E)$ if the respective metric current $T \in \mathcal{M}_1(E)$ defined by (3.1) is as well. Such vector fields and currents will in what follows be called *laminated*.

4. Transportable measures and their flows

Define $\beta_t \colon C(\mathbb{R}^+; E) \times \mathbb{R}^+ \to E$ by the formula

$$\beta_t(\theta, s) := \theta(t + s)$$

so that, in particular, $\beta_0(\theta, s) = \theta(s)$. Consider also the projection

$$q \colon (\theta, s) \in C(\mathbb{R}^+; E) \times \mathbb{R}^+ \mapsto \theta \in C(\mathbb{R}^+; E).$$

Following the idea presented in the introduction, we give the following definition.

DEFINITION 4.1. A finite positive Borel measure μ over E will be called *transportable* by a σ-finite Borel measure η over $C(\mathbb{R}^+; E)$ through a finite Borel measure σ over $C(\mathbb{R}^+; E) \times \mathbb{R}^+$, if

$$\left.\begin{aligned} \beta_{0\#}\sigma &= \mu, \\ q_\#\sigma &\ll \eta. \end{aligned}\right\} \tag{4.1}$$

We will also say that μ is transportable by η if there is a σ satisfying (4.1). The family of measures $\{\mu_t\}_{t \in \mathbb{R}^+}$ defined by

$$\mu_t := \beta_{t\#}\sigma$$

will then be called the *flow of measures* corresponding to η.

Furthermore, the measure μ will be called transportable by a laminated measurable vector field X (or, equivalently, by the respective current $T \in \mathcal{M}_1(E)$) if there is a σ-finite Borel measure η over $C(\mathbb{R}^+; E)$ representing X (respectively, T) without cancellation of mass, which does not charge constant curves, such that μ is transportable by η.

Assume that a Borel measure μ over E is transportable by a Borel measure η over $C(\mathbb{R}^+; E)$ through a Borel measure σ over $C(\mathbb{R}^+; E) \times \mathbb{R}^+$ (i.e. the latter satisfies (4.1)). Clearly, with this definition one has $\mu_t(E) = \mu(E)$, $\mu_0 = \mu$ and the family $\{\mu_t\}$ forms a narrowly continuous curve in the space of finite Borel measures over E. Furthermore, by the disintegration theorem we have

$$\sigma = \mu \otimes \eta_x,$$

where $\{\eta_x\}_{x \in E}$ is a family of Borel probability measures over $C(\mathbb{R}^+; E) \times \mathbb{R}^+$ concentrated over

$$\beta_0^{-1}(x) := \{(\theta, s) \in C(\mathbb{R}^+; E) \times \mathbb{R}^+ : \theta(s) = x\}.$$

Setting $h_{x,t} := \beta_{t\#}\eta_x$, one has in particular that

$$\mu_t = \beta_{t\#}\sigma = \mu \otimes h_{x,t},$$

and for every $t \in \mathbb{R}^+$ the map $x \in E \mapsto h_{x,t}$ is weakly measurable. In other words,

$$\mu_t(B) := \int_E h_{x,t}(B) \, \mathrm{d}\mu(x)$$

for every Borel $B \subset E$.

We now study the notion of transportability of the measures introduced above. The following easy statement is valid.

PROPOSITION 4.2. *Assume that a Borel measure μ over E is transportable by a Borel measure η over $C(\mathbb{R}^+; E)$. Then every finite measure $\psi \ll \mu$ is transportable by η.*

Proof. By the Radon–Nikodym theorem we may write $\psi = \alpha\mu$ with a measurable $\alpha \colon E \to \mathbb{R}^+$. Let σ be a Borel measure over $C(\mathbb{R}^+; E) \times \mathbb{R}^+$ satisfying (4.1). We define the Borel measure τ over $C(\mathbb{R}^+; E) \times \mathbb{R}^+$ by setting $\tau := (\alpha \circ \beta_0)\sigma$. Note that τ is a σ-finite Borel measure (in fact, it is finite if ψ is), because

$$\int_{\beta_0^{-1}(B)} \alpha \circ \beta_0 \, \mathrm{d}\sigma = \int_B \alpha \, \mathrm{d}\beta_{0\#}\sigma = \int_B \alpha \, \mathrm{d}\mu = \psi(B) \tag{4.2}$$

for every Borel $B \subset E$. Therefore, $\tau \ll \sigma$, and hence $q_{\#}\tau \ll q_{\#}\sigma \ll \eta$. On the other hand, by construction one has that $\beta_{0\#}\tau = \psi$ by (4.2). Therefore, ψ is transportable by η through τ. □

Note that in definition 4.1 in the notion of transportability by a laminated measurable vector field X it has been requested that a measure η over curves representing X without cancellation of mass does not charge constant curves (i.e. curves, the images of which are singletons). If this requirement were dropped, then every measure μ would be transported by *every* laminated measurable vector field X. In fact, let η be an arbitrary σ-finite Borel measure over $C(\mathbb{R}^+; E)$ representing X without cancellation of mass, and let $f \colon E \to C(\mathbb{R}^+; E)$ be defined by $f(x)(t) := x$ (i.e. f maps a point $x \in E$ to the constant curve $\theta(t) = x$). Letting $\eta' := f_{\#}\mu$ we have that $\eta + \eta'$ still represents the same measurable vector field without cancellation of mass, and μ is transportable by $\eta + \eta'$, say, by a measure $\sigma := \eta' \otimes \delta_0$ over $C(\mathbb{R}^+; E) \times \mathbb{R}^+$.

However, the respective flow is in fact absolutely unreasonable since one has $\mu_t = \mu$ for all $t \in \mathbb{R}^+$, which has nothing to do with the vector field in consideration. Thus, we have to sacrifice the possibility of transporting all of the measures in order to have reasonable flows. In fact, not all the measures are transportable by a given measurable vector field, as the following statement shows.

PROPOSITION 4.3. *If μ is a measure over E transportable by a laminated measurable vector field X (or, equivalently, by the respective current $T \in \mathcal{M}_1(E)$), then μ is concentrated over $\operatorname{supp} T$ (i.e. $\operatorname{supp} \mu \subset \operatorname{supp} T$).*

Proof. Let η be a σ-finite Borel measure over $C(\mathbb{R}^+; E) \times \mathbb{R}^+$ representing T without cancellation of mass, and not charging constant curves, such that μ is transportable by η through some measure σ over $C(\mathbb{R}^+; E) \times \mathbb{R}^+$. By remark A.9, one has then that $\theta \subset \operatorname{supp} T$ in the sense of traces for η-a.e. θ, i.e. $\theta(t) \in \operatorname{supp} T$ for η-a.e. $\theta \in C(\mathbb{R}^+; E)$ and all $t \in \mathbb{R}^+$, and hence for σ-a.e. $(\theta, t) \in C(\mathbb{R}^+; E) \times \mathbb{R}^+$. Thus $\beta_{0\#}\sigma = \mu$ is concentrated over $\operatorname{supp} T$. $\qquad\square$

We now give some examples of transportable measures.

EXAMPLE 4.4. Assume that η is a finite measure. Then $\mu := \operatorname{start}_\# \eta = e_{0\#}\eta$ is transportable by η through $\sigma := \eta \otimes \delta_0$. In fact,

$$
\begin{aligned}
\int_E u(x) \, \mathrm{d}(\beta_{0\#}\sigma)(x) &= \int_E u(x) \, \mathrm{d}\beta_{0\#}(\eta \otimes \delta_0)(x) \\
&= \int_{C(\mathbb{R}^+; E) \times \mathbb{R}^+} u(\beta_0(\theta, s)) \, \mathrm{d}(\eta \otimes \delta_0)(\theta, s) \\
&= \int_{C(\mathbb{R}^+; E) \times \mathbb{R}^+} u(\theta(s)) \, \mathrm{d}(\eta \otimes \delta_0)(\theta, s) \\
&= \int_{C(\mathbb{R}^+; E)} u(\theta(0)) \, \mathrm{d}\eta(\theta) \\
&= \int_E u(x) \, \mathrm{d}(e_{0\#}\eta)(x) \\
&= \int_E u(x) \, \mathrm{d}\mu(x)
\end{aligned}
$$

for every Borel function u over E, so that $\beta_{0\#}\sigma = \mu$, and hence σ satisfies (4.1).

We have now that each η_x is concentrated over

$$
e_0^{-1}(x) \times \{0\} = \{\theta \in C(\mathbb{R}^+; E) \colon \theta(0) = x\} \times \{0\} \subset C(\mathbb{R}^+; E) \times \mathbb{R}^+,
$$

and

$$
\begin{aligned}
h_{x,t}(B) &= \eta_x(\{(\theta, s) \in C(\mathbb{R}^+; E) \times \mathbb{R}^+ \colon \theta(s + t) \in B\}) \\
&= \eta_x(\{\theta \in C(\mathbb{R}^+; E) \colon \theta(t) \in B\} \times \{0\}) \\
&= \eta_x(e_t^{-1}(B) \times \{0\}).
\end{aligned}
$$

Finally, one has $\mu_t = e_{t\#}\eta$. In fact,

$$\mu_t(B) = \int_E h_{x,t}(B)\,\mathrm{d}\mu(x) = \int_E \eta_x(e_t^{-1}(B) \times \{0\})\,\mathrm{d}\mu(x)$$
$$= (\eta \otimes \delta_0)(e_t^{-1}(B) \times \{0\})$$
$$= \eta(e_t^{-1}(B))$$
$$= (e_{t\#}\eta)(B)$$

for all Borel $B \subset E$.

Another important example is the following one.

EXAMPLE 4.5. Let η be a σ-finite measure over $C(\mathbb{R}^+; E)$ such that η-a.e. $\theta \in C(\mathbb{R}^+; E)$ is parametrized by arc length and

$$\int_{C(\mathbb{R}^+;E)} \ell(\theta)\,\mathrm{d}\eta(\theta) < +\infty.$$

Then the finite measure μ over E defined by

$$\mu := \eta \otimes \theta_\#(\mathcal{L}^1 \llcorner [0, \ell(\theta)]), \quad \text{i.e. } \mu(B) = \int_{C(\mathbb{R}^+;E)} \mathcal{L}^1(\theta^{-1}(B) \cap [0, \ell(\theta)])\,\mathrm{d}\eta(\theta), \quad (4.3)$$

for every Borel set $B \subset E$, is transportable by η. In fact, setting $\sigma := \eta \otimes \mathcal{L}^1 \llcorner [0, \ell(\theta)]$, one has

$$(\beta_{0\#}\sigma)(B) = \sigma(\{(\theta, s) \in C(\mathbb{R}^+; E) \times \mathbb{R}^+ : \theta(s) \in B\})$$
$$= \int_{C(\mathbb{R}^+;E)} \mathcal{L}^1(\theta^{-1}(B) \cap [0, \ell(\theta)])\,\mathrm{d}\eta(\theta)$$
$$= \mu(B)$$

for every Borel $B \subset E$.

In particular, if η represents some laminated current $T \in \mathcal{M}_1(E)$ without cancellation of mass, while η-a.e. $\theta \in C(\mathbb{R}^+; E)$ is parametrized by arc length, then the mass measure m_T is transportable by η through $\sigma := \eta \otimes \mathcal{L}^1 \llcorner [0, \ell(\theta)]$, because clearly in this case one has $\mu = m_T$.

It is worth mentioning that the measure μ defined by (4.3) can be interpreted as the *transport density*. This notion is frequently used in the context of the Monge–Kantorovich optimal transportation problem (i.e. when η and/or T represent the solution to the latter problem) [1, 10].

Note that example 4.5 provides a measure transportable by a measurable vector field by any measure η representing the latter without cancellation of mass. In general this is not the case, i.e. if a measure is transportable by a measurable vector field, it is not true that it is transportable by all measures η representing the latter without cancellation of mass, as the following example shows.

EXAMPLE 4.6. If $E := \mathbb{R}$, $T := [\![[0, 1]]\!]$, then the measure $\mu := \delta_0$ is transportable by $\eta := \delta_\theta$ with $\theta := [0, 1]$ (i.e. $\theta(t) := t$ for all $t \in [0, 1]$) by example 4.4. However,
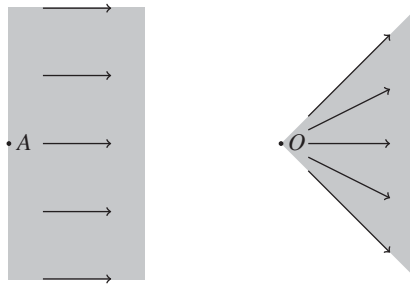
Figure 1. The measure δ_A is not transportable, while δ_O is transportable.

μ is not transportable by $\eta' := \sum_{k=1}^{\infty} \delta_{\theta_k}$ (note that this measure is not finite) with $\theta_k := [1/2^{k+1}, 1/2^k]$ (i.e. $\theta_k(t) := (1 - t)/2^{k+1} + t/2^k$ for all $t \in [0, 1]$), while both η and η' represent T without cancellation of mass.

We will now give an example of a measure not transportable by a given measure η over $C(\mathbb{R}^+; E)$.

EXAMPLE 4.7. Let $E := \mathbb{R}^2$ and $\eta := \mathcal{L}^1 \llcorner [0, 2] \otimes \delta_{\theta_y}$, where $\theta_y(t) := (t \wedge 1, y)$ for each $y \in [0, 2]$ (with the parameter $t \in \mathbb{R}^+$). This measure represents the current $(\mathcal{L}^2 \llcorner [0, 1] \times [0, 2]) \wedge \bar{e}_1$ without cancellation of mass. Let $A := (0, 1) = \theta_1(0)$ (see figure 1). Then $\mu := \delta_A$ is not transportable by η. In fact, suppose the contrary, i.e. the existence of a measure σ over $C(\mathbb{R}^+; E) \times \mathbb{R}^+$ satisfying (4.1). Writing $\sigma := \tilde{\eta} \otimes \lambda_\theta$, where $\tilde{\eta} := q_\# \sigma$, we get that $\tilde{\eta}$ is concentrated over $\{\theta_y : y \in [0, 2]\}$ (because $\tilde{\eta} \ll \eta$) and $\lambda_\theta = 0$ for all $\theta \neq \theta_1$ because $\mu = \beta_{0\#}\sigma$ is concentrated over a singleton A that is the starting point of θ_1. This means that $\sigma = 0$, giving the obvious contradiction $\mu = 0$.

On the contrary, again letting $E := \mathbb{R}^2$ and

$$\eta' := \mathcal{L}^1 \llcorner [-1, 1] \otimes \delta_{\theta'_k},$$

where $\theta'_k(t) := (t \wedge 1, k(t \wedge 1))$ with the parameter $t \in \mathbb{R}^+$ and $k \in [-1, 1]$, we have that η' represents some normal current T' (the one solving the optimal mass transportation problem of minimizing $\mathbb{M}(T)$ among normal currents T satisfying $\partial T = \mathcal{H}^1 \llcorner I - 2\delta_O$, where I stands for the segment with endpoints $(1, -1)$ and $(1, 1)$). The latter current is concentrated over the triangle with vertices O, $(1, -1)$ and $(1, 1)$ and has the field of directions v given by

$$v(x, y) = \frac{(x, y/x)}{|(x, y/x)|}$$

(see figure 1). Then δ_O is transportable by η' by example 4.4.

REMARK 4.8. It is worth remarking that if the finite Borel measure μ is transportable by a measurable vector field corresponding to a laminated current T, then there is a normal current $S \leqslant T$, $S \neq 0$, and a finite Borel measure ν over E such that $\partial S = \nu - \tilde{\mu}$, where $\tilde{\mu}$ is equivalent to μ. In fact, in this case there is a σ-finite Borel measure η over $C(\mathbb{R}^+; E)$ representing T without cancellation of mass and a

Borel measure σ over $C(\mathbb{R}^+; E) \times \mathbb{R}^+$ satisfying (4.1). Disintegrating $\sigma = \alpha\eta \otimes \lambda_\theta$, where $\alpha \in L^1(C(\mathbb{R}^+; E), \eta)$ and each λ_θ is a Borel probability measure over $\mathbb{R}^+$, define

$$\tilde{\eta} := \frac{\alpha}{|\alpha|}\eta = \mathbf{1}_{\{\alpha>0\}}\eta, \qquad \tilde{\sigma} := \tilde{\eta} \otimes \lambda_\theta,$$

$$S := \int_{C(\mathbb{R}^+;E)\times\mathbb{R}^+} [\![\theta \llcorner [s, +\infty)]\!] \, d\tilde{\sigma}(\theta, s),$$

and keeping in mind that $\tilde{\eta}$ represents the laminated current $T_{\tilde{\eta}}$ without cancellation of mass by lemma A.15, we get from lemma A.17 (applied with $T_{\tilde{\eta}}$ and $\tilde{\sigma}$ instead of T and σ, respectively) that $S \leqslant T_{\tilde{\eta}} \leqslant T$ (the latter inequality being due to the fact that $\tilde{\eta} \leqslant \eta$). Finally, one has

$$\partial S = \int_{C(\mathbb{R}^+;E)\times\mathbb{R}^+} (\delta_{\text{end}(\theta)} - \delta_{\theta(s)}) \, d\tilde{\sigma}(\theta, s)$$

$$= \int_{C(\mathbb{R}^+;E)} \delta_{\text{end}(\theta)} \, d\tilde{\eta}(\theta) - \int_{C(\mathbb{R}^+;E)\times\mathbb{R}^+} \delta_{\theta(s)} \, d\tilde{\sigma}(\theta, s),$$

but

$$\int_E f(x) \, d(\partial S)(x) = \int_{C(\mathbb{R}^+;E)} f(\text{end}(\theta)) \, d\tilde{\eta}(\theta) - \int_{C(\mathbb{R}^+;E)\times\mathbb{R}^+} f(\theta(s)) \, d\tilde{\sigma}(\theta, s)$$

$$= \int_E f(x) \, d(\text{end}_\# \, \tilde{\eta} - \beta_{0\#}\tilde{\sigma})(x)$$

for every $f \in C_{\text{b}}(E)$, so that $\partial S = \text{end}_\# \, \tilde{\eta} - \beta_{0\#}\tilde{\sigma}$, and it remains to observe that for $\tilde{\mu} := \beta_{0\#}\tilde{\sigma}$ one has $\tilde{\mu} \ll \mu \ll \tilde{\mu}$ as claimed.

