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INSTANTONS BEYOND TOPOLOGICAL THEORY. I

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To Pierre Schapira

Abstract Many quantum field theories in one, two and four dimensions possess remarkable limits in which the instantons are present, the anti-instantons are absent, and the perturbative corrections are reduced to one-loop. We analyse the corresponding models as full quantum field theories, beyond their topological sector. We show that the correlation functions of all, not only topological (or BPS), observables may be studied explicitly in these models, and the spectrum may be computed exactly. An interesting feature is that the Hamiltonian is not always diagonalizable, but may have Jordan blocks, which leads to the appearance of logarithms in the correlation functions. We also find that in the models defined on Kähler manifolds the space of states exhibits holomorphic factorization. We conclude that in dimensions two and four our theories are logarithmic conformal field theories.

In Part I we describe the class of models under study and present our results in the case of one-dimensional (quantum mechanical) models, which is quite representative and at the same time simple enough to analyse explicitly. Part II will be devoted to supersymmetric two-dimensional sigma models and four-dimensional Yang–Mills theory. In Part III we will discuss non-supersymmetric models.

Keywords: instanton; quantum mechanics; topological field theory; Morse theory; tempered distribution; Epstein–Glaser regularization

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Contents

1. Introduction

For a large class of models of quantum field theory there is a particular limit in which the theory may be analysed exactly in the presence of instanton effects. The simplest are the (twisted) supersymmetric models, in which the path integral measure is defined in a straightforward way. Classically, these models are described by first-order Lagrangians. The corresponding path integral localizes on certain finite-dimensional moduli spaces of classical (instanton) configurations. Therefore, in the path integral description the correlation functions of the corresponding quantum system are given by integrals over these moduli spaces. Such correlation functions have been studied in the literature, but attention has been focused almost exclusively on the correlation functions of the *BPS observables*, which represent cohomology of a supersymmetry charge of the theory. These correlation functions comprise the BPS (or topological) sector of the model and give rise to important invariants, such as the Gromov–Witten and Donaldson invariants (in two and four dimensions, respectively). However, the knowledge of the topological sector is not sufficient for understanding the full quantum field theory.

In this paper we go beyond the topological field theory of these models and investigate the correlation functions of more general—non-BPS, or 'off-shell'—observables in the presence of instantons, i.e. non-perturbatively. We show that in the special limit of the coupling constant that we are considering (namely, $\bar{\tau} \to \infty$, see below) the quantum model may be analysed and solved explicitly, both in the Lagrangian (or path integral) formalism and the Hamiltonian formalism. We describe the space of states of the quantum theory and show that a large class of observables (satisfying certain analytic properties) may be realized as operators acting on this space. Their correlation functions are then represented by the matrix elements of these operators. These matrix elements agree with the path integral representation of the correlation functions (given by integrals over the moduli spaces of instantons), and they also satisfy the usual identities, such as factorization over the intermediate states.

We find some interesting and unexpected features in our models. One of them is the fact that the Hamiltonian is *non-diagonalizable* on the space of states, but has Jordan blocks. This leads to the appearance of logarithmic terms in the correlation functions. Another feature is *holomorphic factorization* of the space of states in models defined on Kähler manifolds. In particular, we find that two-dimensional supersymmetric sigma models and four-dimensional super-Yang–Mills models are *logarithmic conformal field theories* in our limit.

1.1. Description of the models

We begin by describing in more detail the class of models that will be discussed in this paper. These models appear in one, two and four space–time dimensions and are described by the actions which are written below (note that all of our actions are written in Euclidean signature).

One dimension

We start with the supersymmetric quantum mechanics on a compact Kähler manifold X, with the Kähler metric λg, equipped with a holomorphic vector field ξ. We will assume that ξ comes from a holomorphic $\mathbb{C}^{\times}$-action on X with a non-empty set of fixed points which are all isolated. We modify the standard action [**38**] by adding the topological term $-\mathrm{i}\vartheta\int A$, where A is the 1-form obtained by contracting g with the vector field $\xi+\bar{\xi}$ (our assumptions imply that $A=\mathrm{d}f$, where f is a Morse function on X). We allow ϑ to be complex.

We then set $\tau=\vartheta+\mathrm{i}\lambda$, $\bar{\tau}=\vartheta-\mathrm{i}\lambda$ (note that they are not necessarily complex conjugate to each other because ϑ is complex). Consider the limit in which $\bar{\tau}\to-\mathrm{i}\infty$, but τ is kept finite (which means that $\lambda\to+\infty$ and ϑ is adjusted accordingly, so that it has a large imaginary part). In this limit the model is described by the following first-order action on a worldline I:

$$S=-\mathrm{i}\int_I\biggl(p_a\biggl(\frac{\mathrm{d}X^a}{\mathrm{d}t}-v^a\biggr)+\bar{p}_a\biggl(\frac{\mathrm{d}\bar{X}^a}{\mathrm{d}t}-\bar{v}^a\biggr) \\ -\pi_a\biggl(\frac{\mathrm{d}\psi^a}{\mathrm{d}t}-\frac{\partial v^a}{\partial X^b}\psi^b\biggr)-\bar{\pi}_a\biggl(\frac{\mathrm{d}\bar{\psi}^a}{\mathrm{d}t}-\frac{\partial\bar{v}^a}{\partial\bar{X}^b}\bar{\psi}^b\biggr)\biggr)\,\mathrm{d}t-\mathrm{i}\tau\int A, \quad (1.1)$$

where the X^a are complex coordinates on X and $\xi=v^a(\partial/\partial X^a)$.

The quantum model is described by the path integral $\int\mathrm{e}^{-S}$ over all maps $I\to X$. This path integral represents the 'delta-form' supported on the moduli space of *gradient trajectories*, satisfying

$$\frac{\mathrm{d}X^a}{\mathrm{d}t}=v^a. \quad (1.2)$$

This is the instanton moduli space in this case.

Two dimensions

We start with the twisted (type A) $\mathcal{N}=(2,2)$ supersymmetric sigma model with the target compact Kähler manifold X with the Kähler metric λg and the B-field $B=B_{a\bar{b}}\,\mathrm{d}X^a\wedge\mathrm{d}X^{\bar{b}}$, which is a closed (complex) 2-form on X.

We then set

$$\tau_{a\bar{b}}=B_{a\bar{b}}+\tfrac{1}{2}\mathrm{i}\lambda g_{a\bar{b}},\qquad \bar{\tau}_{a\bar{b}}=B_{a\bar{b}}-\tfrac{1}{2}\mathrm{i}\lambda g_{a\bar{b}}.$$

In the limit when $\bar{\tau}_{a\bar{b}}\to-\mathrm{i}\infty$, but the $\tau_{a\bar{b}}$ are kept finite (which means that $\lambda\to\infty$ and the $B_{a\bar{b}}$ have a large imaginary part), this model is described by the following first-order action on a worldsheet Σ:

$$-\mathrm{i}\int_\Sigma(p_a\partial_{\bar{z}}X^a+\bar{p}_a\partial_z\bar{X}^a-\pi_a\partial_{\bar{z}}\psi^a-\bar{\pi}_a\partial_z\bar{\psi}^a)\,\mathrm{d}^2z+\int_\Sigma\tau_{a\bar{b}}\,\mathrm{d}X^a\wedge\mathrm{d}\bar{X}^b. \quad (1.3)$$

Thus, this model is a particular modification of the 'infinite radius limit', achieved by adding to the conventional second-order action of the sigma model the topological term (B-field) with a large imaginary part.

The path integral $\int\mathrm{e}^{-S}$ over all maps $\Sigma\to X$ localizes on the moduli space of *holomorphic maps*, satisfying

$$\partial_{\bar{z}}X^a=0.$$

Four dimensions

We start with twisted $\mathcal{N} = 2$ supersymmetric gauge theory on a four-manifold $\boldsymbol{M}^4$ with compact gauge group G, coupling constant g_{YM} and theta-angle ϑ, which we allow to be complex.

We then set

$$\tau = \frac{\vartheta}{2\pi} + \frac{4\pi\mathrm{i}}{g_{\mathrm{YM}}^2}, \qquad \bar{\tau} = \frac{\vartheta}{2\pi} - \frac{4\pi\mathrm{i}}{g_{\mathrm{YM}}^2}.$$

In the limit when $\bar{\tau} \to -\mathrm{i}\infty$, but τ is kept finite (i.e. $g_{\mathrm{YM}} \to 0$ and ϑ has a large imaginary part), this model is described by the first-order action

$$S = -\mathrm{i} \int_{\boldsymbol{M}^4} (\operatorname{tr} P^+ \wedge F_A + \tau \operatorname{tr} F_A \wedge F_A + \text{fermions}).$$

This model is a particular modification of the 'weak coupling limit' achieved by adding to the conventional second order the topological term

$$-\frac{\mathrm{i}\vartheta}{2\pi} \int_{\boldsymbol{M}^4} \operatorname{tr} F \wedge F,$$

where ϑ has a large imaginary part.

The path integral $\int \mathrm{e}^{-S}$ over all connections on $\boldsymbol{M}^4$ localizes on the moduli space of *anti-self dual connections* satisfying the equations

$$F_A^+ = 0.$$

Thus, to obtain these models we start with the standard (second-order) action and add to it a *topological term*: $-\mathrm{i}\vartheta \int_I \mathrm{d}f$ in one dimension, the B-field $\int_\Sigma B_{a\bar{b}} \, \mathrm{d}X^a \wedge \mathrm{d}X^{\bar{b}}$ in two dimensions, and the ϑ-angle term

$$-\frac{\mathrm{i}\vartheta}{2\pi} \int_{\boldsymbol{M}^4} \operatorname{tr} F \wedge F$$

in four dimensions. We then allow ϑ and $B_{a\bar{b}}$ to be complex and take the limit $\bar{\tau} \to \infty$ as described above.* In this limit the instanton contributions are present while the anti-instanton contributions vanish, and because of that the correlation functions simplify dramatically.

The resulting models are described by the first-order Lagrangians written above.† The corresponding path integral represents the 'delta-form' supported on the instanton

* In two-dimensional sigma models such limits, with large imaginary B-field, have been studied in the literature since the early days of the theory of instantons: see, for example, the papers [**11**] and [**5**] and references cited therein.

† There are also similar models in three dimensions, but they fall into the class of non-supersymmetric field theories, which generically become massive upon inclusion of the instantons; in this paper we consider such models only briefly in § 6.3.

moduli space. This moduli space has components labelled by the appropriate 'instanton numbers', which are finite dimensional (after dividing by the appropriate gauge symmetry group). Therefore, the correlation functions are expressed in terms of integrals over these finite-dimensional components of the moduli space.

When we move away from the special point $\bar{\tau} = \infty$ (with fixed τ), both instantons and anti-instantons start contributing to the correlation functions. The path integral becomes a Mathai–Quillen representative of the Euler class of an appropriate vector bundle over the instanton moduli space, which is 'smeared' around the moduli space of instantons (like Gaussian distribution) (see, for example, [**10**]). Therefore, general correlation functions are no longer represented by integrals over the finite-dimensional instanton moduli spaces and they become much more complicated.

There is however an important class of observables, called the *BPS observables* whose correlation functions are independent of $\bar{\tau}$. They commute with the supersymmetry charge Q of the theory and comprise the *topological sector* of the theory. The perturbation away from the point $\bar{\tau} = \infty$ (that is back to a finite radius in two dimensions or to a non-zero coupling constant in four dimensions) is given by a Q-exact operator, and therefore the correlation functions of the BPS observables (which are Q-closed) remain unchanged when we move away from the special point. This is the secret of success of the computation of the correlation functions of the BPS observables achieved in recent years in the framework of topological field theory: the computation is actually done in the theory at $\bar{\tau} = \infty$, but because of the special properties of the BPS observables the answer remains the same for other values of the coupling constant [**42**]. But for general observables the correlation functions do change in a rather complicated way when we move away from the special point.

Our goal in this paper is to go *beyond the topological sector* and consider more general correlation functions of non-BPS observables. We are motivated, first of all, by the desire to understand non-supersymmetric quantum field theories with instantons. It is generally believed that realistic quantum field theories should be viewed as non-supersymmetric phases of supersymmetric ones. This means that the observables of the original theory may be realized as observables of a supersymmetric theory, but they are certainly not going to be BPS observables. Therefore, we need to develop methods for computing correlation functions of such observables.

In particular (and this was another motivation), developing this theory in two dimensions may help in elucidating the pure spinor approach to superstring theory [**4**], where 'curved $\beta\gamma$-systems' play an important role [**32**].

The third motivation comes from the realization that the correspondence between the full quantum field theory and its topological sector is analogous to the correspondence between a differential graded algebra (DGA) and its cohomology. The cohomology certainly contains a lot of useful information about the DGA, but far from all. For example, there are higher (Massey) operations on the cohomologies, which can only be detected if we use the full DGA structure. In particular, the cohomology of a manifold does not determine its geometric type, but the differential graded algebra of differential forms does (at least, its rational homotopy type).* Likewise, we expect that the passage from the topological sector to the full quantum field theory will reveal a lot of additional information. In particular, while the correlation functions in the topological sector give rise to invariants of the underlying manifold, such as the Gromov–Witten and Donaldson invariants, the correlation functions of the full quantum field theory may allow us to detect some finer information about its geometry.

Since our goal now is to understand the full quantum field theory, and not just its topological sector, it is reasonable to try to describe the theory first at the special value of the coupling constant $\bar{\tau} = \infty$ (and finite τ), where the correlation functions simplify so dramatically due to the vanishing of the anti-instanton contributions. One can then try to extend these results to a neighbourhood of this special value by perturbation theory.

Our approach should be contrasted with the standard perturbation theory approach to quantum field theory, which consists of expanding around a Gaussian fixed point. This approach has many virtues, but it cannot be universally applied. In particular, there are issues with the zero radius of convergence, but more importantly, such perturbation theory is unlikely to shed light on hard dynamical questions such as confinement. One can speculate that the reason for this is that expansion around a Gaussian point does not adequately represent the nonlinearity of the spaces of fields and symmetries.

The expansion around the point $\bar{\tau} = \infty$ that we propose in this paper may be viewed as an alternative to the Gaussian perturbation theory. Here the topology (and perhaps, even the geometry) of the space of fields is captured by the appropriate moduli space of instantons. Therefore, this approach may be beneficial for understanding some of the questions that have proved to be notoriously difficult in the conventional formalism.

We note that the consideration of the theory at $\bar{\tau} = \infty$ has already proved to be very useful in the recent interpretation [**18**] of mirror symmetry for toric varieties via an intermediate I-model and the recent proof [**33**, **34**] of the Seiberg–Witten solution of the $\mathcal{N} = 2$ supersymmetric gauge theories.

1.2. Some puzzles

Before summarizing our results we wish to motivate them by pointing out some 'puzzles' which naturally arise when one considers the models described above. It is natural to start with the one-dimensional case of supersymmetric quantum mechanics. It already contains most of the salient features of the models that we are interested in, and yet is simple enough to allow us to analyse it explicitly.

Let us first look at the classical theory described by the action (1.1). We can include it into a one-parameter family of theories depending on a coupling constant λ by adding

* In the words of Deligne *et al.* [**12**]:

> To understand cohomology and maps on cohomology one need deal only with closed forms, but to detect the finer homotopy theoretic information one also needs to use non-closed forms. Differential geometric aspects of this philosophy have been given by Chern and Simons: 'The manner in which a closed form which is zero in cohomology actually becomes exact contains geometric information'.

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the term $\frac{1}{2}\lambda^{-1}g_{a\bar{b}}p_a\bar{p}_b$. For finite values of λ^{-1} we may substitute the corresponding equations of motion and obtain the second-order action

$$\int_I \left(\tfrac{1}{2}\lambda g_{a\bar{b}}\frac{\mathrm{d}X^a}{\mathrm{d}t}\frac{\mathrm{d}\bar{X}^b}{\mathrm{d}t} + \tfrac{1}{2}\lambda g^{a\bar{b}}\frac{\partial f}{\partial X^a}\frac{\partial f}{\partial \bar{X}^b} - \mathrm{i}\vartheta\frac{\mathrm{d}f}{\mathrm{d}t} + \text{fermions}\right)\mathrm{d}t, \tag{1.4}$$

where $\vartheta = \tau - \mathrm{i}\lambda$. Here f is a Morse function on X, whose gradient is equal to $v = \xi + \bar{\xi}$, so that we have

$$\mathrm{d}f = g_{a\bar{b}}v^a\overline{\mathrm{d}X^b} + g_{\bar{a}b}\bar{v}^a\,\mathrm{d}X^b.$$

This function is also the hamiltonian of the $U(1)$-vector field $\mathrm{i}(\xi - \bar{\xi})$ with respect to the Kähler structure. It is shown in [**16**] that f always exists under our assumption that ξ comes from a holomorphic $\mathbb{C}^\times$-action on X with a non-empty set of fixed points.

The term $-\mathrm{i}\vartheta\,\mathrm{d}f$ (which plays the role of the B-field of the two-dimensional sigma model) is very important, as we will see below. Its role is to distinguish the instanton contributions to the path integral from the anti-instanton contributions. This allows us to keep the instantons and at the same time get rid of the anti-instantons in the limit $\bar{\tau} = \vartheta - \mathrm{i}\lambda \to -\mathrm{i}\infty$ with $\tau = \vartheta + \mathrm{i}\lambda$ being fixed.

The limit $\bar{\tau} \to -\mathrm{i}\infty$ with finite τ is achieved by taking $\lambda \to +\infty$ and $\vartheta \to -\mathrm{i}\infty$ in such a way that $\lambda - |\vartheta|$ is kept finite and fixed. For simplicity we will consider now the case when $\vartheta = -\mathrm{i}\lambda$ (so that $\tau = 0$). We will therefore view the limit $\bar{\tau} \to -\mathrm{i}\infty$ as the limit $\lambda \to \infty$.

At finite values of λ we have the theory with the action (1.4) such that the classical Hamiltonian is bounded from below. The corresponding quantum Hamiltonian is conjugate to a second-order differential operator equal to $-\frac{1}{2}(\lambda^{-1}\Delta + \lambda|\mathrm{d}f|^2 + K_f)$ (the Witten Laplacian [**38**]) acting on the Hilbert space which is the completion of the de Rham complex $\Omega^\bullet(X)$ with respect to the L_2 norm. This Hamiltonian has non-negative spectrum (see § 2.1 for more details).

Now consider the theory in the limit $\lambda \to \infty$. Here the classical Hamiltonian corresponding to the action (1.1) is not bounded from below, which seems to indicate trouble: unbounded spectrum of the quantum Hamiltonian. In fact, the naive quantization of the Hamiltonian is the first-order operator $\mathcal{L}_v = \mathcal{L}_\xi + \mathcal{L}_{\bar{\xi}}$ (where $\mathcal{L}$ denotes the Lie derivative). Under our assumptions on the vector field ξ, the only smooth eigenstates are the constant functions. It is not clear at all how $\mathcal{L}_v$ could possibly be realized as the Hamiltonian of a quantum mechanical model.

This constitutes the first puzzle that we encounter when analysing the model (1.1) (and its higher-dimensional analogues).

The second puzzle has to do with holomorphic factorization. The action (1.1) manifestly splits into the sum of holomorphic and anti-holomorphic terms (unlike the action at $\lambda^{-1} \neq 0$, because the term $\lambda^{-1}g_{a\bar{b}}p_a\bar{p}_b$ is mixed). So naively one expects the same kind of holomorphic factorization for the space of states and for the correlation functions. However, the only globally defined holomorphic functions (for compact X) are constants. There may also be some holomorphic differential forms, but only a finite-dimensional space of those. So it seems that a holomorphic factorization is impossible due to the absence of globally defined holomorphic differential forms.

This makes us wonder that perhaps the structure of the space of states of the theory with the action (1.4) should change when $\lambda \to \infty$ in such a way that the new space of states is the tensor product of chiral and anti-chiral sectors, the Hamiltonian is diagonalizable and has bounded spectrum.

In fact, we will show that this is 'almost' true: the new space of states has the following structure

$$\mathcal{H} = \bigoplus_{\alpha \in A} \mathcal{F}_\alpha \otimes \bar{\mathcal{F}}_\alpha$$

(where A is the finite set of zeros of ξ, or equivalently, critical points of the Morse function f), where $\mathcal{F}_\alpha$ and $\bar{\mathcal{F}}_\alpha$ may be viewed as the chiral and anti-chiral blocks of the model. They have transparent geometric interpretation as spaces of 'delta-forms' supported on the ascending manifolds of the Morse function. We also find that the spectrum of the Hamiltonian is non-negative, but the Hamiltonian is not diagonalizable: it splits into a sum of Jordan blocks (of finite sizes bounded by $\dim_{\mathbb{C}} X + 1$). These are the phenomena typically associated with two-dimensional logarithmic conformal field theory. It is quite curious that we observe these phenomena already at the level of quantum mechanics!

1.3. Summary of the results

We now explain our results in more detail, starting with the one-dimensional case. Let X be a compact Kähler manifold, equipped with a holomorphic vector field ξ which comes from a holomorphic $\mathbb{C}^\times$-action ϕ on X with isolated fixed points. Let us denote those points by x_α, $\alpha \in A$. We will assume that the set A is non-empty (it is necessarily finite).

