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Optimal payment of mortgages

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This article provides a borrower's optimal strategies to terminate a mortgage with a fixed interest rate by paying the outstanding balance all at once. The problem is modelled as a free boundary problem for the appropriate analogue of the Black-Scholes pricing equation under the assumption of the Vasicek model for the short-term rate of investment. Here the free boundary provides the optimal time at which the mortgage contract is to be terminated. A number of integral identities are derived and then used to design efficient numerical codes for computing the free boundary. For numerical simulation, parameters for the Vasicek model are estimated via the method of maximum likelihood estimation using 40 years of data from US government bonds. The asymptotic behaviour of the free boundary for the infinite horizon is fully analysed. Interpolating this infinite horizon behaviour and a known near-expiry behaviour, two simple analytical approximation formulas for the optimal exercise boundary are proposed. Numerical evidence shows that the enhanced version of the approximation formula is amazingly accurate; in general, its relative error is less than 1%, for all time before expiry.

1 Introduction

In this article, we consider a mortgage contract and the problem of finding the optimal time for the mortgage borrower to terminate the mortgage by prepaying it with a lump sum.

The mortgage contract under consideration has an expiration date T, a fixed mortgage interest rate c (year^{-1}) and a constant continuous rate of payment of m (\$/year). At any time t during the term of the mortgage, the outstanding balance owed, $M(t)$, is reduced in the time period $[t, t+dt)$ by $dM(t) = cM(t)dt - mdt$, where $cM(t)dt$ is the interest accrued on the balance and mdt is the payment. For the mortgage to be retired at $t = T$, the condition $M(T) = 0$ applies so that

$$M(t) = \frac{m}{c}\{1 - e^{c(t-T)}\} \qquad \forall\, t \leqslant T.$$

In this contract, the borrower is allowed to terminate the contract at any time t ($t < T$) of his choice by paying a lump sum $M(t)$ to the contract issuer. This decision for the borrower to terminate the contract depends on the alternate investment strategy (e.g. risk-free bonds) available to him.

In this article, we use the Vasicek model [11] for the risk-free short-term market return rate, r_t, described by the stochastic differential equation

$$dr_t = k(\theta - r_t)dt + \sigma\, dW_t, \tag{1.1}$$

where k, θ and σ are assumed to be positive known constants and W_t is the standard Wiener process. Here the units for k, θ, σ and W_t are year^{-1}, year^{-1}, year$^{-3/2}$ and year$^{1/2}$, respectively. To address the fact that the Vasicek model is not sufficient to describe the whole term structure, here we assume for simplicity that in this model, the market price of risk has been incorporated into the drift $k(\theta - r_t)$; that is to say, the probability associated with the Brownian motion $\{W_t\}$ is the risk-neutral probability [1, 12].

Intuitively, if an overall market return rate is expected to be low (relative to c) for a certain amount of time, one should choose to terminate the contract early. On the other hand, if the market return rate is strictly higher than c or if an overall market return rate is expected to be higher than c for a certain amount of time, one should choose to defer the closing date by an investment in the market of the capital $M(t)$ for less than the obligatory payment of m per unit time. Hence, at every moment that the contract is in effect, the borrower must monitor the market return rate and decide whether to immediately close the contract. Statistically, there is an optimal strategy in making such a decision.

To find such a strategy, we introduce a function $V(r,t)$ being the (arbitrage-free) price of the contract at time t and current market return rate $r_t = r$. This value can be regarded as an asset that the contract issuer (the mortgage company) possesses, or a fair price that a buyer would offer to the contract issuer in taking over the contract, say, in an issuer's restructuring or liquidation process. The value of V is calculated according to the borrower's optimal decision; that is, the issuer is a passive player. Since the borrower can terminate the contract by paying $M(t)$ at any time t, we have $0 \leqslant V(r,t) \leqslant M(t)$ for every $r \in \mathbb{R}$ and $t \leqslant T$. This automatically implies that $V(\cdot, T) \equiv 0$.

According to general mathematical finance theory [12], for every $r \in \mathbb{R}$ and $t < T$, we have

$$V(r,t) = \min\{M(t), \quad m\,dt + \mathbb{E}[V(r + dr_t, t + dt)e^{-rdt} \mid r_t = r]\,\} \qquad \forall\, r \in \mathbb{R}, t < T$$

where $\mathbb{E}$ stands for conditional expectation (under the risk-neutral probability). Assuming appropriate regularity on V and using Itô Lemma, we obtain from the above equation that

$$V(r,t) = \min\{M(t), \ \ V(r,t) + [\mathscr{L}V(r,t) + m]\, dt\},$$

where

$$\mathscr{L}V(r,t) = \frac{\partial V(r,t)}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V(r,t)}{\partial r^2} + k(\theta - r)\frac{\partial V(r,t)}{\partial r} - rV(r,t).$$

Thus, V is the solution of the variational inequality

$$0 = \min\{M(t) - V(r,t),\ \mathscr{L}V(r,t) + m\}, \qquad 0 \leqslant V(r,t) \quad \forall\, r \in \mathbb{R}, \quad t \leqslant T. \tag{1.2}$$

Using a classical method (such as that used in [3] for an American put option), it is easy to show that the variational problem (1.2) admits a unique solution and the solution has bounded derivatives $V_{rr} := \frac{\partial^2 V}{\partial r^2}$ and $V_t := \frac{\partial V}{\partial t}$. With this regularity, one can construct a delta hedging portfolio (using zero coupon bonds of various maturities) to replicate the mortgage contract and to conclude that at any time t ($t \leqslant T$) and spot rate $r_t = r$, the value of the mortgage contract is $V(r,t)$. In addition, one can show by a comparison principle that $V_r(,t) := \frac{\partial V(r,t)}{\partial r} \leqslant 0$ for every $r \in \mathbb{R}, t \leqslant T$. Therefore, there is a function $R(\cdot) : (-\infty, T) \to [-\infty, \infty)$ such that

$$V(r,t) < M(t) \quad \Longleftrightarrow \quad r > R(t).$$

We call $r = R(t)$ the *optimal boundary for mortgage contract termination.* That is, the best strategy for the borrower is to terminate the mortgage contract at the first time that the spot market return rate r_t is below $R(t)$.

One can further show that $R(T-) = c, R'(t) \geqslant 0$ for all $t < T$, and $R(-\infty) > -\infty$. Hence, (R, V) solves the following free boundary problem:

$$\begin{cases} \mathscr{L}V(r,t) + m = 0 < M(t) - V(r,t) & \forall\, r > R(t), t < T, \\ V(R(t),t) = M(t), \quad V_r(R(t),t) = 0 & \forall\, t \leqslant T, \\ V(r,T) = 0 & \forall\, r \geqslant R(0) = c. \end{cases} \tag{1.3}$$

Once a solution (R, V) of (1.3) is obtained, a solution of (1.2) can be obtained by extending V to $\mathbb{R} \times (-\infty, T]$ by setting $V(r,t) = M(t)$ for every $r < R(t)$ and $t \leqslant T$.

Similar problems have been studied from an option-theoretical viewpoint by Buser and Hendershott [2], Epperson *et al.* [4], Kau *et al.* [8, 9], Pozdena and Iben [10], Kau and Keenan [7], etc. The mathematical analysis for problem (1.3) has been completely carried out by Jiang *et al.* [6]; see also relevant mathematical work by Yuan *et al.* [13]. In [6], the authors proved that the problem is well-posed; namely, problem (1.3) admits a unique solution that is smooth up to the free boundary $r = R(t)$. Also, the free boundary $R(\cdot)$ is a smooth function strictly increasing on $(-\infty, T)$, and has the asymptotic behaviour

$$R(t) \sim c - \sigma\bar{\kappa}\sqrt{T-t} \quad \text{as} \quad t \nearrow T, \qquad \bar{\kappa} = 0.47386\ldots. \tag{1.4}$$

In this article, we consider numerical aspects of this problem. In the course of this study, we provide an analytical solution to the infinite horizon problem and show that

$$R(t) \sim R^* + \rho^* e^{-c(T-t)} \quad \text{as} \quad t \to -\infty, \tag{1.5}$$

where R^* and ρ^* are constants that can be easily calculated by solving an algebraic equation involving Hermite functions. On the basis of the existing near-expiry behavior (1.4) and our new long-term behavior (1.5), we provide global approximations $R(t) \approx R_I(T-t)$ and $R(t) \approx R_{II}(T-t)$ for all $t \leqslant T$, where

$$R_I(\tau) = c - (c - R^*)\sqrt{1 - e^{-b^*\tau}}, \qquad b^* := \left(\frac{0.474\sigma}{c - R^*}\right)^2 \tag{1.6}$$

$$R_{II}(\tau) = c - \frac{0.474\sigma\sqrt{1-e^{-2c\tau}}}{\sqrt{2c}} + \rho^*(e^{-c\tau} - e^{-2c\tau}) + \left[R^* - c + \frac{0.474\sigma}{\sqrt{2c}}\right](1 - e^{-2c\tau}). \quad (1.7)$$

We numerically demonstrate that these approximations are very accurate. In the special case when typical US economy parameters are used, we have

$$\max_{t\leqslant T} \frac{|R(t) - R_I(T-t)|}{R(T) - R(-\infty)} \leqslant 2\%, \qquad \frac{\max_{t\leqslant T} |R(t) - R_{II}(T-t)|}{R(T) - R(-\infty)} < 0.4\%. \quad (1.8)$$

Here $R(T) - R(-\infty) = c - R^*$ is the total oscillation of $R(\cdot)$ on $(-\infty, T]$; see Figures 1(b) and 3.

The article is organized as follows. In § 2, we use the statistical procedure of maximum likelihood estimation (MLE) to determine reasonable values for the parameters k, θ and σ appearing in the Vasicek model to be used for the stochastic market rate of return. Without knowledge of the market price of risk, we can only speculate that these values should be in the vicinity of those values that incorporate the market price of risk. In § 3, we make a change of variables to reduce problem (1.3) to a simpler version in terms of the heat equation. § 4 develops integral identities that are used in § 5 to obtain fast and accurate numerical schemes on the basis of Newton's method. The logic for approximations (1.6) and (1.7) is provided in § 7. Some numerical experiments are given in § 8, with a final conclusion in § 9.