Let us illustrate remark 4.8 by the following easy example.

EXAMPLE 4.9. Let $E := \mathbb{R}$, $T = [\![0, 1]\!]$, $\eta := \delta_\theta$, where $\theta := [0, 1]$, and $\mu = m_T = \mathcal{L}^1 \llcorner [0, 1]$. Set $S := \int_0^1 [\![s, 1]\!] \, ds$. Clearly, $T - S = \int_0^1 [\![0, s]\!] \, ds$, so that

$$\mathbb{M}(S) \leqslant \int_0^1 (1 - s) \, ds = 1/2$$

and analogously $\mathbb{M}(T - S) \leqslant 1/2$, which gives $\mathbb{M}(S) + \mathbb{M}(T - S) \leqslant 1 = \mathbb{M}(T)$; hence $S \leqslant T$. Furthermore,

$$\partial S = \int_0^1 (\delta_1 - \delta_s) \, ds = \delta_1 - \int_0^1 \delta_s \, ds = \delta_1 - \mu,$$

because

$$\int_E f(x) \, d(\partial S)^-(x) = \int_0^1 f(s) \, ds = \int_E f \, d\mu$$

for every $f \in C_{\text{b}}(E)$.

As a partial converse to remark 4.8, we may claim the following easy statement.

PROPOSITION 4.10. *For a normal current $T \in \mathcal{M}_1(E)$, if there is a normal current $S \leqslant T$ such that $\mu \ll (\partial S)^-$, then the measure μ is transportable by T.*

Proof. Let T and $S \leqslant T$ be normal one-dimensional currents. It is enough to show that $(\partial S)^-$ is transportable by T and then refer to proposition 4.2. Let η be a finite Borel measure over $C(\mathbb{R}^+; E)$ representing S without cancellation of mass and concentrated on curves parametrized, say (for the sake of definiteness), over $[0, 1]$ (i.e. being constant outside of this interval). By example 4.4, one has that $e_{0\#}\eta$ is transportable by η. But since $\partial S = e_{1\#}\eta - e_{0\#}\eta$, we have $(\partial S)^- \leqslant e_{0\#}\eta$, and thus $(\partial S)^-$ is transportable by η again by proposition 4.2. Now let η' be a finite Borel measure over $C(\mathbb{R}^+; E)$ representing $R := T - S$ without cancellation of mass. Then, clearly, $\eta + \eta'$ represents T without cancellation of mass, and since $(\partial S)^-$ is transportable by η, it is also transportable by any bigger measure, in particular, by $\eta + \eta'$, and hence is transportable by T, which concludes the proof. $\qquad\square$

In a particular case in which the metric space E is isometrically embedded in a strictly convex Banach space with the Radon–Nikodym property, we are able to give the following more precise characterization of the transportability condition and the flow it generates.

PROPOSITION 4.11. *Let $T \in \mathcal{M}_1(E)$ be a laminated current, where E is isometrically embedded into a strictly convex Banach space with the Radon–Nikodym property, let μ be a finite Borel measure over E transportable by T through a finite Borel measure σ over $C(\mathbb{R}^+; E) \times \mathbb{R}^+$ such that*

$$\int_{C(\mathbb{R}^+; E)} \ell(\theta) \, \mathrm{d}(q_{\#}\sigma)(\theta) < +\infty,$$

the respective flow of μ being given by the family of measures $\{\mu_t\}$ with $\mu_0 = \mu$. Then there is a $\gamma \in L^1(E; m_T)$ such that $T \llcorner \gamma$ is a normal current, and for some finite Borel measure η over $C(\mathbb{R}^+; E)$ representing $T \llcorner \gamma$ without cancellation of mass one has $\mu_t = e_{t\#}\eta$ for all $t \in \mathbb{R}^+$.

Proof. Since $q_{\#}\sigma$ is absolutely continuous with respect to some measure representing T without cancellation of mass, by lemma A.18 there is a $\gamma \in L^1(E, m_T)$ such that for an $S \in \mathcal{M}_1(E)$ defined by (A 9) one has that S is normal and $S = T \llcorner \gamma$. Defining $\varphi \colon C(\mathbb{R}^+; E) \times \mathbb{R}^+ \to C(\mathbb{R}^+; E)$ by $\varphi(\theta, s)(t) := \theta(s + t)$, we have that $\eta := \varphi_{\#}\sigma$ represents S without cancellation of mass. Finally,

$$e_{t\#}\eta = e_{t\#}(\varphi_{\#}\sigma) = (e_t \circ \varphi)_{\#}\sigma = \beta_{t\#}\sigma = \mu_t,$$

concluding the proof. $\qquad\square$

Note that the flow μ_t may be not unique, as the following example shows.

EXAMPLE 4.12. Let $E := \mathbb{R}^2$, $\eta := \delta_\theta$, where θ is the self-intersecting curve shown in figure 2. The parametrization of θ is such that the point A of self-intersection is attained at two instants t_1 and t_2 with $t_2 > t_1$. Let $\sigma_i := \eta \otimes \delta_{t_i}$, $i = 1, 2$. Then $\mu := \delta_A$ is transportable by η both through σ_1 and through σ_2, but the flows μ_t generated by σ_1 and by σ_2 move the measure δ_A along different branches of the curve θ, namely, in the first case $\mu_t = \delta_{A_t^1}$, while in the second case $\mu_t = \delta_{A_t^2}$ (see figure 2).
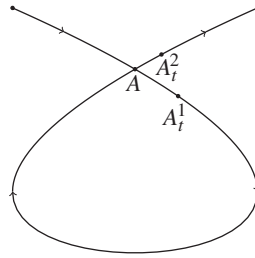
Figure 2. Non-uniqueness of the flow of δ_A in example 4.12.

Note that some (but of course not all) of the non-uniqueness of the flows is due to the fact that the same curves may be parametrized in several different ways. In fact, if we reparametrize in some measurable way all the curves in the support of the measure η transporting the given measure μ, we will get a new flow of the same measure. However, as the following easy statement shows, the measure μ will still remain transportable.

PROPOSITION 4.13. *Assume that η is a σ-finite Borel measure on $C(\mathbb{R}^+; E)$ and let μ be a Borel measure over E transportable by η. Suppose that there exists a Borel reparametrization map*

$$\psi \colon C(\mathbb{R}^+; E) \times \mathbb{R}^+ \to \mathbb{R}^+$$

such that $\psi(\theta, \cdot)$ is surjective, non-decreasing and $\theta(t_1) = \theta(t_2)$ whenever $\psi(\theta, t_1) = \psi(\theta, t_2)$. Then μ is transportable by $g_\# \eta$, where $g \colon C(\mathbb{R}^+; E) \to C(\mathbb{R}^+; E)$ is a Borel map uniquely defined by

$$g(\theta)(\psi(\theta, t)) = \theta(t) \quad \text{for all } t \in \mathbb{R}^+. \tag{4.4}$$

Proof. Since $\psi(\theta, \cdot)$ is surjective, there exists a map $\psi^{-1}(\theta, \cdot)$ such that

$$\psi(\theta, \psi^{-1}(\theta, t')) = t'$$

for all $t' \in \mathbb{R}^+$. The map $\psi^{-1}(\theta, \cdot)$ must be strictly increasing and one can prove that $t' \mapsto \theta(\psi^{-1}(t'))$ is continuous. Hence, (4.4) is satisfied by a unique map $g \colon C(\mathbb{R}^+; E) \to C(\mathbb{R}^+; E)$ defined by $g(\theta)(t') = \theta(\psi^{-1}(\theta, t'))$ (note that although ψ^{-1} clearly may be non-unique, the map g is unique because $\theta(t_1) = \theta(t_2)$ whenever $\psi(\theta, t_1) = \psi(\theta, t_2)$).

We also define $f \colon C(\mathbb{R}^+; E) \times \mathbb{R}^+ \to C(\mathbb{R}^+; E) \times \mathbb{R}^+$ by $f(\theta, t) = (g(\theta), \psi(\theta, t))$, so that $\beta_0 \circ f = \beta_0$ in view of (4.4). Moreover, $q \circ f = g \circ q$ since $q(f(\theta, t)) = g(\theta) = g(q(\theta, t))$. If μ is transportable by η, then by definition 4.1 there is a Borel measure σ over $C(\mathbb{R}^+; E) \times \mathbb{R}^+$ satisfying (4.1). Setting $\sigma' := f_\# \sigma$ one has

$$\beta_{0\#} \sigma' = \beta_{0\#}(f_\# \sigma) = (\beta_0 \circ f)_\# \sigma = \beta_{0\#} \sigma = \mu$$

and

$$q_\# \sigma' = q_\#(f_\# \sigma) = (q \circ f)_\# \sigma = (g \circ q)_\# \sigma = g_\#(q_\# \sigma) \ll g_\# \eta,$$

which concludes the proof. $\qquad\square$

REMARK 4.14. The most natural example of the situation in which proposition 4.13 applies is given by the arc-length reparametrization of curves, i.e. $g(\theta)$ standing for the arc-length reparametrization of θ and

$$\psi(\theta, t) := \begin{cases} \ell(\theta \llcorner [0, t]) & \text{if } \ell(\theta \llcorner [0, t]) < \ell(\theta), \\ \ell(\theta) + t - \bar{t}(\theta) & \text{otherwise,} \end{cases}$$

where $\bar{t}(\theta) := \inf\{t \colon \ell(\theta \llcorner [0, t]) = \ell(\theta)\}$, which is Borel in θ and continuous in t.

REMARK 4.15. It is worth mentioning that always when η represents some current T without cancellation of mass, and a Borel map $g \colon C(\mathbb{R}^+; E) \to C(\mathbb{R}^+; E)$ is such that $g(\theta)$ is a reparametrization of θ for η-a.e. $\theta \in C(\mathbb{R}^+; E)$, the measure $g_\# \eta$ represents the same current, also without cancellation of mass.

In view of the above remarks the following immediate corollary to proposition 4.13 holds true.

COROLLARY 4.16. *Let μ be transportable by some laminated current $T \in \mathcal{M}_1(E)$. Then there is a σ-finite Borel measure η representing T without cancellation of mass and concentrated over arc-length parametrized curves in $C(\mathbb{R}^+; E)$ such that μ is transportable by η.*

A measure η singled out in corollary 4.16 is of course not unique (since the current T may be represented by different measures over curves without cancellation of mass, and μ may be transportable by several of the latter); nevertheless, in most situations such measures are the most interesting.

5. Representation of currents and vector fields

The following result on representation of laminated currents (and hence also of measurable vector fields) will be important in what follows.

THEOREM 5.1. *Let E be a Banach space, let $T \in \mathcal{M}_1(E)$ be a laminated current represented by a σ-finite Borel measure η over $C(\mathbb{R}^+; E)$ without cancellation of mass such that η-a.e. θ is norm differentiable almost everywhere (in particular, this holds when E has the Radon–Nikodym property since then all absolutely continuous curves are so). Then there is a Borel function $V_T \colon E \to E$ with $\|V_T(y)\| = 1$ for m_T-a.e. $y \in E$ satisfying*

$$T(f \, d\pi) = \int_E f(y) \langle V_T(y), d\pi(y) \rangle \, dm_T(y) \tag{5.1}$$

when $(f, \pi) \in L^1(E; m_T) \times Q^1(E)$. Furthermore, for η-a.e. $\theta \in C(\mathbb{R}^+; E)$ and a.e. $t \in \mathbb{R}^+$ one has

$$\dot{\theta}(t) \in V_T(\theta(t))'' |\dot{\theta}|(t) \tag{5.2}$$

(where V_T' is defined by (B 1)), and, in particular, when the norm in E is strictly convex,

$$\dot{\theta}(t) = V_T(\theta(t)) |\dot{\theta}|(t). \tag{5.3}$$

If, moreover, η-a.e. $\theta \in C(\mathbb{R}^+; E)$ are parametrized by arc length, then for η-a.e. $\theta \in C(\mathbb{R}^+; E)$ and a.e. $t \in [0, \ell(\theta))$ one has

$$\dot{\theta}(t) \in V_T(\theta(t))'', \tag{5.4}$$

and, in particular, when the norm in E is strictly convex,

$$\dot{\theta}(t) = V_T(\theta(t)). \tag{5.5}$$

Moreover, the claims (5.2) and (5.4) hold for every V_T satisfying (5.1).

REMARK 5.2. If E is a finite-dimensional normed space, then for every (not necessarily laminated) $T \in \mathcal{M}_1(E)$ a Borel map V_T satisfying (5.1) exists by lemma A.3. The representation (5.1) together with proposition A.8 establishes the natural correspondence between one-dimensional metric currents in a Euclidean space, measurable vector fields and vector-valued Borel maps. In fact, according to proposition A.8, for every $T \in \mathcal{M}_1(E)$ there is a unique measurable vector field X_T satisfying

$$T(f \, \mathrm{d}\pi) = \int_E f(y)(X_T \pi)(y) \, \mathrm{d}m_T(y)$$

for all $f \, \mathrm{d}\pi \in D^1(E)$, and so $(X_T \pi)(y) = \langle V_T(y), \mathrm{d}\pi(y) \rangle$ for m_T-a.e. $y \in E$. Clearly, if T is normal, then additionally

$$\mathrm{div}_{m_T} X_T = \mathrm{div}\, V_T m_T$$

is a signed Borel measure of finite total variation (note the different meanings of the two divergence symbols in the above relationship: it is the abstract divergence operator applied to measurable vector fields on the left-hand side and the usual distributional divergence on the right-hand side). Note that the map V_T satisfying (5.1) is unique up to m_T-almost everywhere equality by remark A.2. Theorem 5.1 therefore gives just the additional information on every representative of V_T in the class of m_T-almost everywhere equal Borel maps in the case in which T is laminated, or, equivalently, on the structure of measures over curves representing the laminated current T.

Proof. We may further assume without loss of generality E to be separable (substituting it, if necessary, with the linear span of supp m_T, which is separable since m_T is assumed to be tight). Let σ be a finite measure over $C(\mathbb{R}^+; E) \times \mathbb{R}^+$ equivalent to $\eta \otimes \mathcal{L}^1$. Disintegrate the measure $\eta \otimes \mathcal{L}^1$ over $C(\mathbb{R}^+; E) \times \mathbb{R}^+$ as

$$\eta \otimes \mathcal{L}^1 = \mu \otimes \tau^y, \quad \text{where } \mu := \beta_{0\#}\sigma,$$

so that μ is equivalent to $\beta_{0\#}(\eta \otimes \mathcal{L}^1)$ and τ^y are σ-finite measures concentrated over $\beta_0^{-1}(y) \subset C(\mathbb{R}^+; E) \times \mathbb{R}^+$, and define

$$V(y) := \int_{C(\mathbb{R}^+; E) \times \mathbb{R}^+} \dot{\theta}(t) \, \mathrm{d}\tau^y(\theta, t) = \int_{\beta_0^{-1}(y)} \dot{\theta}(t) \, \mathrm{d}\tau^y(\theta, t), \tag{5.6}$$

the integrals of functions with values in E here and below being intended in the sense of Bochner. Note that for μ-a.e. $y \in E$ the measure τ^y does not charge $\eta \otimes \mathcal{L}^1$-nullsets, and thus for τ^y-a.e. (θ, t) the derivative $\dot{\theta}(t)$ is well defined. Moreover, the

map $(\theta, t) \in C(\mathbb{R}^+; E) \times \mathbb{R}^+ \mapsto \dot{\theta}(t) \in E$ is $\eta \otimes \mathcal{L}^1$-almost everywhere equal to a Borel map by lemma B.2, and hence τ^y-measurable, and so is the map $(\theta, t) \in C(\mathbb{R}^+; E) \times \mathbb{R}^+ \mapsto |\dot{\theta}|(t) \in E$ (say, because $|\dot{\theta}|(t) = \|\dot{\theta}(t)\|$ for a.e. $t \in \mathbb{R}^+$). The calculation

$$\int_E \mathrm{d}\mu(y) \int_{C(\mathbb{R}^+; E) \times \mathbb{R}^+} |\dot{\theta}|(t) \, \mathrm{d}\tau^y(\theta, t) = \int_{\mathbb{R}^+} \mathrm{d}t \int_{C(\mathbb{R}^+; E)} |\dot{\theta}|(t) \, \mathrm{d}\eta(\theta)$$
$$= \int_{C(\mathbb{R}^+; E)} \ell(\theta) \, \mathrm{d}\eta(\theta)$$
$$= \mathbb{M}(T), \tag{5.7}$$

shows then that $V(y)$ is well defined for μ-a.e. $y \in E$.