Our first result concerns the structure of the spaces of states of the quantum theory with the action (1.1). Under our assumptions, there is a *Bialynicki–Birula decomposition* [**6**,**8**]

$$X = \bigsqcup_{\alpha \in A} X_\alpha$$

of X into complex submanifolds X_α, defined as follows:

$$X_\alpha = \Big\{ x \in X \;\Big|\; \lim_{t \to 0} \phi(t) \cdot x = x_\alpha \Big\}.$$

Under the above assumptions it is proved in [**16**] that there exists a Morse function f on X, whose gradient is the vector field $v = \xi + \bar{\xi}$. The points x_α, $\alpha \in A$, are the critical points of f, and the submanifolds X_α may be described as the 'ascending manifolds' of f. Furthermore, each submanifold X_α is isomorphic to $\mathbb{C}^{n_\alpha}$, where the index of the critical point x_α is $2(\dim_{\mathbb{C}} X - n_\alpha)$ (see [**6**,**8**]). In what follows we will assume that the above decomposition of X is a stratification, that is the closure of each X_α is a union of the X_β.

Now let $\mathcal{H}_\alpha$ be the space of *delta-forms supported on* X_α. An example of such delta-forms is the distribution (or current) on the space of differential forms on X which is defined by the following formula:

$$\langle \Delta_\alpha, \eta \rangle = \int_{X_\alpha} \eta|_{X_\alpha}, \quad \eta \in \Omega^\bullet(X). \tag{1.5}$$

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All other delta-forms supported on X_α may be obtained by applying to Δ_α differential operators defined on a small neighbourhood of X_α. The space $\mathcal{H}_\alpha$ is graded by the degree of the differential form.

We then have a holomorphic factorization

$$\mathcal{H}_\alpha = \mathcal{F}_\alpha \otimes \bar{\mathcal{F}}_\alpha,$$

where $\mathcal{F}_\alpha$ (respectively, $\bar{\mathcal{F}}_\alpha$) is the space of holomorphic (respectively, anti-holomorphic) delta-forms supported on X_α. For example, if $n_\alpha = \dim X$, so that $X_\alpha \simeq \mathbb{C}^{n_\alpha}$ is an open subset of X, then $\mathcal{H}_\alpha$ is the space of differential forms on $\mathbb{C}^{n_\alpha}$, and so it factorizes into the tensor product of holomorphic and anti-holomorphic differential forms. On the other hand, if $n_\alpha = 0$, so that $X_\alpha = x_\alpha$ is a point, then $\mathcal{H}_\alpha$ is the space of distributions supported at x_α. It factorizes into the tensor product of the derivatives with respect to holomorphic and anti-holomorphic vector fields. In the intermediate cases the space $\mathcal{F}_\alpha$ is generated from the delta-form Δ_α supported on X_α under the action of holomorphic differential forms along X_α and holomorphic vector fields in the transversal directions. Thus, $\mathcal{F}_\alpha$ is the space of global sections of a $\mathcal{D}_X$-module, where $\mathcal{D}_X$ is the sheaf of holomorphic differential operators on X.

Now we set

$$\mathcal{H} = \bigoplus_{\alpha \in A} \mathcal{H}_\alpha = \bigoplus_{\alpha \in A} \mathcal{F}_\alpha \otimes \bar{\mathcal{F}}_\alpha.$$

We claim that this space $\mathcal{H}$ is isomorphic to the *space of states* of the quantum mechanical model described by the action (1.1).

The reader may wonder how the space of states of the theory for finite λ described by the action (1.4), which is essentially the space of differential forms on X, could possibly turn into something like this. We explain this in detail below. Here we will only point out that the procedure of taking the limit $\lambda \to \infty$ is quite non-trivial. Before passing to the limit we need to multiply the wave functions of the quantum Hamiltonian of the theory (1.4) by $\mathrm{e}^{\lambda f}$ (this corresponds to adding the term $-\lambda\, \mathrm{d}f$ to the action (1.4)). Standard semi-classical analysis shows that after this multiplication the wave functions with eigenvalues that remain finite in the limit $\lambda \to \infty$ tend to the delta-forms which give us a monomial basis in the spaces $\mathcal{H}_\alpha$ for different αs. In particular, the exact supersymmetric vacua (in other words, the BPS states), which are known to be in bijection with the critical points of f [**38**], become in the limit $\lambda \to \infty$ the delta-forms Δ_α on X_α defined by formula (1.5).

Next, we consider the Hamiltonian. Naively we expect that it is equal to

$$H_{\text{naive}} = \mathcal{L}_\xi + \mathcal{L}_{\bar{\xi}},$$

acting on the above space $\mathcal{H}$. However, we claim that it is actually equal to

$$H = H_{\text{naive}} + 4\pi \sum_{\alpha,\beta} a_{\alpha\beta} \delta_{\alpha\beta} \otimes \bar{\delta}_{\alpha\beta},$$

where the summation is over all α, β such that X_β is a codimension 1 stratum in the closure of X_α. Here $\delta_{\alpha\beta}$ is the *Grothendieck–Cousin (GC) operator* and $\bar{\delta}_{\alpha\beta}$ is its complex

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conjugate (the $a_{\alpha\beta}$ are some non-zero real numbers). The GC operator acting from $\mathcal{F}_\alpha$ to $\mathcal{F}_\beta$ corresponds to taking the singular part of a holomorphic differential form on X_α along this divisor (see, for example, [**26**]).

In particular, we find that the Hamiltonian is not diagonalizable; rather, it has Jordan blocks!

In order to test these predictions, we investigate the factorization of correlation functions over intermediate states. Suppose for simplicity that $X = \mathbb{CP}^1$ and f is the standard 'height' function (see § 3.5). In this case there is one non-trivial component of the moduli space of gradient trajectories, which consists of the trajectories going from the north pole to the south pole. It is isomorphic to $\mathbb{C}^\times \subset \mathbb{CP}^1$, hence its natural compactification is $\mathbb{CP}^1$. Typical observables of our theory are smooth differential forms. We know from the path integral description of the model that the correlation function of observables $\hat{\omega}_1, \dots, \hat{\omega}_n$ corresponding to differential forms $\omega_1, \dots, \omega_n$ is equal to

$$\langle \hat{\omega}_1(t_1)\hat{\omega}_2(t_2)\cdots\hat{\omega}_n(t_n)\rangle = \int_{\mathbb{CP}^1} \omega_1 \wedge \phi(\mathrm{e}^{-(t_1-t_2)})^*(\omega_2) \wedge \cdots \wedge \phi(\mathrm{e}^{-(t_{n-1}-t_n)})^*(\omega_n),$$

where ϕ is the standard $\mathbb{C}^\times$-action on $\mathbb{CP}^1$ and $\phi(q)^*$ denotes the pullback of a differential form under the action of $q \in \mathbb{C}^\times$. Consider the simplest case when ω_1 is a smooth 2-form ω and ω_2 is a smooth function F on $\mathbb{CP}^1$, which we will assume to be non-constant. Then we have

$$\langle \hat{\omega}(t_1)\hat{F}(t_2)\rangle = \int_{\mathbb{CP}^1} \omega\phi(\mathrm{e}^{-t})^*(F),$$

where $t = t_1 - t_2$.

On the other hand, we expect this two-point function to factorize into the sum of one-point functions over all possible intermediate states:

$$\langle \hat{\omega}(t_1)\hat{F}(t_2)\rangle = \langle \hat{\omega}\mathrm{e}^{-tH}\hat{F}\rangle = \sum_{\nu}\langle 0|\hat{\omega}\mathrm{e}^{-tH}|\Psi_\nu\rangle\langle\Psi_\nu^*|\hat{F}|0\rangle. \tag{1.6}$$

If the hamiltonian were diagonalizable, the right-hand side would be the sum of monomials q^α, where $q = \mathrm{e}^{-t}$ and α runs over the spectrum of the Hamiltonian (in our case it consists of non-negative integers). However, consider the following simple example: let

$$F = \frac{1}{1+|z|^2}, \qquad \omega = \frac{1}{(1+|z|^2)^2}\frac{\mathrm{d}^2 z}{\pi}.$$

We find that

$$\int_{\mathbb{CP}^1} \omega\phi(q)^*(F) = \frac{1}{1-q^2} + \frac{2q^2}{(1-q^2)^2}\log q.$$

The appearance of the logarithmic function indicates that the operator H is not diagonalizable, but has Jordan blocks of length two, in agreement with our prediction.

Thus, the logarithmic nature of our model is revealed by elementary calculation of an integral over the simplest possible moduli space of instantons. But it is important to stress that in order to see the logarithm function in a correlation function it is necessary that at least one of the observables involved not be Q-closed. The action of Q on the above

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observables $\hat{\omega}_i$ corresponds to the action of the de Rham differential on the differential forms ω_i. Thus, in the above calculation the 2-form $\hat{\omega}$ is necessarily Q-closed, and so it is a BPS observable. But F is not Q-closed due to our assumption that F is not constant. Therefore, $\hat{F}$ is not a BPS observable. If both F and ω were Q-closed, then the one-point functions appearing on the right-hand side of formula (1.6) would be non-zero only when the intermediate states are vacuum states. On such states the Hamiltonian is diagonalizable, so we would not be able to observe the logarithmic terms. The same argument applies to n-point correlation functions. Thus, we can discover the structure of the space of states of the theory, and in particular, the existence of the Jordan blocks of the Hamiltonian, *only* if we consider correlation functions of non-BPS observables. It is impossible to see these structures within the topological sector of our model. This is yet another reason why it is important to go beyond the topological sector.

Part I contains a detailed and motivated exposition of our results describing the structure of our quantum mechanical models at the special point $\lambda = \infty$ (or, equivalently, $\bar{\tau} = \infty$). We hope that the models corresponding to finite values of λ may be studied by λ^{-1}-perturbation theory around the point $\lambda = \infty$. We will present some sample calculations below which provide some evidence that this is indeed possible.

In Part II we will apply our approach to two-dimensional $\mathcal{N} = (2,2)$ supersymmetric sigma models and four-dimensional $\mathcal{N} = 2$ supersymmetric Yang–Mills theory. Part III will be devoted to generalization to non-supersymmetric models.

We now discuss briefly what happens in dimensions two and four, thus giving a preview of Part II.

1.4. Two-dimensional sigma models

Let us consider first the supersymmetric (type A twisted) two-dimensional sigma model [**41**,**42**] in the $\bar{\tau} \to \infty$ limit (these limits have been previously discussed in [**3**,**18**,**29**]). The first step is to recast these models in the framework of the quantum mechanical models that we have studied above. For a fixed Riemann surface Σ the space of bosonic fields in the supersymmetric sigma model with the target manifold X is $\mathrm{Maps}(\Sigma, X)$, the space of maps $\Sigma \to X$. If we choose Σ to be the cylinder $I \times \mathbb{S}^1$, then we may interpret $\mathrm{Maps}(\Sigma, X)$ as the space of maps from the interval I to the loop space $LX = \mathrm{Maps}(\mathbb{S}^1, X)$. Thus, we may think of the two-dimensional sigma model on the cylinder with the target X as the quantum mechanical model on the loop space LX. Hence it is natural to try to write the Lagrangian of the sigma model in such a way that it looks exactly like the Lagrangian of the quantum mechanical model on LX with a Morse function f.

It turns out that if X is a Kähler manifold, this is 'almost' possible. However, there are two caveats. First of all, the corresponding function f has non-isolated critical points corresponding to the constant loops in LX, so it is in fact a Bott–Morse function. We can deal with this problem by deforming this function so that it only has isolated critical points, corresponding to the constant loops whose values are critical points of a Morse function on X. The second, and more serious, issue is that our function f is not single-valued on LX, but becomes single-valued only after pullback to the universal cover $\widetilde{LX}$ of LX. In other words, it is an example of a Morse–Novikov function, or, more properly, Bott–Morse–Novikov function. Because of that, the instantons are identified with gradient trajectories of the pullback of f to $\widetilde{LX}$.

The universal cover $\widetilde{LX}$ may be described as the space of equivalence classes of maps $\tilde{\gamma}\colon D \to X$, where D is a two-dimensional unit disc, modulo the following equivalence relation: we say that $\tilde{\gamma} \sim \tilde{\gamma}'$ if $\tilde{\gamma}|_{\partial D} = \tilde{\gamma}'|_{\partial D}$ and $\tilde{\gamma}$ is homotopically equivalent to $\tilde{\gamma}'$ in the space of all maps $D \to X$ which coincide with $\tilde{\gamma}$ and $\tilde{\gamma}'$ on the boundary circle ∂D. We have the obvious map $\widetilde{LX} \to LX$, which realizes $\widetilde{LX}$ as a covering of LX. The group of deck transformations is naturally identified with $H_2(X, \mathbb{Z})$. The corresponding Morse function, which goes back to the work of Floer [**15**] is given by the formula

$$f(\tilde{\gamma}) = \int_D \tilde{\gamma}^*(\omega_K), \tag{1.7}$$

where ω_K is the Kähler form.

Suppose now that $I = \mathbb{R}$ with a coordinate t. In the limits $t \to \pm\infty$ a gradient trajectory tends to the critical points of f on $\widetilde{LX}$, which are the preimages of constant maps in LX. Therefore, a gradient trajectory may be interpreted as a map of the cylinder, compactified by two points at $\pm\infty$, to X, or equivalently, a map $\mathbb{CP}^1 \to X$. Moreover, the condition that it corresponds to a gradient trajectory of f simply means that this map is *holomorphic*. Thus, we obtain that the instantons of the two-dimensional sigma model are holomorphic maps $\mathbb{CP}^1 \to X$, and more generally, $\Sigma \to X$, where Σ is an arbitrary compact Riemann surface.

In our infinite radius limit (which corresponds to $\bar{\tau} \to -\mathrm{i}\infty$, as explained above) we obtain the theory governed by first-order action (1.3). Therefore, the corresponding path integral localizes on the moduli space of holomorphic maps $\Sigma \to X$. Because we are dealing with a Morse–Novikov function, this moduli space now has infinitely many connected components labelled by $\beta \in H_2(X, \mathbb{Z})$, all of which are finite dimensional (the component is non-empty only if the integral of the Kähler class ω of X over β is non-negative). The simplest observables of this model are the evaluation observables corresponding to differential forms on X. Their correlation functions are given by integrals of their pullbacks to the moduli spaces of holomorphic maps (more precisely, the Kontsevich moduli spaces of stable maps) under the evaluation maps.

Such correlation functions have been studied extensively in the literature in the case of the *BPS observables*, corresponding to closed differential forms on X. They are expressed in terms of the *Gromov–Witten invariants* of X. Our goal is to go beyond the BPS (or topological) sector of the model and study correlation function of more general, non-BPS, observables. These observables include evaluation observables corresponding to differential forms on X that are not closed, as well as differential operators on X. The correlation functions of the non-BPS observables reveal a lot more about the structure of the theory. In particular, as we have seen previously in quantum mechanical models, they essentially allow us to reconstruct the spaces of states of the theory in the limit $\bar{\tau} \to -\mathrm{i}\infty$.

In Part II we will describe in detail the structure of the spaces of states of the supersymmetric two-dimensional sigma models in our infinite radius limit. First, we will generalize our results obtained in Part I to the case of multivalued Morse functions. (Actually, examples of such functions arise already for finite-dimensional real manifolds with non-trivial fundamental groups.) We will show how to modify our results in this case. Essentially, this amounts to considering the universal cover of our manifold, which is $\widetilde{LX}$ in the case of two-dimensional sigma model. One also needs to impose an equivariance condition on the states of the model corresponding to the action of the (abelianized) fundamental group $H_1(X,\mathbb{Z})$ on $\widetilde{LX}$. Because of this the corresponding spaces of states acquire an additional parameter which is familiar from the construction of 'ϑ-vacua'.

Besides those changes, the structure of the space of states in the limit $\bar{\tau} \to \infty$ is similar to the one that we have observed above in our analysis of quantum mechanical models. There are spaces of 'in' and 'out' states, and each of them is isomorphic to the direct sum of certain spaces of 'delta-forms' supported on the strata of the decomposition of $\widetilde{LX}$ into the ascending and descending manifolds (however, this direct sum decomposition is not canonical). Let us modify our function f given by formula (1.7), as follows: $f \mapsto f_H$, where

$$f_H(\tilde{\gamma}) = \int_D \tilde{\gamma}^*(\omega_K) - \int_{\mathbb{S}^1} \gamma^*(H)\, \mathrm{d}\sigma,$$

where H is a Morse function on X. Then f_H is a Morse function on $\widetilde{LX}$, with isolated critical points: constant maps with values at the critical points x_α, $\alpha \in A$, of H on X.

The corresponding ascending manifolds in $\widetilde{LX}$ are isomorphic to infinite-dimensional vector spaces, which are roughly of half the dimension of the entire loop space. Let us denote them by $X_{\alpha,(\infty/2)+i}$, $\alpha \in A$, $i \in \mathbb{Z}$. Our space of states $\hat{\mathcal{H}}_\tau$, which now depends on the choice of $\tau \in \mathbb{C}$, is realized as the subspace of those states Ψ in

$$\hat{\mathcal{H}} = \bigoplus_{\alpha \in A,\, \gamma \in H_2(X,\mathbb{Z})} \mathcal{F}_{\alpha,(\infty/2)+\gamma} \otimes \bar{\mathcal{F}}_{\alpha,(\infty/2)+\gamma},$$

where $\mathcal{F}_{\alpha,(\infty/2)+\gamma}$ is the space of holomorphic 'delta-forms' supported on $X_{\alpha,(\infty/2)+\gamma}$, and $\mathcal{F}_{\alpha,(\infty/2)+\gamma}$ is its complex conjugate, which satisfy the condition $T(\Psi) = \mathrm{e}^{\mathrm{i}\int_\gamma \tau}\Psi$. Here T is the shift operator $\mathcal{H}_{\alpha,(\infty/2)+\gamma} \to \mathcal{H}_{\alpha,(\infty/2)+\gamma+1}$. Thus, $\hat{\mathcal{H}}_\tau$ is (non-canonically) isomorphic to the direct sum as above with fixed $\gamma = \gamma_0$.

The spaces of delta-forms may then be identified with the familiar Fock representations of the chiral and anti-chiral $\beta\gamma, bc$-systems. The Hamiltonian and the supersymmetry charges may be identified with explicit operators acting on the spaces of states. Thus, the spaces of states are essentially isomorphic to the direct sums of finitely many tensor products of this form, like in a conformal field theory. In particular, there is a large chiral algebra, which is nothing but the chiral de Rham complex of X. However, we find that the Hamiltonian is not diagonalizable and computing matrix elements of observables acting on the space of states, we see the appearance of the logarithm function. Thus, we find that the two-dimensional supersymmetric model in our infinite radius limit is a *logarithmic conformal field theory*.

We stress again that to see this structure it is crucial that we consider the correlation functions of non-BPS observables. The hamiltonian is diagonalizable (in fact, is identically equal to zero) on the BPS states. Therefore, correlation functions of the BPS observables which have been extensively studied in the literature (and which are closely related to the Gromov–Witten invariants) do not contain logarithms. The hamiltonian is also diagonalizable on all purely chiral (and anti-chiral) states; thus, the chiral algebra of the theory is free of logarithms. Logarithmic CFTs of this type have been considered, for example, in [**35**].

We remark that in the case when X is the flag manifold of a simple Lie group, the above semi-infinite stratification of $\widetilde{LX}$ and the corresponding spaces of holomorphic delta-forms have been considered in [**14**]. These spaces are representations of the affine Kac–Moody algebra $\hat{\mathfrak{g}}$ with level 0, which are closely related to the Wakimoto modules. In the non-supersymmetric version the level 0 algebra $\hat{\mathfrak{g}}$ gets replaced by the $\hat{\mathfrak{g}}$ at the critical level $-h^\vee$ (see [**14, 17**]). Therefore, we expect that the corresponding models are closely related to the geometric Langlands correspondence. This will be discussed in Part III.

1.5. Four-dimensional Yang–Mills theory

Finally, we discuss (twisted) $\mathcal{N} = 2$ supersymmetric Yang–Mills theory with gauge group G on a four-dimensional manifold $\boldsymbol{M}^4$ [**40**]. Suppose that $\boldsymbol{M}^4 = \mathbb{R} \times M^3$, where M^3 is a compact three-dimensional manifold. Let t denote the coordinate along the $\mathbb{R}$ factor. Then the Yang–Mills theory may be interpreted as quantum mechanics on the space $\mathcal{A}/\mathcal{G}$ of gauge equivalence classes of G-connections on M^3, with the Morse–Novikov function being the Chern–Simons functional [**1, 40**]. However, there is again a new element, compared to the previously discussed theories, and that is the appearance of gauge symmetry. The quotient $\mathcal{A}/\mathcal{G}$ has complicated singularities because the gauge group $\mathcal{G}$ has non-trivial stabilizers in the space $\mathcal{A}$. For this reason we should consider the *gauged Morse theory* on the space $\mathcal{A}$ of connections itself.

This theory is defined as follows. Let X be a manifold equipped with an action of a group G and a G-invariant Morse function f. Then the gradient vector field $v^\mu \partial_{x^\mu}$, where $v^\mu = h^{\mu\nu}\partial_{x^\nu} f$ commutes with the action of G. Denote by $\mathcal{V}_a^\mu \partial_{x^\mu}$ the vector fields on X corresponding to basis elements J^a of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. We define a gauge theory generalization of the gradient trajectory: it is a pair $(x(t)\colon \mathbb{R} \to X,\ A_t(t)\,\mathrm{d}t \in \Omega^1(\mathbb{R}, \mathfrak{g}^*))$, which is a solution of the equation

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}t} = v^\mu(x(t)) + \mathcal{V}_a^\mu(x(t)) A_t^a(t). \tag{1.8}$$

The group of maps $g(t)\colon \mathbb{R} \to G$ acts on the space of solutions by the formula

$$g\colon (x(t), A_t(t)\,\mathrm{d}t) \mapsto (g(t)\cdot x(t), g^{-1}(t)\partial_t g(t) + g^{-1}(t)A_t(t)g(t)),$$

and the moduli space of gradient trajectories is, naively, the quotient of the space of solutions of (1.8) by this action. However, because this action has non-trivial stabilizers and the ensuing singularities of the quotient, it is better to work equivariantly with the moduli space of solutions of the equations (1.8).