2 Calibration of the vasicek model using MLE

To determine the numerical values of the parameters k, θ and σ in the Vasicek model (1.1), we use the method of maximum likelihood. Starting from an initial rate $r_\tau = x$, at a later time t ($t > \tau$), the probability density p for the rate r_t to be equal to y is given by

$$p(\tau, x; t, y) := \frac{\text{Probability } (r_\tau = x, r_t \in (y, y+dy))}{dy} = \frac{\sqrt{k}\ \exp\left(-\frac{k[(y-\theta)-(x-\theta)e^{-k(t-\tau)}]^2}{\sigma^2(1-e^{-2k(t-\tau)})}\right)}{\sqrt{\pi\sigma^2(1-e^{-2k(t-\tau)})}}.$$

Suppose $\{(t_i, r_i)\}_{i=0}^n$ is a list of sample rates where $r_i = r_{t_i}$. Assume that all $\Delta t = t_{i+1} - t_i$ are positive and equal. Using $d(e^{-kt}(r_t - \theta)) = \sigma e^{-kt} dW_t$, we can show that $\{(r_{t_{i+1}} - \theta) - (r_{t_i} - \theta)e^{-k\Delta t}\}_{i=1}^n$ are independent and identically distributed random variables. Hence, we can define the maximum likelihood function $\Phi(k, \theta, \sigma) := \prod_{i=1}^n p(t_{i-1}, r_{i-1}; t_i, r_i)$. Consequently, the maximum likelihood estimators (MLEs) for k, θ and σ are defined as the maximizer of the function $\Phi(\cdot,\cdot,\cdot)$. Routine calculation gives the following MLEs (maximizer of Φ):

$$k = -\frac{1}{\Delta t}\log b, \quad \theta = \frac{\bar{Y} - b\bar{X}}{1-b}, \quad \sigma^2 = 2k\,\frac{n-1}{n}\,\frac{\text{Cov}[Y,Y] - b^2\text{Cov}[X,X]}{1-b^2},$$

where

$$\bar{X} = \tfrac{1}{n}\textstyle\sum_{i=1}^n r_{i-1}, \quad \bar{Y} = \tfrac{1}{n}\sum_{i=1}^n r_i, \quad b = \tfrac{\text{Cov}[X,Y]}{\text{Cov}[X,X]},$$
$$\text{Cov}[X,Y] = \tfrac{1}{n-1}\textstyle\sum_{i=1}^n (r_i - \bar{Y})(r_{i-1} - \bar{X}),$$
$$\text{Cov}[X,X] = \tfrac{1}{n-1}\textstyle\sum_{i=1}^n (r_{i-1} - \bar{X})^2, \quad \text{Cov}[Y,Y] = \tfrac{1}{n-1}\sum_{i=1}^n (r_i - \bar{Y})^2.$$

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Table 1. *Statistics of the yields of US 13-week treasury bills and the maximum likelihood estimators for the parameters in the Vasicek model. Here "mean" and "StdDev" represent the mean and standard deviation of the yield, measured in annual units.*

From -- To	Sample	Mean	(StdDev)	k	θ	σ
1996--2006	Daily	0.035	(0.017)	0.101	0.032	0.007
1996--2006	Weekly	0.035	(0.017)	0.119	0.032	0.008
1996--2006	Monthly	0.036	(0.017)	0.100	0.033	0.007
1986--2006	Daily	0.045	(0.019)	0.108	0.039	0.008
1986--2006	Weekly	0.045	(0.019)	0.120	0.040	0.009
1986--2006	Monthly	0.045	(0.019)	0.103	0.039	0.008
1976--2006	Daily	0.060	(0.031)	0.148	0.058	0.017
1976--2006	Weekly	0.060	(0.031)	0.169	0.059	0.018
1976--2006	Monthly	0.060	(0.031)	0.177	0.059	0.018
1966--2006	Daily	0.059	(0.028)	0.178	0.059	0.017
1966--2006	Weekly	0.059	(0.028)	0.208	0.059	0.018
1966--2006	Monthly	0.059	(0.028)	0.229	0.059	0.019

Table 1 is a summary of the MLEs from US 13-week treasury bills. We use three different time intervals $\Delta t = t_{i+1} - t_i$: daily, weekly and monthly. We calculated these data using 10-, 20-, 30- and 40-year periods.

3 A transformation and well-posedness

For simplicity, in the sequel we use subscripts to denote partial derivatives. We consider (1.2). First by a maximum principle, the condition $V(r,t) \geqslant 0$ can be replaced by $V(\cdot, T) \equiv 0$, if we are working in the class of bounded solutions. Next, introduce the change of variables

$$x = \frac{\sqrt{k}e^{k(T-t)}}{\sigma}\left[r + \frac{\sigma^2}{k^2} - \theta\right], \qquad s = e^{2k(T-t)},$$

$$u(x,s) = \frac{2ck\sqrt{\pi k}}{m\sigma}\{M(t) - V(r,t)\}\ \exp\left\{-\frac{k}{\sigma^2}\left[r + \frac{\sigma^2}{2k^2} - \theta\right]^2 - \left[k + \frac{\sigma^2}{2k^2} - \theta\right](T-t)\right\}.$$

Then problem (1.2) is transformed to

$$\begin{cases} \min\{u, \quad u_s - \frac{1}{4}u_{xx} - f\} = 0 \quad \text{in } \mathbb{R} \times (1, \infty), \\ u(x,1) = 0 \quad \forall x \in \mathbb{R} \end{cases} \tag{3.1}$$

where

$$f(x,s) := \sqrt{\pi}(s^\gamma - 1)s^{-\nu-1}(x - \beta\sqrt{s})e^{-(\frac{x}{\sqrt{s}}-\alpha)^2},$$
$$\alpha := \frac{\sigma}{2k^{3/2}}, \quad \gamma := \frac{c}{2k}, \quad \beta := \frac{\sqrt{k}}{\sigma}\left(c - \theta + \frac{\sigma^2}{k^2}\right), \quad \nu := 1 + \frac{\sigma^2}{4k^3} + \frac{c-\theta}{2k}. \tag{3.2}$$

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Using a standard theory of variational inequalities (e.g. [5]), one can show that (3.1) admits a unique solution. In addition, there exists $X \in C([1,\infty))$ such that

$$\begin{cases} u_s - \frac{1}{4}u_{xx} = f(x,s)\,\mathbf{1}_{[X(s),\infty)}(x) & \text{in } \mathbb{R}\times(1,\infty), \\ u(x,s) > 0 \quad \forall x > X(s), s > 1, \\ u(x,1) = 0 \quad \forall x \in \mathbb{R}, \qquad u(x,s) = 0 \quad \forall s > 1, x \leqslant X(s) \end{cases} \tag{3.3}$$

where

$$\mathbf{1}_{[z,\infty)}(x) = 1 \text{ if } x \geqslant z, \qquad \mathbf{1}_{[z,\infty)}(x) = 0 \quad \text{if} \quad x < z.$$

Here the differential equation for u is in the L^p sense, that is, both u_s and u_{xx} are in $L^p_{loc}(\mathbb{R}\times[0,\infty))$ for any $p \in (1,\infty)$. Note that the existence of u_{ss} implies the free boundary conditions

$$u(X(s),s) = 0, \quad u_s(X(s),s) = 0 \quad \forall\, s \geqslant 1.$$

Once we find $X(\cdot)$, the optimal boundary $r = R(t)$ for terminating the mortgage is given by

$$R(t) = c + \frac{\sigma}{\sqrt{k}}\left(\frac{X(e^{2k(T-t)})}{e^{k(T-t)}} - \beta\right). \tag{3.4}$$

It is shown in [6] that $R \in C^\infty((-\infty,T))$ so $X \in C^\infty((1,\infty))$.

4 Integral equations

The fundamental solution associated with the heat operator $\partial_s - \frac{1}{4}\partial^2_{xx}$ is denoted by

$$\Gamma(x,s) := \frac{e^{-x^2/s}}{\sqrt{\pi s}}.$$

Using Green's identity, the solution u to the differential equation in (3.3) can be expressed as

$$u(x,s) = \int_1^s d\varsigma \int_{X(\varsigma)}^\infty \Gamma(x-y,s-\varsigma)f(y,\varsigma)\,dy \quad \forall x \in \mathbb{R}, s \geqslant 1. \tag{4.1}$$

4.1 The integral identities

In this section, we derive the following three integral identities for the unknown free boundary function $X(\cdot)$ defined on $(1,\infty)$:

$$0 = \int_1^s d\varsigma \int_{X(\varsigma)}^\infty \Gamma(X(s)-y,s-\varsigma)f(y,\varsigma)\,dy = 0 \quad \forall s > 1, \tag{4.2}$$

$$0 = \int_1^s d\varsigma \int_{X(\varsigma)}^\infty \Gamma_x(X(s)-y,s-\varsigma)f(y,\varsigma)\,dy = 0 \quad \forall\, s > 1, \tag{4.3}$$

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$$2f(X(s),s) = -\int_1^s \Gamma_x(X(s)-X(\varsigma),s-\varsigma)f(X(\varsigma),\varsigma)d\varsigma$$
$$+\int_s^1\int_{X(\varsigma)}^\infty \Gamma_x(X(s)-y,s-\varsigma)f_y(y,\varsigma)\,dyd\varsigma \quad \forall s>1. \tag{4.4}$$

These identities correspond to the facts $u(X(s),s)=0, u_x(X(s),s)=0$ and $u_{xx}(X(s)+,s)-u_{xx}(X(s)-,s)=-4f(X(s),s)$, respectively.

4.2 The first integral representation

Setting $x=X(s)$ in (4.1), we immediately obtain the first integral equation (4.2) for the unknown $X(\cdot)$.

Although $u(x,s)=0$ for all $x\leqslant X(s)$, equation (4.2) always produces the correct free boundary, as shown in the following theorem.

Theorem 4.1 *Suppose* $X: s\in[1,\infty)\to\mathbb{R}$ *is a continuous function satisfying* (4.2). *Define* u *as in* (4.1). *Then* (X,u) *solves* (3.3) *and* u *is the unique solution to* (3.1). *In addition,*

$$X(s)<\beta\sqrt{s} \quad \forall s>1. \tag{4.5}$$

Proof Since X is continuous and f is smooth and bounded, the function u defined in (4.1) satisfies the differential equation in (3.3). In the domain $\{(x,s)\mid s>1, x<X(s)\}$, u satisfies the heat equation $u_s=\frac{1}{4}u_{xx}$ and the zero boundary condition so $u\equiv 0$ in the domain. After transforming to the original variable (r,V), one can show that $V_r\leqslant 0$. From this, we can derive that $u>0$ when $x>X(s)$ and $s>1$. Hence, (X,u) solves (3.3).

Next we prove (4.5). Let U be the solution to

$$\begin{cases} U_s-\frac{1}{4}U_{xx}=f(x,s), & (x,s)\in\Omega:=\{(x,s)\mid s>1, x>\beta\sqrt{s}\},\\ U=0 \text{ on } \partial_p\Omega:=[\beta,\infty)\times\{0\}\cup\{(\beta\sqrt{s},s)\mid s>1\}. \end{cases}$$

Since $f>0$ in Ω, we have $U>0$ in Ω and $U_x(\beta\sqrt{s},s)>0$ for all $s>1$. Comparing u and U on $\bar\Omega$, we see that $u\geqslant U$ on $\bar\Omega$. Since $u_x(X(s),s)=0$, Hopf's lemma implies that $u>U$ when $x=\beta\sqrt{s}, s>1$. Thus, $X(s)<\beta\sqrt{s}$ for all $s>1$.

Finally, notice that $f<0$ whenever $x<\beta\sqrt{s}$, or whenever $x<X(s)$, so that u satisfies the variational inequality (3.1). It is a known fact that for any given smooth bounded f, (3.1) admits a unique solution; see, for example, Friedman [5]. This completes the proof. □

We remark that (4.2) is derived from $u(X(s),s)=0$. Since both $u_x(X(s),s)=0$ and $u_s(X(s),s)=0$, it would not be easy to find a stable and efficient scheme based solely on (4.2) and the standard Newton's method. We shall derive numerical schemes on the basis of alternate integral equations for $X(\cdot)$.