Let $\pi \in Q^1(E)$. Observe that

$$T(f \, \mathrm{d}\pi) = \int_{C(\mathbb{R}^+; E)} [\![\theta]\!](f \, \mathrm{d}\pi) \, \mathrm{d}\eta(\theta)$$
$$= \int_{C(\mathbb{R}^+; E)} \mathrm{d}\eta(\theta) \int_{\mathbb{R}^+} f(\theta(t)) \langle \dot{\theta}(t), \mathrm{d}\pi(\theta(t)) \rangle \, \mathrm{d}t. \tag{5.8}$$

Recalling that $f \in C_{\mathrm{b}}(E)$ and that $\|\mathrm{d}\pi(y)\|' \leqslant \mathrm{Lip}\,\pi$, we get

$$\int_{C(\mathbb{R}^+; E)} \mathrm{d}\eta(\theta) \int_{\mathbb{R}^+} |f(\theta(t)) \langle \dot{\theta}(t), \mathrm{d}\pi(\theta(t)) \rangle| \, \mathrm{d}t$$
$$\leqslant \|f\|_\infty \, \mathrm{Lip}\,\pi \int_{C(\mathbb{R}^+; E)} \mathrm{d}\eta(\theta) \int_{\mathbb{R}^+} |\dot{\theta}|(t) \, \mathrm{d}t$$
$$= \|f\|_\infty \, \mathrm{Lip}\,\pi \int_{C(\mathbb{R}^+; E)} \ell(\theta) \, \mathrm{d}\eta(\theta)$$
$$= \|f\|_\infty \, \mathrm{Lip}\,\pi \mathbb{M}(T)$$
$$< +\infty,$$

and thus we may change the order of integration in (5.8) obtaining

$$T(f \, \mathrm{d}\pi) = \int_{C(\mathbb{R}^+; E) \times \mathbb{R}^+} f(\theta(t)) \langle \dot{\theta}(t), \mathrm{d}\pi(\theta(t)) \rangle \mathrm{d}(\eta \otimes \mathcal{L}^1)(\theta, t)$$
$$= \int_E \mathrm{d}\mu(y) \int_{C(\mathbb{R}^+; E) \times \mathbb{R}^+} f(\theta(t)) \langle \dot{\theta}(t), \mathrm{d}\pi(\theta(t)) \rangle \, \mathrm{d}\tau^y(\theta, t)$$
$$= \int_E f(y) \left\langle \left(\int_{C(\mathbb{R}^+; E) \times \mathbb{R}^+} \dot{\theta}(t) \, \mathrm{d}\tau^y(\theta, t) \right), \mathrm{d}\pi(y) \right\rangle \mathrm{d}\mu(y)$$
$$= \int_E f(y) \langle V(y), \mathrm{d}\pi(y) \rangle \, \mathrm{d}\mu(y). \tag{5.9}$$

The latter also implies that $m_T = \|V\|\mu$ by lemma A.1. Defining

$$V_T(y) := \frac{V(y)}{\|V(y)\|}$$

with the convention $0/0 := 0$, we note that V_T may be assumed to be Borel up to changing it on a set of zero measure m_T, that $\|V_T(y)\| = 1$ for m_T-a.e. $y \in E$

and (5.1) holds. Moreover,

$$\|V(y)\| = \left\| \int_{C(\mathbb{R}^+;E)\times\mathbb{R}^+} \dot{\theta}(t)\,\mathrm{d}\tau^y(\theta,t) \right\| = \int_{C(\mathbb{R}^+;E)\times\mathbb{R}^+} |\dot{\theta}|(t)\,\mathrm{d}\tau^y(\theta,t) \qquad (5.10)$$

for μ-a.e. $y \in E$, in view of (5.7) and the equality $\int_E \|V(y)\|\,\mathrm{d}\mu(y) = \mathbb{M}(T)$. Therefore, for such $y \in E$ one has

$$\dot{\theta}(t) \in \frac{(\int_{C(\mathbb{R}^+;E)\times\mathbb{R}^+} \dot{\theta}(s)\,\mathrm{d}\tau^y(\theta,s))''}{\int_{C(\mathbb{R}^+;E)\times\mathbb{R}^+} |\dot{\theta}|(s)\,\mathrm{d}\tau^y(\theta,s)}|\dot{\theta}|(t)$$

by lemma B.1, or, in other words,

$$\dot{\theta}(t) \in \frac{V(y)''}{\|V(y)\|}|\dot{\theta}|(t) = V_T(y)''|\dot{\theta}|(t) \qquad (5.11)$$

for τ^y-a.e. $(\theta,t) \in C(\mathbb{R}^+;E) \times \mathbb{R}^+$, so that η-a.e. $\theta \in C(\mathbb{R}^+;E)$ satisfies (5.2) for a.e. $t \in \mathbb{R}^+$. Note that (5.11) makes sense since for μ-a.e. $y \in E$ such that $V(y) = 0$ one has $|\dot{\theta}|(t) = 0$ for τ^y-a.e. $(\theta,t) \in C(\mathbb{R}^+;E) \times \mathbb{R}^+$. Therefore, also

$$\eta \otimes \mathcal{L}^1(\{(\theta,t) \in C(\mathbb{R}^+;E) \times \mathbb{R}^+ : V(\theta(t)) = 0 \text{ but } |\dot{\theta}|(t) \neq 0\}) = 0$$

(where by writing $|\dot{\theta}|(t) \neq 0$ for brevity we mean that $|\dot{\theta}|(t)$ is either defined and non-zero or undefined). Relationships (5.3)–(5.5) now follow straightforwardly from (5.11).

It remains to show that (5.4) and (5.2) hold for every V_T satisfying (5.1). We start with (5.4). Suppose that $T = \tilde{V}_T \wedge m_T$. Then $V_T(x) = \tilde{V}_T(x)$ for m_T-a.e. $x \in E$ by remark A.2. We show that then for η-a.e. $\theta \in C(\mathbb{R}^+;E)$ and a.e. $t \in [0,\ell(\theta))$, relationship (5.4) is valid with $\tilde{V}_T$ instead of V_T. This is true since $m_T = \beta_{0\#}(\eta \otimes \mathcal{L}^1 \llcorner [0,\ell(\theta)))$, and hence

$$\eta \otimes \mathcal{L}^1 \llcorner [0,\ell(\theta))(\{(\theta,t) \in C(\mathbb{R}^+;E) \times \mathbb{R}^+ : V_T(\theta(t)) \neq \tilde{V}_T(\theta(t))\})$$
$$= m_T(\{x \in E : V_T(x) \neq \tilde{V}_T(x)\})$$
$$= 0,$$

proving the claim.

Finally, we show that (5.2) holds for every V_T satisfying (5.1). Let $g(\theta)$ stand for the arc-length reparametrization of θ, that is, $g(\theta)(t) := \theta(\varphi_\theta(t \wedge \ell(\theta)))$, where $\varphi_\theta(s) := \inf\{t \in \mathbb{R}^+ : \psi_\theta(t) = s\}$, $\psi_\theta(t) := \ell(\theta \llcorner [0,t])$. Defining $\eta' := g_\#\eta$, we get from (5.4), for η'-a.e. $\gamma \in C(\mathbb{R}^+;E)$ and a.e. $t \leqslant \ell(\gamma)$, the inclusion

$$\dot{\gamma}(t) \in V_T(\gamma((t)))''. \qquad (5.12)$$

But since for $\gamma = g(\theta)$ one has

$$\dot{\gamma}(t) = \dot{\theta}(\varphi_\theta(t)) \cdot \dot{\varphi}_\theta(t) = \frac{\dot{\theta}(\varphi_\theta(t))}{|\dot{\theta}|(\varphi_\theta(t))}$$

for a.e. $t \leqslant \ell(\gamma) = \ell(\theta)$, we get from (5.12) the relationship

$$\frac{\dot{\theta}(\varphi_\theta(t))}{|\dot{\theta}|(\varphi_\theta(t))} \in V_T(\theta(\varphi_\theta(t)))''.$$

This means that the set of $t \in \mathbb{R}^+$ such that $\dot{\psi}_\theta(t) = |\dot{\theta}|(t) \neq 0$ and

$$\frac{\dot{\theta}(t)}{|\dot{\theta}|(t)} \notin V_T(\theta(t))''$$

has zero Lebesgue measure (in view of the area formula), which implies (5.2) for η-a.e. $\theta \in C(\mathbb{R}^+; E)$ and a.e. $t \in \mathbb{R}^+$, and hence concludes the proof for the case in which η-a.e. $\theta \in C(\mathbb{R}^+; E)$ are norm differentiable almost everywhere. $\qquad\square$

It is not difficult to show that when the norm in E is not strictly convex, the inclusion in (5.2) (or (5.3)) does not necessarily become an equality, as the following easy though instructive example shows.

EXAMPLE 5.3. Equip $E := \mathbb{R}^2$ with the norm $\| \cdot \|_1$ defined by $\|(x_1, x_2)\|_1 := |x_1| + |x_2|$. Define η_i, $i = 1, 2$, by the formulae

$$\eta_1 := \mathcal{L}^1 \llcorner [0,1] \otimes \delta_{\theta_{x_2}}, \quad \text{where } \theta_{x_2}(t) := (t \wedge 1, x_2) \text{ for each } x_2 \in [0,1],$$
$$\eta_2 := \mathcal{L}^1 \llcorner [0,1] \otimes \delta_{\theta_{x_1}}, \quad \text{where } \theta_{x_1}(t) := (x_1, t \wedge 1) \text{ for each } x_1 \in [0,1]$$

(with the parameter $t \in \mathbb{R}^+$). Clearly, $T_{\eta_i} = \bar{e}_i \wedge \mathcal{L}^2 \llcorner ([0,1] \times [0,1])$, $\mathbb{M}(T_{\eta_i}) = 1$. Setting $\eta := \eta_1 + \eta_2$, one has

$$T := T_{\eta_1} + T_{\eta_2} = (\bar{e}_1 + \bar{e}_2) \wedge \mathcal{L}^2 \llcorner Q,$$

and thus $\mathbb{M}(T) = 2$; hence, $T_{\eta_i} \leqslant T$ and η represents T without cancellation of mass. One has $m_T = 2\mathcal{L}^2 \llcorner Q$, $V_T = (\bar{e}_1 + \bar{e}_2)/2$, and $\dot{\theta}(t) \in V_T''$ for η-a.e. θ and for all $t \in [0,1]$ (because $V_T'' = \{\lambda_1 \bar{e}_1 + \lambda_2 \bar{e}_2 : \lambda_i \geqslant 0, \ i = 1, 2\}$ in this case), but $\dot{\theta}(t) \neq V_T$.

It is worth remarking, however, that if one equips E instead with the usual Euclidean norm, then one still has $\mathbb{M}(T_{\eta_i}) = 1$, but $\mathbb{M}(T) = \sqrt{2}$, and hence $T_{\eta_i} \not\leqslant T$, $i = 1, 2$; moreover,

$$\int_{C(\mathbb{R}^+; E)} \mathbb{M}(\llbracket \theta \rrbracket) \, \mathrm{d}\eta = 2 > \mathbb{M}(T),$$

and thus η does not represent T without cancellation of mass.

A few remarks regarding theorem 5.1 and its proof have to be made.

REMARK 5.4. If E is a Banach space and η is just some σ-finite measure over $C(\mathbb{R}^+; E)$ concentrated over absolutely continuous almost everywhere norm-differentiable curves and one has

$$\left\| \int_{C(\mathbb{R}^+; E) \times \mathbb{R}^+} \dot{\theta}(t) \, \mathrm{d}\tau^y(\theta, t) \right\| = \int_{C(\mathbb{R}^+; E) \times \mathbb{R}^+} |\dot{\theta}|(t) \, \mathrm{d}\tau^y(\theta, t)$$

for μ-a.e. $y \in E$, or, equivalently (by lemma B.1), with the notation of the proof,

$$\dot{\theta}(t) \in \frac{V(y)''}{\|V(y)\|} |\dot{\theta}|(t)$$

for τ^y-a.e. $(\theta, t) \in C(\mathbb{R}^+; E) \times \mathbb{R}^+$ and for μ-a.e. $y \in E$, then η represents $T_\eta \in \mathcal{M}_1(E)$ without cancellation of mass. In fact, since by (5.9)

$$T_\eta(f \, d\pi) = \int_E f(y) \langle V(y), d\pi(y) \rangle \, d\mu(y),$$

where $\pi \in Q^1(E)$, we have

$$\mathbb{M}(T_\eta) = \int_E \|V(y)\| \, d\mu(y)$$

in view of lemma A.1. Thus,

$$
\begin{aligned}
\mathbb{M}(T_\eta) &= \int_E d\mu(y) \left\| \int_{C(\mathbb{R}^+; E) \times \mathbb{R}^+} \dot{\theta}(t) \, d\tau^y(\theta, t) \right\| \\
&= \int_E d\mu(y) \int_{C(\mathbb{R}^+; E) \times \mathbb{R}^+} |\dot{\theta}|(t) \, d\tau^y(\theta, t) \\
&= \int_{\mathbb{R}^+} dt \int_{C(\mathbb{R}^+; E)} |\dot{\theta}|(t) \, d\eta(\theta) \\
&= \int_{C(\mathbb{R}^+; E)} \ell(\theta) \, d\eta(\theta),
\end{aligned}
$$

showing the claim.

REMARK 5.5. If under the conditions of theorem 5.1 one has additionally that η is finite (and thus the current T is normal), and η-a.e. θ is parametrized by arc length, then

$$V_T(\theta(t)) = \mathbb{E}(\dot{\theta}; \beta_0^{-1}(\mathscr{B}(E)))(t),$$

where $\mathbb{E}(\cdot; \Sigma)$ stands for the conditional expectation with respect to the σ-algebra Σ. To see this, one recalls that $\beta_{0\#}(\eta \otimes \mathcal{L}^1 \llcorner [0, \ell(\theta)]) = m_T$ (example 4.5) and calculates that

$$
\begin{aligned}
\int_{\beta_0^{-1}(e)} & \mathbb{E}(\dot{\theta}; \beta_0^{-1}(\mathscr{B}(E)))(t) \, d(\eta \otimes \mathcal{L}^1 \llcorner [0, \ell(\theta)])(\theta, t) \\
&= \int_E dm_T(y) \int_{\beta_0^{-1}(y)} \mathbf{1}_{\beta_0^{-1}(B)}(\theta, t) \mathbb{E}(\dot{\theta}; \beta_0^{-1}(\mathscr{B}(E)))(t) \, d\bar{\tau}^y(\theta, t) \\
&= \int_E \mathbf{1}_B(y) \, dm_T(y) \int_{\beta_0^{-1}(y)} \mathbb{E}(\dot{\theta}; \beta_0^{-1}(\mathscr{B}(E)))(t) \, d\bar{\tau}^y(\theta, t) \\
&= \int_B dm_T(y) \int_{\beta_0^{-1}(y)} \mathbb{E}(\dot{\theta}; \beta_0^{-1}(\mathscr{B}(E)))(t) \, d\bar{\tau}^y(\theta, t)
\end{aligned}
$$

for every $B \in \mathscr{B}(E)$, where $\bar{\tau}^y$ is a Borel probability measure concentrated over $\beta_0^{-1}(y)$ and satisfying the disintegration identity

$$\eta \otimes \mathcal{L}^1 \llcorner [0, \ell(\theta)] = m_T \otimes \bar{\tau}^y.$$

Keeping in mind that

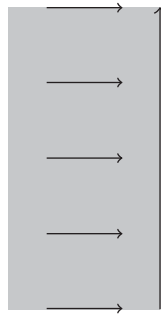$$\mathbb{E}(\dot{\theta}; \beta_0^{-1}(\mathscr{B}(E)))(t) = \lambda(\beta_0(\theta, t)) = \lambda(\theta(t))$$

Figure 3. The situation in example 5.7.

for some Borel vector function of λ over E, we have that over $\beta_0^{-1}(y)$ the latter conditional expectation depends only on y (and not on θ and t), so that

$$\int_{\beta_0^{-1}(B)} \mathbb{E}(\dot{\theta}; \beta_0^{-1}(\mathscr{B}(E)))(t)\, \mathrm{d}(\eta \otimes \mathcal{L}^1 \llcorner [0, \ell(\theta)]) = \int_B \lambda(y)\, \mathrm{d}m_T(y).$$

But, on the other hand, by definition of the conditional expectation one has

$$\int_{\beta_0^{-1}(B)} \mathbb{E}(\dot{\theta}; \beta_0^{-1}(\mathscr{B}(E)))(t)\, \mathrm{d}(\eta \otimes \mathcal{L}^1 \llcorner [0, \ell(\theta)])(\theta, t)$$
$$= \int_B \mathrm{d}m_T(y) \int_{\beta_0^{-1}(y)} \dot{\theta}(t)\, \mathrm{d}\bar{\tau}^y(\theta, t)$$
$$= \int_B V_T(y)\, \mathrm{d}m_T(y),$$

recalling (5.5). Thus, $\lambda(y) = V_T(y)$ for m_T-a.e. $y \in E$, which is exactly the claim being proven.

REMARK 5.6. It is important to remark that when η-a.e. $\theta \in C(\mathbb{R}^+; E)$ is parametrized by arc length, one can assert that (5.4) (or, in a particular case, (5.5)) holds for η-a.e. $\theta \in C(\mathbb{R}^+; E)$ only for a.e. $t \in [0, \ell(\theta))$ and not for a.e. $t \in \mathbb{R}^+$, as the following simple example shows.

EXAMPLE 5.7. Let $E := \mathbb{R}^2$ and $\eta := \mathcal{L}^1 \llcorner [0, 2] \otimes \delta_{\theta_y} + \delta_{\bar{\theta}}$, where $\theta_y(t) := (t \wedge 1, y)$ for each $y \in [0, 2]$ and $\bar{\theta}(t) := (1, t \wedge 2)$ (with the parameter $t \in \mathbb{R}^+$). This measure represents the current $(\mathcal{L}^2 \llcorner [0, 1] \times [0, 2]) \wedge \bar{e}_1 + (\mathcal{H}^1 \llcorner I) \wedge \bar{e}_2$ without cancellation of mass, where I stands for the vertical segment with endpoints $(1, 0)$ and $(1, 2)$ (see figure 3). Clearly, one has $V_T(y) = \bar{e}_1$ for $\mathcal{L}^2$-a.e. $y \in (0, 1) \times (0, 2)$ and $V_T(y) = \bar{e}_2$ for $\mathcal{H}^1$-a.e. $y \in I$. Thus, for instance, for all $y \in (1, 2]$ one has $\dot{\theta}_y(t) = 0$ for all $t \in (1, 2)$ but $V_T(\theta_y(t)) = V_T((1, y)) = \bar{e}_2 \neq 0$ for $\mathcal{L}^1$-a.e. $y \in (1, 2]$, and the measure η of this set of curves is positive.

6. Flows generated by measurable vector fields

The following theorem gives the characterization of flows of a measure transportable by some measurable vector field (or, equivalently, one-dimensional current) in a Banach space.