In Part II we will develop a suitable formalism of equivariant integration on the moduli space of gradient trajectories of the gauged Morse theory. We then apply this formalism to the case when $X = \mathcal{A}$, the space of connections on a three-manifold M^3 and f is the Chern–Simons functional (note that this formalism is also used in gauged sigma models in two dimensions). In this case the corresponding equivariant integrals give us the correlation functions of evaluation observables of the Yang–Mills theory in our weak coupling limit $\bar{\tau} \to -\mathrm{i}\infty$. In the case of the BPS observables these correlation functions are the *Donaldson invariants* [**40**]. They comprise the topological (or BPS) sector of the theory. We will obtain more general (off-shell) correlation functions by considering more general, i.e. non-BPS, evaluation observables. We will present some sample computations of these off-shell correlation functions which exhibit the same effects as in one- and two-dimensional models considered above. In particular, we will observe the appearance of the logarithm (and more generally, polylogarithm) function in the correlation functions. This indicates that, just like the two-dimensional sigma models, the four-dimensional supersymmetric Yang–Mills theory in the $\bar{\tau} \to \infty$ limit is a logarithmic conformal field theory.

1.6. Plan of the paper

The paper is organized as follows. In § 2 we give a pedagogical description of the Lagrangian formalism of our quantum mechanical models. We then discuss the path integral in the limit when the metric of the manifold and the Morse function are both multiplied by the same constant λ which tends to infinity (this corresponds to the limit $\bar{\tau} \to \infty$ with $\tau = 0$). We show that in this limit the path integral localizes on the moduli spaces of gradient trajectories of the Morse function (the instantons of the quantum mechanical models). We introduce the observables of the theory and discuss the analogy between their correlation functions and the Gromov–Witten theory.

In § 3 we start developing the Hamiltonian formalism for our models. We are interested, in particular, with the structure of the space of states of the model in the limit $\lambda \to \infty$. This limit is highly singular and the description of the spaces of states requires special care. We discuss in detail the examples of the flat space $\mathbb{C}$ and of the simplest 'curved' manifold $\mathbb{CP}^1$. We show that the space of states decomposes into the spaces of 'in' and 'out' states, each having a simple and geometrically meaningful description in terms of the stratifications of our manifold by the ascending and descending manifolds of our Morse function. Furthermore, we show that the spaces of states exhibit holomorphic factorization that is absent for finite values of λ. This leads to a great simplification of the correlation functions in the limit $\lambda \to \infty$.

Next, in § 4 we give a more precise description of the spaces of states. We show that states are naturally interpreted as *distributions* (or *currents*) on our manifold X. Because some of these distributions require regularization (reminiscent of the Epstein–Glaser regularization [**13**] in quantum field theory), the action of the Hamiltonian on them becomes non-diagonalizable. We explain this in detail in the case of the $\mathbb{CP}^1$ model. For a general Kähler manifold we compute the action of the Hamiltonian on the spaces of 'in' and 'out' states, as well as the action of the supercharges, in terms of the so-called *Grothendieck–Cousin operators* associated to the stratification of our manifold by the ascending and descending manifolds. We also compute the cohomology of the supercharges using the GC complex.

In § 5 we will realize the observables of the model as linear operators acting on the spaces of states. We will then be able to obtain the correlation functions as matrix elements of these operators and to test our predictions by comparing these matrix elements with the integrals over the moduli spaces of gradient trajectories which were obtained in the path integral approach of § 2. We will see that analytic properties of the observables play an important role in the limit $\lambda \to \infty$. We will also see that factorization of the correlation functions over intermediate states leads to some non-trivial identities on analytic differential forms. In particular, the appearance of logarithm in the correlation functions will be seen as the manifestation of the non-diagonal nature of the Hamiltonian and as the ultimate validation of our description of the space of states.

Finally, in § 6 we discuss possible generalizations of our results. We consider the question of how to relate the spaces of states of our models for finite and infinite values of λ, first in the case when $X = \mathbb{C}$ and then for $X = \mathbb{CP}^1$. We then discuss the computation of correlation functions in λ^{-1} perturbation theory. Next, we consider non-supersymmetric analogues of our models. We discuss, in particular, the computation of the cohomology of the anti-chiral supercharge $\bar{\partial}$ in the 'half-supersymmetric' models, which are one-dimensional analogues of the $(0, 2)$ supersymmetric two-dimensional sigma models. We make contact with the GC complexes of arbitrary (holomorphic) vector bundles on Kähler manifolds and the results of [**39**,**46**] on holomorphic Morse theory. We also discuss briefly the generalization in which a Morse functions is replaced by a Morse–Bott function having non-isolated critical points.

2. Supersymmetric quantum mechanics

In this section we begin our investigation of the quantum mechanical models in the limit $\bar{\tau} \to \infty$. The natural context for these models is the physical realization of Morse theory due to Witten [**38**], which we recall briefly at the beginning of the section. We describe the Lagrangians of these models and the corresponding path integral. We then discuss the path integral in the limit when the metric of the manifold and the Morse function are both multiplied by the same constant λ which tends to infinity (which corresponds to the limit $\bar{\tau} \to \infty$ with $\tau = 0$, discussed in § 1). We show that in this limit the path integral localizes on the moduli spaces of gradient trajectories of the Morse function. We introduce the observables of the theory and discuss the analogy between their correlation functions and the Gromov–Witten theory.

2.1. Recollections on Morse theory

Morse theory associates to a compact smooth Riemannian manifold X and a Morse function f (i.e. a function with isolated non-degenerate critical points) a complex $C^\bullet$, whose cohomology coincides with the de Rham cohomology $H^\bullet(X)$. The ith term C^i of the complex is generated by the critical points of f of index i (the index of a critical point is the number of negative squares in the Hessian quadratic form at the critical point).

The differential $\mathrm{d}\colon C^i \to C^{i+1}$ is obtained by summing over the gradient trajectories connecting critical points.

Witten [**38**] has given the following interpretation of Morse theory. Consider the supersymmetric quantum mechanics on a Riemannian manifold X (in other words, quantum mechanics on ΠTX). The space of states is the Hilbert space $\Omega^\bullet(X)$, the space of complex-valued L_2 differential forms on X with the hermitian inner product

$$\langle\alpha|\beta\rangle = \int_X (\star\bar{\alpha}) \wedge \beta, \tag{2.1}$$

where $\overline{(\cdots)}$ denotes the complex conjugation, and $\star$ is the Hodge star operator.

The supersymmetry algebra is generated by the operators:

$$\mathcal{Q} = \mathrm{d}_\lambda = \mathrm{e}^{-\lambda f}\,\mathrm{d}\mathrm{e}^{\lambda f} = \mathrm{d} + \lambda\,\mathrm{d}f\wedge, \tag{2.2}$$

$$\mathcal{Q}^* = (\mathrm{d}_\lambda)^* = \frac{1}{\lambda}\mathrm{e}^{\lambda f}\,\mathrm{d}^*\mathrm{e}^{-\lambda f} = \frac{1}{\lambda}\,\mathrm{d}^* + \iota_{\nabla f}. \tag{2.3}$$

Here the operator d^* is defined as the adjoint of d with respect to a fixed metric g on X. But $\mathcal{Q}^*$ is the adjoint of $\mathcal{Q} = \mathrm{d}_\lambda$ with respect to the metric λg, which explains the overall factor λ^{-1}.

Their anti-commutator $H = \frac{1}{2}\{\mathcal{Q}, \mathcal{Q}^*\}$ is the Hamiltonian*:

$$H = H_\lambda = \tfrac{1}{2}(-\lambda^{-1}\Delta + \lambda\|\mathrm{d}f\|^2 + K_f), \tag{2.4}$$

where $K_f = (\mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f})$. Recall that for a vector field ξ we denote by $\mathcal{L}_\xi$ the Lie derivative acting on differential forms.

The supersymmetry generators $\mathcal{Q}$, $\mathcal{Q}^*$ are nilpotent, $\mathcal{Q}^2 = 0$, $(\mathcal{Q}^*)^2 = 0$. For compact X, the cohomology of the operator $\mathcal{Q}$ coincides with the cohomology of the operator d, since they are related by the similarity transformation. Standard Hodge theory argument shows that the span of the ground states of H (i.e. those annihilated by H) is isomorphic to the cohomology of $\mathcal{Q}$, and hence to $H^\bullet(X)$.

Indeed, among the $\mathcal{Q}$-closed differential forms α representing a given cohomology class we choose a representative α_{har} which minimizes the norm $\|\alpha\|^2$ with respect to the inner product (2.1). This representative α_{har} is annihilated by $\mathcal{Q}^*$, in addition to $\mathcal{Q}$. As a consequence, α_{har} is annihilated by H. Conversely, if α is annihilated by H, then

$$0 = \langle\alpha|H|\alpha\rangle = \|\mathcal{Q}\alpha\|^2 + \|\mathcal{Q}^*\alpha\|^2, \tag{2.5}$$

hence $\mathcal{Q}\alpha = 0$, $\mathcal{Q}^*\alpha = 0$.

The first step in Witten's approach to Morse theory [**38**] is constructing the approximate ground states of H in the limit $\lambda \to \infty$. According to the semi-classical analysis, they are given by the differential forms localized near the critical points x, such that $\mathrm{d}f_x = 0$. Near such a point the Hamiltonian (2.4) may be approximated by that of supersymmetric harmonic oscillator. We will discuss this in more detail below. Now we just mention that for each critical point of index i one finds a ground state of the Hamiltonian, which is

* In our conventions the Laplacian $\Delta = -\{\mathrm{d}, \mathrm{d}^*\}$ is negative definite.

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a differential form ω_i of degree i approximately equal to a Gaussian distribution around this critical point. The simplest of these are the 0-form $C_\lambda e^{-\lambda f}$ localized at the absolute minimum of f and the top form $C'_\lambda e^{\lambda f}\, d\mu$, localized at the absolute maximum of f (here $d\mu$ is the volume form induced by the metric and C_λ, C'_λ are the constants making the norms of these forms equal to 1).

The eigenvalues of H on these approximate ground states ω_i tend to zero very fast as $\lambda \to \infty$. Therefore, for large λ their span 'splits off' as a subcomplex of the de Rham complex, equipped with the twisted differential $\mathcal{Q} = d_\lambda$. Since cohomology classes may be represented by ground states, as we have seen above, we obtain that the cohomology of this subcomplex is equal to the cohomology of the entire de Rham complex. This homology is in turn isomorphic the cohomology of X. By construction, the dimension of the ith group of this subcomplex is equal to the number of critical points of f of index i, and using this fact we obtain estimates on the ranks of the cohomology groups of X, i.e. the Betti numbers of X. This way Witten proved in [**38**] the Morse inequalities relating the Betti numbers of X to the numbers of critical points of f of various indices (see also [**21**, **22**]).

Perturbatively, each ω_i is annihilated by the supersymmetry charge $\mathcal{Q}$. The next step in Witten's construction is the computation of the instanton corrections to $\mathcal{Q}$ on the ω_i due to the tunnelling transitions. Witten argued that this way one obtains the Morse complex of f. (This is indeed the case if a certain 'Morse–Smale transversality condition' is satisfied.) Thus, one obtains an interpretation of Morse theory in terms of supersymmetric quantum mechanics.

2.2. Important special case

A special case of this construction occurs when X is a compact Kähler manifold, and the Morse function f is the hamiltonian corresponding to a $U(1)$-action on X. Its complexification gives us a $\mathbb{C}^\times$-action on X. In this case, the gradient vector field $v = \nabla f$ may be split as the sum of a holomorphic vector field ξ and its complex conjugate $\bar{\xi}$. On the other hand, the vector field $i(\xi - \bar{\xi})$ generates the $U(1)$ action.

In the main body of this paper we will focus exclusively on this case, because it is the one-dimensional analogue of the two-dimensional supersymmetric sigma models and the super-Yang–Mills theory that we are interested in. Some important simplifications occur in this case. For example, all ground states in the limit $\lambda = \infty$ may be deformed to ground states for finite λ. In other words, there are no instanton corrections to the action of the supercharge $\mathcal{Q}$ (see § 2.5 for more details). These are also the models exhibiting holomorphic factorization in the limit $\lambda = \infty$, as we will see below.

2.3. Path integral and gradient trajectories

Let us discuss the Lagrangian version of the theory. The space of states of our theory is the space of functions on the supermanifold ΠTX. Introduce the corresponding coordinates x^μ, ψ^μ and the momenta p_μ, π_μ. The configuration space is the space of maps $I \to X$, where $I = I_{t_i,t_f}$ is the 'worldline', which could be a finite interval $[t_i, t_f]$, or half-line $(-\infty, t_f)$, $[t_i, +\infty)$ or the entire line $(-\infty, +\infty)$. The standard action is given by the

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formula [**38**]

$$S = \int_I \Big(\tfrac{1}{2}\lambda g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t} + \tfrac{1}{2}\lambda g^{\mu\nu}\frac{\partial f}{\partial x^\mu}\frac{\partial f}{\partial x^\nu} + \mathrm{i}\pi_\mu \mathrm{D}_t\psi^\mu - \mathrm{i}g^{\mu\nu}\frac{\mathrm{D}^2 f}{\mathrm{D}x^\nu\,\mathrm{D}x^\alpha}\pi_\mu\psi^\alpha + \tfrac{1}{2}\lambda^{-1}R^{\mu\nu}_{\alpha\beta}\pi_\mu\pi_\nu\psi^\alpha\psi^\beta \Big)\,\mathrm{d}t. \quad (2.6)$$

Here $\mathrm{D}/\mathrm{D}x^\mu$ is the covariant derivative on X corresponding to the Levi-Civita connection, and D_t is its pullback to I under the map $x\colon I \to X$. It is easy to the see that the hamiltonian corresponding to this action is the quasi-classical limit of the hamiltonian (2.4).

The correlation functions of the theory are given by path integrals:

$$\langle x_\mathrm{f}|\mathrm{e}^{(t_n-t_\mathrm{f})H}\mathcal{O}_n\mathrm{e}^{(t_{n-1}-t_n)H}\cdots\mathrm{e}^{(t_1-t_2)H}\mathcal{O}_1\mathrm{e}^{(t_\mathrm{i}-t_1)H}|x_\mathrm{i}\rangle = \int_{I\to X;\,x(t_\mathrm{i})=x_\mathrm{i},\,x(t_\mathrm{f})=x_\mathrm{f}} \mathcal{O}_1(t_1)\cdots\mathcal{O}_n(t_n)\mathrm{e}^{-S}. \quad (2.7)$$

Here $\mathcal{O}_1,\dots,\mathcal{O}_n$ are observables which we will discuss in detail below. More precisely, the right-hand side gives the integral kernel for this correlation function with respect to x_i and $x_\mathrm{f} \in X$. In other words, if Ψ, Ψ^* are states in the Hilbert space of the theory (which is the space of L_2 differential forms on X), then

$$\langle\Psi^*|\mathrm{e}^{(t_n-t_\mathrm{f})H}\mathcal{O}_n\mathrm{e}^{(t_{n-1}-t_n)H}\cdots\mathrm{e}^{(t_1-t_2)H}\mathcal{O}_1\mathrm{e}^{(t_\mathrm{i}-t_1)H}|\Psi\rangle = \int_{X^2}\Psi^*(x_\mathrm{f})\Psi(x_\mathrm{i})\int_{I\to X;\,x(t_\mathrm{i})=x_\mathrm{i},\,x(t_\mathrm{f})=x_\mathrm{f}} \mathcal{O}_1(t_1)\cdots\mathcal{O}_n(t_n)\mathrm{e}^{-S}. \quad (2.8)$$

We will now discuss in detail how to pass to the limit $\lambda \to \infty$ in such a way that we keep the instanton contributions, but get rid of the anti-instantons. The procedure will be similar in two and four dimensions.

We start with the trivial, but crucial observation (sometimes called the 'Bogomolny trick') that the bosonic part of the action may be rewritten as follows:

$$\int_I \Big(\tfrac{1}{2}\lambda|\dot{x} \mp \nabla f|^2 \pm \lambda\frac{\mathrm{d}f}{\mathrm{d}t}\Big)\,\mathrm{d}t, \quad (2.9)$$

where

$$(\nabla f)^\mu = g^{\mu\nu}\frac{\partial f}{\partial x^\nu}.$$

It is clear from this formula that the absolute minima of the action, with fixed boundary conditions $x(t_\mathrm{i}) = x_\mathrm{i}$, $x(t_\mathrm{f}) = x_\mathrm{f}$, will be achieved on the gradient trajectories of f (appearing below with the + sign) or the gradient trajectories of $-f$ (with the − sign): $\dot{x} = \pm\nabla f$, or equivalently,

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}t} = \pm g^{\mu\nu}\frac{\partial f}{\partial x^\nu}$$

(provided that gradient trajectories connecting x_i and x_f exist). These are the *instantons* and *anti-instantons* of our model, respectively. The former realize maps for which

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$f(x(t_{\rm f})) > f(x(t_{\rm i}))$ and the latter realize maps for which $f(x(t_{\rm f})) < f(x(t_{\rm i}))$. Both contribute to the path integral with the same weight factor $\mathrm{e}^{-\lambda|f(x(t_{\rm f}))-f(x(t_{\rm i}))|}$. As $\lambda \to \infty$ this factor goes to 0 exponentially fast, and this is the reason why instanton and anti-instanton contributions are negligible compared to the contributions of small fluctuations around the constant maps.

Now we wish to modify our Lagrangian in such a way that we retain the instantons and make anti-instantons disappear altogether in the $\lambda \to \infty$ limit. This is achieved by adding the term

$$-\int_I \lambda\,\mathrm{d}f = -\int_I \lambda\frac{\mathrm{d}f}{\mathrm{d}t}\,\mathrm{d}t = \lambda(f(x_{\rm i}) - f(x_{\rm f})) \tag{2.10}$$

to the action (2.6). The resulting action reads

$$\int_I (\lambda|\dot{x} - \nabla f|^2 + \text{fermions})\,\mathrm{d}t. \tag{2.11}$$

The effect is that now the instantons, i.e. the gradient trajectories $\dot{x} = \nabla f$, become the absolute minima of the action. The action on them is equal to 0, so all of them make contributions to the path integral of the finite order (independent of λ). In contrast, the action on anti-instantons is now $2\lambda|f(x_{\rm f}) - f(x_{\rm i})|$. They do not correspond to the absolute minima of the action any more. Accordingly, their contribution to the path integral is even more suppressed than before: now they occur in the path integral with the weight factor $\mathrm{e}^{-2\lambda|f(x_{\rm f})-f(x_{\rm i})|}$. Therefore, in the limit $\lambda \to \infty$ instantons will make finite contributions to the path integral (on par with the fluctuations around the constant maps), but anti-instantons will not contribute to the path integral at all.*

No matter how large λ is though, both instantons and anti-instantons make contributions to general correlation functions.† Therefore, if we want to eliminate completely the anti-instanton contributions from the correlation functions, we really have to take the limit $\lambda \to \infty$.

In order to achieve that, we first rewrite the action in terms of a first-order Lagrangian as follows:

$$\begin{aligned} S_\lambda = \int_I \biggl(&- \mathrm{i}p_\mu\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}t} + g^{\mu\nu}\frac{\partial f}{\partial x^\nu}\right) + \tfrac{1}{2}\lambda^{-1}g^{\mu\nu}p_\mu p_\nu \\ &+ \mathrm{i}\pi_\mu\left(\mathrm{D}_t\psi^\mu - g^{\mu\nu}\frac{\mathrm{D}^2 f}{\mathrm{D}x^\nu\,\mathrm{D}x^\alpha}\psi^\alpha\right) + \tfrac{1}{2}\lambda^{-1}R^{\mu\nu}_{\alpha\beta}\pi_\mu\pi_\nu\psi^\alpha\psi^\beta\biggr)\,\mathrm{d}t. \end{aligned} \tag{2.12}$$

For finite values of λ, by eliminating the momenta variables using the equations of motion, we obtain precisely the action (2.11). Therefore, the two actions are equivalent for finite

* Note that by adding to the Lagrangian the term $\lambda\,\mathrm{d}f$ instead, we would retain the anti-instantons and get rid of the instantons.

† With the exception of some λ-independent correlation functions of the topological sector of the theory discussed below.

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values of λ. But now we can take the limit $\lambda \to \infty$ in the new action. The resulting action is

$$S_\infty = -\mathrm{i} \int_I \left(p_\mu \left(\frac{\mathrm{d}x^\mu}{\mathrm{d}t} + g^{\mu\nu} \frac{\partial f}{\partial x^\nu} \right) - \pi_\mu \left(\mathrm{D}_t \psi^\mu - g^{\mu\nu} \frac{\mathrm{D}^2 f}{\mathrm{D}x^\nu \, \mathrm{D}x^\alpha} \psi^\alpha \right) \right) \mathrm{d}t. \tag{2.13}$$

Now the equations

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}t} = g^{\mu\nu} \frac{\partial f}{\partial x^\nu} \tag{2.14}$$

are the equation of motion. Thus, the instantons (gradient trajectories of f), which in the original theory corresponded to absolute minima of the action, but were *not* solutions of the equations of motion, have now become solutions of the equations of motion. At the same time anti-instantons (gradient trajectories of $-f$) have disappeared.

We note that in the sum of the terms

$$-\int_I g^{\mu\nu} \left(p_\mu \frac{\partial f}{\partial x^\nu} - \pi_\mu g^{\mu\nu} \frac{\mathrm{D}^2 f}{\mathrm{D}x^\nu \, \mathrm{D}x^\alpha} \psi^\alpha \right) \mathrm{d}t$$

in the above action we can replace $\mathrm{d}t$ by an arbitrary connection A_t on a principal $\mathbb{R}$-bundle on I. This observation will be very useful in the context of two-dimensional sigma models.