4.3 The second integral identity

For every $x \in \mathbb{R}$ and $s \geqslant 1$, a differentiation with respect to x for u in (4.1) gives

$$u_x(x,s) = \int_1^s d\varsigma \int_{X(\varsigma)}^{\infty} \Gamma_x(x-y, s-\varsigma) f(y,\varsigma)\, dy.$$

Such differentiation is permitted since f is bounded and smooth, and

$$\int_1^s \int_{\mathbb{R}} |\Gamma_x(x-y, s-\varsigma)|\, dy\, d\varsigma = \frac{4\sqrt{s-1}}{\sqrt{\pi}} < \infty.$$

The condition $u_x(X(s), s) = 0$ immediately gives us the second integral equation (4.3).

For the same reason as before, although $u_x(x,t) = 0$ for all $x \leqslant X(s)$, a solution to (4.3) always provides us the correct answer.

Theorem 4.2 *Suppose* $X : s \in [1,\infty) \to \mathbb{R}$ *is continuous and satisfies* (4.3). *Then it is unique and the function* u *defined in* (4.1) *solves* (3.1) *and* (X, u) *solves* (3.3).

The proof is analogous to that for Theorem 4.1, and hence is omitted.

4.4 The third integral identity

To take another derivative, we use integration by parts to write

$$u_x(x,s) = \int_1^s \left\{ \Gamma(x - X(\varsigma), s-\varsigma) f(X(\varsigma),\varsigma) + \int_{X(\varsigma)}^{\infty} \Gamma(x-y, s-\varsigma) f_y(y,\varsigma)\, dy \right\} d\varsigma.$$

Assume that $X(\cdot)$ is continuous. Then for $x \neq X(s)$, we can interchange the order of differentiation and integration to obtain

$$u_{xx}(x,s) = \int_1^s \left\{ \Gamma_x(x - X(\varsigma), s-\varsigma) f(X(\varsigma),\varsigma) + \int_{X(\varsigma)}^{\infty} \Gamma_x(x-y, s-\varsigma) f_y(y,\varsigma)\, dy \right\} d\varsigma.$$

Suppose that $[X(s) - X(\varsigma)]/(s-\varsigma)^{3/2}$ is integrable over $\varsigma \in (1, s)$. Then

$$\int_1^s |\Gamma_x(X(s) - X(\varsigma), s-\varsigma) f(X(\varsigma),\varsigma)| d\varsigma = O(1) \int_1^s \frac{|X(s) - X(\varsigma)|}{(s-\varsigma)^{3/2}}\, d\varsigma < \infty.$$

As f is smooth, we derive that

$$\begin{aligned} \lim_{\varepsilon \to 0+} u_{xx}(X(s) \pm \varepsilon, s) = \mp 2 f(X(s), s) + \int_1^s \Bigg\{ & \Gamma_x(X(s) - X(\varsigma), s-\varsigma) f(X(\varsigma),\varsigma) \\ & + \int_{X(\varsigma)}^{\infty} \Gamma_x(X(s)-y, s-\varsigma) f_y(y,\varsigma)\, dy \Bigg\} d\varsigma. \end{aligned}$$

Consequently, since $u_{xx}(x,s) = 0$ for all $x < X(s)$, we have $u_{xx}(X(s)+, s) = -4f(X(s), s)$ and the integral identity (4.4).

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The fact that $u_{xx}(X(s)+,s) > 0$ allows us to devise a stable and efficient Newton's iteration scheme to solve X from equation (4.3), which comes from $u_x(\cdot,s) = 0$. As we see, the identity (4.4) plays an important role in simplifying our scheme.

5 A Newton iteration scheme

5.1 The derivation

We propose to numerically solve X from equation (4.3). For this, we define an operator Q from $\rho \in C^1((1,\infty))$ to $Q[\rho]$ by

$$\begin{aligned} Q[\rho](s) &:= \int_s^1 \int_{\rho(\varsigma)}^\infty \Gamma_x(\rho(s)-y, s-\varsigma) f(y,\varsigma)\, dy d\varsigma \\ &= \int_s^1 \int_0^\infty \Gamma_x(\rho(s)-\rho(\varsigma)-z, s-\varsigma) f(\rho(\varsigma)+z,\varsigma)\, dz d\varsigma \quad \forall s > 1. \end{aligned}$$

Thus, our problem is the following:

(P) *Find* $X \in C([1,\infty)) \cap C^\infty((1,\infty))$ *such that* $Q[X] \equiv 0$.

To solve **(P)** numerically, we use Newton's method. To implement this method, we need to calculate the first variation of $Q[\rho]$. For every smooth function ζ, we compute

$$\begin{aligned} Q'[\rho,\zeta](s) &= \lim_{\varepsilon \searrow 0} \frac{Q[\rho+\varepsilon\zeta](s) - Q[\rho](s)}{\varepsilon} \\ &= \int_1^s \int_0^\infty \big\{ (\zeta(s)-\zeta(\varsigma))\Gamma_{xx}(\rho(s)-\rho(\varsigma)-z, s-\varsigma) f(\rho(\varsigma)+z,\varsigma) \\ &\quad + \zeta(\varsigma)\Gamma_x(\rho(s)-\rho(\varsigma)-z, s-\varsigma) f_y(\rho(\varsigma)+z,\varsigma) \big\}\, dz\, d\varsigma \\ &= \zeta(s) \int_s^1 \Big\{ \Gamma_x(\rho(s)-\rho(\varsigma), s-\varsigma) f(\rho(\varsigma),\varsigma) + \int_{\rho(\varsigma)}^\infty \Gamma_x(\rho(s)-y, s-\varsigma) f_y(y,\varsigma)\, dy \Big\}\, d\varsigma \\ &\quad - \int_1^s \zeta(\varsigma)\Gamma_x(\rho(s)-\rho(\varsigma), s-\varsigma) f_y(\rho(\varsigma),\varsigma)\, d\varsigma. \end{aligned}$$

In particular, when $\rho = X$, we can use (4.4) to simplify the expression as

$$Q'[X,\zeta](s) = -2f(X(s),s)\zeta(s) - \int_s^1 \zeta(\varsigma)\Gamma_x(X(s)-X(\varsigma), s-\varsigma) f(X(\varsigma),\varsigma)\, d\varsigma.$$

Let Δs, representing a particular mesh size, be small. Suppose $\zeta \equiv 0$ on $[1, s-\Delta s]$. Then

$$\begin{aligned} Q'[X,\zeta](s) &= -2f(X(s),s)\zeta(s) - \int_{s-\Delta s}^s \zeta(\varsigma)\Gamma_x(X(s)-X(\varsigma), s-\varsigma) f(X(\varsigma),\varsigma)\, d\varsigma \\ &= -2f(X(s),s)\zeta(s) + o(1)\|\zeta\|_{L^\infty([s-\Delta s,s])}. \end{aligned} \tag{5.1}$$

Here we have assumed that the improper integral $\int_1^s |\Gamma_x(X(s)-X(\varsigma), s-\varsigma)| d\varsigma$ is convergent.

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5.2 The Newton iteration

Now we use Newton's method to devise an iteration scheme for the unknown function X.

Suppose we have already found X in $[1, s-\Delta s]$ and want to find X on $(s-\Delta s, s]$. Picking an initial guess $X^{old}(s)$, say $X^{old} \equiv X(s-\Delta s)$ on $[s-\Delta s, s]$. We can find an iterative updating scheme from X^{old} to X^{new} according to the following rationale. Let $\zeta = X(s) - X^{old}(s)$ be the amount of unknown correction needed. Then $X^{old} = X - \zeta$, and using $Q[X](s) = 0$ and (5.1), we have

$$Q[X^{old}](s) = Q[X-\zeta](s) - Q[X](s) \approx 2f(X(s), s)\zeta(s).$$

This gives us the approximation formula for the correction $\zeta(s) \approx \frac{Q[X^{old}](s)}{2f(X(s),s)}$. Thus, we have the following Newton scheme in a continuous setting:

$$X^{new}(\varsigma) = X^{old}(\varsigma) + \frac{Q[X^{old}](\varsigma)}{2f(X^{old}(\varsigma), \varsigma)} \quad \forall\, \varsigma \in (s-\Delta s, s].$$

We remark that in the interval $(1, 1+\Delta s]$, one could pick the very first initial guess $X^{old} \equiv \beta$.

5.3 The operator Q

Since $Q[X](s) = u_x(X(s), s)$ involves a double integral over $y \in (X(\varsigma), \infty)$ and $\varsigma \in (1, s)$, to reduce the amount of calculation needed, we reduce the double integral to a single integral. We begin with

$$\begin{aligned} u_x(x,s) &= \int_s^1 \int_{X(\varsigma)}^{\infty} \Gamma_x(x-y, s-\varsigma) f(y,\varsigma)\, dy\, d\varsigma \\ &= \int_1^s \frac{(\varsigma^{\sigma}-1)}{s\varsigma^{v}\sqrt{s-\varsigma}} \int_{X(\varsigma)}^{\infty} \frac{2s(y-x)(y-\beta\sqrt{\varsigma})}{(s-\varsigma)\varsigma} e^{-\frac{(x-y)^2}{s-\varsigma} - \frac{(y-\alpha\sqrt{\varsigma})^2}{\varsigma}}\, dy\, d\varsigma. \end{aligned}$$

Using integration by parts, we can derive that

$$\begin{aligned} u_x(x,s) &= \int_1^s \frac{G_1(x, X(\varsigma), s, \varsigma)}{\sqrt{s-\varsigma}} d\varsigma + \int_1^s G_2(x, X(\varsigma), s, \varsigma)\, d\varsigma \quad \forall\, x \in \mathbb{R}, s > 1, \\ Q[X](s) &= \int_1^s G_2(X(s), X(\varsigma), s, \varsigma)\, d\varsigma - 2\int_1^s G_1(X(s), X(\varsigma), s, \varsigma)\, d\sqrt{s-\varsigma} \quad \forall s > 1 \end{aligned}$$

where

$$\begin{aligned} G_1(x,y,s,\varsigma) &:= \frac{\varsigma^{\gamma}-1}{s\varsigma^{v}} \{ y - \beta\sqrt{\varsigma} - \tfrac{s-\varsigma}{s}(x - \alpha\sqrt{\varsigma}) \} e^{-\frac{(x-y)^2}{s-\varsigma} - \frac{(y-\alpha\sqrt{\varsigma})^2}{\varsigma}}, \\ G_2(x,y,s,\varsigma) &:= \frac{\sqrt{\pi}(\varsigma^{\gamma}-1)e^{-(x-\alpha\sqrt{\varsigma})^2/s}}{s^{3/2}\varsigma^{v-1/2}} \left\{ \tfrac{1}{2} - \left(\tfrac{x}{\sqrt{\varsigma}} - \alpha\right)\left(\tfrac{x\sqrt{\varsigma}}{s} + \tfrac{s-\varsigma}{s}\alpha - \beta\right) \right\} \\ &\quad \times \mathrm{Erfc}\left(\sqrt{\tfrac{s}{(s-\varsigma)\varsigma}}(y-x) + \left(\tfrac{x}{\sqrt{\varsigma}} - \alpha\right)\sqrt{\tfrac{s-\varsigma}{s}} \right) \quad \left(\mathrm{Erfc}(z) := \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-t^2} dt \right), \end{aligned}$$