THEOREM 6.1. *Assume that a finite measure μ over a Banach space E is transportable by a σ-finite measure η, concentrated over some separable subset of $C(\mathbb{R}^+;$ $E)$ consisting only of absolutely continuous curves, through a measure σ over $C(\mathbb{R}^+;$ $E) \times \mathbb{R}^+$ such that $\tilde{\eta} := q_\# \sigma$ satisfies*

$$\int_{C(\mathbb{R}^+;E)} \ell(\theta)\,\mathrm{d}\tilde{\eta}(\theta) < +\infty, \tag{6.1}$$

and η-a.e. θ is norm differentiable almost everywhere (in particular, the latter holds when E has the Radon–Nikodym property since then all absolutely continuous curves are so). Then the following assertions are valid.

(A) *Representing by disintegration $\sigma = \mu_t \otimes \nu_t^y$, where $\mu_t := \beta_{t\#}\sigma$, so that each ν_t^y is a probability measure concentrated over*

$$\beta_t^{-1}(y) = \{(\theta, s) \in C(\mathbb{R}^+; E) \times \mathbb{R}^+ : \theta(s+t) = y\},$$

and defining

$$v_t(y) := \int_{C(\mathbb{R}^+;E)\times\mathbb{R}^+} \dot{\theta}(s+t)\,\mathrm{d}\nu_t^y(\theta, s) = \int_{\beta_t^{-1}(y)} \dot{\theta}(s+t)\,\mathrm{d}\nu_t^y(\theta, s), \tag{6.2}$$

we have that v_t is Bochner integrable with respect to μ_t for a.e. $t \in \mathbb{R}^+$ (and even more, the function $v \colon E \times \mathbb{R}^+ \to E$ defined by $v(t,x) := v_t(x)$ is Bochner integrable with respect to $\mathrm{d}t \otimes \mu_t$), and the continuity equation

$$\frac{\partial \mu_t}{\partial t} + \operatorname{div} v_t \mu_t = 0 \tag{6.3}$$

is valid in the weak sense, i.e.

$$-\int_{\mathbb{R}^+} \dot{\psi}(t)\,\mathrm{d}t \int_E \varphi(y)\,\mathrm{d}\mu_t(y) - \int_{\mathbb{R}^+} \psi(t)\,\mathrm{d}t \int_E \langle v_t(y), \mathrm{d}\varphi(y)\rangle\,\mathrm{d}\mu_t(y) = 0 \tag{6.4}$$

for every $\varphi \in Q^1(E) \cap C_{\mathrm{b}}(E)$ and $\psi \in C_0^1(\mathbb{R}^+)$.

(B) *If, moreover, η represents some laminated current $T \in \mathcal{M}_1(E)$ without cancellation of mass, then theorem 5.1 holds. In particular, if in this case η-a.e. $\theta \in C(\mathbb{R}^+; E)$ is parametrized by arc length, then*

$$\dot{\theta}(t) \in V_T(\theta(t))'' \tag{6.5}$$

(with the equality $\dot{\theta}(t) = V_T(\theta(t))$ if E is strictly convex) for η-a.e. $\theta \in C(\mathbb{R}^+; E)$ and for a.e. $t \in [0, \ell(\theta))$. If, in addition, E is strictly convex, then one also has

$$v_t(y) = V_T(y)(1 - \varphi_t(y)) \tag{6.6}$$

for a.e. $t \in \mathbb{R}^+$ and μ_t-a.e. $y \in E$, where $\varphi_t(y) := \nu_t^y(S_t^y)$ stands for the proportion of mass that is stopped at point y after time t,

$$\begin{aligned} S_t^y &:= \{(\theta, s) \in C(\mathbb{R}^+; E) \times \mathbb{R}^+ : \theta(s+t) = y,\ s+t \geqslant \ell(\theta)\} \\ &= \{(\theta, s) \in C(\mathbb{R}^+; E) \times \mathbb{R}^+ : \theta(s+t) = y = \operatorname{end}(\theta)\}, \end{aligned}$$

and thus, in particular,

$$\frac{\partial \mu_t}{\partial t} + \operatorname{div} V_T (1 - \varphi_t) \mu_t = 0 \tag{6.7}$$

in the weak sense.

It is worth remarking that the construction (6.2) of the vector field v_t as the superposition of the speeds of the relevant curves is quite similar to the construction of [17, theorem 7]. Note that the term containing φ_t must appear in (6.7) in view of remark 5.6. It is also worth emphasizing that the relationship (6.6) is just a μ_t-almost everywhere equality. For instance, in example 5.7 with $\mu := \mathcal{H}^1 \llcorner (\{0\} \times [0,2]) + \delta_{(1,0)}$ one gets $\mu_t = \mathcal{H}^1 \llcorner (\{t \wedge 1\} \times [0,2]) + \delta_{(1,t \wedge 2)}$, and, when $t > 1$, one has $v_t(x) = \bar{e}_2$ for $x = (1, t \wedge 2)$ and $v_t(x) = 0$ elsewhere.

Proof. We divide the proof into several steps in order to simplify the reading.

STEP 1. First we show that (6.2) is a correct definition. To this end, let $N_\theta \in \mathbb{R}^+$ for a given $\theta \in C(\mathbb{R}^+; E)$ stand for the set

$$N_\theta := \{ s \in \mathbb{R}^+ \colon \dot{\theta}(s) \text{ is undefined} \}.$$

Representing by disintegration $\sigma = \tilde{\eta} \otimes \lambda_\theta$ with λ_θ Borel probability measures over $\mathbb{R}^+$, we get from lemma 6.2 (applied with $\mu := \lambda_\theta$, $B := N_\theta$) that $\lambda_\theta(N_\theta - t) = 0$ for a.e. $t \in \mathbb{R}$ and $\tilde{\eta}$-a.e. $\theta \in C(\mathbb{R}^+; E)$. Hence, observing that

$$N_\theta - t = \{ s \in \mathbb{R}^+ \colon \dot{\theta}(s+t) \text{ is undefined} \},$$

we get that

$$\sigma(\{ (\theta, s) \in C(\mathbb{R}^+; E) \times \mathbb{R}^+ \colon \dot{\theta}(s+t) \text{ is undefined} \}) = 0$$

for a.e. $t \in \mathbb{R}^+$, which implies that for such $t \in \mathbb{R}^+$ the derivative $\dot{\theta}(s+t)$ is well defined for ν_t^y-a.e. $(\theta, s) \in C(\mathbb{R}^+; E) \times \mathbb{R}^+$ and μ_t-a.e. $y \in E$. Furthermore, the function $(\theta, s) \in C(\mathbb{R}^+; E) \times \mathbb{R}^+ \mapsto \dot{\theta}(s+t)$ is $\tilde{\eta} \otimes \mathcal{L}^1$-almost everywhere equal to a Borel map by lemma B.2, and hence ν_t^y-measurable (and automatically separable valued by an assumption on η).

Now, the estimate

$$\int_{\mathbb{R}^+} \mathrm{d}t \int_E \mathrm{d}\mu_t(y) \left(\int_{C(\mathbb{R}^+; E) \times \mathbb{R}^+} |\dot{\theta}|(s+t) \, \mathrm{d}\nu_t^y(\theta, s) \right)$$

$$= \int_{C(\mathbb{R}^+; E) \times \mathbb{R}^+} \mathrm{d}\sigma(\theta, s) \int_{\mathbb{R}^+} |\dot{\theta}|(s+t) \, \mathrm{d}t = \int_{C(\mathbb{R}^+; E) \times \mathbb{R}^+} \mathrm{d}\sigma(\theta, s) \int_s^{+\infty} |\dot{\theta}|(\tau) \, \mathrm{d}\tau$$

$$= \int_{C(\mathbb{R}^+; E) \times \mathbb{R}^+} \ell(\theta \llcorner [s, +\infty)) \, \mathrm{d}\sigma(\theta, s)$$

$$\leqslant \int_{C(\mathbb{R}^+; E) \times \mathbb{R}^+} \ell(\theta) \, \mathrm{d}\sigma(\theta, s)$$

$$= \int_{C(\mathbb{R}^+; E)} \ell(\theta) \, \mathrm{d}\tilde{\eta}(\theta)$$

$$< +\infty \tag{6.8}$$

shows that the Bochner integral in (6.2) is well defined for a.e. $t \in \mathbb{R}^+$ and μ_t-a.e. $y \in E$, and, moreover, that for such t one has that v_t is Bochner integrable with respect to μ_t for a.e. $t \in \mathbb{R}^+$ (and even more, the function $v \colon E \times \mathbb{R}^+ \to E$ defined by $v(t, x) := v_t(x)$ is Bochner integrable with respect to $\mathrm{d}t \otimes \mu_t$), because

$$\int_{\mathbb{R}^+} \mathrm{d}t \int_E \|v_t(y)\| \, \mathrm{d}\mu_t(y)$$
$$\leqslant \int_{\mathbb{R}^+} \mathrm{d}t \int_E \mathrm{d}\mu_t(y) \left(\int_{C(\mathbb{R}^+; E) \times \mathbb{R}^+} |\dot{\theta}|(s + t) \, \mathrm{d}\nu_t^y(\theta, s) \right)$$
$$< +\infty.$$

STEP 2. We show now the validity of the continuity equation (6.3). Let $\varphi \in Q^1(E) \cap C_{\mathrm{b}}(E)$. Since, in the notation of step 1, for $\tilde{\eta}$-a.e. $\theta \in C(\mathbb{R}^+; E)$ the equality

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} \right) \varphi(\theta(t)) = \langle \dot{\theta}(t), \mathrm{d}\varphi(\theta(t)) \rangle$$

is valid for all $t \in \mathbb{R}^+ \setminus \tilde{N}_\theta$, where $\mathcal{L}^1(\tilde{N}_\theta) = 0$, from lemma 6.2 (applied with $\mu := \lambda_\theta$, $B := \tilde{N}_\theta$) we have that $\lambda_\theta(\tilde{N}_\theta - t) = 0$ for a.e. $t \in \mathbb{R}^+$ and $\tilde{\eta}$-a.e. $\theta \in C(\mathbb{R}^+; E)$, and thus

$$\sigma\left(\left\{ (\theta, s) \in C(\mathbb{R}^+; E) \times \mathbb{R}^+ : \frac{\mathrm{d}}{\mathrm{d}t} \varphi(\theta(s + t)) \neq \langle \dot{\theta}(s + t), \mathrm{d}\varphi(\theta(s + t)) \rangle \right\} \right) = 0$$

for a.e. $t \in \mathbb{R}^+$. Hence, for all such $t \in \mathbb{R}^+$ for μ_t-a.e. $y \in E$ one has

$$\nu_t^y\left(\left\{ (\theta, s) \in C(\mathbb{R}^+; E) \times \mathbb{R}^+ : \frac{\mathrm{d}}{\mathrm{d}t} \varphi(\theta(s + t)) \neq \langle \dot{\theta}(s + t), \mathrm{d}\varphi(\theta(s + t)) \rangle \right\} \right) = 0,$$

which implies that

$$\int_{C(\mathbb{R}^+; E) \times \mathbb{R}^+} \frac{\mathrm{d}}{\mathrm{d}t} \varphi(\theta(s + t)) \, \mathrm{d}\sigma(\theta, s)$$
$$= \int_E \mathrm{d}\mu_t(y) \left(\int_{C(\mathbb{R}^+; E) \times \mathbb{R}^+} \frac{\mathrm{d}}{\mathrm{d}t} \varphi(\theta(s + t)) \, \mathrm{d}\nu_t^y(\theta, s) \right)$$
$$= \int_E \mathrm{d}\mu_t(y) \left(\int_{C(\mathbb{R}^+; E) \times \mathbb{R}^+} \langle \dot{\theta}(s + t), \mathrm{d}\varphi(\theta(s + t)) \rangle \, \mathrm{d}\nu_t^y(\theta, s) \right)$$
$$= \int_E \mathrm{d}\mu_t(y) \left\langle \left(\int_{C(\mathbb{R}^+; E) \times \mathbb{R}^+} \dot{\theta}(s + t) \, \mathrm{d}\nu_t^y(\theta, s) \right), \mathrm{d}\varphi(y) \right\rangle$$
$$= \int_E \mathrm{d}\mu_t(y) \langle v_t(y), \mathrm{d}\varphi(y) \rangle.$$

Note that the above calculation makes sense since

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \varphi(\theta(s + t)) \right| \leqslant \mathrm{Lip}\, \varphi |\dot{\theta}|(s + t),$$

so that the above integral is defined in view of (6.8). Multiplying the above equation by $\psi(t)$, integrating over $\mathbb{R}^+$ and using integration by parts in $\mathrm{d}t$, we get

$$\int_{\mathbb{R}^+} \psi(t)\,\mathrm{d}t \int_E \langle v_t(y), \mathrm{d}\varphi(y) \rangle\,\mathrm{d}\mu_t(y)$$

$$= \int_{\mathbb{R}^+} \psi(t)\,\mathrm{d}t \int_{C(\mathbb{R}^+;E)\times\mathbb{R}^+} \frac{\mathrm{d}}{\mathrm{d}t}\varphi(\theta(s+t))\,\mathrm{d}\sigma(\theta,s)$$

$$= \int_{C(\mathbb{R}^+;E)\times\mathbb{R}^+} \mathrm{d}\sigma(\theta,s) \int_{\mathbb{R}^+} \psi(t)\frac{\mathrm{d}}{\mathrm{d}t}\varphi(\theta(s+t))\,\mathrm{d}t$$

$$= -\int_{C(\mathbb{R}^+;E)\times\mathbb{R}^+} \mathrm{d}\sigma(\theta,s) \int_{\mathbb{R}^+} \dot{\psi}(t)\varphi(\theta(s+t))\,\mathrm{d}t$$

$$= -\int_{\mathbb{R}^+} \dot{\psi}(t)\,\mathrm{d}t \int_{C(\mathbb{R}^+;E)\times\mathbb{R}^+} \varphi(\theta(s+t))\,\mathrm{d}\sigma(\theta,s)$$

$$= -\int_{\mathbb{R}^+} \dot{\psi}(t)\,\mathrm{d}t \int_E \varphi(x)\,\mathrm{d}\mu_t(y),$$

so (6.3) (in the sense of (6.4)) is proven.

STEP 3. Assume now that η represents some laminated current T without cancellation of mass. The relationship (6.5) follows from theorem 5.1. Under the additional assumption that η-a.e. $\theta \in C(\mathbb{R}^+;E)$ is parametrized by arc length, we have $\dot{\theta}(t) \in V_T(\theta(t))''$, and, if E is also strictly convex, then $\dot{\theta}(t) = V_T(\theta(t))$ for η-a.e. $\theta \in C(\mathbb{R}^+;E)$ and for a.e. $t \in [0, \ell(\theta))$ by theorem 5.1. In other words, in the latter case the set

$$M := \{(\theta,\tau) \in C(\mathbb{R}^+;E) \times \mathbb{R}^+ : \dot{\theta}(\tau) \neq V_T(\theta(\tau)),\ \tau < \ell(\theta)\}$$

satisfies $(\eta \otimes \mathcal{L}^1)(M) = 0$. Setting $M_\theta := \{\tau \in \mathbb{R}^+ : (\theta,\tau) \in M\}$, we have $\mathcal{L}^1(M_\theta) = 0$ for η-a.e. $\theta \in C(\mathbb{R}^+;E)$. From lemma 6.2 applied with $\mu := \lambda_\theta$, $B := M_\theta$ we conclude that $\lambda_\theta(M_\theta - t) = 0$ for a.e. $t \in \mathbb{R}^+$ and η-a.e. $\theta \in C(\mathbb{R}^+;E)$. Hence, observing that

$$M_\theta - t = \{s \in \mathbb{R}^+ : \dot{\theta}(s+t) \neq V_T(\theta(s+t)),\ s+t < \ell(\theta)\},$$

we obtain

$$\sigma(\{(\theta,s) \in C(\mathbb{R}^+;E) \times \mathbb{R}^+ : \dot{\theta}(s+t) \neq V_T(\theta(s+t)),\ s+t < \ell(\theta)\}) = 0 \quad (6.9)$$

for a.e. $t \in \mathbb{R}^+$, because $\sigma = \tilde{\eta} \otimes \lambda_\theta$ with $\tilde{\eta} \ll \eta$. Now, (6.9) implies that $\dot{\theta}(s+t) = V_T(\theta(s+t))$ for ν_t^y-a.e. $(\theta,s) \in C(\mathbb{R}^+;E) \times \mathbb{R}^+$ such that $s+t < \ell(\theta)$ and μ_t-a.e. $y \in E$, for a.e. $t \in \mathbb{R}^+$. Thus, from (6.2) we get

$$v_t(y) = \int_{(S_t^y)^c} \dot{\theta}(s+t)\,\mathrm{d}\nu_t^y(\theta,s) = \int_{(S_t^y)^c} V_T(\theta(s+t))\,\mathrm{d}\nu_t^y(\theta,s)$$

$$= \int_{\beta_t^{-1}(y)\setminus S_t^y} V_T(y)\,\mathrm{d}\nu_t^y(\theta,s)$$

$$= V_T(y)(1 - \varphi_t(y)),$$

which shows (6.6) and hence concludes the proof. $\qquad\square$

LEMMA 6.2. *Let $B \subset \mathbb{R}$ be a Borel set with $\mathcal{L}^1(B) = 0$ and let μ be a finite Borel measure over $\mathbb{R}$. Then $\mu(B - t) = 0$ for a.e. $t \in \mathbb{R}$.*

Proof. Otherwise there is a Borel set $\Delta \subset \mathbb{R}$ with $\mathcal{L}^1(\Delta) > 0$ such that $\mu(B - t) > 0$ for all $t \in \Delta$. This implies that

$$I := \int_{\mathbb{R}} \mathbf{1}_\Delta(t) \, \mathrm{d}t \int_{\mathbb{R}} \mathbf{1}_B(x + t) \, \mathrm{d}\mu(x) > 0,$$

but on the other hand, by the Tonelli theorem one has

$$I = \int_{\mathbb{R}} \mathrm{d}\mu(x) \int_{\mathbb{R}} \mathbf{1}_\Delta(t) \mathbf{1}_B(x + t) \, \mathrm{d}t = \int_{\mathbb{R}} \mathrm{d}\mu(x) \int_{\mathbb{R}} \mathbf{1}_\Delta(t) \mathbf{1}_{B-x}(t) \, \mathrm{d}t = 0$$