We now describe how the coordinate invariance is realized in the above action. The bosonic variables x^μ and p_μ transform as functions and 1-forms on X, respectively. The fermionic variables π_μ, ψ^μ transform as sections of the cotangent and tangent bundles to X, respectively. Note that we have

$$\mathrm{D}_t \psi^\lambda = \frac{\mathrm{d}\psi^\lambda}{\mathrm{d}t} + \Gamma^\lambda_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}t} \psi^\nu,$$

where Γ is the Levi-Civita connection on the tangent bundle TX. Therefore, if we redefine p_μ as follows:

$$p'_\mu = p_\mu + \Gamma^\lambda_{\mu\nu} \psi^\nu \pi_\lambda, \tag{2.15}$$

we absorb the connection operators into p'_μ and obtain the following formula for the action*:

$$S_\infty = -\mathrm{i} \int_I \left(p'_\mu \left(\frac{\mathrm{d}x^\mu}{\mathrm{d}t} - g^{\mu\nu} \frac{\partial f}{\partial x^\nu} \right) - \pi_\mu \left(\frac{\mathrm{d}\psi^\mu}{\mathrm{d}t} - \frac{\partial}{\partial x^\alpha} \left(g^{\mu\nu} \frac{\partial f}{\partial x^\nu} \right) \psi^\alpha \right) \right) \mathrm{d}t. \tag{2.16}$$

However, the new momenta p'_μ no longer transform as 1-forms.

Indeed, we have under the coordinate transformation $x^\mu \mapsto \tilde{x}^\nu$:

$$\psi^\mu \mapsto \tilde{\psi}^\mu = \psi^\nu \frac{\partial \tilde{x}^\nu}{\partial x^\mu}, \qquad \pi_\mu \mapsto \tilde{\pi}_\mu = \pi_\nu \frac{\partial x^\nu}{\partial \tilde{x}^\mu}.$$

* More generally, we could use another connection in formula (2.15).

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This transformation law forces p'_μ to transform inhomogeneously:

$$p'_\mu \mapsto \tilde{p}'_\mu = p'_\nu \frac{\partial x^\nu}{\partial \tilde{x}^\mu} + \frac{\partial^2 x^\alpha}{\partial \tilde{x}^\mu \partial \tilde{x}^\nu} \frac{\partial \tilde{x}^\nu}{\partial x^\beta} \pi_\alpha \psi^\beta. \tag{2.17}$$

The action (2.16) is invariant under the supersymmetry generated by the supercharges Q and Q^* defined by the formulae

$$\begin{aligned} Qx^\mu &= \psi^\mu, & Q\psi^\mu &= 0, \\ Q\pi_\mu &= p'_\mu, & Qp'_\mu &= 0, \\ Q^* x^\mu &= 0, & Q^*\psi^\mu &= g^{\mu\nu} \frac{\partial f}{\partial x^\nu}, \\ Q^*\pi_\mu &= 0, & Q^* p'_\mu &= 0. \end{aligned}$$

They correspond to the de Rham differential and the contraction operator $\imath_{\nabla f}$, respectively.

In particular, we find that the Lagrangian is Q-exact:

$$L = -\mathrm{i}Q \cdot \pi_\mu \left(\frac{\mathrm{d}x^\mu}{\mathrm{d}t} - g^{\mu\nu} \frac{\partial f}{\partial x^\nu} \right).$$

Recall that the deformation from $\lambda = \infty$ back to finite λ is achieved by adding the terms

$$\tfrac{1}{2}\lambda^{-1} g^{\mu\nu} p_\mu p_\nu + \tfrac{1}{2}\lambda^{-1} R^{\mu\nu}_{\alpha\beta} \pi_\mu \pi_\nu \psi^\alpha \psi^\beta$$

to Lagrangian (2.13). It is important to note that, just like the Lagrangian (2.13), this expression is Q-exact and equal to

$$Q \cdot \tfrac{1}{2}\lambda^{-1} g^{\mu\nu} \pi_\mu p'_\nu. \tag{2.18}$$

2.4. Correlation functions as integrals over moduli spaces of instantons

We now discuss the correlation functions in our model in the Lagrangian, i.e. path integral formalism. It turns out that these correlation functions may be represented by integrals over finite-dimensional moduli spaces of gradient trajectories.

The first question to ask is what are the observables of our theory. Typically, observables in quantum mechanics on a manifold Y are obtained by quantizing functions on T^*Y, which in our case is $T^*(\Pi TX)$. The simplest are the observables corresponding to the functions on ΠTX, which are the same as differential forms on X. These are quantized in a straightforward way in the coordinate polarization. In the original quantum mechanical model, for finite λ, the operator, corresponding to a differential n-form ω on X (which is the same thing as a function on ΠTX, which is a degree n polynomial in the fermionic variables), is the operator of multiplication by ω. The correlation functions of these observables are easiest to compute in the $\lambda \to \infty$ limit by using the path integral.

To see that, consider the following finite-dimensional model situation: a vector space $\mathbb{R}^M$ and functions f^a, $a = 1, \ldots, N$, defining a codimension N submanifold $C \subset \mathbb{R}^M$.

Then the delta-like differential form supported on this subvariety has the following integral representation:

$$\delta_C = \int \prod_a \mathrm{d}p_a \, \mathrm{d}\pi_a \, \mathrm{e}^{\mathrm{i}p_a f^a + \mathrm{i}\pi_a \, \mathrm{d}f_a}.$$

This delta-form may be viewed as the limit, when $\lambda \to \infty$, of the regularized integral

$$\delta_{C,\lambda} = \int \prod_a \mathrm{d}p_a \, \mathrm{d}\pi_a \, \mathrm{e}^{\mathrm{i}p_a f^a + \mathrm{i}\pi_a \, \mathrm{d}f_a - \lambda^{-1} p_a p_{\bar{a}}}.$$

Comparing these formulae to (2.13) and (2.12), we see that the path integral

$$\int \mathrm{D}p \, \mathrm{D}\pi \, \mathrm{e}^{-S_\infty}$$

looks like the delta-like form supported on the gradient trajectories, solutions of the equations (2.14), while $\int \mathrm{D}p \, \mathrm{D}\pi \, \mathrm{e}^{-S_\lambda}$ may be viewed as its regularized version. More precisely, the integral $\int \mathrm{D}p \, \mathrm{D}\pi \, \mathrm{e}^{-S_\lambda}$ should be viewed as the Mathai–Quillen representative of the Euler class of an appropriate vector bundle over the space of gradient trajectories (see [**2**, **10**, **27**]).

In a similar way one shows that in the limit $\lambda \to \infty$ the correlation functions in our theory will be equal to integrals of differential forms over the moduli spaces of gradient trajectories. Note that the integral over fluctuations around the instanton solutions contributes only the one-loop determinants, which cancel each other out for bosonic and fermionic degrees of freedom, up to a sign. Moreover, this sign disappears in the case that we are most interested in: when X is a Kähler manifold and the Morse function f satisfies the conditions listed in § 3.6 below.

In particular, the kernel of the evolution operator in our theory is just the delta-form supported on the submanifold of those pairs $(x_\mathrm{i}, x_\mathrm{f}) \in X \times X$ which are connected by the gradient trajectories $x(t)\colon I_{t_\mathrm{i},t_\mathrm{f}} \to M$ such that $x(t_\mathrm{i}) = x_\mathrm{i}$ and $x(t_\mathrm{f}) = x_\mathrm{f}$.

From now on we will focus on the case of the infinite line $I = \mathbb{R}$.

The gradient trajectories $\mathbb{R} \to X$ necessarily start and end at the critical points of f (recall that we have assumed that they are isolated). The corresponding moduli space is therefore a union of connected components labelled by pairs of critical points of f, x_- and x_+, which play the role of the boundary conditions in the path integral. Let $\mathcal{M}_{x_-,x_+}$ be the moduli space of the gradient trajectories, that is solutions $x(t)$ to (2.14), which obey

$$x(t) \to x_\pm, \quad t \to \pm\infty. \tag{2.19}$$

We have evaluation maps

$$\left.\begin{aligned} \mathrm{ev}\colon \mathcal{M}_{x_-,x_+} \times \mathbb{R} \to X, &\qquad \mathrm{ev}_t\colon \mathcal{M}_{x_-,x_+} \to X \\ \mathrm{ev}(m,t) = x_m(t), &\qquad \mathrm{ev}_t(m) = x_m(t). \end{aligned}\right\} \tag{2.20}$$

The simplest observables of our theory correspond to differential forms on X. They are called the *evaluation observables*. The correlation function of the evaluation observables

$\hat{\omega}_i$ corresponding to differential forms ω_i, $i = 1, \ldots, n$, on X, in the sector corresponding to the boundary conditions $x_\pm$ is given by the integral

$$ {}_{x_-}\langle \hat{\omega}_1(t_1)\hat{\omega}_2(t_2)\cdots\omega_k(t_k)\rangle_{x_+} = \int_{\mathcal{M}_{x_-,x_+}} \mathrm{ev}_{t_1}^*\,\omega_1 \wedge \mathrm{ev}_{t_2}^*\,\omega_2 \wedge \cdots \wedge \mathrm{ev}_{t_k}^*\,\omega_k. \tag{2.21} $$

If the forms ω_i have definite cohomological degrees, then, according to formula (2.21), the above integral is non-vanishing only if the following selection rule (fermionic charge conservation) is obeyed:

$$ \sum_{i=1}^{n} \deg\omega_i = \dim\mathcal{M}_{x_-,x_+} = n_{x_+} - n_{x_-}, \tag{2.22} $$

where n_x is the index of the critical point x.

Let us note that the correlation function (2.21) is invariant with respect to the time shift $t \mapsto t+\mathrm{const}$. This invariance is verified by the expression (2.21) due to the fact that the time shifts act on $\mathcal{M}_{x_-,x_+}$. Indeed, if $x_m(t)$ is the gradient trajectory corresponding to a point $m \in \mathcal{M}_{x_-,x_+}$, then so is

$$ x_{m^s}(t) = x_m(t+s). $$

Thus, we obtain an action of the transformations $g^s\colon m \mapsto m^s$, $s \in \mathbb{R}$, on $\mathcal{M}_{x_-,x_+}$.

Since the integral (2.21) is not changed by the changes of the integration variables, the simultaneous time shift $t_i \mapsto t_i + s$, which can be absorbed into the change of moduli $m \mapsto m^s$, does not affect the correlation function.

Note that if we wish to distinguish contributions of different types of instantons, running between different critical points, we may also add a finite term

$$ \Delta S_\tau = -\mathrm{i}\tau \int_{-\infty}^{+\infty} \mathrm{d}f \tag{2.23} $$

to the action (2.16). Then the above correlation function will get multiplied by the factor $\mathrm{e}^{\mathrm{i}\tau(f(x_+)-f(x_-))}$.

A natural way to obtain the term (2.23) is as follows: introduce an additional parameter ϑ, the 'ϑ angle'. Let us set

$$ \tau = \vartheta + \mathrm{i}\lambda, \qquad \bar{\tau} = \vartheta - \mathrm{i}\lambda. $$

Let us add to the second-order action (2.6), instead of (2.10), the term

$$ -\mathrm{i}\vartheta \int_I \mathrm{d}f = -\lambda \int_I \mathrm{d}f - \mathrm{i}\tau \int_I \mathrm{d}f. $$

Consider the limit when $\lambda \to +\infty$, $\vartheta \to -\mathrm{i}\infty$ so that $\tau = \mathrm{i}(\lambda - |\vartheta|)$ remains finite, but $\bar{\tau} \to -\mathrm{i}\infty$. In this limit we recover the first-order Lagrangian (2.16) that we have previously obtained in the $\lambda \to \infty$ limit, but with the term (2.23) added.

The additional coupling constant ϑ is the precursor of the B-field in two-dimensional sigma models and the ϑ angle in four-dimensional Yang–Mills theory (this will be discussed in detail in Part II). The interpretation viewing the $\lambda \to \infty$ limit as the limit $\bar{\tau} \to \infty$ has a direct generalization to quantum field theories in two and four dimensions, which we consider in Part II. However, in the context of quantum mechanics (on a simply connected manifold), this does not add much extra value. Indeed, since the instanton moduli space has finitely many components (labelled by pairs of critical points x_-, x_+), separating the contributions of different components with the weight factor $\mathrm{e}^{\mathrm{i}\tau(f(x_-)-f(x_+))}$ does not make much of a difference (unlike the two-dimensional and four-dimensional models, where the instanton moduli spaces have infinitely many components).

2.5. Topological sector

Let us suppose now that our forms ω_i are closed, $\mathrm{d}\omega_i = 0$. Since the supersymmetry charge Q of our model at $\lambda = \infty$ corresponds to the de Rham differential, this means that the corresponding observables $\hat{\omega}_i$ are Q-closed. The correlation functions (2.21) simplify considerably in this case.

This simplification is particularly drastic in the case when X is a Kähler manifold, and so we will focus on this case from now on. There are two reasons for that. The first reason is that in the calculations below we would like to use the fact that the integral of an exact differential form over $\mathcal{M}_{x_-,x_+}$ is equal to 0. But $\mathcal{M}_{x_-,x_+}$ is not compact, and so this statement is not true in general. Note that $\mathcal{M}_{x_-,x_+}$ is the intersection of the descending manifold X^{x_+} of x_+ and the ascending manifold X_{x_-} of x_-. Recall that X^{x_+} consists of the possible values at $t = 0$ of the gradient trajectories $[0, +\infty) \to X$ whose value at $t = +\infty$ is the critical point x_+. Likewise, X_{x_-} consists of the possible values at $t = 0$ of the gradient trajectories $(-\infty, 0] \to X$ whose value at $t = -\infty$ is x_-.

Now suppose that X is a compact Kähler manifold and the Morse function f is the hamiltonian of a vector field corresponding to a $U(1)$-action. Let us write this vector field as $\mathrm{i}(\xi - \bar{\xi})$, where ξ is a holomorphic vector field on X. Then the gradient of f is the vector field $\xi + \bar{\xi}$. In this case the descending and ascending manifolds are isomorphic to $\mathbb{C}^n$. Let us assume in addition that the manifolds X^{x_+} and X_{x_-} form transversal stratifications of X. Then X^{x_+} has a natural compactification $\bar{X}^{x_+}$ which is just the closure of X^{x_+} inside X. Since the descending manifolds form a stratification of X, this closure is the union of the descending manifolds $X^{x'_+}$, where x'_+ runs over a subset of the critical points which are 'above' x_+. Likewise, X_{x_-} has a compactification which is the union of $X_{x'_-}$ with x'_- running over the set of critical points that are 'below' x_-. But then the moduli space $\mathcal{M}_{x_-,x_+}$ has a natural compactification obtained by 'gluing in' the moduli spaces $\mathcal{M}_{x'_-,x'_+}$, where x'_- and x'_+ are critical points that lie 'below' x_- and 'above' x_+, respectively.* Hence the moduli space $\mathcal{M}_{x_-,x_+}$ is also a complex manifold and its complement in the compactification has real codimension 2. Therefore, the integral of an exact form over $\mathcal{M}_{x_-,x_+}$ is equal to 0.

* Note that the evaluation maps extend naturally to this compactification.

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The second reason why an additional simplification occurs for Kähler manifolds is that in this case all ground states of the quantum hamiltonian at the $\lambda \to \infty$ limit are Q-closed, and hence they deform to true ground states for finite values of λ. Indeed, the action of the supersymmetry charge Q on the ground states is given by the Morse differential, as shown in [**38**]. Since the indices of the critical points are even if X is a Kähler manifold, its action is equal to 0 in this case.

Let us now derive some properties of the Q-closed observables corresponding to closed differential forms on a Kähler manifold X. First of all, they are independent of the individual times t_i, $i = 1, \dots, n$. Indeed, using the Cartan formula

$$\mathcal{L}_v = \{\mathrm{d}, \imath_v\} \tag{2.24}$$

for the vector field $v = \nabla f$, we find

$$\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{ev}_t^* \omega_i = -\operatorname{ev}_t^* \mathcal{L}_v \omega_i = -\mathrm{d}(\operatorname{ev}_t^* \imath_v \omega_i),$$

hence the t-derivative of the integral (2.21) is equal to zero.

Another important property is that if all the ω_i are closed and at least one of them is exact: $\omega_j = \mathrm{d}\eta_j$, then the corresponding correlation function vanishes. Indeed, we then find that

$$\int_{\mathcal{M}_{x_-,x_+}} \operatorname{ev}_{t_1}^* \omega_1 \wedge \cdots \wedge \operatorname{ev}_{t_k}^* \omega_{\alpha_k} = \int_{\mathcal{M}_{x_-,x_+}} \mathrm{d}(\operatorname{ev}_{t_1}^* \omega_{\alpha_1} \wedge \cdots \wedge \operatorname{ev}_{t_j}^* \eta_j \wedge \cdots \wedge \operatorname{ev}_{t_k}^* \omega_k) = 0.$$

This is also clear from the point of view of the original path integral, because the Lagrangian of our theory is Q-exact.

This has an important consequence: consider the theory at finite values of λ. As we explained above, it can be viewed as a deformation of the theory at $\lambda = \infty$ obtained by adding to the Lagrangian the expression (2.18). Since this expression is Q-exact, the corresponding correlation function of Q-closed observables will be independent of this deformation, and so the answer that we obtain in the theory at $\lambda = \infty$ will remain valid at finite values of λ (at least in some neighbourhood of $\lambda^{-1} = 0$).

Here it is important to note that the supersymmetry charge Q of the theory with the action (2.11) is independent of λ and corresponds to the de Rham differential (in particular, it is the same for finite λ as for $\lambda = \infty$). But the supersymmetry charge of the 'physical' theory with the action (2.6) differs from it by conjugation with $\mathrm{e}^{-\lambda f}$. However, this conjugation does not change the evaluation observables corresponding to the differential forms, and therefore these observables are Q-closed in both theories.

Thus, we arrive at the following conclusion: there is a sector of the 'physical' theory which is independent of λ. It comprises the Q-closed observables, corresponding to closed differential forms on X. The correlation functions of these observables (on the infinite line and with the boundary conditions $x_{\mathrm{i}} = x_-$, $x_{\mathrm{f}} = x_+$) are given, for all values of λ, by integrals over the finite-dimensional moduli spaces of instantons $\mathcal{M}_{x_-,x_+}$.* The

* More precisely, because the Lagrangian (2.11) differs from the 'physical' Lagrangian (2.6) (for finite values of λ) by the term $\lambda\,\mathrm{d}f$, the correlation functions of the 'physical' theory will be equal to the correlation functions of the first-order theory at $\lambda = \infty$ times $\mathrm{e}^{-\lambda(f(x_+)-f(x_-))}$.

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correlation functions of the Q-closed observables $\hat{\omega}_i$ do not depend on the closed forms ω_i themselves, but only on their cohomology classes. For this reason this sector of the theory is called the 'topological sector' and the corresponding theory is referred to as 'topological field theory'. Alternatively, the Q-closed observables are referred to as the 'BPS observables' and the topological sector is called the 'BPS sector'.

2.6. Analogy with the Gromov–Witten theory

It is instructive to note the analogy between the topological sector of the Morse quantum mechanical model considered above and the Gromov–Witten theory. This analogy will become more clear in Part II when we discuss the two-dimensional sigma models. This material is discussed in more detail in [**9**].

Let X be a compact Kähler manifold and Σ a compact Riemann surface. The analogues of the moduli spaces $\mathcal{M}_{x_-,x_+}$ in Gromov–Witten theory are the moduli spaces $\mathcal{M}_\Sigma(X,\beta)$ of holomorphic maps $\Phi\colon \Sigma \to X$ of a fixed degree $\beta \in H_2(X)$. For a point $p \in \Sigma$ we have evaluation maps $\mathrm{ev}_p\colon \mathcal{M}_\Sigma(X,\beta) \to X$. Now, given an n-tuple of points $p_1,\dots,p_n$ and a collection of differential forms $\omega_1,\dots,\omega_n$ on X, we can consider the integral

$$\int_{\mathcal{M}_\Sigma(X,\beta)} \mathrm{ev}_{p_1}^*(\omega_1)\wedge\cdots\wedge \mathrm{ev}_{p_n}^*(\omega_n). \tag{2.25}$$

We will assume for simplicity that $(\Sigma,(p_{\mathrm{i}}))$ does not admit any continuous automorphisms. This integral* is analogous to the integrals (2.21). They are equal to correlation functions of evaluation observables of the two-dimensional supersymmetric sigma model with the target X in the infinite radius limit (we will consider this model in more detail in Part II).

Let $\mathcal{M}_{g,n}(X,\beta)$ be the moduli space of data $(\Sigma,(p_{\mathrm{i}}),\Phi)$, where Σ is a genus g Riemann surface. Then we have a projection $\pi_{g,n}\colon \mathcal{M}_{g,n}(X,\beta) \to \mathcal{M}_{g,n}$, and $\mathcal{M}_\Sigma(X,\beta)$ is the fibre of $\pi_{g,n}$ at $(\Sigma,(p_{\mathrm{i}})) \in \mathcal{M}_{g,n}$. We have natural evaluation maps $\mathrm{ev}_i\colon \mathcal{M}_{g,n}(X,\beta) \to X$. The general Gromov–Witten invariants are the integrals

$$\int_{\mathcal{M}_{g,n}(X,\beta)} \mathrm{ev}_1^*(\omega_1)\wedge\cdots\wedge \mathrm{ev}_n^*(\omega_n). \tag{2.26}$$

These are the correlation functions of what is often referred to as the 'sigma model coupled to gravity', and the observables are the 'cohomological descendants' of the evaluation observables.