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$$G_1(x,x,s,s) = \frac{(s^\gamma - 1)(x-\beta\sqrt{s})e^{-(x/\sqrt{s}-\alpha)^2}}{s^{\nu+1}} = \frac{1}{\sqrt{\pi}} f(x,s),$$

$$G_2(x,x,s,s) = \frac{\sqrt{\pi}(s^\gamma - 1)e^{-(x/\sqrt{s}-\alpha)^2}}{s^{\nu+1}} \left\{ \frac{1}{2} - \left(\frac{x}{\sqrt{s}} - \alpha\right)\left(\frac{x}{\sqrt{s}} - \beta\right) \right\}.$$

5.4 The standard numerical scheme

Suppose we use mesh points $\{s_i\}_{i=0}^{\infty}$, where $1 = s_0 < s_1 < s_2 < \cdots$. We denote the approximation of $X(s_i)$ by X_i. One can check that

$$X_0 = X(s_0) = X(1) = \beta.$$

We can use the trapezoid rule to discretize the integral for $Q[X](s_n)$:

$$\begin{aligned} Q[X](s_n) \approx & \sum_{i=1}^{n} (\sqrt{s_n - s_{i-1}} - \sqrt{s_n - s_i})(G_1(X_n, X_i, s_n, s_i) + G_1(X_n, X_{i-1}, s_n, s_{i-1})) \\ & + \sum_{i=1}^{n} (s_i - s_{i-1}) \frac{G_2(X_n, X_i, s_n, s_i) + G_2(X_n, X_{i-1}, s_n, s_{i-1})}{2}. \end{aligned}$$

Consider $z = X_n$ as an unknown. Numerically, we solve for it from the equation $Q_n(z) = 0$. Since $G_1(\cdot,\cdot,\cdot,1) \equiv 0$ and $G_2(\cdot,\cdot,\cdot,1) \equiv 0$, we have

$$\begin{aligned} Q_1(z) &:= \sqrt{s_1 - 1}\, G_1(z,z,s_1,s_1) + \frac{s_1 - 1}{2} G_2(z,z,s_1,s_1), \\ Q_n(z) &:= \sqrt{s_n - s_{n-1}} G_1(z,z,s_n,s_n) + \frac{s_n - s_{n-1}}{2} G_2(z,z,s_n,s_n) \\ & \quad + \sum_{i=1}^{n-1} \left\{ (\sqrt{s_n - s_{i-1}} - \sqrt{s_n - s_{i+1}}) G_1(z, X_i, s_n, s_i) + \frac{(s_{i+1} - s_{i-1})}{2} G_2(z, X_i, s_n, s_i) \right\} \end{aligned}$$

for $n \geqslant 2$.

Suppose $X_0, X_1, \cdots, X_{n-1}$ are known. We solve for $X_n = z$ from $Q_n(z) = 0$ by the following iteration:

$$\begin{cases} z_0 = X_{n-1} + \frac{X_{n-1} - X_{n-2}}{s_{n-1} - s_{n-2}} (s_n - s_{n-1}), \\ z_{q+1} = z_q + \dfrac{Q_n(z_q)}{2f(z_q, s_n)}, \quad q = 0, 1, 2, \cdots, \\ X_n = z_{q+1} \quad \text{if } |z_{q+1} - z_q| \leqslant \varepsilon, \text{a given tolerance.} \end{cases} \tag{5.2}$$

Here z_0 is an initial guess derived from a linear interpolation. We point out that Newton's method is quite efficient. For instance, in the example summarized on the left-hand side of Table 2, when 1,024 evenly distributed division points are used for the interval $[1, e^{2kT}] \ni s$ with $T = 1$ (year) and the tolerance is set to be $\varepsilon = 5 \times 10^{-7}$, the sum of all the q's in the 1,024 steps are 275; that is, the average number q of iterations is about 0.3, which means $q = 0$ in most updating steps from X_{n-1} to X_n.

Table 2. *Rate of convergence for the standard numerical scheme (left) and the upgraded scheme (right). Here "grid" stands for the number of grids, "iteration" is the total Newton iterations, "solution" is the value of X at $s = e^{2k\tau}$ with $\tau = T - t = 1$ (year), "improvement" is the difference between the current solution with that in the previous row and "rate" is the ratio of the consecutive improvements.*

Standard scheme		tolerance 5.0×10^{-7}			Upgraded scheme		(Tolerance 5.0×10^{-7})		
Grid	Iteration	Solution	Improvement	Rate	Grid	Iteration	Solution	Improvement	Rate
8	61	0.2161798	3.0×10^{-2}	3.1	8	21	0.2436451	2.9×10^{-4}	0.0
16	90	0.2303882	1.4×10^{-2}	2.1	16	37	0.2438225	1.8×10^{-4}	1.7
32	127	0.2373004	6.9×10^{-3}	2.1	32	59	0.2439030	8.1×10^{-5}	2.2
64	183	0.2406784	3.4×10^{-3}	2.0	64	85	0.2439357	3.3×10^{-5}	2.5
128	238	0.2423363	1.7×10^{-3}	2.0	128	142	0.2439484	1.3×10^{-5}	2.6
256	353	0.2431532	8.2×10^{-4}	2.0	256	266	0.2439531	4.7×10^{-6}	2.7
512	326	0.2435571	4.0×10^{-4}	2.0	512	253	0.2439548	1.7×10^{-6}	2.7
1024	275	0.2437574	2.0×10^{-4}	2.0	1024	213	0.2439555	6.3×10^{-7}	2.8

Numerical simulation shows that this numerical scheme has an error of size $X(s_n) - X_n = O((\Delta s))$, where Δs is the mesh size. That is to say, when the mesh size is halved, the error reduces by half.

5.5 Upgraded numerical scheme

In general, one can improve the rate of convergence for numerical integration by using higher order quadrature rules. Since in the current situation singular integrals are involved, higher order quadrature rules are not very effective. Here we introduce a *modified trapezoid rule* designed specifically for the singular integrals at hand.

Notice that for any constants $a < b \leqslant s$ and linear function $g(x)$ on $[a, b]$, we have

$$\begin{aligned}\int_a^b \frac{g(x)}{\sqrt{s-x}}dx &= \int_a^b \frac{(b-x)g(a) + (x-a)g(b)}{(b-a)\sqrt{s-x}}dx \\ &= \frac{2(b-a)}{3(\sqrt{s-a}+\sqrt{s-b})^2}\{[\sqrt{s-a}+2\sqrt{s-b}]g(a) + [2\sqrt{s-a}+\sqrt{s-b}]g(b)\}.\end{aligned}$$

Thus, we can use the following discretization for the function $Q[X](s_n)$. When $n = 1$,

$$\bar{Q}_1(z) = \frac{4\sqrt{s_1-1}}{3}G_1(z, z, s, s) + \frac{s_1-1}{2}G_2(z, z, s, s).$$

When $n \geqslant 2$,

$$\begin{aligned}\bar{Q}_n(z) = &\frac{4\sqrt{s_n - s_{n-1}}}{3}G_1(z, z, s_n, s_n) + \frac{s_n - s_{n-1}}{2}G_2(z, z, s_n, s_n) \\ &+ \sum_{i=1}^{n-1} \frac{s_{i+1} - s_{i-1}}{2}G_2(z, X_i, s_n, s_i)\end{aligned}$$

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$$+\sum_{i=1}^{n-1}\frac{2G_1(z,X_i,s_n,s_i)}{3}\left\{\frac{(s_i-s_{i-1})(\sqrt{s_n-s_i}+2\sqrt{s_n-s_{i-1}})}{(\sqrt{s_n-s_i}+\sqrt{s_n-s_{i-1}})^2}\right.$$
$$\left.+\frac{(s_{i+1}-s_i)(\sqrt{s_n-s_i}+2\sqrt{s_n-s_{i+1}})}{(\sqrt{s_n-s_i}+\sqrt{s_n-s_{i+1}})^2}\right\}.$$

Setting $X_0=\beta$ and a 'ghost' value $X_{-1}=\beta+0.334\sqrt{s_1-1}$, we can calculate $\{X_n\}$ iteratively for $n=1,2,\cdots$ by scheme (5.2) (with Q_n replaced by $\bar{Q}_n$). The rate of convergence is observed by numerical experimentation to be about $O((\Delta s)^{3/2})$: $X(s_n)-X_n=O(\Delta s^{3/2})$. That is, when the mesh size Δs is halved, the error reduces by a factor $2\sqrt{2}=2.8$.

5.6 A numerical example

The rate of convergence for uniform mesh size for two simulations are summarized in Table 2. In this example, we take a typical US economy in 2006, in annual units,

$$c=0.055,\quad \theta=0.05,\quad \sigma=0.015,\quad k=0.15.$$

One notices that the Newton iteration converges very fast; for example, when 1,024 evenly distributed grid points are used for the interval $[1,e^{2kT}]$ with $T=1$ (year), the total number of iterations for the two schemes are 287 and 213, respectively, which means iteration is not needed in most updates. Also, one sees that the upgraded scheme is significantly better than the standard scheme.

Figure 1(a) illustrates the difference between the two schemes. Since the upgraded scheme directly treats the singularity of the integral, the improvement of the solution at the first node is significant.

5.7 Approximations at the first node

The equations for the numerical approximations X_1 at the first node from the two schemes are respectively the following:

$$\text{Standard scheme: } X_1-\beta\sqrt{s_1}=-\frac{\sqrt{\pi}\sqrt{s_1-1}}{2}\left\{\frac{1}{2}-\left(\frac{X_1}{\sqrt{s_1}}-\alpha\right)\left(\frac{X_1}{\sqrt{s_1}}-\beta\right)\right\}.$$
$$\text{Upgraded scheme: } \frac{4}{3}(X_1-\beta\sqrt{s_1})=-\frac{\sqrt{\pi}\sqrt{s_1-1}}{2}\left\{\frac{1}{2}-\left(\frac{X_1}{\sqrt{s_1}}-\alpha\right)\left(\frac{X_1}{\sqrt{s_1}}-\beta\right)\right\}.$$

Hence, we have the following asymptotic expansions:

$$\text{Standard scheme: } X_1\approx\beta-\tfrac{\sqrt{\pi}}{4}\sqrt{s_1-1}\approx\beta-0.443\sqrt{s_1-1}.$$
$$\text{Upgraded scheme: } X_1\approx\beta-\tfrac{3\sqrt{\pi}}{16}\sqrt{s_1-1}\approx\beta-0.332\sqrt{s_1-1}.$$

We note that the true asymptotic expansion is $X(s)=\beta-[0.334\ldots+o(1)]\sqrt{s-1}$ as $s\searrow 1$.