(since the inner integral vanishes), and this contradiction concludes the proof. $\square$

REMARK 6.3. It is worth mentioning that under the conditions of theorem 6.1(B), if η-a.e. $\theta \in C(\mathbb{R}^+; E)$ is parametrized by arc length, then $\|v_t\|\mu_{t_-} \ll m_T$ for a.e. $t \in \mathbb{R}^+$. In fact, if $B \subset E$ is such that $m_T(B) = 0$, then $(\eta \otimes \mathcal{L}^1)(\tilde{M}) = 0$, where

$$\tilde{M} := \{(\theta, \tau) \in C(\mathbb{R}^+; E) \times \mathbb{R}^+ : \theta(\tau) \in B, \ \tau < \ell(\theta)\}.$$

Thus, $\mathcal{L}^1(\tilde{M}_\theta) = 0$, where $\tilde{M}_\theta := \{\tau \in \mathbb{R}^+ : (\theta, \tau) \in M\}$ for η-a.e. $\theta \in C(\mathbb{R}^+; E)$, and hence by lemma 6.2 (applied with λ_θ, $\tilde{M}_\theta$ instead of μ, B, respectively) one has $\lambda_\theta(\tilde{M}_\theta - t) = 0$ for a.e. $t \in \mathbb{R}^+$ and η-a.e. $\theta \in C(\mathbb{R}^+; E)$, so that

$$\sigma(\{(\theta, s) \in C(\mathbb{R}^+; E) \times \mathbb{R}^+ : \theta(s + t) \in B, \ s + t < \ell(\theta)\}) = 0$$

for a.e. $t \in \mathbb{R}^+$, because $\sigma = \tilde{\eta} \otimes \lambda_\theta$ with $\tilde{\eta} \ll \eta$. Now,

$$\begin{aligned}
\int_B \|v_t(y)\| \, \mathrm{d}\mu_t(y) &\leqslant \int_B \mathrm{d}\mu_t(y) \left(\int_{C(\mathbb{R}^+; E) \times \mathbb{R}^+} |\dot{\theta}|(s + t) \, \mathrm{d}\nu_t^y(\theta, s) \right) \\
&= \int_B \mathrm{d}\mu_t(y) \left(\int_{\{(\theta, s) \in C(\mathbb{R}^+; E) \times \mathbb{R}^+ : \, s + t < \ell(\theta)\}} \mathrm{d}\nu_t^y(\theta, s) \right) \\
&= \sigma(\{(\theta, s) \in C(\mathbb{R}^+; E) \times \mathbb{R}^+ : \theta(s + t) \in B, \ s + t < \ell(\theta)\}) \\
&= 0
\end{aligned}$$

for a.e. $t \in \mathbb{R}^+$.

One has to note that if E is not strictly convex, then our construction may provide a lot of 'unreasonable' flows of a given measure. For instance, in the notation of example 5.3, if $\mu := \mathcal{H}^1 \llcorner ([0, 1] \times \{0\})$, then the flow of μ produced by the measure η (through $\sigma := \eta \otimes \delta_0$; see example 4.4) is given by $\mu_t := \mathcal{H}^1 \llcorner ([0, 1] \times \{t \wedge 1\})$, although $T = V_T \wedge m_T$ with $m_T = 2\mathcal{L}^2 \llcorner Q$, $V_T = (\bar{e}_1 + \bar{e}_2)/2$, i.e. this flow is not in the direction of V_T. This does not happen when we equip E with a strictly convex (say, Euclidean) norm.

7. Canonical flows

We present here two particular constructions of the flow that may be considered canonical.

7.1. Acyclic vector fields

Consider a measurable vector field corresponding to an *acyclic* normal current $T \in \mathcal{M}_1(E)$ (i.e. such that $C \in \mathcal{M}_1(E)$, $C \leqslant T$, and $\partial C = 0$ implies that $C = 0$). In this case there is a measure η over $C(\mathbb{R}^+; E)$ representing T without cancellation of mass and satisfying $(\partial T)^+ = \mathrm{end}_\# \eta$ and $(\partial T)^- = \mathrm{start}_\# \eta$. We may now claim the following result.

PROPOSITION 7.1. *Let μ be a finite measure over E with $\mu \perp (\partial T)^+$. The following assertions hold true.*

  (i) *If μ is transportable by η, then there is a normal current $S \leqslant T$ with μ equivalent to $(\partial S)^-$. Furthermore, one has that μ_t is equivalent to $e_{t\#}\bar{\eta}$ for some $\bar{\eta}$ representing S without cancellation of mass.*

  (ii) *If η is concentrated over curves parametrized by arc length (which can always be assumed without loss of generality) and almost everywhere differentiable, and E is a strictly convex Banach space, then, for any flow μ_t corresponding to η, if there is an interval $(0, \tau)$ such that $\mu_t \perp (\partial T)^+ = 0$ for $t \in (0, \tau)$, one has*

$$\frac{\partial \mu_t}{\partial t} + \mathrm{div}\, V_T \mu_t = 0 \qquad (7.1)$$

  in the weak sense over $(0, \tau)$.

Proof. To prove (i), recall that by remark 4.8, if μ is transportable by η, then there is a normal current $S \leqslant T$ such that $\partial S = \nu - \tilde{\mu}$ with ν equivalent to $\mathrm{end}_\# \eta$ and $\tilde{\mu}$ equivalent to μ. But by assumption on η one has $\mathrm{end}_\# \eta = (\partial T)^+$, and hence $\mu \perp \mathrm{end}_\# \eta$ (by the assumption on μ), and therefore also $\tilde{\mu} \perp \mathrm{end}_\# \eta$. Hence, $(\partial S)^- = \tilde{\mu}$. Furthermore, in the notation of remark 4.8 we have that S is represented by a measure $\bar{\eta}$ over $C(\mathbb{R}^+; E)$ without cancellation of mass, where

$$\bar{\eta}(B) := \tilde{\sigma}(\{(\theta, s) \subset C(\mathbb{R}^+; E) \times \mathbb{R}^+ : s \in \mathbb{R}^+,\ r_s(\theta) \in B\}),$$

where $r_s(\theta)(t) = e_t(r(\theta, s)) = \theta(t + s)$, $r \colon C(\mathbb{R}^+; E) \times \mathbb{R}^+ \to C(\mathbb{R}^+; E)$ is defined by $r(\theta, s)(t) := \theta(t + s)$, and $B \subset C(\mathbb{R}^+; E)$ is an arbitrary Borel set (in other words, $\bar{\eta} := r_\# \tilde{\sigma}$). Thus, one has $\beta_{t\#} \tilde{\sigma} = e_{t\#} \bar{\eta}$, and recalling that μ_t is equivalent to $\beta_{t\#} \tilde{\sigma}$, one concludes the proof of (i).

Now, under the assumption of (ii) we have that $\mu_t \perp \mathrm{end}_\# \eta$ for all $t \leqslant \tau$. Hence, for μ_t-a.e. $y \in E$ one has, in the notation of theorem 6.1(C), that $S_t^y = \emptyset$, which gives $\varphi_t(y) = 0$ for $t \leqslant \tau$, which proves (ii). $\qquad\square$

7.2. Cyclic vector fields

Consider a flow of a given measure generated by some cyclic, or, in other words, divergence-free vector field, i.e. such that the respective current $T \in \mathcal{M}_1(E)$ is a cycle, $\partial T = 0$. According to our definition this is a flow produced by some measures η over $C(\mathbb{R}^+; E)$ representing T without cancellation of mass and concentrated over a set of curves of finite length. This might lead to a non-natural notion of the flow. For instance, if η-a.e. $\theta \in C(\mathbb{R}^+; E)$ is parametrized by arc length, then $|\dot{\theta}|(t) = 0$ for a $t > \ell(\theta)$, which means that each 'particle' moving along the path θ

stops at instant $\ell(\theta)$. Let, for instance, T be a cycle associated with the clockwise oriented unit circumference in $E = \mathbb{R}^2$, let $\mu := \delta_{(-1,0)}$, and observe that $T = T_\eta$ with $\eta := \delta_{\theta_1} + \delta_{\theta_2}$, θ_1 and θ_2 being arc-length parametrized right and left semi-circumferences, directed from top to bottom and from bottom to top, respectively. Clearly, δ_{θ_2} transports μ through the measure $\sigma := \delta_{\theta_2} \otimes \delta_0$ (see example 4.4), but one has in this situation that $\mu_t = \delta_{(-\cos t, \sin t)}$ when $t \in [0, \pi/2)$, and $\mu_t = \delta_{(0,1)}$ when $t \geqslant \pi/2$, so that the particle stops moving after $\pi/2$. On the contrary, since $\partial T = 0$ what one naturally expects is that the particle never stops moving (because there is 'no boundary to reach'). It is not difficult, however, to assert the existence of such a natural flow, as the following proposition shows. In particular, in the above example the particle under such a flow will endlessly rotate along the unit circumference.

PROPOSITION 7.2. *Let η represent some cycle $T \in \mathcal{M}_1(E)$, $\partial T = 0$, without cancellation of mass and concentrated over arc-wise-parametrized curves of unit length (i.e. parametrized with unit speed over $[0,1]$ and stopping definitely at $t = 1$). Then there is a Borel measure $\hat\eta$ over $C(\mathbb{R}; E)$ (equipped with the topology of uniform convergence over bounded intervals) concentrated over $\mathrm{Lip}_1(\mathbb{R}; E)$ such that*

(a) *$\pi^1_{\#}\hat\eta = \eta$, where $\pi^k \colon C(\mathbb{R}; E) \to C(\mathbb{R}^+; E)$ is the map defined by*

$$\pi^k(\theta)(t) := \theta(t \wedge k), \quad t \in \mathbb{R}^+;$$

(b) *$g^{\pm}_{\#}\hat\eta = \hat\eta$, where $g^{\pm} \colon C(\mathbb{R}; E) \to C(\mathbb{R}; E)$ are the shift maps defined by*

$$g^{\pm}(\theta)(t) := \theta(t \pm 1);$$

(c) *for $\hat\eta$-a.e. $\theta \in C(\mathbb{R}; E)$ one has $\theta(\mathbb{R}) \subset \mathrm{supp}\, T$;*

(d) *$\pi^k_{\#}\hat\eta$ represents the current kT without cancellation of mass;*

(e) *$\hat\eta$-a.e. $\theta \in C(\mathbb{R}; E)$ never stops, namely, it has almost everywhere constant unit velocity.*

If μ is transportable by η through a measure $\sigma = \tilde\eta \otimes \lambda_\theta$ over $C(\mathbb{R}^+; E) \times \mathbb{R}^+$ with $\tilde\eta := q_{\#}\sigma \ll \eta$, $\tilde\eta = \alpha\eta$ for some $\alpha \in L^1(C(\mathbb{R}^+; E), \eta)$, then μ is transportable by $\hat\eta$ through

$$\hat\sigma := ((\alpha \circ \pi^1)\hat\eta) \otimes \lambda_{\pi^1(\theta)}.$$

If E is a strictly convex Banach space, and the η-a.e. curve is almost everywhere differentiable, then the respective flow $\hat\mu_t := \beta_{t\#}\hat\sigma$ satisfies

$$\frac{\partial \hat\mu_t}{\partial t} + \mathrm{div}\, V_T \hat\mu_t = 0 \tag{7.2}$$

in the weak sense and

$$\dot\theta(t) = V_T(\theta(t)) \tag{7.3}$$

for a.e. $t \in \mathbb{R}^+$ and $\hat\eta$-a.e. $\theta \in C(\mathbb{R}^+; E)$.

Proof. Existence of the measure $\hat{\eta}$ with properties (a)–(c) is just proposition 4.2 from [20], property (d) is remark 4.3 from the same paper, and, finally, (e) follows from (a) and the fact that η-a.e. $\theta \in C(\mathbb{R}^+, E)$ is parametrized with unit speed over $[0, 1]$ stopping definitely at $t = 1$. Thus, (7.3) follows from (d) combined with theorem 5.1.

To show that μ is transportable by $\hat{\eta}$ through $\hat{\sigma}$, we note that $q_{\#}\hat{\sigma} = (\alpha \circ \pi^1)\hat{\eta}$ and, recalling (a), we get

$$\int_{C(\mathbb{R}^+;E)} (\alpha \circ \pi^1)(\theta) \, \mathrm{d}\hat{\eta}(\theta) = \int_{C(\mathbb{R}^+;E)} \alpha(\theta) \, \mathrm{d}(\pi^1_{\#}\hat{\eta})(\theta) = \int_{C(\mathbb{R}^+;E)} \alpha(\theta) \, \mathrm{d}\eta(\theta) \leqslant +\infty,$$

because $a \in L^1(C(\mathbb{R}^+; E), \eta)$. Thus $\alpha \circ \pi^1 \in L^1(C(\mathbb{R}^+; E), \eta)$, and hence $q_{\#}\hat{\sigma} \ll \hat{\eta}$. We now calculate $\beta_{0\#}\hat{\sigma}$. For every $f \in C_{\mathrm{b}}(E)$ one has

$$\begin{aligned}
\int_E f(x) \, \mathrm{d}(\beta_{0\#}\hat{\sigma})(x) &= \int_{C(\mathbb{R}^+;E) \times \mathbb{R}^+} f(\theta(s)) \, \mathrm{d}\hat{\sigma}(\theta, s) \\
&= \int_{C(\mathbb{R}^+;E)} \alpha(\pi^1(\theta)) \, \mathrm{d}\hat{\eta}(\theta) \int_{\mathbb{R}^+} f(\theta(s)) \, \mathrm{d}\lambda_{\pi^1(\theta)}(s) \\
&= \int_{C(\mathbb{R}^+;E)} \alpha(\theta) \, \mathrm{d}(\pi^1_{\#}\hat{\eta})(\theta) \int_{\mathbb{R}^+} f(\theta(s)) \, \mathrm{d}\lambda_{\theta}(s),
\end{aligned}$$

the latter relationship being due to the change of variables and the fact that $((\pi^1)^{-1}(\theta))(s) = \theta(s)$ for $s \in [0, 1]$ and that the λ_{θ} are concentrated over $[0, 1]$. Thus, recalling (a), we get

$$\begin{aligned}
\int_E f(x) \, \mathrm{d}(\beta_{0\#}\hat{\sigma})(x) &= \int_{C(\mathbb{R}^+;E)} \alpha(\theta) \, \mathrm{d}\eta(\theta) \int_{\mathbb{R}^+} f(\theta(s)) \, \mathrm{d}\lambda_{\theta}(s) \\
&= \int_{C(\mathbb{R}^+;E) \times \mathbb{R}^+} f(\theta(s)) \, \mathrm{d}\sigma(\theta, s) \\
&= \int_E f(x) \, \mathrm{d}(\beta_{0\#}\sigma)(x) \\
&= \int_E f(x) \, \mathrm{d}\mu(x),
\end{aligned}$$

because μ is transportable through σ. Therefore, $\beta_{0\#}\hat{\sigma} = \mu$, and thus μ is transportable also through $\hat{\sigma}$.

It remains to show (7.2). To this end, observe that

$$\hat{\mu}_t = \beta_{t\#}\hat{\sigma} = \beta_{t\#}\hat{\sigma}^k,$$

where $k \in \mathbb{N}$ is any natural number satisfying $k \geqslant \lceil t \rceil + 1$, and

$$\hat{\sigma}^k := (((\alpha \circ \pi^1) \circ \pi^k)\pi^k_{\#}\hat{\eta}) \otimes \lambda_{\pi^1(\theta)} = ((\alpha \circ \pi^1)\pi^k_{\#}\hat{\eta}) \otimes \lambda_{\pi^1(\theta)}$$

(the latter equality being due to the fact that $\pi^1(\pi^k(\theta)) = \pi^1(\theta)$ for all $\theta \in C(\mathbb{R}^+; E)$). In particular, this means that $\mu = \hat{\mu}_0$ is transportable by $\pi^k_{\#}\hat{\eta}$ through $\hat{\sigma}^k$. Now let $\tau > 0$ be arbitrary and let $k := \lceil t \rceil + 1$. Recalling (d), we get (7.2) for $t \in (0, \tau)$ with the help of theorem 6.1(C) (note that $\varphi_t \equiv 0$, because the $\pi^k_{\#}\hat{\eta}$-a.e. curve does not stop definitely). But since $\tau > 0$ is arbitrary, we get (7.2) for $t \in \mathbb{R}^+$, thus concluding the proof. □

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Appendix A. Metric currents and measurable vector fields

A.1. Metric currents, mass measures and fields of directions

We provide some results on the representation of an arbitrary current $T \in \mathcal{M}_1(E)$ in a Banach space E in the form

$$T(f\,\mathrm{d}\pi) = \int_E f(x)\langle V(x), \mathrm{d}\pi(x)\rangle\,\mathrm{d}\mu(x) \qquad (\mathrm{A}\,1)$$

for some Borel measure μ and some Borel map $V\colon E \to E$, $\|V\| \in L^1(E;\mu)$, where $\pi \in Q^1(E)$. If such a representation exists, we will write (as it is usual in the context of de Rham currents) $T = V \wedge \mu$. Of course, such a representation is not unique; for instance, $2V \wedge \mu/2$ gives the same current. The map V can be viewed as representing the 'field of directions' of the current.