Instead of integrating over $\mathcal{M}_{g,n}(X,\beta)$, we may take the pushforward

$$\pi_{g,n*}(\mathrm{ev}_1^*(\omega_1)\wedge\cdots\wedge \mathrm{ev}_n^*(\omega_n)), \tag{2.27}$$

* The moduli space $\mathcal{M}_\Sigma(X,\beta)$ is not compact, but for compact X this integral is well defined for smooth differential forms ω_i on X under the above assumption on $(\Sigma,(p_{\mathrm{i}}))$.

which is a differential form on $\mathcal{M}_{g,n}$. In particular, (2.25) occurs as a special case when the degree of this differential form is equal to zero. Then its value at $(\Sigma, (p_{\mathrm{i}})) \in \mathcal{M}_{g,n}$ is given by (2.25). More general observables give rise to differential forms of positive degree on $\mathcal{M}_{g,n}$.

More precisely, we need to replace $\mathcal{M}_{g,n}(X, \beta)$ by the Kontsevich's space of stable maps $\bar{\mathcal{M}}_{g,n}(X, \beta)$ and $\mathcal{M}_{g,n}$ by its Deligne–Mumford compactification $\bar{\mathcal{M}}_{g,n}$.

In the quantum mechanical model the analogues of the moduli spaces of holomorphic maps are the moduli spaces $\mathcal{M}_{x_-,x_+}$ of gradient trajectories, and the analogues of the moduli spaces of stable maps are compactifications of $\mathcal{M}_{x_-,x_+}$ discussed above. The integrals (2.21) that we have considered so far are the analogues of the integrals (2.25).

The definition of the analogues of the more general integrals (2.26) is also straightforward. Let $\mathcal{M}_{x_-,x_+,n}$ be the moduli space of data $(p_1, \dots, p_n, x)$, where $p_1, \dots, p_n$ are distinct points of the real line, considered as an affine line, i.e. without a fixed coordinate, and $x \colon \mathbb{R} \to X$ is a gradient trajectory. If we choose a coordinate t on $\mathbb{R}$, then we can replace $(p_1, \dots, p_n)$ by the real numbers $(t_1, \dots, t_n)$ and x by a parametrized map $x(t)$. Other coordinates are obtained by a shift $t \mapsto t + u$. Therefore, we may equivalently consider the data $(t_1, \dots, t_n, x(t))$ modulo the diagonal action of the group $\mathbb{R}$ of translations:

$$(t_1, \dots, t_n, x(t)) \mapsto (t_1 + u, \dots, t_n + u, x(t + u)).$$

Note that the group of translations plays here the same role that the group PGL_2 of Möbius transformation of $\Sigma = \mathbb{CP}^1$ plays in the Gromov–Witten theory. We have a natural map $\pi_n \colon \mathcal{M}_{x_-,x_+,n} \to \mathrm{Conf}_n$, where

$$\mathrm{Conf}_n = (\mathbb{R}^n \backslash \Delta)/\mathbb{R}_{\mathrm{diag}} \simeq \mathbb{R}^{n-1}_{>0}$$

is the configuration space of n points on the real line (here Δ is the union of all diagonals). It plays the role of $\mathcal{M}_{g,n}$. There is also a natural relative compactification $\bar{\mathcal{M}}_{x_-,x_+,n}$ of $\mathcal{M}_{x_-,x_+,n}$ defined similarly to the moduli spaces of stable maps (see [**9**]).

We have the evaluation maps $\mathrm{ev}_i \colon \bar{\mathcal{M}}_{x_-,x_+,n} \to X$ corresponding to evaluating the map $x \colon \mathbb{R} \to X$ at the point p_{i}. Now it is clear that the analogues of the general Gromov–Witten invariants (2.27) in Morse theory are obtained as the pushforwards

$$\pi_{n*}(\mathrm{ev}_1^*(\omega_1) \wedge \dots \wedge \mathrm{ev}_n^*(\omega_n)). \tag{2.28}$$

These 'Morse theory invariants' are nothing but differential forms on the configuration space Conf_n. The simplest examples are the 0-form components of these differential forms whose values at fixed points $p_1, \dots, p_n$ are just the integrals (2.21) introduced above. These more general correlation functions may be interpreted as the correlation functions of the $\hat{\omega}_i$ and their 'cohomological descendants', which are constructed following [**42**].

Note that the observable $\hat{\omega}_i$ is a 0-form on the 'worldline' $\mathbb{R}$, i.e. a function. The cohomological descendant of $\hat{\omega}_i$ is the 1-form $\hat{\omega}_i^{(1)}$ on $\mathbb{R}$ defined by the formula

$$\hat{\omega}_i^{(1)} = \widehat{\iota_v \omega_i}\, \mathrm{d}t,$$

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where v is the gradient vector field ∇f. In particular, suppose that ω_i is a closed differential form on X. Since $\hat{\omega}_i$ is obtained from ω_i by pulling back with respect to a gradient trajectory, we have $\mathrm{d}\hat{\omega}_i/\mathrm{d}t = \widehat{\mathcal{L}_v \omega_i}$, where $\mathcal{L}_v$ is the Lie derivative with respect to the gradient vector field v. Since the action of Q corresponds to the action of the de Rham differential d_X along X, we obtain, using the Cartan formula $\mathcal{L}_{\nabla f} = \{\mathrm{d}_X, \imath_{\nabla f}\}$, that

$$Q \cdot \hat{\omega}_i^{(1)}\,\mathrm{d}t = \mathrm{d}_t \hat{\omega}_i,$$

where d_t is the de Rham differential along the worldline. This is analogous to the formula for the cohomological descent in the Gromov–Witten theory [**42**].

The general 'Morse theory invariants' (2.28) may be interpreted as correlation functions of observables of this type alongside the $\hat{\omega}_i$ considered before.

More precisely, when we take the pullback of a differential p-form ω_i on X via $\mathrm{ev}_i\colon \bar{\mathcal{M}}_{x_-,x_+,n} \to X$, we obtain a p-form on $\bar{\mathcal{M}}_{x_-,x_+,n}$, which decomposes locally into the sum of two differential forms. One is a p-form along the fibre of the projection π_n and 0-form along the base: it corresponds to $\hat{\omega}_i$, and the other is a $(p-1)$-form along the fibre and a 1-form along the base: this is $\hat{\omega}_i^{(1)}$. Thus, the correlation function of m 'zero-observables' $\hat{\omega}_i$ and k 'one-observables' $\hat{\omega}_j^{(1)}$ will pick up precisely the k-form component of the general 'Morse theory invariant' on Conf_{m+k} defined by formula (2.28).

From the physical perspective, defining these more general correlation functions corresponds to 'coupling our quantum mechanical model to gravity'. In the path integral formalism it is described as follows. We write our action in the form

$$S = \int (-\mathrm{i}p'\dot{q} + \mathrm{i}\pi\dot{\psi} + H\,\mathrm{d}t),$$

where H is the classical hamiltonian whose quantization gives $\mathcal{L}_v$, and Q^* is its superpartner whose quantization gives $\imath_v$. In order to enforce the invariance under the time reparametrizations and at the same time preserve the Q-symmetry we add the einbein field e and its superpartner $\chi = Q \cdot e$ and consider the action

$$S_{\mathrm{topgrav}} = \int (-\mathrm{i}p'\dot{q} + \mathrm{i}\pi\dot{\psi} + (eH - \chi Q^*)\,\mathrm{d}t). \tag{2.29}$$

The path integrals corresponding to this action may be expressed in terms of the 'Morse theory invariants' (2.28).

Note also that more generally we may consider arbitrary graphs instead of the real line. We then need to assign to each edge of the graph a Morse function and impose the condition that the sum of the functions corresponding to the edges coming out of each vertex is zero. The corresponding integrals are related to the integrals considered by Fukaya [**19**]. They may also be interpreted as the terms of the perturbative expansion of a particular quantum field theory on X. For instance, if we only allow three-valent graphs, this will be the perturbative expansion of the (generalized) Chern–Simons theory on X around a particular background, and can be viewed as a topological open string on T^*X, as in [**36, 43**]. More precisely, if we allow N different Morse functions $f_1, \dots, f_N$, then this will be the Chern–Simons theory with the Lagrangian

$$L = \mathrm{Tr}(A \wedge \mathrm{d}A + \tfrac{2}{3} A \wedge A \wedge A),$$

where A is an $N \times N$-valued differential form on X, and we make the perturbation theory expansion around the 1-form $\mathrm{diag}(\mathrm{d}f_1, \ldots, \mathrm{d}f_N)$. This is however beyond the scope of the present paper.

For more on the topological supersymmetric quantum mechanics, see [**28**, **30**].

2.7. More general observables

As we discussed above, the observables in the BPS sector correspond to closed differential forms on X. However, *any* differential form ω on X gives rise to a legitimate observable in our theory, and the correlation functions of such observables are still given by the same integrals over the moduli spaces $\mathcal{M}_{x_-,x_+}$ (or $\mathcal{M}_{x_-,x_+,n}$) as in the case of closed forms. The difference is, of course, that these correlation functions are no longer independent of λ, so this answer is correct only at $\lambda = \infty$.

Why should we bother considering non-BPS evaluation observables? We have already partially answered this question in the Introduction. In particular, if we only consider the BPS observables, we cannot gain any insights into the structure of the space of states of our theory beyond the ground states. Indeed, their correlation functions only depend on their Q-cohomology classes. One can modify any BPS observable by Q-exact terms so as to make it commute with Q and Q^*. Such a representative transforms a ground state into a ground state. But non-BPS observables transform ground states into excited states, and, as we will see below, we can understand the structure of the space of states by considering their correlation functions.

Besides, considering non-BPS observables allows us to bring into play some important Q-exact observables, which are 'invisible' in the BPS sector.

A general local observable of the quantum mechanical model that we are considering corresponds to the quantization of an arbitrary function $\mathcal{O}(x, p, \psi, \pi)$. Upon quantization they become *differential operators* on $\Omega^\bullet(X)$:

$$\mathcal{O}(x, p, \psi, \pi) \mapsto \hat{\mathcal{O}} = \mathcal{O}\left(x, -\mathrm{i}\frac{\partial}{\partial x}, \psi, -\mathrm{i}\frac{\partial}{\partial \psi}\right).$$

Examples are the differential forms themselves, which we have already considered above, and the Lie derivatives $\mathcal{L}_v$ with respect to vector fields on X. If we write $v = v^\mu \partial/\partial x^\mu$, then

$$\mathcal{L}_v = i\left(v^\mu p'_\mu + \frac{\partial v^\mu}{\partial x^\nu}\psi^\nu \pi_\mu\right).$$

These observables have a transparent path integral interpretation. Namely, inserting this observable at the time t_0 corresponds to infinitesimally deforming the gradient trajectory at the time t_0 along the vector field v. In particular, our hamiltonian is included among these observables.

The observables $\mathcal{L}_v$ are Q-closed, but they are also Q-exact, as follows from the Cartan formula $\mathcal{L}_v = \{\mathrm{d}, \imath_v\}$ that we have already encountered above.

This means that if we insert the operator $\mathcal{L}_v$ into a correlation function of BPS observables, then we will obtain zero. But the observables $\mathcal{L}_v$, and other differential operators, play a very important role in the full theory. Indeed, on a Kähler manifold we often have a large Lie algebra of global holomorphic vector fields, and the corresponding Lie derivatives will be chiral operators of our theory. The algebra of holomorphic differential operators that they generate is the precursor of the chiral de Rham complex of [**31**]. If we wish to understand the role played by the chiral de Rham complex in the two-dimensional sigma models, it is natural to consider its quantum mechanical analogue: the algebra of holomorphic differential operators on X (more precisely, its supersymmetric analogue: the algebra of holomorphic differential operators on ΠTX). But in order to obtain non-trivial correlation functions involving these operators we must consider non-BPS observables.

As we already indicated above, at $\lambda = \infty$ the correlation functions of non-BPS evaluation observables are easy to obtain from the path integral point of view. Our goal now is to give an interpretation of these correlation functions from the Hamiltonian point of view. In other words, we wish to describe explicitly the space of states of the theory at $\lambda = \infty$, represent the general observables as operators acting on this space and represent their correlation functions as matrix elements of these operators. We will take up this task in the next section.

3. Hamiltonian formalism

In the previous section we discussed the Lagrangian (or path integral) formulation of the supersymmetric quantum mechanical model on a Kähler manifold X governed by the first-order action (2.13) in the limit $\lambda = \infty$. This formulation is convenient because it gives a simple answer for the correlation functions of the observables corresponding to differential forms on X: they are given by integrals over finite-dimensional moduli spaces of gradient trajectories.

Now we would like to develop the Hamiltonian formalism for this model. This means that we need to define the space of states of the model and realize our observables as linear operators acting on this space of states. The correlation functions are then given by matrix elements of these operators. Note that the Lagrangian description provides us with an important testing device: these matrix elements should reproduce the finite-dimensional integrals described above.

We will see in this section that the Hamiltonian structure of our model is rather unusual: the quantum Hamiltonian is non-hermitian and even non-diagonalizable, and the space of states decomposes into the spaces of ‘in’ and ‘out’ states. However, these spaces have a simple and geometrically meaningful description in terms of the stratifications by the ascending and descending manifolds of our Morse function. Moreover, the spaces of states exhibit holomorphic factorization that is absent for finite values of λ. This leads to a great simplification of the correlation functions. In § 5 we will establish the equivalence between the results obtained in the Hamiltonian and the Lagrangian formalisms, discovering along the way some interesting identities on integrals of analytic differential forms.

3.1. Supersymmetric quantum mechanics at $\lambda = \infty$

Our task is to describe the Hamiltonian formalism of the theory with the first-order action (2.13). Following the most obvious route, we start with the classical hamiltonian found from this action:

$$H_{\mathrm{class}} = \mathrm{i}\left(p'_\mu g^{\mu\nu}\frac{\partial f}{\partial x^\nu} + \frac{\partial}{\partial x^\alpha}\left(g^{\mu\nu}\frac{\partial f}{\partial x^\nu}\right)\pi_\mu \psi^\alpha\right).$$

Its naive quantization is the operator

$$H_{\mathrm{naive}} = \mathcal{L}_v,$$

which is the Lie derivative with respect to the gradient vector field $v = \nabla f$.*

The corresponding supersymmetry charges are $Q = \mathrm{d}$ and $Q^* = 2\imath_v$. They satisfy the relation

$$H_{\mathrm{naive}} = \tfrac{1}{2}\{Q, Q^*\}$$

according to Cartan's formula (2.24).

The standard Hodge theory, which is at work in the quantum mechanical model at finite values of λ (see § 2.1), deals with the operators like d and d*, which depend explicitly on the metric on the manifold X. Here we choose instead the operators d and $2\imath_v$, for some vector field v, which are metric independent. The problem with this definition is that the Hamiltonian $\mathcal{L}_v$ is a first-order differential operator, which naively has an unbounded spectrum. Also, if we take the time evolution operator

$$\mathrm{e}^{-tH_{\mathrm{naive}}}$$

for large t, it will tend to make the wave functions concentrated near the critical points of the Morse function (where $v = 0$), thus posing some problems with completeness. Besides, the choice of zero eigenstates of H_{naive} seems quite ambiguous, for any differential form (or current) supported on a v-invariant submanifold in X naively leads to such a state.

Finally, in the standard supersymmetric quantum mechanics the operators $\mathcal{Q}$ and $\mathcal{Q}^*$ are adjoint to each other, and as the result, the Hamiltonian is self-adjoint. But this property no longer holds in our case, and so we cannot expect that $\mathcal{L}_v$ is self-adjoint.

3.2. Way out: λ regularization

In order to make sense of all this, we recall how we got the action (2.13) in the first place: we started with the second-order action (2.6) for finite values of λ, then we added the topological term

$$-\int_I \lambda\,\mathrm{d}f = \lambda(f(x_{\mathrm{i}}) - f(x_{\mathrm{f}})), \tag{3.1}$$

passed to the first-order action (2.11), and finally took the limit $\lambda \to \infty$. This suggests that in order to develop the correct Hamiltonian formalism of the theory we should retrace these steps from the Hamiltonian point of view.

* The importance of considering such first-order Hamiltonians has been emphasized by Gerard 't Hooft.

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We recall from § 2.1 that the hamiltonian of the theory with the action (2.6) is given by formula

$$H = \tfrac{1}{2}\{\mathcal{Q}, \mathcal{Q}^*\},$$

where

$$\mathcal{Q} = \mathrm{d} + \lambda\,\mathrm{d}f\wedge, \qquad \mathcal{Q}^* = \frac{1}{\lambda}\mathrm{d}^* + \iota_{\nabla f}.$$

This hamiltonian is Witten's Laplacian given by formula (2.4). The corresponding space of states is just the space of L_2 differential forms on X. This is a Hilbert (super)space with respect to the hermitian inner product (2.1).

Here it is important to note that because we rescale by λ both the Morse function and the *metric* on X (which leads to the overall factor λ^{-1} in the formula for $\mathcal{Q}^*$), we obtain Witten's Laplacian multiplied by λ^{-1}. This difference in normalization is important: in the normalization of [**38**] the smallest non-zero eigenvalue of the Hamiltonian is of the order of λ, and hence only the ground states survive in the limit $\lambda \to \infty$ (recall that we are under the assumption that X is Kähler, and so all ground states at $\lambda = \infty$ deform to true ground states at finite λ). In our normalization not only the ground states, but also the states with the eigenvalues proportional to λ (in the normalization of [**38**]), survive in this limit. These will be the excited states of our theory at $\lambda = \infty$, as we will see below.

The next step is to add the term (3.1) to the action. What does this correspond to from the Hamiltonian point of view? To see that, we recall the correspondence between the correlation functions in the Lagrangian and Hamiltonian formalisms expressed in formula (2.8). When we add the topological term (3.1) to the action, the right-hand side of this formula is multiplied by $\mathrm{e}^{\lambda(f(x_\mathrm{f})-f(x_\mathrm{i}))}$. This means that the correlation function in the new theory (with the term (3.1)) between the 'in' state $\mathrm{e}^{\lambda f(x_\mathrm{i})}\Psi(x_\mathrm{i})$ and the 'out' state $\mathrm{e}^{-\lambda f(x_\mathrm{f})}\Psi^*(x_\mathrm{f})$ is the same as the correlation function of the old theory (without the term (3.1)) between the states $\Psi(x_\mathrm{i})$ and $\Psi^*(x_\mathrm{f})$.

Thus, we obtain that from the Hamiltonian point of view the effect of the addition of the term (3.1) to the action is as follows.

- The 'in' states get multiplied by $\mathrm{e}^{\lambda f(x)}$:

$$\Psi \mapsto \tilde{\Psi} = \mathrm{e}^{\lambda f}\Psi. \tag{3.2}$$

- The 'out' states get multiplied by $\mathrm{e}^{-\lambda f(x)}$:

$$\Psi^* \mapsto \tilde{\Psi}^* = \mathrm{e}^{-\lambda f}\Psi^*. \tag{3.3}$$

- The operators get conjugated:

$$\mathcal{O} \mapsto \tilde{\mathcal{O}} = \mathrm{e}^{\lambda f}\mathcal{O}\mathrm{e}^{-\lambda f}.$$

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The last rule applies, in particular, to the operators $\mathcal{Q}$, $\mathcal{Q}^*$ and H. We find that the new operators are*

$$Q_\lambda = \tilde{\mathcal{Q}} = \mathrm{e}^{\lambda f}\mathcal{Q}\mathrm{e}^{-\lambda f} = \mathrm{d}, \tag{3.4}$$

$$Q^*_\lambda = \tilde{\mathcal{Q}}^* = \mathrm{e}^{\lambda f}\mathcal{Q}^*\mathrm{e}^{-\lambda f} = 2\iota_v + \frac{1}{\lambda}\mathrm{d}^*, \tag{3.5}$$

$$\tilde{H}_\lambda = \mathrm{e}^{\lambda f}H\mathrm{e}^{-\lambda f} = \tfrac{1}{2}\{Q_\lambda, Q^*_\lambda\} = \mathcal{L}_v - \frac{1}{2\lambda}\Delta. \tag{3.6}$$

Finally, we may take the limit $\lambda \to \infty$ and we indeed recover the operators Q, Q^* and H_{naive} that we discussed above.

However, now we obtain a more clear picture of what is happening with the space of states as we perform the above procedure. Namely, to obtain the 'in' space of states we need to choose a suitably normalized basis of eigenfunctions of the Hamiltonian H, then multiply all of them by $\mathrm{e}^{\lambda f}$ and pass to the limit $\lambda \to \infty$. Likewise, to obtain the 'out' space of states we need to multiply those basis eigenfunctions by $\mathrm{e}^{-\lambda f}$ and then pass to the limit. The main question is of course what we mean by 'taking the limit', in particular, in what ambient space the limiting elements 'live'. As we will argue in § 4, the proper ambient space is not the space of functions on X (or, more generally, differential forms), but the space of *generalized functions*, i.e. a suitable space of linear functionals on the space of functions (or, more generally, *currents*, i.e. functionals on the space of differential forms). We will first explain how this works in the flat space, namely, for $X = \mathbb{C}$, and then discuss in detail the first example of a 'curved' space, namely, $X = \mathbb{CP}^1$.