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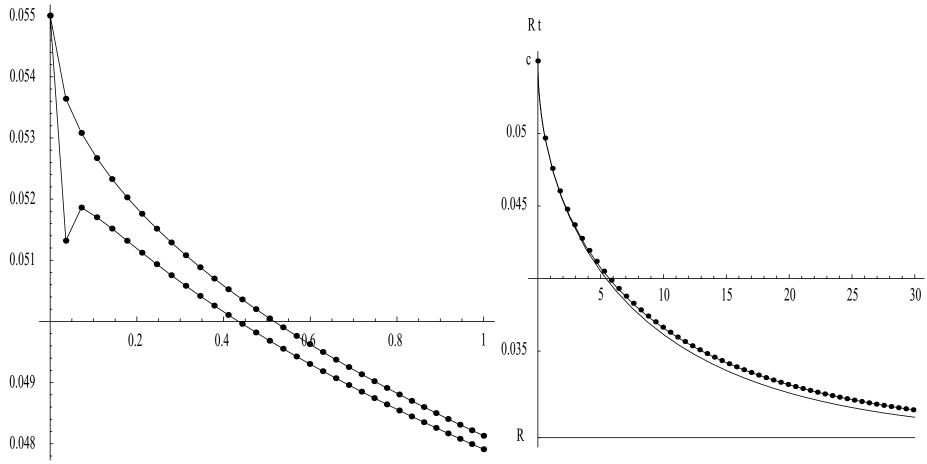

(a) (b)

FIGURE 1. (a) Numerical solutions of the curve $(t, R(t))$ in annual units with 32 grid points. Dots on the top curve are from the upgraded scheme; the dots on the bottom curve are for the standard scheme. (b) Dots represent the 'true' solution, calculated by using 2,048 division points in the upgraded scheme outlined in § 5. The curve 'on the dots' is the second approximation, with maximum relative error 0.004 that is not discernable from the plot. The curve below the dots is the first approximation, with relative error 0.02.

5.8 An exact solution for a special f

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When f in (3.2) is given by $f(x,s) = \gamma\sqrt{\pi}e^{-(\beta-\alpha)^2}(x-\beta)(s-1)$, there is an exact solution of (3.3) as

$$X(s) = \beta - \kappa\sqrt{s-1}, \qquad u(x,s) = \gamma\sqrt{\pi}e^{-(\beta-\alpha)^2}(s-1)^{5/2}g\left(\frac{\beta - x}{\sqrt{s-1}}\right),$$

where g, together with the unknown constant κ, solve the 'free boundary' problem

$$g''(z) + 2zg'(z) - 10g(z) - 4z = 0 \quad \forall z < \kappa, \quad g(\kappa) = g'(\kappa) = 0,$$
$$g > 0 \text{ in } (-\infty, \kappa], \qquad g = 0 \text{ in } (\kappa, \infty), \quad g(z) = O(z) \text{ as } z \to -\infty.$$

We find that the solution to this free boundary problem is given by

$$g(z) = \frac{1}{2}\left\{\frac{\kappa\int_{-\infty}^{z}(z-t)^5 e^{-t^2}dt}{\int_{-\infty}^{\kappa}(\kappa-t)^5 e^{-t^2}dt} - z\right\} \quad \forall z < \kappa,$$

where κ is the unique solution to the transcendental equation

$$5\kappa\int_{-\infty}^{\kappa}(\kappa-t)^4 e^{-t^2}dt = \int_{-\infty}^{\kappa}(\kappa-t)^5 e^{-t^2}dt \Leftrightarrow \quad \int_{\kappa}^{\infty}\frac{15e^{-t^2}dt}{2t^4(5+2t^2)^2} = \sqrt{\pi}.$$

A numerical calculation gives

$$\kappa = 0.3343641440309\ldots.$$

This exact solution of (3.3) can be used to test the accuracy (rate of convergence) of our numerical schemes. In [6], the asymptotic behaviour (1.4) (with $\bar{\kappa} = \sqrt{2}\,\kappa$) is derived by a method equivalent to replacing f by its asymptotic expansion $\gamma\sqrt{\pi}e^{-(\beta-\alpha)^2}(x-\beta)(s-1)$.

6 Asymptotic behaviour of $R(t)$ when $t \to -\infty$

In this section, we prove the following.

Theorem 6.1 *There exist constants* $R^* \in (-\infty, c)$ *and* $\rho^* > 0$ *such that* (1.5) *holds.*

The idea here is to study first the limit $(R^*, V^*(\cdot)) := \lim_{t\to-\infty}(R(t), \frac{c}{m}V(\cdot, t))$, which solves a so-called *infinite horizon problem*, and then the limit $\zeta^*(r) := \lim_{t\to-\infty}\zeta(r,t)$, where

$$\zeta(r,t) := \frac{V_t(r,t)}{\dot{M}(t)} = -\frac{V_t(r,t)}{m\,e^{c(t-T)}}.$$

After deriving the relation

$$\dot{R}(t) = \frac{c\,\sigma^2}{2m}\frac{V_{tr}(R(t)+,t)}{(c-R(t))(1-e^{c(t-T)})} = \frac{ce^{c(t-T)}\,\sigma^2}{2}\frac{\zeta_r(R(t)+,t)}{(R(t)-c)(1-e^{c(t-T)})},$$

we see that

$$\rho^* := \frac{1}{c}\lim_{t\to-\infty}\frac{\dot{R}(t)}{e^{c(t-T)}} = \frac{\sigma^2}{2}\frac{\zeta_r^*(R^*)}{(R^*-c)}.$$

The proof of the theorem is given in the following subsections. In the mean time, we derive formulas for $R^*, V^*(\cdot), \zeta^*(\cdot)$ and ρ^*.

6.1 The infinite horizon problem

In [6], it is shown that $\dot{R}(t) > 0$. Also, one can show that $V_t \leqslant 0$. Hence, there exists

$$\lim_{t\to-\infty}\Big(R(t),\ \frac{c}{m}V(\cdot,t)\Big) = (R^*, V^*(\cdot)). \tag{6.1}$$

From (1.3), one derives that (R^*, V^*) is a solution to the following *infinite horizon problem*:

$$\begin{cases} \left\{\dfrac{\sigma^2}{2}\dfrac{d^2}{dr^2} + k(\theta - r)\dfrac{d}{dr} - r\right\}V^* = -c & \text{in } (R^*, \infty), \\ V^*(R^*) = 1, \quad V_r^*(R^*) = 0, \quad 0 \leqslant V^* \leqslant 1 & \text{in } (R^*, \infty). \end{cases} \tag{6.2}$$

Theorem 6.2 *Assume that* σ, k, θ, c *are positive constants. Then* (6.2) *admits a unique solution. In addition, the solution has the property that* $R^* \in (-\infty, c)$ *and* $V_r^*(r) < 0$ *for all* $r \in (R^*, \infty)$.

In the next two subsections, we prove Theorem 6.2, along with formulas for R^* and $V^*(\cdot)$.

6.2 The homogeneous equation

We begin with the homogeneous equation

$$\left\{\frac{\sigma^2}{2}\frac{d^2}{dr^2} + k(\theta - r)\frac{d}{dr} - r\right\} G(r) = 0, \quad r \in \mathbb{R}.$$

In a self-adjoint form, this equation can be written as

$$\left\{e^{-k(r-\theta)^2/\sigma^2} G_r(r)\right\}_r = \tfrac{2}{\sigma^2}\, e^{-k(r-\theta)^2/\sigma^2} r\, G(r), \quad r \in \mathbb{R}. \tag{6.3}$$

If G_1 and G_2 are two linearly independent solutions, their Wronskian satisfies

$$G_{1r}(r)G_2(r) - G_{2r}(r)G_1(r) = Ce^{k(r-\theta)^2/\sigma^2},$$

where C is a non-zero constant. Thus, if there is a solution bounded at $r = \infty$, it is unique up to a constant multiple. We now find such a solution.

Lemma 1 *Assume that* $\sigma > 0$ *and* $k > 0$. *Then* (6.3) *admits a unique solution satisfying*

$$\lim_{r\to\infty} G(r)e^{r/k}r^{-\mu} = 1, \qquad \mu := \frac{\sigma^2 - 2\theta k^2}{2k^3}.$$

In addition, there exists $r_0 \in [-\infty, 0)$ *such that*

$$G_r < 0 < G \text{ in } (r_0, \infty), \qquad \int_{r_0}^{\infty} re^{-k(r-\theta)^2/\sigma^2} G(r)\, dr = 0. \tag{6.4}$$

In particular, (i) *when* $\sigma^2 \leqslant 2k^2\theta, r_0 = -\infty$; (ii) *when* $\sigma^2 > 2k^2\theta$, $r_0 > -\infty$ *and* $G_r(r_0) = 0$.

Proof Make a change of variables

$$x = \frac{\sqrt{k}}{\sigma}\left(r + \frac{\sigma^2}{k^2} - \theta\right), \quad H(x) = e^{r/k}G(r).$$

Then, $H = H(x)$ satisfies the Hermite equation

$$H_{xx} = 2xH_x - 2\mu H \quad \forall\, x \in \mathbb{R}.$$

A particular solution of this ordinary differential equation (ODE) is the Hermite function defined as

$$H(\mu; x) = \frac{(-1)^m}{\Gamma(m-\mu)} \int_0^{\infty} t^{m-\mu-1}\frac{d^m e^{-t^2-2xt}}{dt^m} dt \quad \forall x, \mu \in \mathbb{C}, \quad m \in \mathbb{N} \cap (\mathrm{Re}(\mu), \infty). \tag{6.5}$$

Here $\Gamma(\cdot)$ is the Gamma function, $\mathbb{N} = \{0, 1, 2, \cdots\}$ is the set of non-negative integers, and, most importantly, the integral on the right-hand side is independent of the integer

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m, and hence $H(\mu;x)$ is an entire function of both variables $\mu \in \mathbb{C}$ and $x \in \mathbb{C}$. The integer m here is introduced so that the integral is uniformly convergent. Without it, one can use contour integrals to express

$$H(\mu;x) = \frac{1}{\Gamma(-\mu)[1-e^{-2\pi\mu\mathbf{i}}]}\int_\omega t^{-\mu-1}e^{-t^2-2xt}dt \qquad \forall\, x \in \mathbb{C},\ \mu \in \mathbb{C}\setminus\mathbb{Z}$$

where ω is any contour starting from $\infty e^{2\pi\mathbf{i}}$, rotating around the origin clockwise without touching the origin and positive real axis, and finally ending at $\infty e^{0\mathbf{i}}$. One can derive the relations

$$\begin{aligned}
H_x(\mu;x) &= 2\mu H(\mu-1;x) && \forall\, x,\mu \in \mathbb{C},\\
H(\mu+1;x) &= 2xH(\mu;x) - 2\mu H(\mu-1;x) && \forall\, x,\mu \in \mathbb{C},\\
H(\mu;x) &\sim (2x)^\mu \quad \text{as} \quad x \to \infty && \forall\, \mu \in \mathbb{C}\\
H(\mu;x) &\sim \frac{\sqrt{\pi}e^{x^2}}{\Gamma(-\mu)(-x)^{\mu+1}} \quad \text{as} \quad x \to -\infty && \forall\, \mu \in \mathbb{C}\setminus\mathbb{N},\\
H(\mu;-x) &= (-1)^\mu H(\mu;x) && \forall\, x \in \mathbb{C},\ \mu \in \mathbb{N}.
\end{aligned}$$

Also, from $[e^{-x^2}H_x]_x = -2\mu e^{-x^2}H$, one can derive that on the real axis, $H(\mu;\cdot) > 0 > H_x(\mu;\cdot)$ when $\mu \leqslant 0$ and $H(\mu;\cdot)$ changes sign when $\mu > 0$.