LEMMA A.1. *Let E be a Banach space endowed with the norm $\|\cdot\|$ and let $T = V \wedge \mu \in \mathcal{M}_1(E)$. Then $m_T = \|V(x)\|\mu$. In particular, if $\mu = m_T$, then $\|V(x)\| = 1$ for m_T-a.e. $x \in E$.*

REMARK A.2. It follows from lemma A.1 that for a given μ the map V providing the representation $(\mathrm{A}\,1)$ is unique up to μ-almost everywhere equality. In fact, if $V \wedge \mu = \tilde{V} \wedge \mu$, then $(V - \tilde{V}) \wedge \mu = 0$, and hence one has $\int_E \|V - \tilde{V}\|\,\mathrm{d}\mu = 0$.

Proof. Since for $\pi \in Q^1(E)$ one has $\|\mathrm{d}\pi(x)\|' \leqslant \mathrm{Lip}\,\pi$ for all $x \in E$, the representation $(\mathrm{A}\,1)$ implies that

$$|T(f\,\mathrm{d}\pi)| \leqslant \int_E |f(x)|\|\mathrm{d}\pi(x)\|'\|V(x)\|\,\mathrm{d}\mu(x) \leqslant \mathrm{Lip}\,\pi \int_E |f(x)|\|V(x)\|\,\mathrm{d}\mu(x) \quad (\mathrm{A}\,2)$$

for $\pi \in Q^1(E)$. We may assume without loss of generality E to be separable (if not, just substitute it for the linear span of the closure of a σ-compact set on which m_T is concentrated). Now, as shown in [14], if $\pi \in \mathrm{Lip}_\mathrm{b}(E)$ with E a separable Banach space, then there is a sequence $\pi_k \in Q^1(E) \cap C_\mathrm{b}(E)$ with $\mathrm{Lip}\,\pi_k \leqslant \mathrm{Lip}\,\pi$ converging to π pointwise as $k \to \infty$, so that $\lim_k T(f\,\mathrm{d}\pi_k) = T(f\,\mathrm{d}\pi)$. This provides $(\mathrm{A}\,2)$ for all $\pi \in \mathrm{Lip}_\mathrm{b}(E)$. Finally, if $\pi \in \mathrm{Lip}(E)$, then defining $\pi_k := (-k) \wedge \pi \vee k$ we have $\pi_k \in \mathrm{Lip}_\mathrm{b}(E)$ with $\mathrm{Lip}\,\pi_k \leqslant \mathrm{Lip}\,\pi$, and $\lim_k \pi_k(x) = \pi(x)$ for every $x \in E$, and therefore $\lim_k T(f\,\mathrm{d}\pi_k) = T(f\,\mathrm{d}\pi)$, which means that $(\mathrm{A}\,2)$ holds for every $f\,\mathrm{d}\pi \in D^1(E)$, thus implying that $m_T \leqslant \|V\|\mu$ by the definition of the mass measure.

To prove the opposite inequality, note that since μ is a tight measure it is concentrated on some separable set $S \subset E$. By [8, lemma 6.10.16], the set $V(S)$ is separable (since V is Borel), and thus we may choose a countable dense subset $\{\alpha_j\} \subset V(S)$. For a given $\varepsilon > 0$ let

$$E_i := \{x \in S \colon \|V(x) - \alpha_i\| \leqslant \varepsilon\},$$

$$D_1 := E_1, \quad D_i := E_i \setminus \bigcup_{j=1}^{i-1} D_i.$$

We have that the function $V_\varepsilon \colon E \to E$ defined by $V_\varepsilon(x) := \sum_i \alpha_i \mathbf{1}_{D_i}(x)$ is Borel and $\|V(x) - V_\varepsilon(x)\| \leqslant \varepsilon$ for μ-a.e. $x \in E$. Now let $a_j \in E'$ be such that $\|a_j\|' = 1$ and $\langle \alpha_j, a_j \rangle = \|\alpha_j\|$, and set $a_\varepsilon(x) := \sum_i a_i \mathbf{1}_{D_i}(x)$ so that $a_\varepsilon \colon E \to E'$ is Borel with $\|a_\varepsilon(x)\|' = 1$ and $\langle V_\varepsilon(x), a_\varepsilon(x) \rangle = \|V_\varepsilon(x)\|$ for μ-a.e. $x \in E$. Let also $\pi_i \colon E \to \mathbb{R}$ stand for the linear functional $\pi_i(x) := \langle x, a_i \rangle$ (so that $\operatorname{Lip} \pi_i = 1$). Then for every Borel $B \subset E$ one gets the estimate

$$
\begin{aligned}
m_T(B) &\geqslant \sum_i T(\mathbf{1}_e \mathbf{1}_{D_i} \, \mathrm{d}\pi_i) \\
&= \int_B \langle V(x), a_\varepsilon \rangle \, \mathrm{d}\mu(x) \\
&\geqslant \int_B \langle V_\varepsilon(x), a_\varepsilon \rangle \, \mathrm{d}\mu(x) - \int_E \|V(x) - V_\varepsilon(x)\| \|a_\varepsilon(x)\|' \, \mathrm{d}\mu(x) \\
&\geqslant \int_B \|V_\varepsilon(x)\| \, \mathrm{d}\mu(x) - \varepsilon \mu(E).
\end{aligned}
$$

Sending $\varepsilon \to 0^+$, we get

$$m_T(B) \geqslant \int_B \|V(x)\| \, \mathrm{d}\mu(x),$$

which concludes the proof since $B \subset E$ is arbitrary. □

LEMMA A.3. *Let E be a finite-dimensional normed space endowed with the norm $\| \cdot \|$, and let $T \in \mathcal{M}_1(E)$. Then $T = V_T \wedge m_T$ for some Borel map $V_T \colon E \to E$ satisfying $\|V_T(x)\| = 1$ for m_T-a.e. $x \in E$.*

Proof. The representation of T in the form (A 1) with $V_T \in L^\infty(E; m_T)$ is due to [23, theorem 1.3] for the case in which $m_T \ll \mathcal{L}^n$. To prove it for the general case, let $T_k \in \mathcal{M}_1(E)$ be such that $T_k \rightharpoonup T$, $m_{T_k} \rightharpoonup m_T$ as $k \to +\infty$, and $m_{T_k} \ll \mathcal{L}^n$ for all $k \in \mathbb{N}$ (such a sequence is constructed, say, by convolutions of T with smooth approximate identity, like, for example, in step 1 of the proof of lemma C.1 in [19]; note that there one only proves that $T_k \rightharpoonup T$, $\mathbb{M}(T_k) \to \mathbb{M}(T)$, and then $m_{T_k} \rightharpoonup m_T$ is lemma A.3 from [20]). Then $T_k = V_k \wedge m_{T_k}$ for some Borel $V_k \colon E \to E$ satisfying $\|V_k(x)\| = 1$ for m_{T_k}-a.e. $x \in E$. Up to a subsequence (not relabelled) we have that the sequence of vector measures $\{V_k m_{T_k}\}$ over E is weakly convergent to some vector measure in E of finite total variation. Denoting the latter limit measure by $V\mu$ for some positive Borel measure μ and some Borel $V \colon E \to \mathbb{R}^n$ satisfying $\|V(x)\| = 1$ for μ-a.e. $x \in E$, we get for $f \, \mathrm{d}\pi \in D^1(E)$ and $\pi \in C^1(E) \cap \operatorname{Lip}(E)$ the

relationship

$$T(f\,\mathrm{d}\pi) = \lim_k T_k(f\,\mathrm{d}\pi) = \int_E f(x)\langle V(x), \mathrm{d}\pi(x)\rangle\,\mathrm{d}\mu(x),$$

so that $T = V \wedge \mu$, and hence $m_T = \|V\|\mu$ by lemma A.1. It suffices now to set $V_T(x) := V(x)/\|V(x)\|$ (with the convention $0/0 := 0$). $\qquad\square$

We find it useful to provide here a rather particular property of laminated currents regarding the representation (A 1), which, although not used elsewhere in the paper, is useful to get a clear idea of how different they are from generic metric currents. We will state it in the particular case in which $E = \mathbb{R}^n$ is a Euclidean space. Recall the following definition of a tangent space with respect to a finite positive Borel measure μ over $\mathbb{R}^n$ introduced in [9] and further studied in [11]. Denote by X_μ the set of vector-valued Borel functions $V \colon E \to E$ such that $|V| \in L^1_{\mathrm{loc}}(E; \mu)$ and the distributional divergence $\operatorname{div} V\mu$, defined as

$$\langle \varphi, \operatorname{div} V\mu \rangle := \int_{\mathbb{R}^n} \nabla\varphi \cdot V\,\mathrm{d}\mu, \quad \varphi \in C_0^\infty(E),$$

is a finite signed Radon measure over E. Then the tangent space Q_μ to a measure μ at each point $x \in \mathbb{R}^n$ is defined as the μ-essential union of $\{V(x)\colon V \in X_\mu\}$, i.e. $Q_\mu \colon \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is the unique μ-measurable closed-valued multifunction such that

(i) if $V \in X_\mu$, then $V(x) \in Q_\mu(x)$ for μ-a.e. $x \in \mathbb{R}^n$; and

(ii) it is minimal among all such multifunctions with respect to the inclusion μ-almost everywhere, i.e. if $P \colon \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is another μ-measurable closed-valued multifunction such that when $V \in X_\mu$, $V(x) \in P(x)$ for μ-a.e. $x \in \mathbb{R}^n$, then necessarily $Q_\mu(x) \subset P(x)$ for μ-a.e. $x \in \mathbb{R}^n$.

PROPOSITION A.4. *If $E = \mathbb{R}^n$ is Euclidean space, then for every laminated current $T \in \mathcal{M}_1(E)$ the representation $T = V_T \wedge m_T$ implies that $V_T(y) \in Q_{m_T}(y)$ for m_T-a.e. $y \in E$.*

REMARK A.5. Note that the representation $T = V_T \wedge m_T$ with V_T tangent to m_T is not true for a generic (not necessarily laminated) current T: in fact, for the current from remark A.12 one has $Q_{m_T}(y) = \{0\}$ for m_T-a.e. $y \in E = \mathbb{R}$, and hence the existence of such a representation would give $T = 0$, which is obviously not the case.

Proof. Let us show that $V_T(y) \in Q_{m_T}(y)$ for m_T-a.e. $y \in E$. In fact, this is clearly true if T is normal, since in this case

$$\langle \operatorname{div} V_T m_T, \varphi \rangle = \int_E \nabla\varphi \cdot V_T\,\mathrm{d}m_T = T(1\,\mathrm{d}\varphi) = \partial T(\varphi), \quad \varphi \in C_0^\infty(E),$$

so that $|\langle \operatorname{div} V\mu, \varphi\rangle| \leqslant \mathbb{M}(\partial T) \cdot \|\varphi\|_\infty$, which in other words means that $V_T \in X_{m_T}$. Now passing to the general case of a laminated current T, let $B_k \subset C(\mathbb{R}^+; E)$ be an increasing sequence of Borel sets $B_k \nearrow C(\mathbb{R}^+; E)$ such that the measures $\eta_k := \eta \llcorner B_k$ are finite. Clearly, one has that $T_k := T_{\eta_k}$ are represented by η_k without

cancellation of mass, are normal (because the η_k are finite), and $T_k \leqslant T_{k+1} \leqslant T$ for every $k \in \mathbb{N}$, and $\mathbb{M}(T - T_k) \to 0$ as $k \to \infty$ (because $\ell(\cdot)\eta$ is a finite measure). Thus, by lemma A.6 one has that $T_k = V \wedge \lambda_k m_T = \lambda_k V \wedge m_T$ for some $\lambda_k \in L^\infty(E; m_T)$, $0 \leqslant \lambda_k \leqslant 1$, and, moreover, the sequence of functions λ_k is pointwise non-decreasing, $\lambda_k \nearrow 1$ as $k \to \infty$. As already proven, one then has that $\lambda_k(y)V_T(y) \in Q_{m_T}(y)$ for m_T-a.e. $y \in E$. Since for each such $y \in E$ there is a $j \in \mathbb{N}$ such that $\lambda_k(y) \geqslant \lambda_j(y) > 0$, we have that $V_T(y) \in Q_{m_T}(y)$, as claimed. $\quad\square$

The following auxiliary statement, which is of certain interest by itself, has been used in the above proof.

LEMMA A.6. *If E is a strictly convex finite-dimensional normed space, $T = V \wedge m_T \in \mathcal{M}_1(E)$, $S \in \mathcal{M}_1(E)$ with $S \leqslant T$, then there is a $\lambda \in L^\infty(E; m_T)$ with $0 \leqslant \lambda(x) \leqslant 1$ for m_T-a.e. $x \in E$ such that $m_S = \lambda m_T$ and $S = V \wedge m_S = V \wedge \lambda m_T$.*

Proof. Since $S \leqslant T$, we have $m_S + m_{T-S} = m_T$, and, in particular, $m_S \leqslant m_T$ (see [19, remark 3.5]), and hence there is a $\lambda \in L^\infty(E; m_T)$ with $0 \leqslant \lambda(x) \leqslant 1$ for m_T-a.e. $x \in E$ such that $m_S = \lambda m_T$ and $m_{T-S} = (1 - \lambda)m_T$. By lemma A.3 one has $S = V_S \wedge \lambda m_T = \lambda V_S \wedge m_T$, with $\|V_S(x)\| = 1$ for λm_T-a.e. $x \in E$, and

$$T - S = V_{T-S} \wedge (1 - \lambda)m_T = (1 - \lambda)V_{T-S} \wedge m_T,$$

with $\|V_{T-S}(x)\| = 1$ for $(1 - \lambda)m_T$-a.e. $x \in E$. On the other hand,

$$T - S = (V - \lambda V_S) \wedge m_T$$

so that $V - \lambda V_S = (1 - \lambda)V_{T-S}$. Thus, for m_T-a.e. $x \in E$ such that $\lambda(x) \neq 0$, either $\lambda(x) = 1$, which implies that $V_S(x) = V(x)$, or $0 < \lambda(x) < 1$, which implies that

$$\|V(x) - \lambda V_S(x)\| = 1 - \lambda(x) = \|(V(x)\| - \|\lambda(x)V_S(x))\|,$$

which is only possible (recalling that $\|V(x)\| = \|V_S(x)\| = 1$) when $V_S(x) = V(x)$, thus concluding the proof. $\quad\square$

REMARK A.7. It is easy to observe from the proof of lemma A.6 that its statement is true in a generic metric space E if one knows *a priori* that both $S = V_S \wedge m_S$ and $T - S = V_{T-S} \wedge m_{T-S}$ for some Borel maps V_S and V_{T-S}.

A.2. Remarks on measurable vector fields

We mention the following easy properties of measurable vector fields.

(1) *Change of measure.* If $\psi \ll \mu$, then a μ-vector field is also a ψ-vector field (because $L^\infty(E; \mu) \subset L^\infty(E; \psi)$ with continuous embedding; to prove weak continuity one just observes that for each $v \in L^1(E; \psi)$ one has $v(\mathrm{d}\psi/\mathrm{d}\mu) \in L^1(E; \mu)$, where $\mathrm{d}\psi/\mathrm{d}\mu$ stands for the respective Radon–Nikodym derivative).

(2) *Locality.* If $f(x) = \text{const.}$ for μ-a.e. $x \in A \subset E$, then $(Xf)(x) = 0$ for μ-a.e. $x \in A$ if X is a μ-vector field (see [16, lemma 13.4]). In particular, if $f(x) = g(x)$ for μ-a.e. $x \in A$, then $(Xf)(x) = (Xg)(x)$ for μ-a.e. $x \in A$.

(3) *Bounds.* One has that $|(Xf)(x)| \leqslant C \operatorname{Lip} f$ for μ-a.e. $x \in E$ (with $C > 0$ depending on X). Moreover, thanks to the locality property, X can be uniquely extended to the whole of $\operatorname{Lip}(E)$ (we will slightly abuse the notation, denoting this extension by the same letter X) with such a bound (see [13, theorem 2.15]). This extension satisfies the weak continuity property in the form $\lim_k X f_k = Xf$ in the weak* sense of $L^\infty(E; \mu)$ whenever $\lim_k f_k = f$ pointwise and with $\operatorname{Lip} f_k$ uniformly bounded. In fact, for a fixed $z \in E$, defining $f^r := \inf_{B_r(z)} f \vee f \wedge \sup_{B_r(z)} f$ for an $f \in \operatorname{Lip}(E)$, we get $f = f^r$ over $B_r(z)$, and hence for an arbitrary $g \in L^1(E; \mu)$ one has

$$\int_E g X f_k \, \mathrm{d}\mu = \int_E g \mathbf{1}_{B_r(z)} X f_k^r \, \mathrm{d}\mu + \int_E g \mathbf{1}_{B_r(z)^\mathrm{c}} X f_k \, \mathrm{d}\mu,$$

$$\int_E g X f \, \mathrm{d}\mu = \int_E g \mathbf{1}_{B_r(z)} X f^r \, \mathrm{d}\mu + \int_E g \mathbf{1}_{B_r(z)^\mathrm{c}} X f \, \mathrm{d}\mu,$$

so that the claim follows from the estimates

$$\left| \int_E g \mathbf{1}_{B_r(z)^\mathrm{c}} X f_k \, \mathrm{d}\mu \right| \leqslant \int_{B_r(z)^\mathrm{c}} |g| \, \mathrm{d}\mu \to 0 \quad \text{as } r \to +\infty \text{ uniformly in } k,$$

$$\left| \int_E g \mathbf{1}_{B_r(z)^\mathrm{c}} X f \, \mathrm{d}\mu \right| \leqslant \int_{B_r(z)^\mathrm{c}} |g| \, \mathrm{d}\mu \to 0 \quad \text{as } r \to +\infty$$

and from the fact that now the f_k^r are uniformly bounded in k (with a bound depending only on $f(z)$, on the bound on the Lipschitz constants of f_k, and on r), and hence $\lim_k \int_E g \mathbf{1}_{B_r(z)} X f_k^r \, \mathrm{d}\mu = \int_E g \mathbf{1}_{B_r(z)} X f^r \, \mathrm{d}\mu$.

We now show the relationship between measurable vector fields and metric currents.