Before proceeding with these examples, we wish to comment on the reason why the spaces of 'in' and 'out' states turn out to be different in the limit $\lambda \to \infty$. This is easiest to explain from the Lagrangian point of view. The 'in' and 'out' states in a general quantum mechanical model may be constructed by acting by local observables on the vacuum and covacuum states, respectively. Namely, let $\mathcal{O}_i(t_i)$, $i = 1, \dots, n$, be observables with $t_1 < t_2 < \cdots < t_n$ and $(\mathcal{O}_n(t_n)\cdots\mathcal{O}_1(t_1)|0\rangle)(x)$ the corresponding state, considered as a differential form on X. Then we have the following symbolic representation of this state using the path integral:

$$(\mathcal{O}_n(t_n)\cdots\mathcal{O}_1(t_1)|0\rangle)(x) = \int_{x(t)\colon(-\infty,0]\to X;\, x(0)=x} \mathcal{O}_1(t_1)\cdots\mathcal{O}_n(t_n)\mathrm{e}^{-S}. \tag{3.7}$$

* Note the difference between this conjugation and the procedure by which we had defined the operators $\mathcal{Q}$ and $\mathcal{Q}^*$ in formulae (2.2), (2.3): there we defined $\mathcal{Q}$ by conjugating d by $\mathrm{e}^{-\lambda f}$, and then defined $\mathcal{Q}^*$ as the *adjoint* of $\mathcal{Q}$, so that $\mathcal{Q}^*$ was obtained by conjugating d^* by $\mathrm{e}^{\lambda f}$ (and dividing by λ). In particular, their anti-commutator $\{\mathcal{Q}, \mathcal{Q}^*\}$ has a very different spectrum from $\{\mathrm{d}, \mathrm{d}^*\}$. Now we are conjugating both $\mathcal{Q}$ and $\mathcal{Q}^*$ by $\mathrm{e}^{\lambda f}$. The resulting operators are $Q = \mathrm{d}$ and

$$Q^* = \frac{1}{\lambda}\mathrm{e}^{2\lambda f}\,\mathrm{d}^*\mathrm{e}^{-2\lambda f}.$$

They are not adjoint to each other any more. But the corresponding anti-commutator $\{Q, Q^*\}$ has the same spectrum as $\{\mathcal{Q}, \mathcal{Q}^*\}$ for any finite value of λ.

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The boundary conditions at $t = -\infty$ in the above integral are determined by the choice of the ground state $|0\rangle$. Likewise, an 'out' state of our model is constructed by the formula

$$(\langle 0|\mathcal{O}'_1(s_1)\cdots\mathcal{O}'_m(s_m))(x) = \int_{x(t):\,[0,+\infty)\to X;\,x(0)=x} \mathcal{O}'_1(s_1)\cdots\mathcal{O}'_m(s_m)\mathrm{e}^{-S}, \qquad (3.8)$$

where the boundary conditions at $t = +\infty$ are determined by the choice of the covacuum state $\langle 0|$.

This construction allows us to define a natural pairing between the two spaces: if we denote the state (3.7) by Ψ and the state (3.8) by Ψ^*, then by definition

$$\langle \Psi^*, \Psi\rangle = \int_{x(t):\,(-\infty,+\infty)\to X} \mathcal{O}_1(t_1)\cdots\mathcal{O}_n(t_n)\mathcal{O}'_1(s_1)\cdots\mathcal{O}'_m(s_m)\mathrm{e}^{-S}, \qquad (3.9)$$

with appropriate boundary conditions understood.

In general, the spaces of 'in' and 'out' states are different. However, suppose that the action of our theory is CPT invariant, that is invariant under the time reversal $t \mapsto -t$ and complex conjugation (this corresponds to the Hamiltonian being a self-adjoint operator). In this case we have a natural anti-linear map from the space of 'in' states to the space of 'out' states: namely, by applying the time reversal and complex conjugation we transform the integral (3.7) into an integral of the form (3.8), hence we transform an 'in' state into an 'out' state. This allows us to identify the two spaces in the case when the action is invariant under the CPT symmetry. Combining this identification and the pairing (3.9), we obtain a hermitian inner product on the resulting (single) space of states.

However, our action is not invariant under the above CPT symmetry. The original action (2.6) is CPT-invariant, but the topological term (3.1) that we have added to it breaks this invariance.* Therefore, there is no natural identification between the spaces of 'in' and 'out' states. For finite values of λ, the CPT invariance is only mildly violated: when we apply the CPT transformation, we shift the action by $2\lambda \int \mathrm{d}f$. This means that there is still a map from the space of 'in' states to the space of 'out' states, but it involves multiplication by $\mathrm{e}^{-2\lambda f}$. More precisely, this map sends

$$\Psi \mapsto \mathrm{e}^{-2\lambda f} \star \bar{\Psi}.$$

However, the operation of multiplication by $\mathrm{e}^{-2\lambda f}$ has no obvious limit $\lambda \to \infty$, and so at $\lambda = \infty$ the two spaces become *non-isomorphic*. This is reflected in the fact that the action (2.13) is not CPT-invariant and the hamiltonian H_{naive} is not self-adjoint. Thus, we arrive at the following conclusion.

In the limit $\lambda \to \infty$ our model has two spaces of states: the space of 'in' states $\mathcal{H}^{\mathrm{in}}$, and the space of 'out' states $\mathcal{H}^{\mathrm{out}}$. The transition amplitudes define a pairing:

$$\mathcal{H}^{\mathrm{out}} \otimes \mathcal{H}^{\mathrm{in}} \to \mathbb{C},$$

but the two spaces are not canonically isomorphic.

* Note that the invariance would have been preserved if λ were purely imaginary, but we need λ to be real!

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3.3. The case of flat space $\mathbb{C}$

Now we analyse the simplest example where we can follow the fate of the states in the Hilbert space while taking the $\lambda \to \infty$ limit. This is the case $X = \mathbb{C}$,

$$f = \tfrac{1}{2}\omega|z|^2, \qquad g = \mathrm{d}z\,\mathrm{d}\bar{z},$$

where ω is a non-zero real number. Thus, $g_{z\bar{z}} = \frac{1}{2}$, $g^{z\bar{z}} = 2$. The corresponding gradient vector field is

$$v = \nabla f = \omega(z\partial_z + \bar{z}\partial_{\bar{z}}). \tag{3.10}$$

The potential is

$$|\mathrm{d}f|^2 = \omega^2|z|^2,$$

and the Hamiltonian has the form

$$H_\lambda = -\frac{2}{\lambda}\partial_z\partial_{\bar{z}} + \frac{\lambda}{2}\omega^2|z|^2 + K_\omega,$$

where

$$K_\omega = \omega(F + \bar{F} - 1),$$

where F and $\bar{F}$ are the fermionic left and right charge operators. Thus, K_ω is equal to $-\omega$ on 0-forms, 0 on 1-forms, and ω on 2-forms. Hence our model is nothing but the two-dimensional supersymmetric harmonic oscillator.

We have the following orthonormal basis of eigenfunctions of H_λ: the 0-forms

$$\Psi_{n,\bar{n}} = \frac{1}{\sqrt{\pi(\lambda\omega)^{(n+\bar{n}-1)}n!\bar{n}!}}\mathrm{e}^{\lambda|\omega|z\bar{z}/2}\partial_z^{\bar{n}}\partial_{\bar{z}}^{n}(\mathrm{e}^{-\lambda|\omega|z\bar{z}}), \quad n,\bar{n} \geqslant 0, \tag{3.11}$$

and the 1-forms and 2-forms obtained by multiplying $\Psi_{n,\bar{n}}$ with $\mathrm{d}z$ and $\mathrm{d}\bar{z}$. The corresponding eigenvalues are

$$E_{n,\bar{n}} = |\omega|(n + \bar{n} + 1) + K_\omega. \tag{3.12}$$

Note that these eigenfunctions have an additional property that they are also eigenfunctions of the operators of $U(1)$ rotation $z \mapsto z\mathrm{e}^{\mathrm{i}\varphi}$.

Now we describe the space of eigenfunctions of the conjugated operator

$$\tilde{H}_\lambda = \mathrm{e}^{\lambda f}H_\lambda\mathrm{e}^{-\lambda f} = \omega(z\partial_z + \bar{z}\partial_{\bar{z}}) - \frac{2}{\lambda}\partial_z\partial_{\bar{z}} + (K_\omega + \omega) \tag{3.13}$$

and its adjoint. They will be basis elements of the 'in' and 'out' spaces of states. The eigenfunctions of $\tilde{H}_\lambda$ are obtained by multiplying the functions $\Psi_{n,\bar{n}}$ (and the corresponding differential forms) with $\mathrm{e}^{\lambda f} = \mathrm{e}^{\lambda\omega z\bar{z}/2}$, and the eigenfunctions of its adjoint are obtained by multiplying with $\mathrm{e}^{-\lambda f} = \mathrm{e}^{-\lambda\omega z\bar{z}/2}$.

At this point the sign of ω becomes crucial. Let us assume first that $\omega > 0$. This means that the point 0 is a 'repulsive' critical point: the gradient trajectories flow away from 0. In this case a basis of the 'in' space is given by the functions

$$\tilde{\Psi}^{\mathrm{in}}_{n,\bar{n}} = \frac{1}{(\lambda\omega)^{n+\bar{n}}}\mathrm{e}^{\lambda\omega z\bar{z}}\partial_z^{\bar{n}}\partial_{\bar{z}}^{n}\mathrm{e}^{-\lambda\omega z\bar{z}}, \quad n,\bar{n} \geqslant 0, \tag{3.14}$$

and the differential forms obtained from them by multiplying with $\mathrm{d}z$ and $\mathrm{d}\bar{z}$. We recall that the 'out' state corresponding to an 'in' state Ψ is $\mathrm{e}^{-2\lambda f} \star \bar{\Psi}$. Therefore, a basis of the 'out' space is given by the 2-forms

$$\tilde{\Psi}^{\text{out}}_{n,\bar{n}} = \frac{\lambda\omega}{2\pi} \frac{1}{n!\bar{n}!} \partial_z^n \partial_{\bar{z}}^{\bar{n}} \mathrm{e}^{-\lambda\omega z\bar{z}} \, \mathrm{d}z \, \mathrm{d}\bar{z}, \quad n, \bar{n} \geqslant 0, \tag{3.15}$$

and the differential forms obtained from them by contracting with the vector fields ∂_z and $\partial_{\bar{z}}$. The eigenvalues of $\tilde{H}_\lambda$ on these functions are given by the same formula (3.12) (so they are independent of λ).

Here and below we use the notation

$$\mathrm{d}z \, \mathrm{d}\bar{z} = \mathrm{d}^2 z = \mathrm{i} \, \mathrm{d}z \wedge \mathrm{d}\bar{z} = 2 \, \mathrm{d}x \wedge \mathrm{d}y, \quad z = x + \mathrm{i}y. \tag{3.16}$$

The normalization in formulae (3.15) and (3.15) is chosen in such a way that these expressions have well-defined limits as $\lambda \to \infty$ (see below) and the pairing between $\tilde{\Psi}^{\text{in}}_{n,\bar{n}}$ and $\tilde{\Psi}^{\text{out}}_{m,\bar{m}}$ is equal to $\delta_{n,m}\delta_{\bar{n},\bar{m}}$. To obtain the 'in' states satisfying this property, we multiply the states $\Psi_{n,\bar{n}}$ by the function

$$A_{n,\bar{n}} = \sqrt{\pi n! \bar{n}!} (\lambda\omega)^{-(n+\bar{n}+1)/2} \mathrm{e}^{\lambda|\omega| z\bar{z}/2},$$

and to obtain the 'out' states we multiply $\Psi_{n,\bar{n}}$ by $(A_{n,\bar{n}})^{-1}$. This suggests that the transformation from the states of the theory at finite λ to the 'in' states of the new theory, normalized as above, is achieved not merely by multiplying the states by $\mathrm{e}^{\lambda f}$, but by applying the operator

$$\Psi \mapsto \tilde{\Psi}^{\text{in}} = \lambda^{-1/2} \mathrm{e}^{\lambda f} (\lambda^{-H/2|\omega|} \cdot \Psi).$$

Likewise, for 'out' states we have

$$\Psi \mapsto \tilde{\Psi}^{\text{out}} = \lambda^{1/2} \mathrm{e}^{-\lambda f} (\lambda^{H/2|\omega|} \cdot \Psi).$$

We now come to the key point of our analysis: finding the limits of the states (3.15) and (3.15) as $\lambda \to \infty$. First, we find from formula (3.15) that in this limit we have

$$\tilde{\Psi}^{\text{in}}_{n,\bar{n}} \to (-1)^{n+\bar{n}} z^n \bar{z}^{\bar{n}},$$

so the wave functions become monomials! On the other have, we find that

$$\tilde{\Psi}^{\text{out}}_{n,\bar{n}} \to \frac{1}{n!\bar{n}!} \partial_z^n \partial_{\bar{z}}^{\bar{n}} \delta^{(2)}(z, \bar{z}) \, \mathrm{d}z \, \mathrm{d}\bar{z}.$$

Thus, the 'out' states become the derivatives of the delta-form supported at $0 \in \mathbb{C}$!

We conclude that the space of 'in' states of our theory at $\lambda = \infty$ is the space of polynomial differential forms on $\mathbb{C}$:

$$\mathcal{H}^{\text{in}} = \mathbb{C}[z, \bar{z}] \otimes \Lambda[\mathrm{d}z, \mathrm{d}\bar{z}], \tag{3.17}$$

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on which the Hamiltonian $H = \tilde{H}_\infty$ simply acts by dilatations:

$$\mathrm{e}^{-tH}\Psi(z,\bar{z},\mathrm{d}z,\mathrm{d}\bar{z}) = \Psi(qz,q\bar{z},q\,\mathrm{d}z,q\,\mathrm{d}\bar{z}),$$

where $q = \mathrm{e}^{-\omega t}$.

There is a unique ground state, $\tilde{\Psi}^{\mathrm{in}}_{\mathrm{vac}} = 1$, and the spectrum of excited states is degenerate, consisting of all positive integers.

The space of 'out' states is the space of 'delta-forms' supported at $z = 0$:

$$\mathcal{H}^{\mathrm{out}} = \Lambda[\mathrm{d}z,\mathrm{d}\bar{z}] \otimes \mathbb{C}[\partial_z,\partial_{\bar{z}}] \cdot \delta^{(2)}(z,\bar{z}), \tag{3.18}$$

on which the evolution operator acts as

$$\mathrm{e}^{-tH}\Upsilon(\mathrm{d}z,\mathrm{d}\bar{z},\partial_z,\partial_{\bar{z}})\delta^{(2)}(z,\bar{z}) = q^2\Upsilon(q^{-1}\,\mathrm{d}z,q^{-1}\,\mathrm{d}\bar{z},q\partial_z,q\partial_{\bar{z}})\delta^{(2)}(z,\bar{z}).$$

We see that H acts on $\mathcal{H}^{\mathrm{in}}$ as $\mathcal{L}_v$, where v is the gradient vector field given by formula (3.10), and on $\mathcal{H}^{\mathrm{out}}$ as $-\mathcal{L}_v$, yet the spectra of these two seemingly opposite operators are identical. This is how the 'self-adjoint' nature of the Hamiltonian is realized in the $\lambda \to \infty$ limit.

There is a natural pairing between the 'in' and 'out' spaces defined by the formula

$$\langle \tilde{\Psi}^{\mathrm{out}}, \tilde{\Psi}^{\mathrm{in}} \rangle = \int \tilde{\Psi}^{\mathrm{out}} \wedge \tilde{\Psi}^{\mathrm{in}}.$$

This pairing is well defined because $\tilde{\Psi}^{\mathrm{out}}$ is a distribution (more precisely, a current) supported at $0 \in \mathbb{C}$ and $\tilde{\Psi}^{\mathrm{in}}$ is a differential form that is smooth in the neighbourhood of 0. This completes the analysis of the spaces of states in the case when $\omega > 0$.

Now consider the case when $\omega < 0$, which corresponds to an 'attractive' critical point $0 \in \mathbb{C}$. Then the roles of $\mathcal{H}^{\mathrm{in}}$ and $\mathcal{H}^{\mathrm{out}}$ are reversed. Thus, $\mathcal{H}^{\mathrm{in}}$ is the space of delta-forms with support at $0 \in \mathbb{C}$ and $\mathcal{H}^{\mathrm{out}}$ is the space of polynomials functions on $\mathbb{C}$.

3.4. The kernel of the evolution operator in the limit $\lambda \to \infty$

It is instructive to analyse how the kernel of the evolution operator at finite λ becomes the delta-form supported on the gradient trajectories in the limit $\lambda \to \infty$.

Suppose we study supersymmetric quantum mechanics on an n-dimensional manifold X, and the space of states is the space of L_2 differential forms on X. Then the kernel K_t of the evolution operator is an n-form on $X \times X$ defined by the formula

$$\langle \Psi^* | \mathrm{e}^{-tH} | \Psi \rangle = \int_{X\times X} K_t(x,y) \wedge \Psi(x) \wedge \star\bar{\Psi}^*(y).$$

It is normalized so that K_0 is the delta-form (of degree n) supported on the diagonal in $X \times X$.

Suppose that we have a complete basis $\{\Psi_\gamma\}$ of normalized eigenfunctions of the Hamiltonian H, with the eigenvalues $\{E_\gamma\}$. Then we have the following formula for K_t:

$$K_t = \sum_\gamma p_{\mathrm{i}}^*(\Psi_\gamma(x)) p_{\mathrm{f}}^*(\star\overline{\Psi_\gamma^*}(y)) q^{E_\gamma}, \tag{3.19}$$

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where $q = \mathrm{e}^{-t}$ and p_{i}, p_{f} are the projections $X \times X \to X$ on the first and the second factors, respectively.

Let us compute the kernel of the evolution operator in the theory on $X = \mathbb{C}$ at finite λ, before conjugation by $\mathrm{e}^{\lambda f}$. We have the complete basis

$$\Psi_{n,\bar{n},p,\bar{p}} = \Psi_{n,\bar{n}}(\mathrm{d}z)^p(\mathrm{d}\bar{z})^{\bar{p}}, \quad n, \bar{n} \geqslant 0,\ p, \bar{p} = 0, 1.$$

where $\Psi_{n,\bar{n}}$ is given by formula (3.11). Thus, the degree of this state considered as a differential form on $\mathbb{C}$ is $p + \bar{p}$. These states are eigenstates of the hamiltonian H_λ with the eigenvalues $|\omega|(n + \bar{n} + p + \bar{p})$. In addition, they are eigenstates of the rotation operator P, which is the Lie derivative with respect to the vector field $|\omega|(z\partial_z - \bar{z}\partial_{\bar{z}})$. The corresponding eigenvalues are $|\omega|(n - \bar{n} + p - \bar{p})$.

Instead of considering the kernel K_t of the evolution operator e^{-tH} we will consider the kernel $K_{t,\bar{t}}$ of the modified evolution operator $\mathrm{e}^{-t(H+P)/2-\bar{t}(H-P)/2}$ (it reduces to K_t if $\bar{t} = t$). Denote $q = \mathrm{e}^{-|\omega|t}$, $\bar{q} = \mathrm{e}^{-|\omega|\bar{t}}$. Let us consider the $(p, \bar{p}) = (1, 1)$-form component of K_t, which we will denote by $K^{(1,1)}$ (for other components formulae are similar). We have an analogue of formula (3.19), from which we find that

$$K^{(1,1)}_{t,\bar{t}} = \sum_{n,\bar{n}} \Psi_{n,\bar{n},1,1}(z, \bar{z})\overline{\Psi_{n,\bar{n},1,1}(w, \bar{w})}q^n\bar{q}^{\bar{n}}\, \mathrm{d}(qz - w) \wedge \mathrm{d}(\bar{q}\bar{z} - \bar{w}). \tag{3.20}$$

Denote this expression by $U_{t,\bar{t}}\, \mathrm{d}(qz - w) \wedge \mathrm{d}(\bar{q}\bar{z} - \bar{w})$. Using formula (3.11), we find that

$$\begin{aligned} U_{t,\bar{t}} &= \sum_{n,\bar{n}} \frac{q^n\bar{q}^{\bar{n}}}{n!\bar{n}!\pi(\lambda\omega)^{n+\bar{n}-1}} \mathrm{e}^{\lambda|\omega|(z\bar{z}+w\bar{w})/2}(\partial_z^{\bar{n}}\partial_{\bar{z}}^{n}\mathrm{e}^{-\lambda|\omega|z\bar{z}})\partial_{\bar{w}}^{\bar{n}}\partial_w^n\mathrm{e}^{-\lambda|\omega|w\bar{w}} \\ &= \frac{\lambda\omega}{\pi}\mathrm{e}^{\lambda|\omega|(z\bar{z}+w\bar{w})/2}\exp\left(\frac{q}{\lambda|\omega|}\partial_{\bar{z}}\partial_w + \frac{\bar{q}}{\lambda|\omega|}\partial_z\partial_{\bar{w}}\right)\cdot \mathrm{e}^{-\lambda|\omega|(z\bar{z}+w\bar{w})}. \end{aligned}$$

Substituting the formula

$$\mathrm{e}^{-\lambda|\omega|z\bar{z}} = \int \frac{\mathrm{d}k\,\mathrm{d}\bar{k}}{\lambda|\omega|\pi}\exp\left(-\frac{k\bar{k}}{\lambda|\omega|} + \mathrm{i}(k\bar{z} + \bar{k}z)\right)$$

in the above expression, we obtain

$$U_{t,\bar{t}} = \frac{\lambda|\omega|}{\pi(1 - q\bar{q})}\exp\left(\tfrac{1}{2}\lambda|\omega|(z\bar{z} - w\bar{w}) - \frac{\lambda|\omega|}{1 - q\bar{q}}(z - \bar{q}w)(\bar{z} - q\bar{w})\right).$$

Before we pass to the limit $\lambda \to \infty$, we need to multiply $U_{t,\bar{t}}$ by $\mathrm{e}^{\lambda(f(z,\bar{z})-f(w,\bar{w}))}$, where $f(z, \bar{z}) = \frac{1}{2}\omega z\bar{z}$. Suppose that $\omega > 0$ (for $\omega < 0$ the calculation is similar). Then we find that

$$U_{t,\bar{t}} \mapsto \tilde{U}_{t,\bar{t}} = \frac{\lambda|\omega|}{\pi(1 - q\bar{q})}\exp\left(-\frac{\lambda|\omega|}{1 - q\bar{q}}(w - qz)(\bar{w} - \bar{q}\bar{z})\right).$$

It is clear that when $\lambda \to \infty$, this expression tends to the delta-function supported on the shifted diagonal $w = qz$, $\bar{w} = \bar{q}\bar{z}$:

$$\delta^{(2)}(w - qz, \bar{w} - \bar{q}\bar{z}).$$

Therefore, the kernel $K_{t,\bar{t}}$ of the (modified) evolution operator tends to

$$K_{t,\bar{t}} \to \delta^{(2)}(w - qz, \bar{w} - \bar{q}\bar{z})\,\mathrm{d}(qz - w) \wedge \mathrm{d}(\bar{q}\bar{z} - \bar{w}). \tag{3.21}$$

As expected, this is precisely the delta-form supported on the shifted diagonal $w = qz$, which corresponds to the flow along the gradient trajectory $z \mapsto zq$.