Now going back to the original variable, we find that

$$G(r)e^{-k(r-\theta)^2/\sigma^2} = e^{-x^2+x\sigma k^{-3/2}-\theta/k}H(\mu;x).$$

It follows that

$$\lim_{|r|\to\infty}\{|G_r(r)| + |G(r)|\}e^{-k(r-\theta)^2/\sigma^2}e^{|r|/(2k)} = 0. \tag{6.6}$$

Integrating (6.3) over $\mathbb{R}$, we obtain $\int_{\mathbb{R}} re^{-k(r-\theta)^2/\sigma^2}G(r)dr = 0$, where the improper integer is uniformly convergent. Finally, we have the following:

(1) When $\sigma^2 > 2k^2\theta$, we have $\mu > 0$. As $H(\mu;\cdot)$ changes sign, so does $G(\cdot)$. Thus, there exists a finite real r_0 such that $G_r(r_0) = 0$ and $G_r < 0$ in (r_0,∞). This implies that $G > 0$ in $[r_0,\infty)$. After integrating (6.3) over $[r_0,\infty)$, we obtain the integral identity in (6.4).

(2) When $\sigma^2 \leqslant 2k^2\theta$, we have $\mu \leqslant 0$, so that $H(\mu;x) > 0$ for all $x \in \mathbb{R}$. Thus, $G > 0$ in $\mathbb{R}$. As $[e^{-k(r-\theta)^2/\sigma^2}]_r$ is positive in $(0,\infty)$ and negative in $(-\infty,0)$, in view of (6.6), we derive $G_r < 0$ on $\mathbb{R}$. Hence, (6.4) holds with $r_0 = -\infty$. This completes the proof. □

6.3 Proof of Theorem 6.2

We divide the proof into several steps. Suppose (R^*,V^*) solves (6.2). We first establish certain properties of (R^*,V^*) and then derive a formula for it, thereby obtaining both existence and uniqueness.

1. First, we show that $V_r^* < 0$ in (R^*,∞).

Suppose otherwise. Then $V_r^*(r_1) \geqslant 0$ at some $r_1 > R^*$. Since $V^*(R^*) = 1$ is a global maximum, $r_2 := \sup\{r \in (R^*,r_1) \mid V_r^*(r) < 0\}$ is well defined and by continuity $V_r^*(r_2) = 0$. The case $V_{rr}^*(r_2) < 0$ is impossible since it would imply $V_r^* > 0$ in $(r_2-\varepsilon,r_2)$ for some small

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positive ε, contradicting the definition of r_2. The case $V^*_{rr}(r_2) = 0$ is also impossible since it would imply by the ODE for V^* that $r_2 V^*(r_2) = c > 0$ and $\frac{\sigma^2}{2} V^*_{rrr}(r_2) = V^*(r_2) > 0$, so that $V^*_r > 0$ in $(r_2 - \varepsilon, r)$ for some small positive ε. Hence, $V^*_{rr}(r_2) > 0$ and, by the ODE, $r_2 V^*(r_2) > c$. Set $r_3 = \sup\{r > r_2 \mid V^*_r > 0 \text{ in } (r_2, r)\}$. Then for every $r \in (r_2, r_3)$, $rV^*(r) > r_2 V(r_2) > c$ and $[e^{-k(r-\theta)^2/\sigma^2} V_r(r)]_r = (rV - c)e^{-k(r-\theta)^2/\sigma^2} > 0$. That is, $e^{-k(r-\theta)^2/\sigma^2} V_r$ is a strictly increasing function on $[r_2, r_3)$. This implies $r_3 = \infty$ and $\lim_{r\to\infty} e^{-k(r-\theta)^2/\sigma^2} V_r > 0$, which further implies $\lim_{r\to\infty} V^*_r = \infty$, contradicting the boundedness of V^*. Thus, we must have $V^*_r < 0$ in (R^*, ∞). Consequently, $0 < V^* < 1$ in (R^*, ∞).

2. Next, we show that $R^* > r_0$. For this, consider the weighted Wronskian

$$W(r) = \{V^*_r(r)G(r) - V^*(r)G_r(r)\}e^{-k(r-\theta)^2/\sigma^2}.$$

It satisfies $\frac{\sigma^2}{2} W_r = -ce^{-k(r-\theta)^2/\sigma^2} G$. Integrating this equation over (r, ∞) gives

$$W(r) = \frac{2c}{\sigma^2} \int_r^\infty Ge^{-k(r-\theta)^2/\sigma^2} dt \quad \forall r \geqslant R^*. \tag{6.7}$$

First, consider the case $r_0 > -\infty$. Should $R \leqslant r_0$, we would have $0 < W(r_0) = V^*_r(r_0)G(r_0)e^{-k(r-\theta)^2/\sigma^2} \leqslant 0$, a contradiction, since $G_r(r_0) = 0$ and $G > 0$ on $[r_0, \infty)$.

Next, we consider the case $r_0 = -\infty$. Then $G > 0$ on $\mathbb{R}$. Should $R^* = -\infty$, the boundedness of V^* implies that along a sequence $R_j \to -\infty$, $V_r(R_j) \to 0$ so that, in view of (6.6), $W \to 0$ along the sequence $\{R_j\}$, thus contradicting (6.7). Thus, we must have $R^* > r_0$.

3. Now we show that R^* needs to satisfy the following **solvability condition** for R^*:

$$\int_{R^*}^\infty (r - c)G(r)e^{-k(r-\theta)^2/\sigma^2} dr = 0, \qquad R^* > r_0. \tag{6.8}$$

In fact, substituting $V^*(R^*) = 1$ and $V^*_r(R^*) = 0$ into (6.7) at $r = r^*$ gives

$$e^{-k(r-\theta)^2/\sigma^2} G_r(R^*) = -\frac{2c}{\sigma^2} \int_{R^*}^\infty Ge^{-k(r-\theta)^2/\sigma^2} dt.$$

Equation (6.8) then follows by noting that

$$e^{-k(r-\theta)^2/\sigma^2} G_r(R^*) = \int_\infty^{R^*} [e^{-k(r-\theta)^2/\sigma^2} G_r(r)]_r dr = -\frac{2}{\omega^2} \int_{R^*}^\infty re^{-k(r-\theta)^2/\sigma^2} G(r)\, dr.$$

4. Here we show that (6.8) has a unique solution R^*. Since $\int_{r_0}^\infty rG(r)e^{-k(r-\theta)^2/\sigma^2} dr = 0$ and $G > 0$ on $[r_0, \infty)$, we see that $r_0 < 0$ and that the function

$$\Psi(c, r) := \int_r^\infty (t - c)G(t)e^{-k(t-\theta)^2/\sigma^2} dt, \quad c > 0, r \in \mathbb{R}$$

has the property

$$\Psi(c, \infty) = 0, \quad \Psi_r(c, \cdot) < 0 \text{ in } (c, \infty), \quad \Psi_r(c, \cdot) > 0 \text{ in } (r_0, c), \quad \Phi(r_0) < 0.$$

It then follows that the algebraic equation $\Psi(c, \cdot) = 0$ has a unique root in (r_0, ∞). Thus,

R^* is the unique root to (6.8) and

$$r_0 < R^* < c, \qquad \lim_{c \searrow 0} R^* = r_0 \in [-\infty, 0).$$

One notices that $\Psi(c, r) > 0$ for all $r > R^*$.

5. We are ready now to derive a formula for V^*. Integrating over $[R^*, r)$, equation (6.7) multiplied by $e^{k(r-\theta)^2/\sigma^2} G^{-2}$ and using $V^*(R^*) = 1$, we obtain

$$V^*(r) := G(r)\left\{\frac{1}{G(R^*)} + \frac{2c}{\sigma^2}\int_{R^*}^{r} \frac{e^{k(t-\theta)^2/\sigma^2}}{G^2(t)} \int_t^\infty G(s) e^{-k(s-\theta)^2/\sigma^2} ds\, dt\right\}. \tag{6.9}$$

Using

$$\frac{1}{G(R^*)} - \frac{1}{G(r)} = \int_{R^*}^{r} \frac{G_r(t)}{G^2(t)} dt = -\frac{2}{\sigma^2}\int_{R^*}^{r} \frac{e^{k(t-\theta)^2/\sigma^2}}{G^2(t)} \int_t^\infty s\, G(s) e^{-k(s-\theta)^2/\sigma^2} ds\, dt,$$

we can write the above expression as

$$V^*(r) = 1 - \frac{2G(r)}{\sigma^2}\int_{R^*}^{r} \frac{e^{k(t-\theta)^2/\sigma^2}}{G^2(t)} \int_t^\infty (s-c)\, G(s) e^{-k(s-\theta)^2/\sigma^2} ds\, dt. \tag{6.10}$$

In conclusion, if (R^*, V^*) solves (6.2), then R^* is the unique root to (6.8) and V^* is given by (6.9), which is equivalent to (6.10).

6. Finally, from (6.9), we see that $V^* > 0$ on $[R^*, \infty)$. Also, as $\Psi(c, r) > 0$ for all $r > R^*$, we see from (6.10) that $V^* < 1$ in (R^*, ∞) and that $V^*(R^*) = 1$, and $V_r(R^*) = 0$. It is then an easy exercise to show that V^* in (6.9) satisfies the ODE in (6.2). Thus, (R^*, V^*) obtained in this manner is indeed a solution to (6.2). We have hence established the existence of a unique solution to (6.2), thereby completing the proof of Theorem 6.2. □

6.4 Asymptotic behaviour of $R(t)$ as $t \to \infty$

Recall that [6]

$$V_r(R(t), t) = 0, \qquad V_t(R(t), t) = \dot{M}(t) = -me^{c(t-T)}.$$

This implies, by the partial differential equation (PDE) for V in (1.3) and by differentiating $V_r(R(t), t) = 0$, that

$$\begin{aligned} V_{rr}(R(t)+, t) &= \frac{2}{\sigma^2}\{rM(t) - m - \dot{M}\} = \frac{2m}{c\sigma^2}(r-c)(1-e^{c(T-t)}), \\ \dot{R}(t) &= -\frac{V_{rt}(R(t)+, t)}{V_{rr}(R(t)+, t)} = \frac{c\,\sigma^2}{2m}\frac{V_{tr}(R(t)+, t)}{(c-R(t))(1-e^{c(T-t)})}. \end{aligned}$$

Hence, to find the asymptotic behaviour of $\dot{R}(t)$ as $t \to -\infty$, it suffices to find the asymptotic behaviour of $V_{tr}(R(t)+, t)$ as $t \to -\infty$. For this, we consider the function V_t whose boundary value at $r = R(t)$ is known to be $V_t = \dot{M}(t) = -me^{c(t-T)}$. Also, V_t satisfies

$$\left\{\frac{\partial}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial r^2} + k(\theta - r)\frac{\partial}{\partial r} - r\right\} V_t = 0, \quad r > R(t), t < T.$$

For the leading order expansion of V_t as $t \to -\infty$, it is natural to consider

$$\zeta(r,t) := \frac{V_t(r,t)}{\dot{M}(t)} = -\frac{V_t}{m\, e^{c(t-T)}}.$$

Then ζ satisfies the following problem:

$$\begin{cases} \left\{ \dfrac{\partial}{\partial t} + \dfrac{\sigma^2}{2}\dfrac{\partial^2}{\partial r^2} + k(\theta - r)\dfrac{\partial}{\partial r} + (c-r) \right\} \zeta(r,t) = 0, & r > R(t), t < T, \\ \zeta(r,t) = 1 \quad \forall r \leqslant R(t),\ t < T, \qquad \zeta(r,T) = 1 \quad \forall r \in \mathbb{R}. \end{cases} \tag{6.11}$$