PROPOSITION A.8. *Every μ-vector field X over E defines a metric current $T \in \mathcal{M}_1(E)$ by the formula*

$$T(f \, \mathrm{d}\pi) := \int_E f X \pi \, \mathrm{d}\mu, \tag{A 3}$$

with $m_T \leqslant C\mu$. Conversely, for every metric current $T \in \mathcal{M}_1(E)$ with $m_T \leqslant C\mu$ there is a unique μ-vector field X over E such that (A 3) holds.

Proof. One immediately verifies that if X is a μ-vector field, then T defined by (A 3) is a one-dimensional metric current, and

$$|T(f \, \mathrm{d}\pi)| := C \operatorname{Lip} \pi \int_E |f| \, \mathrm{d}\mu,$$

so that $m_T \leqslant C\mu$. On the other hand, if $T \in \mathcal{M}_1(E)$, then for every $\pi \in \operatorname{Lip}_\mathrm{b}(E)$ define $X_T \pi \in L^\infty(E; m_T)$ by the relationship

$$\int_E f X_T \pi \, \mathrm{d}m_T := T(f \, \mathrm{d}\pi)$$

for every $f \in L^1(E; m_T)$. It is immediate to verify that X_T is an m_T-derivation. If $\mu \neq m_T$, then $m_T \ll \mu$ implies by the Radon–Nikodym theorem that $m_T = \alpha\mu$ for

some Borel $\alpha\colon E\to\mathbb{R}^+$. Letting $X\pi:=\alpha X_T\pi$, we get

$$\int_E f X\pi\,\mathrm{d}\mu = \int_E f\alpha(X_T\pi)\,\mathrm{d}\mu = \int_E f X_T\pi\,\mathrm{d}m_T = T(f\,\mathrm{d}\pi),$$

and since this is valid for every $f\in L^1(E;\mu)$, and hence for every $f\in L^1(E;m_T)$, we have $X\pi\in L^\infty(E;\mu)$. The uniqueness of the vector field X follows from the immediate observation that $\int_E f X\pi\,\mathrm{d}\mu = 0$ for every $f\in L^1(E;\mu)$, which means that $X\pi = 0$ for every $\pi\in\mathrm{Lip}(E)$, and hence $X = 0$. $\qquad\square$

Note that if one decides to use the language of measurable vector fields instead of that of currents, then the notion of the boundary of a current has to be substituted by the equivalent one of the divergence of a μ-vector field X defined as a linear continuous functional over $\mathrm{Lip}_\mathrm{b}(E)$ (equipped with the norm $\|u\|_{\mathrm{Lip}} := \|u\|_\infty + \mathrm{Lip}\,u$) by the formula

$$\langle f, \mathrm{div}_\mu X\rangle := \int_E Xf\,\mathrm{d}\mu = T(1\,\mathrm{d}f) = \partial T(f), \tag{A 4}$$

where $f\in\mathrm{Lip}_\mathrm{b}(E)$, $T\in\mathcal{M}_1(E)$, and $\langle\cdot,\cdot\rangle$ stands for the duality between $\mathrm{Lip}_\mathrm{b}(E)$ and its continuous dual. If X is normal, then in fact the above defined divergence is a signed Radon measure of finite total variation $|\mathrm{div}_\mu X|(E)\leqslant\mathbb{M}(\partial T)$, and hence definition (A 4) may be extended to $f\in C_\mathrm{b}(E)$, and $\langle\cdot,\cdot\rangle$ may be interpreted as the standard duality between $C_\mathrm{b}(E)$ and the space of signed Radon measure of finite total variation. If $E=\mathbb{R}^n$ and μ is the Lebesgue measure, when $V\colon\mathbb{R}^n\to\mathbb{R}^n$ is a bounded measurable map (i.e. a finite-dimensional vector field in the classical sense), and X is defined as the classical directional derivative along V, i.e. $Xf := V\cdot\nabla f$ for smooth functions f, we have that $\mathrm{div}_\mu X$ is identified with the usual distributional divergence of V.

A.3. Representation of currents by measures over curves

The following remarks are worth being made.

REMARK A.9. If a non-negative σ-finite Borel measure η over $C(\mathbb{R}^+;E)$ is such that

$$T(\omega) = \int_{C(\mathbb{R}^+;E)} [\![\theta]\!](\omega)\,\mathrm{d}\eta(\theta) \quad\text{for all } \omega\in D^1(E), \tag{A 5}$$

which we commonly write as $T = T_\eta$, then

$$\mathbb{M}(T) \leqslant \int_{C(\mathbb{R}^+;E)} \mathbb{M}([\![\theta]\!])\,\mathrm{d}\eta(\theta) \leqslant \int_{C(\mathbb{R}^+;E)} \ell(\theta)\,\mathrm{d}\eta(\theta). \tag{A 6}$$

In fact,

$$|T(f\,\mathrm{d}\pi)| \leqslant \int_{C(\mathbb{R}^+;E)} \left(\mathrm{Lip}\,\pi\int_E |f|\,\mathrm{d}m_{[\![\theta]\!]}\right)\mathrm{d}\eta(\theta),$$

which gives for a Borel set $B\subset E$ the inequality

$$m_T(B) \leqslant \int_{C(\mathbb{R}^+;E)} m_{[\![\theta]\!]}(B)\,\mathrm{d}\eta(\theta), \tag{A 7}$$

and hence, in particular, (A 6) (plugging in $B := E$). The inequality in (A 6) may eventually be strict, as is seen even with the trivial example of θ_1 being any rectifiable curve parametrized, say, over $[0, 1]$, and $\theta_2(t) := \theta_1(1 - t)$, so that $T = 0$ satisfies (A 5) with $\eta := \delta_{\theta_1} + \delta_{\theta_2}$.

However, if η represents $T \in \mathcal{M}_1(E)$ without cancellation of mass, then the inequalities in (A 6) become equalities,

$$\mathbb{M}(T) = \int_{C(\mathbb{R}^+; E)} \mathbb{M}(\llbracket \theta \rrbracket) \, \mathrm{d}\eta(\theta) = \int_{C(\mathbb{R}^+; E)} \ell(\theta) \, \mathrm{d}\eta(\theta),$$

and hence, in particular,

$$m_T(B) = \int_{C(\mathbb{R}^+; E)} m_{\llbracket \theta \rrbracket}(B) \, \mathrm{d}\eta(\theta) = \int_{C(\mathbb{R}^+; E)} \mathrm{d}\eta(\theta) \int_{\theta^{-1}(B)} |\dot{\theta}|(t) \, \mathrm{d}t \qquad \text{(A 8)}$$

for every Borel $B \subset E$ (because the equality holds when $B = E$).

An easy consequence of the latter is that η-a.e. non-constant $\theta \in C(\mathbb{R}^+; E)$ belong to the support of T, i.e. $\theta \subset \operatorname{supp} T$ (in the sense of traces). In particular, note that then every measure η representing $T \in \mathcal{M}_1(E)$ without cancellation of mass, just under our assumption on tightness of m_T, is necessarily tight since it is concentrated on a separable space $C(\mathbb{R}^+; \operatorname{supp} T)$ (recalling that a tight measure is concentrated over its support and the latter is separable).

The following statement, which combines the results from [19, 20] extending the representation theorems from [21] for currents over a Euclidean space to metric currents, shows in particular that every normal current is laminated (although, of course, there are laminated currents that are not normal).

PROPOSITION A.10. *The current $T \in \mathcal{M}_1(E)$ is normal if and only if it is represented by some* finite *Borel measure η over $C(\mathbb{R}^+; E)$ without cancellation of mass.*

Proof. The 'only if' part (i.e. the existence of a finite Borel measure η representing T without cancellation of mass) is [20, corollary 3.3], while the 'if' part follows from [19, theorem 4.2]. $\qquad \square$

REMARK A.11. The representation of laminated (or even normal) current by some measure over curves is of course not unique: for instance, the current $\llbracket [0, 1] \rrbracket$ (in $\mathbb{R}$) is represented both by the measure $\delta_{\theta_1} + \delta_{\theta_2}$ and by $\delta_{\theta_1 \circ \theta_2}$, where $\theta_1 := [0, 1/2]$, $\theta_2 := [1/2, 1]$ and $\theta_1 \circ \theta_2$ stands for the usual composition of curves, i.e. $\theta_1 \circ \theta_2 = [0, 1]$. However, if $T \in \mathcal{M}_1(E)$ is an *acyclic* normal current, i.e. $C \leqslant T$ with $\partial C = 0$ implies that $C = 0$, then there is a representation of T by a finite Borel measure η over $C(\mathbb{R}^+; E)$ that may be considered canonical, namely, η-a.e. $\theta \in C(\mathbb{R}^+; E)$ is an injective curve stopping definitely at $t = 1$ (and hence, in particular, it may be considered without loss of generality to be parametrized over $[0, 1]$), and $e_{1\#}\eta = (\partial T)^+$, $e_{0\#}\eta = (\partial T)^-$, where $e_i(\theta) := \theta(i)$ (see [19, theorem 5.1]).

REMARK A.12. Not every $T \in \mathcal{M}_1(E)$ can be represented by a σ-finite Borel measure η over $C(\mathbb{R}^+; E)$, and hence the class of laminated currents does not coincide with $\mathcal{M}_1(E)$. In the context of classical Whitney flat chains in a Euclidean space, the example of a flat chain that is not laminated in the sense of definition 3.2 was

provided in [10]. We consider here its analogue in the setting of metric currents. Let, as in example 1.1, $K \subset [0,1]$ be a Cantor set of positive Lebesgue measure (actually every closed totally disconnected set of positive Lebesgue measure will suit) and define the current $T \in \mathcal{M}_1(\mathbb{R})$ by setting

$$T(f \, \mathrm{d}\pi) := \int_K f(x) \, \mathrm{d}\pi(x).$$

Clearly, $T \neq 0$ and $\operatorname{supp} T \subset K$. On the other hand, T cannot be represented by any σ-finite Borel measure η over $C(\mathbb{R}^+; E)$, because otherwise η-a.e. θ would be supported over K by (A 8), and hence would be constant (its trace being a singleton) implying that $[\![\theta]\!] = 0$, which would give $T = 0$ by (A 5).

REMARK A.13. It is worth mentioning that the representation (3.2) does not really depend on the distance in E, but rather on the lengths of the curves it generates. In fact, if the distances d_1 and d_2 over E have the same geodesic distance d (this means, in particular, that the lengths of every absolutely continuous curve θ for all of these distances coincide, and hence will be denoted by $\ell(\theta)$), and η represents $T \in \mathcal{M}_1(E_1)$ without cancellation of mass, then it also represents the same current but viewed as $T \in \mathcal{M}_1(E_2)$ without cancellation of mass, where E_1, E_2 and E stand for E equipped with distances d_1, d_2 and d, respectively. In fact,

$$d_1(\operatorname{start}(\theta), \operatorname{end}(\theta)) \leqslant \ell(\theta)$$

for every absolutely continuous curve θ, which implies that $d \geqslant d_1$, and hence $\mathbb{M}(T) \geqslant \mathbb{M}_1(T)$, where $\mathbb{M}_1$, $\mathbb{M}_2$ and $\mathbb{M}$ stand for the masses in $\mathcal{M}_1(E_1)$, $\mathcal{M}_1(E_2)$ and $\mathcal{M}_1(E)$, respectively. But then

$$\mathbb{M}(T) \leqslant \int_{C(\mathbb{R}^+;E)} \ell(\theta) \, \mathrm{d}\eta(\theta) = \mathbb{M}_1(T) \leqslant \mathbb{M}(T),$$

the first inequality being valid in view of (A 6), and hence all the above inequalities are in fact equalities. Thus, also

$$\mathbb{M}_2(T) = \int_{C(\mathbb{R}^+;E)} \ell(\theta) \, \mathrm{d}\eta(\theta),$$

and hence η represents T in $\mathcal{M}_1(E_2)$ without cancellation of mass.

REMARK A.14. If η represents T_η without cancellation of mass and $0 \leqslant \tilde{\eta} \leqslant \eta$, then $\tilde{\eta}$ represents $T_{\tilde{\eta}}$ also without cancellation of mass. In fact, one has that $T - T_{\tilde{\eta}} = T_{\eta-\tilde{\eta}}$, and hence by remark A.9 one gets

$$\mathbb{M}(T_{\tilde{\eta}}) \leqslant \int_{C(\mathbb{R}^+;E)} \ell(\theta) \, \mathrm{d}\tilde{\eta}(\theta), \qquad \mathbb{M}(T_{\eta-\tilde{\eta}}) \leqslant \int_{C(\mathbb{R}^+;E)} \ell(\theta) \, \mathrm{d}(\eta - \tilde{\eta})(\theta),$$

so that

$$\mathbb{M}(T) \leqslant \mathbb{M}(T_{\tilde{\eta}}) + \mathbb{M}(T - T_{\tilde{\eta}}) = \mathbb{M}(T_{\tilde{\eta}}) + \mathbb{M}(T_{\eta-\tilde{\eta}}) \leqslant \int_{C(\mathbb{R}^+;E)} \ell(\theta) \, \mathrm{d}\eta(\theta) = \mathbb{M}(T),$$

which means that the all the inequalities above are in fact equalities.

We provide also the following lemmas on measures over curves representing some currents without cancellation of mass.

LEMMA A.15. *Suppose that η is a σ-finite Borel measure over $C(\mathbb{R}^+; E)$ that represents some laminated current $T \in \mathcal{M}_1(E)$ without cancellation of mass, and let $\tilde{\eta}$ be another σ-finite Borel measure over $C(\mathbb{R}^+; E)$ having $\tilde{\eta} \ll \eta$ and*

$$\int_{C(\mathbb{R}^+;E)} \ell(\theta) \, \mathrm{d}\tilde{\eta}(\theta) < +\infty.$$

Then $\tilde{\eta}$ represents $T_{\tilde{\eta}}$ without cancellation of mass.

Proof. One has $\tilde{\eta} = \alpha\eta$ for some $\alpha \in L^1(C(\mathbb{R}^+; E), \eta)$, $\alpha \geqslant 0$. The statement is clear when α is a simple (i.e. finite-valued) function. In fact, denoting $\bar{\alpha}$ as the norm of α in $L^\infty(C(\mathbb{R}^+; E), \eta)$, we have that $\tilde{\eta} \leqslant \bar{\alpha}\eta$, while the latter measure represents $T_{\bar{\alpha}\eta} = \bar{\alpha}T_\eta$ without cancellation of mass; hence $\tilde{\eta}$ represents $T_{\tilde{\eta}} \leqslant T_{\bar{\alpha}\eta}$ without cancellation of mass by remark A.14.

If α is generic, we approximate it as a supremum of simple functions α_k converging to α in $L^1(C(\mathbb{R}^+; E), \eta)$ as $k \to \infty$. Then for every $\omega = f \, \mathrm{d}\pi \in D^1(E)$ one has

$$T_{\alpha_k\eta}(\omega) - T_{\alpha\eta}(\omega) = \int_{C(\mathbb{R}^+;E)} [\![\theta]\!](\omega)(\alpha(\theta) - \alpha_k(\theta)) \, \mathrm{d}\eta(\theta),$$

and therefore

$$\mathbb{M}(T_{\alpha_k\eta} - T_{\alpha\eta}) \leqslant \int_{C(\mathbb{R}^+;E)} \ell(\theta)(\alpha(\theta) - \alpha_k(\theta)) \, \mathrm{d}\eta(\theta),$$

the latter integral vanishing as $k \to \infty$ by the Beppo Levi theorem. Hence, $T_{\alpha_k\eta} \to T_{\alpha\eta} = T_{\tilde{\eta}}$ in mass as $k \to \infty$, and, in particular,

$$\begin{aligned}
\mathbb{M}(T_{\tilde{\eta}}) = \lim_k \mathbb{M}(T_{\alpha_k\eta}) &= \lim_k \int_{C(\mathbb{R}^+;E)} \ell(\theta)\alpha_k(\theta) \, \mathrm{d}\eta(\theta) \\
&= \int_{C(\mathbb{R}^+;E)} \ell(\theta)\alpha(\theta) \, \mathrm{d}\eta(\theta) \\
&= \int_{C(\mathbb{R}^+;E)} \ell(\theta) \, \mathrm{d}\tilde{\eta}(\theta),
\end{aligned}$$

again by the Beppo Levi theorem, showing the statement. $\qquad \square$

LEMMA A.16. *Let E be isometrically embedded into a strictly convex Banach space with the Radon–Nikodym property. Under the conditions of lemma A.15 one has then that $T_{\tilde{\eta}} = T \llcorner \gamma$ for some $\gamma \in L^1(E, m_T)$ (and with $m_{T_{\tilde{\eta}}} = \gamma m_T$).*

Proof. Since the statement is stable with respect to isometric embeddings of the metric space E, we may assume E to be a strictly convex Banach space with the Radon–Nikodym property. The proof will be provided in several steps.

STEP 1. Assume that $\tilde{\eta} \leqslant \eta$. Then $T_{\tilde{\eta}} \leqslant T_\eta = T$. Hence, recalling that both $T_{\tilde{\eta}}$ and $T - T_{\tilde{\eta}} = T_{\eta-\tilde{\eta}}$ are laminated currents represented by $\tilde{\eta}$ and $\eta - \tilde{\eta}$, respectively, without cancellation of mass by remark A.14, and hence satisfying the conditions

of remark A.7 in view of theorem 5.1, we apply remark A.7 to get the existence of a $\beta \in L^1(E, m_T)$, with $0 \leqslant \beta \leqslant 1$, such that $S = T \llcorner \beta$.