This completes our analysis of the spaces of states of the model defined on $X = \mathbb{C}$.

3.5. The case of $\mathbb{CP}^1$: ground states

Now we consider the first non-trivial 'curved' manifold, namely, $X = \mathbb{CP}^1$. We will choose the Fubini–Study metric

$$g = \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{(1 + z\bar{z})^2}, \tag{3.22}$$

and the Morse function

$$f = \frac{1}{4}\frac{z\bar{z} - 1}{z\bar{z} + 1}.$$

The corresponding gradient vector field is the Euler vector field

$$v = z\partial_z + \bar{z}\partial_{\bar{z}},$$

and so it has the form $v = \xi + \bar{\xi}$, where $\xi = z\partial_z$ is the holomorphic vector field on $\mathbb{CP}^1$. This vector field generates the standard $\mathbb{C}^\times$ action: $z \mapsto zq$, $q \in \mathbb{C}^\times$.

The hamiltonian (before conjugation by $\mathrm{e}^{\lambda f}$) is given by formula (2.4), which in this case reads

$$H_\lambda = -\frac{2}{\lambda}(1 + z\bar{z})^2\partial_z\partial_{\bar{z}} + \frac{\lambda}{2}\frac{z\bar{z}}{(1 + z\bar{z})^2} - \frac{z\bar{z} - 1}{z\bar{z} + 1}(F + \bar{F} - 1). \tag{3.23}$$

Our Morse function has two critical points, $z = 0$ and $z = \infty$ (see Figure 1). Near $z = 0$ we have

$$f = -\tfrac{1}{4} + \tfrac{1}{2}z\bar{z} + \cdots, \tag{3.24}$$

while near $z = \infty$ we have

$$f = \tfrac{1}{4} - \tfrac{1}{2}w\bar{w} + \cdots, \tag{3.25}$$

where $w = z^{-1}$ is a local coordinate near the point ∞.

Thus, $z = 0$ is a 'repulsive' critical point, and $z = \infty$ is an 'attractive' critical point. This indicates that both scenarios discussed in the case of the flat space $X = \mathbb{C}$ should somehow be realized in the $\mathbb{CP}^1$ model.

What is the structure of the spaces of states of our theory? We start with the 'in' space $\mathcal{H}^{\mathrm{in}}$. For finite values of λ the space of states is the space of L_2 differential forms on $\mathbb{CP}^1$. It is easy to find the ground states of H_λ. There are two of them, and they are localized near the critical points. The one corresponding to $z = 0$ is the function

$$_0\Psi_{\mathrm{vac}} = \sqrt{\frac{\lambda}{\pi(\mathrm{e}^{\lambda/2} - \mathrm{e}^{-\lambda/2})}}\,\mathrm{e}^{-\lambda f}, \tag{3.26}$$

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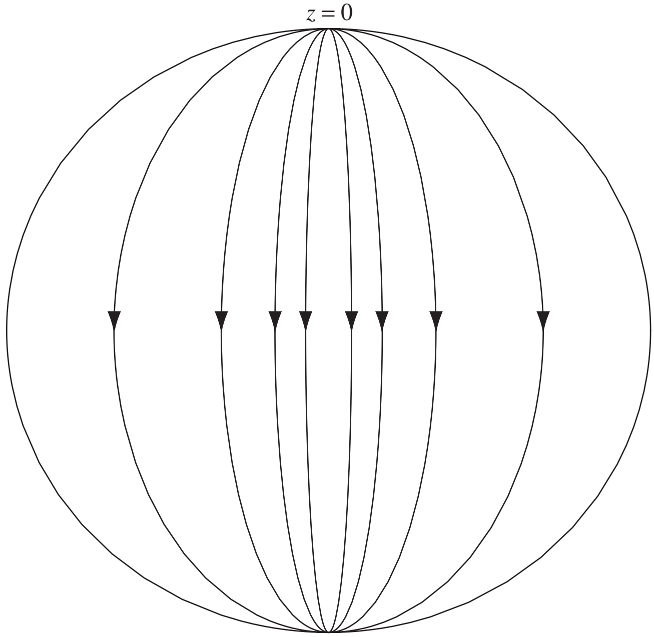

Figure 1. Morse theory on $\mathbb{CP}^1$.

and the one corresponding to $z = \infty$ is the 2-form

$$ {}_{\infty}\Psi_{\mathrm{vac}} = \sqrt{\frac{\lambda}{\pi(\mathrm{e}^{\lambda/2} - \mathrm{e}^{-\lambda/2})}}\mathrm{e}^{\lambda f}\omega_{\mathrm{FS}}, \tag{3.27} $$

where

$$ \omega_{\mathrm{FS}} = \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{(1 + z\bar{z})^2} $$

is the Fubini–Study Kähler form. To see that these are ground states, we check that they are annihilated by both supersymmetry charges. In each case, we obtain that one of the supercharges obviously annihilates it by counting the degree of the differential form, and it is a straightforward calculation to show that the other one does as well. We have normalized these states in such a way that they have unit norm with respect to our hermitian inner product (2.1).

Now we change the theory by adding the term (3.1) to the action. As explained in § 3.2, this amounts to multiplying the 'in' states by the function $\mathrm{e}^{\lambda f}$. In the case of ground states, we obtain the following states of the new theory:

$$ {}_0\tilde{\Psi}^{\mathrm{in}}_{\mathrm{vac}} = 1, $$
$$ {}_{\infty}\tilde{\Psi}^{\mathrm{in}}_{\mathrm{vac}} = \frac{\lambda}{\pi(\mathrm{e}^{\lambda/2} - \mathrm{e}^{-\lambda/2})}\mathrm{e}^{2\lambda f}\omega_{\mathrm{FS}}. $$

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We have changed the normalization factors so as to ensure that the above expressions have well-defined limits as $\lambda \to \infty$ (as distributions).

The limit of ${}_0\tilde{\Psi}^{\text{in}}_{\text{vac}}$ is just the constant function 1. This is not surprising, because this ground state corresponds to the point $z = 0$, which is a 'repulsive' critical point. So by analogy with the case of $X = \mathbb{C}$ we should expect that this ground state, appropriately rescaled, becomes the constant function. On the other hand, ${}_\infty\tilde{\Psi}^{\text{in}}_{\text{vac}}$ corresponds to the 'attractive' critical point $z = \infty$. Again, our analysis in the case of $X = \mathbb{C}$ suggests that in this case we should expect the limit of this ground state to be the delta-form supported at $z = \infty$ (or, equivalently, $w = 0$). This is exactly what happens: the above formula, interpreted as a distribution on $\mathbb{CP}^1$, has a well-defined limit which is equal to $\delta^{(2)}(w, \bar{w})\,\mathrm{d}w\,\mathrm{d}\bar{w}$.

Likewise, we obtain the 'out' ground states by multiplying the suitably normalized ground states by $\mathrm{e}^{-\lambda f}$. It is clear that mapping $f \mapsto -f$ we interchange the 'in' and 'out' ground states. However, under this map the critical point $z = 0$ becomes 'attractive' while the critical point $z = \infty$ becomes 'repulsive'. Therefore, their roles get interchanged, so that the 'out' ground state corresponding to $z = 0$ is the delta-form supported at 0, $\delta^{(2)}(z, \bar{z})\,\mathrm{d}z\,\mathrm{d}\bar{z}$, while the 'out' ground state corresponding to $z = \infty$ is the function 1.

3.6. Ground states for other Kähler manifolds

The calculation of the previous section has a natural generalization to other Kähler manifolds. Suppose that we have a compact Kähler manifold X with a holomorphic vector field ξ, which comes from a $\mathbb{C}^\times$-action ϕ on X with isolated fixed points. Let us denote the fixed points of ϕ by x_α, $\alpha \in A$. We will assume that the set A is non-empty. According to [**16**], there exists a Morse function f whose gradient is the vector field $v = \xi + \bar{\xi}$. The critical points of f coincide with the fixed points of ϕ and the zeroes of ξ and v.*

Under these assumptions, we have the Bialynicki–Birula decompositions [**6**]

$$X = \bigsqcup_{\alpha \in A} X_\alpha = \bigsqcup_{\alpha \in A} X^\alpha \tag{3.28}$$

of X into complex submanifolds X_α and X^α, defined as follows:

$$X_\alpha = \Bigl\{x \in X \Bigm| \lim_{t \to -\infty} \phi(\mathrm{e}^t) \cdot x = x_\alpha\Bigr\}, \tag{3.29}$$

$$X^\alpha = \Bigl\{x \in X \Bigm| \lim_{t \to +\infty} \phi(\mathrm{e}^t) \cdot x = x_\alpha\Bigr\}. \tag{3.30}$$

The submanifolds X_α and X^α are the *ascending* and *descending* manifolds of our Morse function f, respectively, introduced in § 2.5. Each submanifold X_α is isomorphic to $\mathbb{C}^{n_\alpha}$, and X^α is isomorphic to $\mathbb{C}^{n-n_\alpha}$, where the index of the critical point x_α is $2(n - n_\alpha)$. In what follows we will assume, for simplicity, that the following *Morse–Smale condition* holds: the strata X_α and X^β intersect transversely for all α, β. Then, according to [**7**],

* Note also that f is the hamiltonian of the vector field $\mathrm{i}(\xi - \bar{\xi})$, corresponding to the subgroup $U(1) \subset \mathbb{C}^\times$, with respect to the Kähler structure on X.

the two decompositions (3.28) of X are in fact stratifications: that is, the closure of each X_α is a union of the X_β (and similarly for the X^α).

In the case when $X = \mathbb{CP}^1$ both stratifications consist of two 'cells': the ascending manifolds are the one-dimensional cell $\mathbb{C}_0 = \mathbb{CP}^1 \backslash \infty$ and the point $\infty \in \mathbb{CP}^1$, and the descending manifolds are the one-dimensional cell $\mathbb{C}_\infty = \mathbb{CP}^1 \backslash 0$ and the point $0 \in \mathbb{CP}^1$ (they satisfy the transversality condition). The above calculation shows that the ground states of our theory may be viewed as the delta-forms supported on these cells. For an open cell it is just the function 1, and for a one-point cell it is the delta 2-form supported at that point.

In general, for each stratum X_α or X^α of our decompositions we can construct a similar delta-form, which we will denote by Δ_α, or Δ^α, respectively. For the sake of definiteness, consider the case of ascending manifolds X_α. Then Δ_α is the constant function along X_α, extended as a delta-form (of degree equal to the codimension of X_α, that is $2(n-n_\alpha)$) in the transversal directions. More precisely, this is a distribution (or current) on the space of differential forms on X which is defined by the following formula:

$$\langle \Delta_\alpha, \eta \rangle = \int_{X_\alpha} \eta|_{X_\alpha}, \quad \eta \in \Omega^\bullet(X) \tag{3.31}$$

(the integral converges because η is well defined on X, which is assumed to be compact).

Under our assumptions, all critical points have even indices, and the semi-classical analysis of [**38**] (see also [**21**, **22**]) shows that ground states are also in one-to-one correspondence with the critical points of the Morse function. As in the case of $X = \mathbb{CP}^1$, it is easy to write down explicit formulae for the ground states corresponding to the minimum and maximum of f, which are going to be a function and a top form, respectively.* For the other ground states one can write approximate semi-classical formulae for large λ, which are essentially given by the Gaussian distributions around the critical points of the form

$$_\alpha\Psi_{\mathrm{vac}} \sim \exp\biggl(-\lambda \sum_{i=1}^{n} |\mu_i|\,|z_i|^2\biggr)\, \mathrm{d}^2 z_{n_\alpha+1} \wedge \cdots \wedge \mathrm{d}^2 z_n, \tag{3.32}$$

where the z_i are normal holomorphic coordinates around x_α with respect to which the Hessian of f is the diagonal matrix with the eigenvalues μ_i, $i = 1, \dots, n$, each occurring with multiplicity two (recall our convention (3.16) for the differentials). We order them in such a way that the eigenvalues μ_i are positive for $i = 1, \dots, n_\alpha$ and negative for $i = n_\alpha + 1, \dots, n$.

On a general (real) manifold X, because of the instanton corrections, only some linear combinations of these states give rise to true ground states for finite values of λ. In particular, because of the instanton corrections the supercharge Q, acting on these states, becomes the differential of the Morse complex [**38**].† But in our case, since the critical

* While these are legitimate states in the models under consideration, we note that in more general models of quantum field theory (and even in quantum mechanics on a non-compact manifold) the analogous wave functions do not belong to the physical spectrum (see [**45**]).

† Note, however, that they become ground states at $\lambda = \infty$, even though they are not annihilated by Q, reflecting the non-Hodge nature of the algebra of supercharges in this limit.

points have only even indices, each of the functions ${}_\alpha\Psi_{\rm vac}$ corresponds to a true ground state of the Hamiltonian H_λ for finite λ.

Now we perform the operation of multiplication by the function $\mathrm{e}^{\lambda f}$. Near x_α we have

$$f \sim f(x_\alpha) + \sum_{i=1}^{n} \mu_i |z_i|^2 + \cdots .$$

Therefore, after multiplication by $\mathrm{e}^{\lambda f}$ the ground states become (up to an overall factor)

$${}_\alpha\Psi^{\rm in}_{\rm vac} \mapsto {}_\alpha\tilde{\Psi}^{\rm in}_{\rm vac} \sim \exp\biggl(-2\lambda \sum_{i=n_\alpha+1}^{n} |\mu_i|\,|z_i|^2 \biggr)\, \mathrm{d}^2 z_{n_\alpha+1} \wedge \cdots \wedge \mathrm{d}^2 z_n. \tag{3.33}$$

In other words, the terms with positive eigenvalues μ_i get canceled, while the terms with negative μ_i get doubled. The resulting form (after including an appropriate λ-dependent normalization constant) tends to the delta-form Δ_α.

For example, if $X = \mathbb{CP}^1$, then the 'delta-form' on the open orbit $\mathbb{C}_0$ is just the function 1, and the delta-form corresponding to ∞ is $\delta^{(2)}(w,\bar{w})\,\mathrm{d}^2 w$.

Thus, we claim that the suitably normalized ground states become, after multiplication by $\mathrm{e}^{\lambda f}$ and taking the limit $\lambda \to \infty$, the delta-forms Δ_α.

It is instructive to derive this result using the path integral approach. As already noted above, we may construct states of the theory by using path integral over half-line $(-\infty, 0]$, see formula (3.7). In particular, the vacuum state ${}_\alpha\tilde{\Psi}^{\rm in}_{\rm vac}$, viewed as a differential form on X, may be represented symbolically by the path integral

$${}_\alpha\tilde{\Psi}^{\rm in}_{\rm vac}(x) = \int_{x(t)\colon(-\infty,0]\to X;\, x(0)=x} \mathrm{e}^{-S}$$

(where the action S includes the term (3.1)). The question is which boundary condition to take at $t = -\infty$. Recall that in the limit $\lambda = \infty$ that we are considering the path integral localizes on the gradient trajectories. But the value of a gradient trajectory $x(t)\colon (-\infty, 0] \to X$ at $t = -\infty$ is necessarily a critical point. It is clear therefore that the boundary condition that we need to take in order to obtain the ground state ${}_\alpha\tilde{\Psi}^{\rm in}_{\rm vac}(x)$ in the limit $\lambda = \infty$ is $x(-\infty) = x_\alpha$.

Thus, we find that ${}_\alpha\tilde{\Psi}^{\rm in}_{\rm vac}(x)$ is given by the integral of e^{-S} over the gradient trajectories $x(t)\colon (-\infty, 0] \to X$ connecting x_α at $t = -\infty$ and x at $t = 0$. But such trajectories exist only if x belongs to the ascending manifold X_α, and if it does, then there is exactly one such trajectory! So from this perspective it is clear that ${}_\alpha\tilde{\Psi}^{\rm in}_{\rm vac}(x)$ has to be supported on X_α. One needs to work a little harder and analyse the fermionic contribution to the path integral to see that ${}_\alpha\tilde{\Psi}^{\rm in}_{\rm vac}(x)$ is in fact the delta-form Δ_α, but since we have already obtained this result from the Hamiltonian perspective, we will not do this here.

The same analysis applies to the 'out' ground states. Here we need to consider the path integral on the half-line from 0 to $+\infty$, so the ground states are supported on the submanifold of those $x \in X$ which could be connected to the critical point x_α by a gradient trajectory $x(t)\colon [0, +\infty) \to X$ such that $x(0) = x$ and $x(+\infty) = x_\alpha$. This is

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precisely the descending manifold X^α. From the Hamiltonian point of view this is also clear, because the 'out' state corresponding to (3.32) is

$$_\alpha\tilde{\Psi}^{\text{out}}_{\text{vac}} = \mathrm{e}^{-2\lambda f} \star \overline{_\alpha\tilde{\Psi}^{\text{in}}_{\text{vac}}} \sim \exp\biggl(-2\lambda \sum_{i=1}^{n_\alpha} |\mu_i|\, |z_i|^2 \biggr) \,\mathrm{d}^2 z_1 \wedge \cdots \wedge \mathrm{d}^2 z_{n_\alpha},$$

which tends to the delta form Δ^α supported on X^α.

To summarize, we have now determined the ground states of our quantum mechanical model at $\lambda = \infty$, for any Kähler manifold equipped with a Morse function satisfying the conditions listed above.

The 'in' ground states are the delta-forms Δ_α supported on the closures of the ascending manifolds X_α of the Morse function, and the 'out' ground states are the delta-forms Δ^α supported on the closures of the descending manifolds X^α.

3.7. Back to $\mathbb{CP}^1$: excited states

What about the excited states of the theory? For finite values of λ explicit formulae for those are unknown, but for large λ we can understand the behaviour of 'low-lying' eigenfunctions of the hamiltonian H_λ qualitatively using the standard semi-classical methods. Here 'low-lying' means that the eigenvalue remains a finite number as $\lambda \to \infty$. In general, there will be other eigenfunctions as well (for example, of order λ), but since we are interested in the limit $\lambda \to \infty$, we will ignore them.

Let us consider the case of $X = \mathbb{CP}^1$ first. Semi-classical approximation tells us that as far as the low-lying excited states are concerned, the situation is as follows: the eigenfunctions are localized in the neighbourhoods of the critical points, and the picture around each critical point is qualitatively the same as in the case of the flat space $X = \mathbb{C}$ discussed in the previous section. Thus, we have two sets of eigenfunctions, corresponding to the points 0 and ∞, which will be indicated by a subscript. In both cases we also have to take into account the degree of the state considered as a differential form.

Consider first the states corresponding to the critical point $z = 0$ (we will mark them with a left subscript 0). This is a 'repulsive' critical point as can be seen from the expansion (3.24) of f near $z = 0$. Therefore, the structure of the low-lying eigenfunctions localized near $z = 0$ will be the same as that of the eigenfunctions obtained in the case of $X = \mathbb{C}$ and a 'repulsive' critical point. Let us start with the 0-forms. Recalling the formulae obtained in § 3.3, we find that those are labelled by two integers n, $\bar{n} \geqslant 0$, and near zero are approximately equal to

$$_0\Psi_{n,\bar{n},0,0} = z^n \bar{z}^{\bar{n}} \mathrm{e}^{-\lambda f} + O(\lambda^{-1}), \quad n, \bar{n} \geqslant 0.$$

The first of them, $_0\Psi_{0,0,0,0}$ is in fact the ground state $_0\Psi_{\text{vac}}$ given by formula (3.26). However, we have now changed our normalization, so as to make the limits of these functions, multiplied by $\mathrm{e}^{\lambda f}$, well defined. We also have excited states that are 1-forms and 2-forms:

$$_0\Psi_{n,\bar{n},p,\bar{p}} = z^n \bar{z}^{\bar{n}} \mathrm{e}^{-\lambda f} (\mathrm{d}z)^p (\mathrm{d}\bar{z})^{\bar{p}} + \cdots, \quad n, \bar{n} \geqslant 0, \;\; p, \bar{p} = 0, 1.$$

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After we multiply these eigenfunctions by $\mathrm{e}^{\lambda f}$, they become polynomial differential forms on $\mathbb{C}_0$, which is the ascending manifold corresponding to the critical point $z = 0$.

Thus, we conclude that in the $\lambda \to \infty$ limit the space of 'in' states contains a piece

$$\mathcal{H}^{\mathrm{in}}_{\mathbb{C}_0} = \mathbb{C}[z, \bar{z}] \otimes \Lambda[\mathrm{d}z, \mathrm{d}\bar{z}],$$

as in the case of $X = \mathbb{C}$ and a 'repulsive' critical point.