Here the initial and boundary data for ζ follow from the fact that $V(r,t) = M(t)$ for all $r \leqslant R(t)$ and that $V(\cdot, T) = 0$. We prove in a subsequent subsection that there is a limit

$$\lim_{t\to-\infty} \zeta(r,t) = \zeta^*(r) \qquad \forall\, r > R^* \tag{6.12}$$

that satisfies the ODE problem

$$\begin{cases} \left\{ \dfrac{\sigma^2}{2}\dfrac{d^2}{dr^2} + k(\theta - r)\dfrac{d}{dr} + (c-r) \right\} \zeta^*(r) = 0 \quad \forall\, r > R^*, \\ \zeta^*(R^*) = 1, \quad \sup_{r \geqslant R^*} \zeta^*(r) < \infty. \end{cases} \tag{6.13}$$

For this, we have the following:

Lemma 2 *Problem* (6.13) *has a unique solution, and the solution satisfies*

$$\zeta^*_r(r) < 0, \quad 0 < \zeta^*(r) \leqslant 1 \quad \forall\, r \geqslant R^*.$$

In addition, in terms of the Hermite function, it is given by

$$\zeta^*(r) = \frac{e^{(R^*-r)/k} H(\mu + c/k; x)}{H(\mu + c/k; x^*)},$$

$$x := \frac{\sqrt{k}}{\sigma}\left(r + \frac{\sigma^2}{k^2} - \theta\right), \quad x^* := \frac{\sqrt{k}}{\sigma}\left(R^* + \frac{\sigma^2}{k^2} - \theta\right), \quad \mu := \frac{\sigma^2 - 2k^2\theta}{2k^3}.$$

Now we can calculate

$$\begin{aligned} \lim_{t\to-\infty} \dot{R}(t) e^{-c(t-T)} &= \frac{c\,\sigma^2}{2} \lim_{t\to-\infty} \frac{\zeta_r(R(t)+,t)}{(R(t)-c)(1-e^{c(t-T)})} \\ &= \frac{c\sigma^2 \zeta^*_r(R^*)}{2(R^*-c)} = \frac{c\,\sigma\sqrt{k}}{2(c-R^*)} \left\{ \frac{\sigma}{k\sqrt{k}} - \frac{H_x(\mu + c/k; x^*)}{H(\mu + c/k; x^*)} \right\}. \end{aligned}$$

Consequently, using

$$R(t) = R^* + \int_{-\infty}^{t} \{\dot{R}(\hat{t}) e^{-c(\hat{t}-T)}\} e^{c(\hat{t}-T)} d\hat{t},$$

we obtain the asymptotic expansion $R(t) \sim R^* + \rho^* e^{c(t-T)}$ for large negative t, as stated in

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Theorem 6.1, where

$$\rho^* := \frac{\sigma^2 \zeta_r^*(R^*)}{2(R^* - c)} = \frac{\sigma\sqrt{k}}{2\,(c - R^*)} \left\{ \frac{\sigma}{k\sqrt{k}} - \frac{H_x(\mu + c/k; x^*)}{H(\mu + c/k; x^*)} \right\}. \tag{6.14}$$

□

Now to complete the proof of Theorem 6.1, it remains to prove Lemma 2 and (6.12), which are the subjects of the next two subsections.

6.5 The problem (6.13)

The ODE in (6.13) is homogeneous and has two linearly independent solutions, at least one of which is unbounded near $r = \infty$ (by using the Wronskian). Hence, if (6.13) has a solution, it is unique. Consider

$$\hat{G}(r) = e^{-r/\theta} H(\mu + c/\theta; x).$$

It satisfies

$$\left\{ \frac{\sigma^2}{2} \frac{d^2}{dr^2} + k(\theta - r)\frac{d}{dr} + (c - r) \right\} \hat{G} = 0 \quad \forall\, r \in \mathbb{R}, \quad \hat{G}(\infty) = 0.$$

We show that $\hat{G}_r < 0$ on $[R^*, \infty)$. For this, notice that V^* satisfies

$$\left\{ \frac{\sigma^2}{2} \frac{d^2}{dr^2} + k(\theta - r)\frac{d}{dr} + (c - r) \right\} V^* = c(V^* - 1) < 0 \quad \forall r > R^*.$$

Thus, the Wronskian of $\hat{G}$ and V^* satisfies

$$\frac{d}{dr}\{e^{-k(r-\theta)^2/\sigma^2}[V_r^* \hat{G} - \hat{G}_r V^*]\} = c(V^* - 1)\hat{G}e^{-k(r-\theta)^2/\sigma^2}.$$

Suppose that $\hat{G}_r < 0$ on $[R^*, \infty)$ is not true. Then there exists $r_1 \geqslant R^*$ such that $\hat{G}_r(r_1) = 0$ and $\hat{G}_r < 0$ on (r_1, ∞). However, this would imply $\hat{G} > 0$ on $[r_1, \infty)$ and that, since $V_r^*(r_1) \leqslant 0$,

$$0 \geqslant V_r^* \hat{G} - \hat{G}_r V^* \Big|_{r=r_1} = e^{k(r_1-\theta)^2/\sigma^2} \int_{r_1}^{\infty} c(1 - V^*)\hat{G}e^{-k(r-\theta)^2/\sigma^2} dr > 0,$$

a contradiction. Thus, $\hat{G}_r < 0$ on $[R^*, \infty)$. Consequently, $0 < \hat{G}(r) < \hat{G}(R^*)$ for all $r > R^*$, and

$$\zeta^*(r) = \frac{\hat{G}(r)}{\hat{G}(R^*)} = \frac{e^{(R^*-r)/k} H(\mu + c/k; x)}{H(\mu + c/k; x^*)}$$

is the unique solution to (6.13). This completes the proof of Lemma 2. □

6.6 The limit of ζ as $t \to -\infty$

Here we verify (6.12).

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1. Since $V(r,t) = M(t)$ for all $r \leqslant R(t)$, we have $V_t(r,t) = M_t(t)$ for all $r \leqslant R(t)$. Also since $V(\cdot, T) = 0$, we know from the PDE in (1.3) that $V_t(r,T) = -m$ for all $r > c = R(0)$. Thus,

$$\zeta(r,t) = 1 \quad \forall\, r \leqslant R(t), t \leqslant T, \qquad \zeta(r,T) = 1 \quad \forall r \in \mathbb{R}.$$

In addition, 0 is a subsolution and $e^{(c-R^*)(T-t)}$ is a supersolution to ζ, so that

$$0 < \zeta(r,t) < e^{(c-R^*)(T-t)} \qquad \forall\, r \geqslant R^*, t < T.$$

This implies that for each $t \leqslant T$, $\zeta(\cdot,t)$ is a bounded function.

2. Let ζ^* be the unique solution to (6.13), as stated in Lemma 2. Now using $\zeta(R(t),t) = 1 > \zeta^*(R(t))$ for all $t \leqslant T$ and comparing the function ζ and ζ^* on $\{(r,t) \mid r \geqslant R(t), t \leqslant T\}$, we see that $\zeta(r,t) > \zeta^*(r)$ for all $r \geqslant R(t)$. As $\zeta(r,t) = 1 > \zeta^*(r)$ for $r \in (R^*, R(t)]$, we see that

$$\zeta(r,t) > \zeta^*(r) \quad \forall r > R^*, t \leqslant T.$$

3. To estimate the upper bound, let

$$G_1(r) = \zeta^*(r)\left\{1 + \int_{R^*}^{r} \frac{e^{k(t-\theta)^2/\sigma^2}}{\zeta^{*2}(t)} dt\right\} \quad \forall r \in \mathbb{R}.$$

This is another solution to the ODE in (6.13) that satisfies $\lim_{r\to\infty} G_1(r) = \infty$. Define

$$\delta(t) := \inf\{\delta > 0 \mid \zeta(r,t) \leqslant \zeta^*(r) + \delta G_1(r) \quad \forall r \geqslant R^*\}, \quad \forall t \leqslant T.$$

Since $\zeta(\cdot,t)$ is bounded and $G_1(\infty) = \infty$, $\delta(t)$ is positive and finite. In addition,

$$\zeta(r,t) \leqslant \zeta^*(r) + \delta(t) G_1(r) \quad \forall\, r \geqslant R^*, t \leqslant T.$$

Furthermore, since $\dot{R} > 0$, we have $\zeta(r) + \delta(\hat{t}) G_1(r)|_{r=R(t)} \geqslant \zeta(R(t),\hat{t}) = 1$ for all $t < \hat{t}$. Hence, comparing $\zeta(r,t)$ and $\zeta^*(r) + \delta(\hat{t}) G_1(r)$ on $\{(r,t) \mid r \geqslant R(t),\ t \leqslant \hat{t}\}$, we have

$$\zeta(r,t) < \zeta^*(r) + \delta(\hat{t}) G_1(r) \quad \forall\, r > R(t), t < \hat{t} \leqslant T.$$

Hence, $0 < \delta(t) < \delta(\hat{t})$ for all $t < \hat{t}_1 \leqslant T$. Consequently, there exists

$$\delta_* := \lim_{t\to-\infty} \delta(t) \in [0,\infty).$$

4. Here we show that $\delta_* = 0$. Suppose on the contrary that $\delta_* > 0$.

(a) On the spatially bounded domain $\{(r,t) \mid r \in [R^*, c+2], t < T\}$, let $\hat{\zeta}$ be the solution to the boundary value problem

$$\begin{cases} \left\{\dfrac{\partial}{\partial t} + \dfrac{\sigma^2}{2}\dfrac{\partial^2}{\partial r^2} + k(\theta - r)\dfrac{\partial}{\partial r} + (c-r)\right\}\hat{\zeta}(r,t) = 0, \quad r \in (R(t), c+2), t < T, \\ \hat{\zeta}(c+2,t) = \zeta^*(c+2) + \delta(t) G_1(c+2) \quad \forall t < T, \\ \hat{\zeta}(r,T) = 1 \quad \forall r \in [R^*, c+2], \quad \hat{\zeta}(r,t) = 1 \quad \forall r \in [R^*, R(t)], t \leqslant T. \end{cases}$$

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By comparison,

$$\zeta(r,t) \leqslant \hat{\zeta}(r,t) \quad \forall\, r \in [R^*, c+2], \quad t \leqslant T.$$

Also, using an elementary PDE analysis, say the Fourier series, one can show that uniformly in $r \in [R^*, c+2]$, $\lim_{t\to-\infty} \hat{\zeta}(r,t) = \hat{\zeta}(r,-\infty)$, where $\hat{\zeta}(r,-\infty)$ is the solution to the ODE in (6.13) on $[R^*, c+2]$ with the boundary value

$$\hat{\zeta}(R^*,-\infty) = 1, \quad \hat{\zeta}(c+2,-\infty) = \zeta^*(c+2) + \delta_* G_1(c+2).$$

By comparison, it is easy to see that $\hat{\zeta}(r,-\infty) < \zeta^*(r) + \delta_* G_1(r)$ for all $r \in [R^*, c+2)$. Thus, there exists $\delta_1 \in (0, \delta_*)$ such that

$$\hat{\zeta}(r,-\infty) < \zeta^*(r) + \delta_1 G_1(r) \quad \forall r \in [R^*, c+1].$$

This also implies that there exists $t_1 \ll -1$ such that

$$\zeta(r,t) < \zeta^*(r) + \delta_1 G_1(r) \quad \forall r \in [R^*, c+1],\ t \leqslant t_1.$$