STEP 2. Assume that $\tilde{\eta} = \alpha \eta$, where $\alpha \in L^1(C(\mathbb{R}^+; E), \eta)$ is a simple function, i.e. there is a finite partition $P := \{B_k\}$ of η-almost all $C(\mathbb{R}^+; E)$ consisting of Borel subsets $B_j \subset C(\mathbb{R}^+; E)$ such that α is a non-negative constant α_j over each B_j. Then $T_{\tilde{\eta}}(\omega) = \sum_k \alpha_k S_k(\omega)$, where

$$S_k(\omega) := \int_{C(\mathbb{R}^+; E)} [\![\theta]\!](\omega) \, \mathrm{d}\left(\frac{1}{\alpha_k} \tilde{\eta} \llcorner B_k\right)(\theta)$$

for every $\omega \in D^1(E)$. Applying the result of step 1 to each S_k, we get the existence of $\beta_k \in L^1(E, m_T)$ satisfying $0 \leqslant \beta_k \leqslant 1$ such that $S_k = T \llcorner \beta_k$, and hence $S = T \llcorner \gamma$ with $\gamma := \sum_k \alpha_k \beta_k$.

The following observation regarding this construction is important. Let $\tilde{\eta}' = \alpha' \eta$, where $\alpha' \in L^1(C(\mathbb{R}^+; E), \eta)$ is a simple function with $\alpha' \leqslant \alpha$. Then $T_{\tilde{\eta}'} = T \llcorner \gamma'$ with $\gamma' \leqslant \gamma$. In fact, there is a finite partition $Q := \{B_k\}$ of η-almost all $C(\mathbb{R}^+; E)$ consisting of Borel subsets $B_j \subset C(\mathbb{R}^+; E)$ such that both α and α' are constant over each B_j. Defining for brevity $\alpha'_j := \alpha' \llcorner B_j$, we get $T_{\tilde{\eta}'}(\omega) = \sum_k \alpha'_k S'_k(\omega)$, where

$$\begin{aligned}
S'_k(\omega) &:= \int_{C(\mathbb{R}^+; E)} [\![\theta]\!](\omega) \, \mathrm{d}\left(\frac{1}{\alpha'_k} \tilde{\eta}' \llcorner B_k\right)(\theta) \\
&= \int_{C(\mathbb{R}^+; E)} [\![\theta]\!](\omega) \, \mathrm{d}(\eta \llcorner B_k)(\theta) \\
&= \int_{C(\mathbb{R}^+; E)} [\![\theta]\!](\omega) \, \mathrm{d}\left(\frac{1}{\alpha_k} \tilde{\eta} \llcorner B_k\right)(\theta) \\
&= S_k(\omega) \\
&= T \llcorner \beta_k(\omega)
\end{aligned}$$

for every $\omega \in D^1(E)$. Thus, $T_{\tilde{\eta}'} = T \llcorner \gamma'$ with $\gamma := \sum_k \alpha'_k \beta_k$, which implies that $\gamma' \leqslant \gamma$ because $\alpha'_k \leqslant \alpha_k$ for all k.

STEP 3. Consider now the general case $\tilde{\eta} = \alpha \eta$ with an arbitrary non-negative $\alpha \in L^1(C(\mathbb{R}^+; E), \eta)$. Letting $\alpha_k \in L^1(C(\mathbb{R}^+; E), \eta)$ be a non-decreasing sequence of simple functions with $\alpha_k \leqslant \alpha$ and $\alpha_k \nearrow \alpha$ pointwise, we get as in the proof of lemma A.15 that $T_{\alpha_k \eta} \to T_{\alpha \eta} = T_{\tilde{\eta}}$ in mass as $k \to \infty$. But according to the result of step 2, one has $T_{\alpha_k \eta} = T \llcorner \gamma_k$ for some non-negative $\gamma_k \in L^1(E, m_T)$, with γ_k a pointwise non-decreasing sequence. Clearly, one also has $T_{\alpha_k \eta} \leqslant T_{\alpha \eta} = T_{\tilde{\eta}}$ so that, in particular,

$$\int_E \gamma_k \, \mathrm{d}m_T = \mathbb{M}(T_{\alpha_k \eta}) \leqslant \mathbb{M}(T_{\tilde{\eta}}) < +\infty.$$

Defining $\gamma := \sup_k \gamma_k$, by the Beppo Levi theorem we get

$$\mathbb{M}(T_{\alpha_k \eta}) = \int_E \gamma_k \, \mathrm{d}m_T \to \int_E \gamma \, \mathrm{d}m_T \quad \text{as } k \to +\infty,$$

but since $\lim_k \mathbb{M}(T_{\alpha_k \eta}) = \mathbb{M}(T_{\tilde{\eta}})$, we get $\mathbb{M}(T_{\tilde{\eta}}) = \int_E \gamma \, \mathrm{d}m_T$. Finally,

$$\lim_k T_{\alpha_k \eta}(f \, \mathrm{d}\pi) = \lim_k T(\gamma_k f \, \mathrm{d}\pi) = T(\gamma f \, \mathrm{d}\pi)$$

for all $f\,\mathrm{d}\pi \in D^1(E)$, because $\gamma_k f \to \gamma f$ in $L^1(E, m_T)$, and hence $T_{\tilde{\eta}} = T \llcorner \gamma$, proving the claim. $\qquad\square$

Finally, the lemmas below provide a construction that we use in analysing the transportability condition.

LEMMA A.17. *Let σ be a σ-finite Borel measure over $C(\mathbb{R}^+; E) \times \mathbb{R}^+$ such that $q_\#\sigma$ represents some laminated current $T \in \mathcal{M}_1(E)$ without cancellation of mass. Then for $S \in \mathcal{M}_1(E)$ defined by*

$$S := \int_{C(\mathbb{R}^+;E)\times\mathbb{R}^+} [\![\theta\llcorner[s, +\infty)]\!]\,\mathrm{d}\sigma(\theta, s) \qquad\qquad (A\,9)$$

one has $S \leqslant T$ and

$$\mathbb{M}(S) = \int_{C(\mathbb{R}^+;E)\times\mathbb{R}^+} \ell(\theta\llcorner[s, +\infty))\,\mathrm{d}\sigma(\theta, s). \qquad\qquad (A\,10)$$

Proof. Letting $R := \int_{C(\mathbb{R}^+;E)\times\mathbb{R}^+} [\![\theta\llcorner[0, s]]\!]\,\mathrm{d}\sigma(\theta, s)$, we get

$$S + R = \int_{C(\mathbb{R}^+;E)\times\mathbb{R}^+} [\![\theta]\!]\,\mathrm{d}\sigma(\theta, s) = \int_{C(\mathbb{R}^+;E)} [\![\theta]\!]\,\mathrm{d}(q_\#\sigma)(\theta) = T,$$

and

$$\mathbb{M}(S) \leqslant \int_{C(\mathbb{R}^+;E)\times\mathbb{R}^+} \ell(\theta\llcorner[s, +\infty))\,\mathrm{d}\sigma(\theta, s),$$

$$\mathbb{M}(R) \leqslant \int_{C(\mathbb{R}^+;E)\times\mathbb{R}^+} \ell(\theta\llcorner[0, s])\,\mathrm{d}\sigma(\theta, s),$$

so that

$$\mathbb{M}(S) + \mathbb{M}(R) \leqslant \int_{C(\mathbb{R}^+;E)\times\mathbb{R}^+} \ell(\theta)\,\mathrm{d}\sigma(\theta, s) = \int_{C(\mathbb{R}^+;E)} \ell(\theta)\,\mathrm{d}(q_\#\sigma)(\theta) = \mathbb{M}(T),$$

which concludes the proof. $\qquad\square$

LEMMA A.18. *Let E be isometrically embedded in a strictly convex Banach space with the Radon–Nikodym property, let $T \in \mathcal{M}_1(E)$ be a laminated current represented by some σ-finite Borel measure η over $C(\mathbb{R}^+; E)$ without cancellation of mass, and let σ be a finite Borel measure over $C(\mathbb{R}^+; E) \times \mathbb{R}^+$ such that $q_\#\sigma \ll \eta$ and*

$$\int_{C(\mathbb{R}^+;E)} \ell(\theta)\,\mathrm{d}(q_\#\sigma)(\theta) < +\infty.$$

Then for an $S \in \mathcal{M}_1(E)$ defined by (A 9) one has that S is a normal current, (A 10) is valid and $S = T\llcorner\gamma$ for some $\gamma \in L^1(E, \alpha m_T)$.

Proof. For the sake of brevity, define $\tilde{\eta} := q_\#\sigma$. By lemma A.15 this measure represents $T_{\tilde{\eta}}$ without cancellation of mass, and by lemma A.16 one has $T_{\tilde{\eta}} = T\llcorner\alpha$ for some $\alpha \in L^1(E, m_T)$. Lemma A.17 says then that $S \leqslant T_{\tilde{\eta}}$ and is a normal current and (A 10) holds. By lemma A.6 we then get $S = T_{\tilde{\eta}}\llcorner\beta$ for some $\beta \in L^\infty(E, m_T)$, with $0 \leqslant \beta \leqslant 1$, and to conclude the proof it is enough to set $\gamma := \alpha\beta$. $\qquad\square$

Appendix B. Some auxiliary statements

In this section E stands for an arbitrary Banach space.

B.1. Duality of Banach spaces

For a $v \in E$ we set

$$v' := \{\xi \in E' \colon \langle v, \xi \rangle = \|v\|\|\xi\|'\}. \tag{B 1}$$

Of course, $v' \neq \emptyset$ and $0' = E'$. Clearly, $v \in v''$, where $v'' := (v')'$. But if E is strictly convex, then $u \in v''$, $u \in E$ implies that $u = \lambda v$ for some $\lambda \in \mathbb{R}^+$. In fact, otherwise there is a $\xi \in v' \subset E'$, with, say, $\|\xi\|' = 1$ such that $u \in \xi'$, and thus $\xi \in u'$. The claim then follows from [15, theorem 2]. Thus, for a strictly convex space E one has $v'' \cap E = \{\lambda v \colon \lambda \in \mathbb{R}^+\}$.

The following easy statement, which identifies the cases in which equality holds in the Jensen inequality, is valid.

LEMMA B.1. *Let* $f \in L^1(\Omega, \mu; E)$, *where* (Ω, Σ, μ) *is a measure space with some positive σ-finite measure μ. If*

$$\left\| \int_\Omega f(x)\, \mathrm{d}\mu(x) \right\| = \int_\Omega \|f(x)\|\, \mathrm{d}\mu(x), \tag{B 2}$$

then either $f = 0$ or

$$f(x) \in \frac{\left(\int_\Omega f(y)\, \mathrm{d}\mu(y) \right)''}{\int_\Omega \|f(y)\|\, \mathrm{d}\mu(y)} \|f(x)\|$$

for μ-a.e. $x \in E$. In particular, if E is strictly convex, then (B 2) implies either $f(x) = 0$ or

$$f(x) = \frac{\int_\Omega f(y)\, \mathrm{d}\mu(y)}{\int_\Omega \|f(y)\|\, \mathrm{d}\mu(y)} \|f(x)\|$$

for μ-a.e. $x \in E$.

Proof. We first consider the case in which $\mu = \mathbb{P}$ a probability measure. For every $\xi \in (\int_\Omega f(y)\, \mathrm{d}\mathbb{P}(y))'$ one has

$$\int_\Omega \langle f(y), \xi \rangle\, \mathrm{d}\mathbb{P}(y) = \left\langle \int_\Omega f(y)\, \mathrm{d}\mathbb{P}(y), \xi \right\rangle = \left\| \int_\Omega f(y)\, \mathrm{d}\mathbb{P}(y) \right\| \|\xi\|'$$
$$= \left(\int_\Omega \|f(y)\|\, \mathrm{d}\mathbb{P}(y) \right) \|\xi\|', \tag{B 3}$$

the latter equality being due to (B 2). But since $\langle f(x), \xi \rangle \leqslant \|f(x)\|\|\xi\|'$, we have $\langle f(x), \xi \rangle = \|f(x)\|\|\xi\|'$ for $\mathbb{P}$-a.e. $x \in E$ (since otherwise (B 3) would become a strict inequality), which shows that

$$f(x) \in \left(\int_\Omega f(y)\, \mathrm{d}\mathbb{P}(y) \right)'' \tag{B 4}$$

for $\mathbb{P}$-a.e. $x \in E$. In the general case, assuming that $f \neq 0$ and letting

$$\mathbb{P} := \frac{\|f\|\, \mathrm{d}\mu}{\int_\Omega \|f(y)\|\, \mathrm{d}\mu(y)},$$

it suffices to apply (B 4) with $f/\|f\|$ instead of f. $\qquad \square$

B.2. Differentiability of curves in Banach spaces

The curve $\theta \in C(\mathbb{R}^+; E)$ is called weakly (respectively, norm) differentiable at $t \in \mathbb{R}^+$ if there is a map $\dot{\theta} \colon \mathbb{R}^+ \to E$ satisfying

$$\lim_{h \to 0} \frac{1}{h} (\theta(t+h) - \theta(t) - \dot{\theta}(t)h) = 0 \tag{B 5}$$

in the sense of weak (respectively, strong, i.e. norm) convergence.

The following more or less folkloric assertions hold true.

LEMMA B.2. *If a curve $\theta \in C(\mathbb{R}^+; E)$ is weakly differentiable at a.e. $t \in \mathbb{R}^+$, then $\dot{\theta} \colon \mathbb{R}^+ \to E$ is almost everywhere equal to a Borel function, and, moreover, locally Bochner integrable (respectively, Bochner integrable) when θ is absolutely continuous (respectively, rectifiable). Moreover, if $S \subset E$ is some closed separable linear subspace of E, and $B \subset C(\mathbb{R}^+; S)$ is a Borel set of curves weakly differentiable almost everywhere, and η is a Borel measure concentrated on B, then the map $(\theta, t) \colon B \times \mathbb{R}^+ \mapsto \dot{\theta}(t)$ is $\eta \otimes \mathcal{L}^1$-almost everywhere equal to a Borel map, and, moreover, is Bochner integrable with respect to $\eta \otimes \mathcal{L}^1$ over $B \times K$, with $K \subset \mathbb{R}^+$ an arbitrary compact set (respectively, $B \times \mathbb{R}^+$) when η-a.e. θ is absolutely continuous and $\theta \mapsto \ell(\theta \llcorner K)$ is η-integrable (respectively, rectifiable and $\theta \mapsto \ell(\theta)$ is η-integrable).*

Proof. For the second part of the statement, we observe that every function $(\theta, t) \in B \times \mathbb{R}^+ \mapsto \langle \dot{\theta}(t), \psi \rangle \in \mathbb{R}$ for every $\psi \in E'$ is equal for all $\theta \in B$ and a.e. $t \in \mathbb{R}^+$ (and hence for $\eta \otimes \mathcal{L}^1$-a.e. $(\theta, t) \in B \times \mathbb{R}^+$) to a Borel map $g_\psi(\theta, t) := \limsup_k k \langle \theta(t + 1/k) - \theta(t), \psi \rangle$, and hence the map $(\theta, t) \colon B \times \mathbb{R}^+ \mapsto \dot{\theta}(t)$ is $\eta \otimes \mathcal{L}^1$-weakly measurable. But since for all $\theta \in B$ one has $\dot{\theta}(t) \in S$ whenever it is defined, i.e. almost everywhere (because S is also weakly closed), and S is separable, this map is also strongly measurable as claimed. Recalling that clearly $\|\dot{\theta}(t)\| \leqslant |\dot{\theta}|(t)$ for a.e. $t \in \mathbb{R}^+$ and every $\theta \in B$ (in fact, it is not difficult to prove that the equality holds, but we do not need it), we have also the claimed Bochner integrability under the respective absolute continuity conditions of η-a.e. $\theta \in B$.

The first part follows from the second one by choosing $B := \{\theta\}$, $\eta := \delta_\theta$, the Dirac measure concentrated on θ, and $S \subset E$ any closed separable linear subspace containing $\theta(\mathbb{R}^+)$ (which exists because the latter set is separable). $\qquad\square$

LEMMA B.3. *If $\theta \in C(\mathbb{R}^+; E)$ is weakly differentiable almost everywhere and the weak derivative $\dot{\theta}$ is locally Bochner integrable, then θ is also norm differentiable almost everywhere. In particular, every absolutely continuous function $\theta \in C(\mathbb{R}^+; E)$ is weakly differentiable almost everywhere if and only if it is norm differentiable almost everywhere.*

Proof. Observe that

$$\langle \theta(b) - \theta(a), \psi \rangle = \int_a^b \langle \dot{\theta}(\tau), \psi \rangle \, \mathrm{d}\tau = \left\langle \int_a^b \dot{\theta}(\tau) \, \mathrm{d}\tau, \psi \right\rangle$$

for every $\psi \in E'$, and hence

$$\theta(b) - \theta(a) = \int_a^b \dot{\theta}(\tau) \, \mathrm{d}\tau$$

for every pair $\{a, b\} \in \mathbb{R}^+$. Thus,

$$\left\| \frac{\theta(t+s) - \theta(t)}{s} - \dot{\theta}(t) \right\| = \left\| \frac{1}{s} \int_t^{t+s} (\dot{\theta}(\tau) - \dot{\theta}(t)) \, \mathrm{d}\tau \right\|$$
$$\leqslant \frac{1}{|s|} \int_{B_{|s|}(\tau)} \|\dot{\theta}(\tau) - \dot{\theta}(t)\| \, \mathrm{d}\tau \to 0$$

as $s \to 0$ whenever $\tau \in \mathbb{R}^+$ is a Lebesgue point of $\dot{\theta}$ (by [7, proposition 5.3], a.e. point is so), thus concluding the proof. $\qquad\square$

We remark for the sake of completeness that in the case in which E is a dual of some Banach space, one can define the notion of weak* differentiability of the curve $\theta \colon \mathbb{R}^+ \to E$ by requiring the limit in (B 5) to be intended in the weak* sense. In this case the first part of lemma B.3 remains valid (with the obvious modification, i.e. substituting 'weak*' for 'weak'), while the second part is not: in fact, as shown in [4], every absolutely continuous curve in a dual of a separable Banach space is weak* differentiable almost everywhere (and the norm of the weak* derivative coincides almost everywhere with the metric derivative), but clearly might not be norm differentiable unless E has the Radon–Nikodym property; therefore, the weak* derivative is not necessarily measurable in the strong sense, and hence the first part of lemma B.2 also fails. The problem is that the weak* derivative is only weakly* measurable, but might fail to be separable valued.

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