(b) Now we compare the function $\zeta(r,t)$ and $\zeta^*(r) + \delta_1 G_1(r)$ on $[c+1,\infty) \times (-\infty, t_1]$. Since $\delta_1 < \delta_*$, for each fixed $t < t_1$, we see from the definition of δ_* that the maximum of

$$\varphi(r,t) := \zeta(r,t) - [\zeta^*(r) + \delta_1 G_1(r)], \quad r \in [R^*, \infty)$$

is positive. As $\varphi(r,t) < 0$ for all $r \in [R^*, c+1]$ and $\varphi(\infty,t) = -\infty$, there exists $\hat{r}(t) \in (c+1,\infty)$ such that $0 < \varphi(\hat{r},t) = \max_{r \geqslant R^*} \varphi(r,t)$. Using $\varphi_r(\hat{r},t) = 0 \geqslant \varphi_{rr}(\hat{r},t)$ and the PDE for φ, we have

$$\begin{aligned} 0 &= \varphi_t + \tfrac{\sigma^2}{2}\varphi_{rr} + (\eta - \theta r)\varphi_r + (c-r)\varphi|_{r=\hat{r}} \\ &\leqslant \varphi_t(\hat{r},t) + (c-\hat{r})\varphi(\hat{r},t). \end{aligned}$$

Hence, denoting $K(t) := \varphi(\hat{r},t) = \max_{r > R(t)} \varphi(r,t)$, we have

$$\begin{aligned} \frac{d}{dt}K(t) &:= \liminf_{h\to 0} \frac{K(t+h)-K(t)}{h} \geqslant \lim_{h\to 0} \frac{\varphi(\hat{r},t+h)-\varphi(\hat{r},t)}{h} = \varphi_t(\hat{r},t) \\ &\geqslant (\hat{r}-c)\varphi(\hat{r},t) \geqslant \varphi(\hat{r},t) = K(t). \end{aligned}$$

Thus, $\frac{d}{dt}[K(t)e^{-t}] \geqslant 0$ for all $t < t_1$. After integration, this gives

$$0 < K(t) \leqslant K(t_1)e^{t-t_1} \quad \forall\, t < t_1, \qquad \lim_{t\to-\infty} K(t) = 0.$$

This implies that for all sufficiently large negative t, $\max_{r\geqslant R^*} \varphi(r,t) = K(t) \leqslant \frac{1}{2}(\delta_* - \delta_1)\min_{r\geqslant R^*} G_1(r)$, so that $\zeta(r,t) \leqslant \zeta^*(r) + \frac{1}{2}(\delta_1 + \delta_*)G_1(r)$ for all $r \geqslant R^*$ and sufficiently large negative t, contradicting the definition of δ_*.

In conclusion, we must have $0 = \delta_* = \lim_{t\to-\infty} \delta(t)$.

5. Denote $K_1(t) = \max_{r\in[R^*,c+2]} |\zeta(r,t) - \zeta^*(r)|$. Then

$$0 \leqslant \lim_{t\to-\infty} K_1(t) \leqslant \sup_{r\in[R^*,c+2]} G_1(r) \lim_{t\to-\infty} \delta(t) = 0.$$

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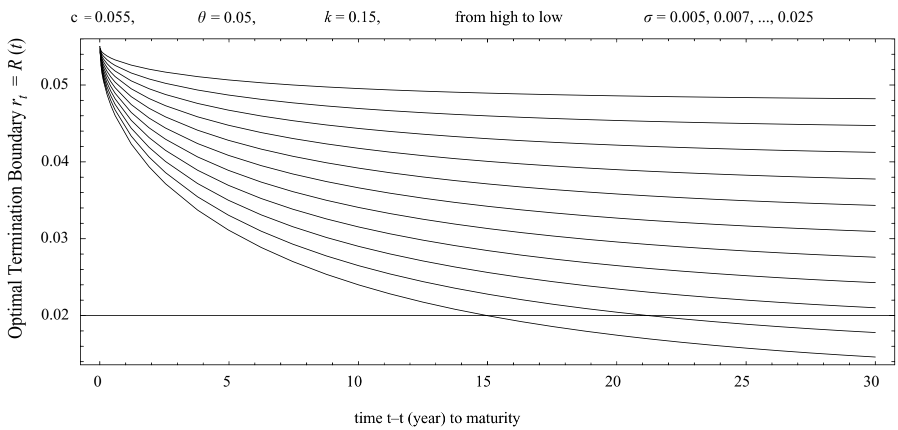

FIGURE 2. Optimal termination boundaries $r_t = R(t)$ as a function of time $T - t$ to maturity. Each curve corresponds to a particular value of σ.

Set

$$K_2(t) := \sup_{r \geqslant c+1} |\zeta(r,t) - \zeta^*| = \lim_{\varepsilon \searrow 0} \max_{r \geqslant c+1} [\zeta(r,t) - \zeta^*(r) - \varepsilon G_1(r)].$$

Using a similar idea as in **4(b)**, one can show that

$$K_2(t) \leqslant K_1(t) + K_2(T)e^{(t-T)} \quad \forall\, t \leqslant T.$$

This implies that $\lim_{t\to-\infty} K_2(t) = 0$. Thus,

$$\lim_{t\to-\infty} \sup_{r \geqslant R^*} |\zeta(r,t) - \zeta^*(r)| = 0.$$

Finally, since $\dot{R}$ is bounded in $(-\infty, T-1]$ (c.f. [6]), one can use a local regularity theory for parabolic equations to show that $\lim_{t\to-\infty} \zeta_r(R(t)+, t) = \zeta_r(R^*+)$. This completes the proof of (6.12) as well as the proof of Theorem 6.1. □

7 Global approximations

7.1 The simple global approximation

We seek a simple approximation formula for $R(T-\tau)$ such that (i) it has asymptotic expansion $c - \bar{\kappa}\sigma\sqrt{\tau}$ for a small positive τ and (ii) it exponentially approaches R^* for a large τ. For this, we seek an approximation of the form

$$R(T-\tau) \approx R_I(\tau) := c - \bar{\kappa}\sigma\sqrt{\frac{1-e^{-b\tau}}{b}}.$$

For any $b > 0$, this approximation has the right asymptotic behaviour for small $\tau = T - t$. To match it with the large τ behaviour, we need $R^* = c - \sigma\bar{\kappa}\sqrt{\frac{1}{b}}$, that is, $b = (\frac{\bar{\kappa}\sigma}{c-R^*})^2$. Hence, we have the **first approximation for** R in (1.6).

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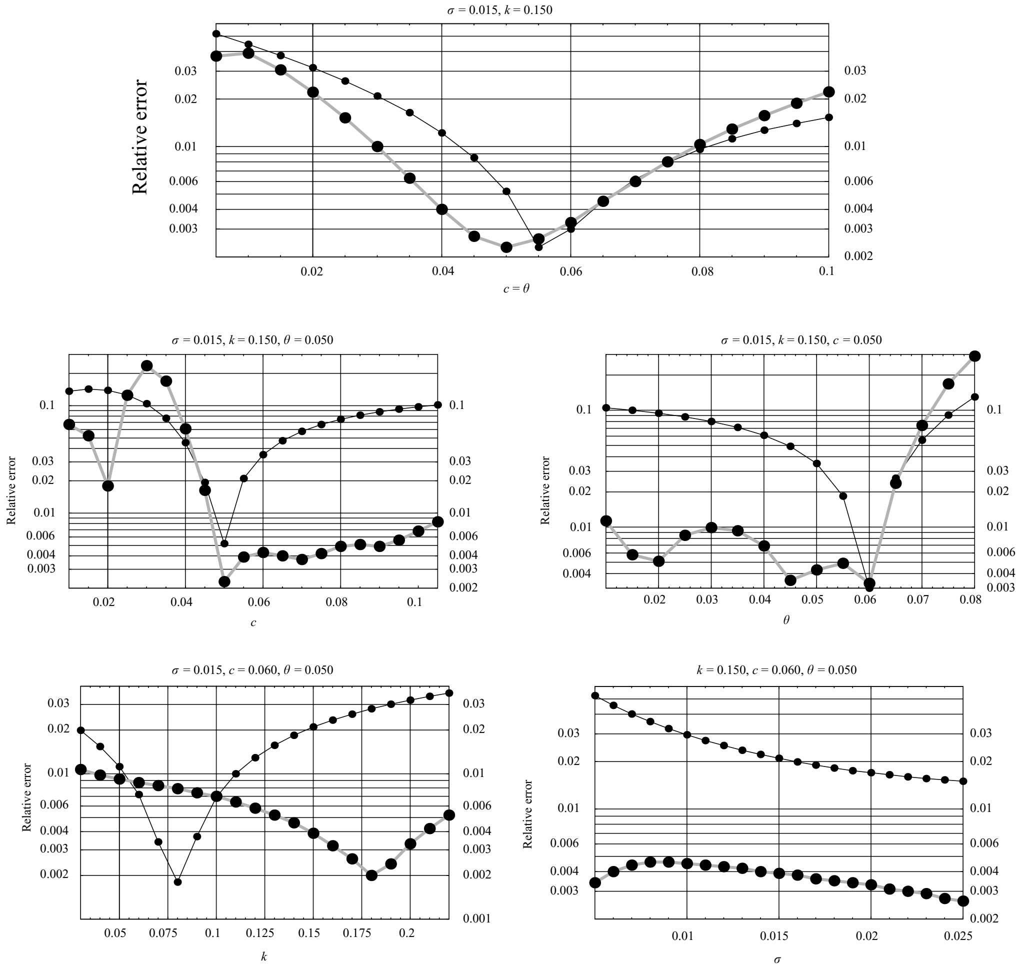

FIGURE 3. Relative errors of the first approximation R_{I} (thin curve) and the second approximation R_{II} (thick curve) as functions of parameters.

7.2 An enhanced approximation

In the above approximation, we only used the information $R(t) \sim R^*$ as $t \to -\infty$. Here we use the more detailed information that $R(t) \sim R^* + \rho^* e^{-c\tau} = O(e^{-2c\tau})$ for $t = T - \tau \to -\infty$ to guess the approximation $R \approx R_{II}$, where R_{II} is as in (1.7).

It is a pleasant surprise to find that for a typical parameter set, the relative errors of the two approximations satisfy (1.8); see Figures 1(b) and 3.

8 Numerical examples

In Figure 2, we display the optimal termination boundaries as σ changes, keeping other parameters fixed. Similar figures (not displayed here) of the boundaries can also be obtained as only one of the four parameters is changing. We can reasonably conclude that the optimal mortgage termination boundary $r_t = R(c, \theta, k, \sigma; t)$ is (i) increasing in c, (ii) decreasing in θ, (iii) increasing in k and (iv) decreasing in σ.

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In Figure 3, we display the relative errors of our two analytical approximation formulas for the optimal boundary, as one of the parameters changes while others are kept fixed. One can see that when $0 < \theta \leqslant c$, both approximations are extremely accurate.

9 Conclusion

We have designed efficient numerical algorithms for calculating the optimal boundary for mortgage termination. In addition, we provide a rigorous analysis for the asymptotic behaviour of the infinite horizon problem. This, together with the earlier result in [6], allows us to provide a quite complete theoretical study of the problem. Most importantly, we provided two simple analytic formulas that provide global approximation of the termination boundary. Our numerical evidence shows that the analytical approximation is extremely accurate across all expiries.

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