Next, we look at the excited states corresponding to the critical point $z = \infty$, or $w = 0$, where $w = z^{-1}$ (we will mark them with a left subscript ∞). This critical point is 'attractive', according to formula (3.25). Following the example of $X = \mathbb{C}$ with an 'attractive' critical point, and recalling the expansion of f near ∞, we find that there will be other eigenfunctions, which near $w = 0$ are equal to

$${}_{\infty}\Psi_{n,\bar{n},1,1} = \frac{\lambda}{\pi(\mathrm{e}^{\lambda/2} - \mathrm{e}^{-\lambda/2})} \mathrm{e}^{-\lambda f} \frac{1}{n!\bar{n}!} \partial_w^n \partial_{\bar{w}}^{\bar{n}} \mathrm{e}^{2\lambda f}\, \mathrm{d}w\, \mathrm{d}\bar{w} + O(\lambda^{-1}), \quad n, \bar{n} \geqslant 0.$$

The first of them is the ground state (3.27). We have again changed our normalization so as to obtain a well-defined limit as $\lambda \to \infty$.

Multiplying these forms by $\mathrm{e}^{\lambda f}$ and taking the limit $\lambda \to \infty$, we obtain the derivatives of the delta-form

$${}_{\infty}\tilde{\Psi}^{\mathrm{in}}_{n,\bar{n},1,1} = \frac{1}{n!\bar{n}!} \partial_w^n \partial_{\bar{w}}^{\bar{n}} \delta^{(2)}(w, \bar{w})\, \mathrm{d}w\, \mathrm{d}\bar{w}, \quad n, \bar{n} \geqslant 0.$$

In addition, there will be 1-forms and 0-forms ${}_{\infty}\tilde{\Psi}^{\mathrm{in}}_{n,\bar{n},p,\bar{p}}$ obtained by contracting the above 2-forms with the vector fields ∂_w and $\partial_{\bar{w}}$. Thus, we obtain that this critical point contributes the piece

$$\mathcal{H}^{\mathrm{in}}_{\infty} = \mathbb{C}[\partial_w, \partial_{\bar{w}}] \delta^{(2)}(w, \bar{w}) \otimes \Lambda[\mathrm{d}w, \mathrm{d}\bar{w}]$$

to the space of 'in' states of the theory at $\lambda = \infty$.

We conclude that the space of 'in' states is isomorphic to the direct sum of two subspaces attached naturally to the critical points:

$$\mathcal{H}^{\mathrm{in}} \simeq \mathcal{H}^{\mathrm{in}}_{\mathbb{C}_0} \oplus \mathcal{H}^{\mathrm{in}}_{\infty}.$$

Naively, the hamiltonian is $H_{\mathrm{naive}} = \mathcal{L}_v$, which naturally acts on this direct sum. It has as basis of eigenstates the obvious monomial basis of this space, and the eigenvalues are non-negative integers.

However, we will see below that $\mathcal{H}^{\mathrm{in}}$ does not canonically decompose into a direct sum, but is rather an extension

$$0 \to \mathcal{H}^{\mathrm{in}}_{\infty} \to \mathcal{H}^{\mathrm{in}} \to \mathcal{H}^{\mathrm{in}}_{\mathbb{C}_0} \to 0.$$

Moreover, as we will explain in § 4, this space is realized as a canonical subspace of the space of distributions on $\mathbb{CP}^1$. The hamiltonian is indeed $\mathcal{L}_v$, but acting on this space it is not diagonalizable. It is diagonalizable on the subspace $\mathcal{H}^{\mathrm{in}}_{\infty}$, but has generalized eigenvectors on $\mathcal{H}^{\text{in}}_{\mathbb{C}_0}$ that are adjoint to eigenvectors in $\mathcal{H}^{\text{in}}_{\infty}$. We will give below explicit formulae for the action of the hamiltonian.

The same analysis applies to the 'out' states of our model. Now the roles of 0 and ∞ get interchanged, so the corresponding space of states is isomorphic to the direct sum

$$\mathcal{H}^{\text{out}} \simeq \mathcal{H}^{\text{out}}_{\mathbb{C}_\infty} \oplus \mathcal{H}^{\text{out}}_0,$$

where

$$\begin{aligned}\mathcal{H}^{\text{out}}_0 &= \mathbb{C}[\partial_z, \partial_{\bar{z}}]\delta^{(2)}(z,\bar{z}) \otimes \Lambda[\mathrm{d}z, \mathrm{d}\bar{z}],\\ \mathcal{H}^{\text{out}}_{\mathbb{C}_\infty} &= \mathbb{C}[w,\bar{w}] \otimes \Lambda[\mathrm{d}w, \mathrm{d}\bar{w}].\end{aligned}$$

The naive hamiltonian is $-\mathcal{L}_v$ which is diagonalized on the monomial elements, with the eigenvalues being non-negative integers.

But in fact we will see that $\mathcal{H}^{\text{out}}_0$ is an extension

$$0 \to \mathcal{H}^{\text{out}}_0 \to \mathcal{H}^{\text{out}} \to \mathcal{H}^{\text{out}}_{\mathbb{C}_\infty} \to 0,$$

which is also realized in the space of distributions on $\mathbb{CP}^1$. The hamiltonian $-\mathcal{L}_v$, acting on this space, has off-diagonal terms that make it non-diagonalizable.

3.8. Generalization to other Kähler manifolds and holomorphic factorization

The discussion of the previous section is generalized in a straightforward way to the case of an arbitrary Kähler manifold satisfying the above conditions. Recall that we have the stratifications of X by descending and ascending manifolds. For each ascending manifold X_α we define the space $\mathcal{H}^{\text{in}}_\alpha$ of all *delta-forms supported on* X_α. In particular, it contains the ground state Δ_α constructed in § 3.6. Moreover, the space $\mathcal{H}^{\text{in}}_\alpha$ is generated from Δ_α under the action of differential operators defined in the neighbourhood of X_α.

To describe the structure of $\mathcal{H}^{\text{in}}_\alpha$ in more concrete terms, we recall that the stratum X_α is isomorphic to $\mathbb{C}^{n_\alpha}$, where the index of the corresponding critical point x_α is $2(n-n_\alpha)$. Let us choose holomorphic coordinates z_i, $i = 1,\dots,n$, in the neighbourhood of $X_\alpha \subset X$ in such a way that the coordinates $z_1,\dots,z_{n_\alpha}$ are holomorphic coordinates along X_α and the holomorphic coordinates $z_{n_\alpha+1},\dots,z_n$ are transversal to X_α (so that X_α is described by the equations $z_i = 0$, $\bar{z}_i = 0$, $i = n_\alpha+1,\dots,n$). Then $\mathcal{H}^{\text{in}}_\alpha$ is spanned by the monomials which may schematically be represented in the form

$$\prod_{1\leqslant i,\bar{i},j,\bar{j}\leqslant n_\alpha} z_i \bar{z}_{\bar{i}}\, \mathrm{d}z_j\, \mathrm{d}\bar{z}_{\bar{j}} \prod_{n_\alpha+1\leqslant k,\bar{k},l,\bar{l}\leqslant n} \partial_{z_k}\partial_{\bar{z}_{\bar{k}}}\imath_{\partial_{z_l}}\imath_{\partial_{\bar{z}_{\bar{l}}}} \cdot \Delta_\alpha \tag{3.34}$$

(here, as before, $\imath_v$ denotes the operator of contraction of a differential form by a vector field v). Thus, we see that $\mathcal{H}^{\text{in}}_\alpha$ is indeed generated from Δ_α under the action of (super)differential operators.

In addition, the space $\mathcal{H}^{\text{in}}_\alpha$ exhibits the following holomorphic factorization:

$$\mathcal{H}^{\text{in}}_\alpha \simeq \mathcal{F}^{\text{in}}_\alpha \otimes \bar{\mathcal{F}}^{\text{in}}_\alpha,$$

where $\mathcal{F}_\alpha^{\text{in}}$ (respectively, $\bar{\mathcal{F}}_\alpha^{\text{in}}$) is the space of holomorphic (respectively, anti-holomorphic) delta-forms supported on X_α.

For example, if $n_\alpha = \dim X$, so $X_\alpha \simeq \mathbb{C}^{n_\alpha}$ is an open subset of X, then $\mathcal{H}_\alpha^{\text{in}}$ is the space of differential forms on $\mathbb{C}^{n_\alpha}$. Therefore, it factorizes into the tensor product of the spaces $\mathcal{F}_\alpha^{\text{in}}$ and $\bar{\mathcal{F}}_\alpha^{\text{in}}$ of holomorphic and anti-holomorphic differential forms, respectively.

On the other hand, if $n_\alpha = 0$, so $X_\alpha = x_\alpha$ is a point, then $\mathcal{H}_\alpha^{\text{in}}$ is the space of distributions supported at x_α. It factorizes into the tensor product

$$\mathcal{H}_\alpha^{\text{in}} \simeq (\mathbb{C}[\partial_{z_i}] \otimes \Lambda[\imath_{\partial_{z_i}}])_{i=1,\dots,n} \otimes (\mathbb{C}[\partial_{\bar{z}_i}] \otimes \Lambda[\imath_{\partial_{\bar{z}_i}}])_{i=1,\dots,n} \cdot \Delta_\alpha$$

(note that the operators $\imath_{\partial_{z_i}}$ anti-commute and hence generate an exterior algebra). Therefore, we may write

$$\mathcal{H}_\alpha^{\text{in}} \simeq \mathcal{F}_\alpha^{\text{in}} \otimes \bar{\mathcal{F}}_\alpha^{\text{in}},$$

where

$$\mathcal{F}_\alpha^{\text{in}} = \mathbb{C}[\partial_{z_i}] \otimes \Lambda[\imath_{\partial_{z_i}}]_{i=1,\dots,n}, \qquad \bar{\mathcal{F}}_\alpha^{\text{in}} = \mathbb{C}[\partial_{\bar{z}_i}] \otimes \Lambda[\imath_{\partial_{\bar{z}_i}}]_{i=1,\dots,n}.$$

A proper interpretation of the space $\mathcal{F}_\alpha^{\text{in}}$ for a general critical point x_α is achieved in the framework of the theory of holomorphic $\mathcal{D}$-modules (see, for example, [**25**]).

Let us return to the case when $n_\alpha = \dim X$ and $X_\alpha \simeq \mathbb{C}^{n_\alpha}$ is an open subset of X. Then $\mathcal{F}_\alpha^{\text{in}}$ is the space of holomorphic differential forms on $\mathbb{C}^{n_\alpha}$. In particular, its subspace of $\mathcal{F}_\alpha^{\text{in},0}$ of degree zero forms consists of holomorphic functions on X_α. The holomorphic differential operators on X_α naturally act on $\mathcal{F}_\alpha^{\text{in},0}$. Since X_α is open and dense in X, $\mathcal{F}_\alpha^{\text{in},0}$ is the space of global sections of a holomorphic $\mathcal{D}_X$-module, where $\mathcal{D}_X$ is the sheaf of holomorphic differential operators on X. Its generator is the function 1, which is annihilated by ∂_{z_i}, $i = 1, \dots, n$.

The entire space $\mathcal{F}_\alpha^{\text{in}}$ may be viewed as the space of global sections of a $\mathcal{D}_{\Pi TX}$-module, where $\mathcal{D}_{\Pi TX}$ is the sheaf of holomorphic differential operators on the supermanifold ΠTX. The constant function 1 is again a generator of this $\mathcal{D}_X$-module.*

Now consider the space $\mathcal{F}_\alpha^{\text{in}}$ in the case when $X_\alpha = x_\alpha$. Then the degree zero part $\mathcal{F}_\alpha^{\text{in},0}$ of $\mathcal{F}_\alpha^{\text{in}}$ may also be interpreted as the space of global sections of a $\mathcal{D}_X$-module, called the $\mathcal{D}_X$-module of 'holomorphic delta-functions with support at x_α', or of 'local cohomology of $\mathcal{O}_X$ with support at x_α'. It is defined as follows: its space of sections on any open subset not containing the point x_α is zero, and the space of sections on an open subset U containing x_α is the space $\mathcal{F}_\alpha^{\text{in},0}$. To define the structure of $\mathcal{D}_X$-module we need to show how to act on $\mathcal{F}_\alpha^{\text{in},0}$ by holomorphic differential operators on U. Without loss of generality we may assume that U is a very small neighbourhood of x_α with coordinates $z_1, \dots, z_n$. Thus, we need to show how to act on $\mathcal{F}_\alpha^{\text{in},0}$ by functions z_i and vector fields ∂_{z_i}. This is done by realizing $\mathcal{F}_\alpha^{\text{in},0}$ as the module over the algebra of polynomial differential operators in z_i, ∂_{z_i}, $i = 1, \dots, n$, generated by a vector annihilated by z_i, $i = 1, \dots, n$. Let us denote this generating vector by $\delta_\alpha^{\text{hol}}$. Informally, we may view $\delta_\alpha^{\text{hol}}$ as a 'holomorphic delta-function', because it satisfies $z_i \cdot \delta_\alpha^{\text{hol}} = 0$ for all $i = 1, \dots, n$.

* Note that there is no natural structure of (left) $\mathcal{D}_X$-module on the subspace $\mathcal{F}_\alpha^{\text{in},i}$ of i-forms in $\mathcal{F}_\alpha^{\text{in}}$, except for $i = 0$. However, the subspace of top forms has a natural structure of right $\mathcal{D}_X$-module.

As for the entire space $\mathcal{F}_\alpha^{\mathrm{in}}$, it may be viewed as a $\mathcal{D}_{\Pi TX}$-module. It is generated by a vector, which we denote by $\Delta_\alpha^{\mathrm{hol}}$, which satisfies the relations

$$z_i \cdot \Delta_\alpha^{\mathrm{hol}} = 0, \quad \mathrm{d}z_i \cdot \Delta_\alpha^{\mathrm{hol}} = 0, \quad i = 1, \dots, n.$$

It is instructive to think of $\Delta_\alpha^{\mathrm{hol}}$ as the 'holomorphic delta-form'. Note that $\delta_\alpha^{\mathrm{hol}} = \imath_{\partial_{z_1}} \cdots \imath_{\partial_{z_n}} \Delta_\alpha^{\mathrm{hol}}$ (since $\Delta_\alpha^{\mathrm{hol}}$ is a top form).

For a more general critical point x_α with index $2(n - n_\alpha)$, the space $\mathcal{F}_\alpha^{\mathrm{in}}$ is a $\mathcal{D}_{\Pi TX}$-module, generated from a 'holomorphic delta-form' $\Delta_\alpha^{\mathrm{hol}}$ supported on X_α. By definition, $\Delta_\alpha^{\mathrm{hol}}$ is annihilated by z_i, $\mathrm{d}z_i$, $i = 1, \dots, n_\alpha$, and by ∂_{z_i}, $\imath_{\partial_{z_i}}$, $i = n_\alpha + 1, \dots, n$. The space $\mathcal{F}_\alpha^{\mathrm{in}}$ is obtained from $\Delta_\alpha^{\mathrm{hol}}$ under the action of holomorphic polynomials along X_α (in the variables $z_1, \dots, z_{n_\alpha}$) and holomorphic vector fields in the transversal directions (that is $\partial_{z_{n_\alpha+1}}, \dots, \partial_{z_n}$), as well as the exterior algebra in $\mathrm{d}z_1, \dots, \mathrm{d}z_{n_\alpha}$, $\imath_{\partial_{z_{n_\alpha+1}}}, \dots, \imath_{\partial_{z_n}}$.

In particular, the degree zero part of $\mathcal{F}_\alpha^{\mathrm{in}}$ is the space of global sections of a $\mathcal{D}_X$-module. This $\mathcal{D}_X$-module is in fact the pushforward of the $\mathcal{D}_{X_\alpha}$-module $\mathcal{O}_{X_\alpha}$ to X under the embedding $X_\alpha \hookrightarrow X$. It may also be realized as the local cohomology $H_{X_\alpha}^{n-n_\alpha}(\mathcal{O}_X)$ of the structure sheaf $\mathcal{O}_X$ on X with support on X_α (for more on this, see § 4.8). The entire space $\mathcal{F}_\alpha^{\mathrm{in}}$ is identified with the local cohomology $H_{X_\alpha}^{n-n_\alpha}(\Omega_{X,\mathrm{hol}})$, where $\Omega_{X,\mathrm{hol}}$ is the sheaf of holomorphic differential forms on X. It is naturally a $\mathcal{D}_{\Pi TX}$-module.

For example, if $X = \mathbb{CP}^1$, then there are two critical points: 0 and ∞. If $x_\alpha = 0$, then the corresponding stratum X_α is $\mathbb{C}_0 = \mathbb{CP}^1 \backslash \infty$. The corresponding algebra of differential operators is generated by z and ∂_z. The 0-form part of $\mathcal{H}_\alpha^{\mathrm{in}}$ is in this case the $\mathcal{D}$-module of functions on $\mathbb{C}_0$. Its space of global sections is $\mathbb{C}[z]$, and the algebra of differential operators naturally acts on it.

If $x_\alpha = \infty$, then we have a coordinate at ∞ that we previously denoted by w. The corresponding algebra of differential operators is generated by w and ∂_w. The 0-form part of $\mathcal{H}_\alpha^{\mathrm{in}}$ is the $\mathcal{D}_{\mathbb{CP}^1}$-module of holomorphic delta-functions with support at ∞. Its space of sections over any open subset $U \subset \mathbb{CP}^1$ containing ∞ may be defined concretely as $\mathbb{C}[w, w^{-1}]/\mathbb{C}[w]$. In other words, it is the quotient of the space of functions defined on $\mathbb{C}_\infty = \mathbb{CP}^1 \backslash 0$ by the subspace of those functions which are regular at ∞ (thus, it is spanned by the 'polar parts' of these functions).* It is generated, under the action of ∂_w, by the vector $1/w$. Inside the quotient $\mathbb{C}[w, w^{-1}]/\mathbb{C}[w]$ this vector is annihilated by w, and hence it may be thought of as a particular realization of the 'holomorphic delta-function' $\delta_\infty^{\mathrm{hol}}$ at ∞.

We have a similar description of the anti-holomorphic factor $\bar{\mathcal{F}}_\alpha^{\mathrm{in}}$.

Based on the semi-classical analysis similar to the one performed in the case of $\mathbb{CP}^1$, we now describe the space of 'in' states as follows.

The space of 'in' states of the theory at $\lambda = \infty$ is isomorphic to the direct sum

$$\mathcal{H}^{\mathrm{in}} \simeq \bigoplus_{\alpha \in A} \mathcal{H}_\alpha^{\mathrm{in}} = \bigoplus_{\alpha \in A} \mathcal{F}_\alpha^{\mathrm{in}} \otimes \bar{\mathcal{F}}_\alpha^{\mathrm{in}}. \tag{3.35}$$

* Equivalently, we could have chosen any open subset U containing ∞ and taken the quotient of holomorphic functions on $U \backslash \infty$ by holomorphic functions on U; or we could take the quotient $\mathbb{C}((w^{-1}))/\mathbb{C}[\![w]\!]$ corresponding to the formal disc around ∞.

Thus, we observe the appearance of 'conformal blocks' (or 'holomorphic blocks') in the space of states reminiscent of the structure of two-dimensional conformal field theory. It is surprising that we observe this structure at the level of quantum mechanics, i.e. one-dimensional quantum field theory. This structure is also reflected in the correlation functions of the observables decomposing into the product of holomorphic and anti-holomorphic parts: they decompose into the sum of products of holomorphic and anti-holomorphic expressions. There are no non-constant smooth observables of this type, but if we allow singularities of special kind, such observables can be easily constructed. For example, in the case of $X = \mathbb{CP}^1$ one can consider observables of the form

$$\left(\sum_{i=1}^{M} \frac{\alpha_i \,\mathrm{d}z}{z - a_i}\right)\left(\sum_{j=1}^{N} \frac{\beta_j \,\mathrm{d}\bar{z}}{\bar{z} - \bar{b}_j}\right),$$

and explicit calculations show that their correlation functions indeed exhibit factorization into 'conformal blocks'.

Such holomorphic factorization certainly cannot be expected in the theory at the finite values of the coupling constant λ, because the action contains the term $\lambda^{-1} g^{a\bar{b}} p_a p_{\bar{b}}$ mixing holomorphic and anti-holomorphic fields (and a similar mixed fermionic term). But for $\lambda = \infty$ this term disappears and we find that both the Lagrangian and the Hamiltonian of our theory are equal to sums of holomorphic and anti-holomorphic parts (see formula (1.1)). Therefore, naively one might expect that the space of states of the theory is the tensor product of holomorphic and anti-holomorphic sectors. However, what we find is a *direct sum* of such tensor products. This is a precursor of the Quillen anomaly familiar from two-dimensional conformal field theory.

Naively, the supercharge Q is the de Rham differential d naturally acting on the spaces $\mathcal{H}^{\mathrm{in}}_\alpha$, and the hamiltonian is $H_{\mathrm{naive}} = \{Q, \imath_v\} = \mathcal{L}_v$. The cohomology of Q on each $\mathcal{H}^{\mathrm{in}}_\alpha$ is one dimensional, occurring in degree $2i$ and represented by the delta-form Δ_α. These are therefore the BPS states of our theory, in agreement with the expectation that the BPS states are identified with the cohomology of X. Indeed, the ascending manifolds give us a decomposition of X into even-dimensional cells, which therefore give a basis in the homology of X. The forms Δ_α give the dual basis in the cohomology.

More precisely, we will see below that, just like in the case of $X = \mathbb{CP}^1$, the spaces $\mathcal{H}^{\mathrm{in}}$ are not canonically isomorphic to the above direct sums of the spaces $\mathcal{H}^{\mathrm{in}}_\alpha$. Rather, there are canonical filtrations with the consecutive subquotients isomorphic to $\mathcal{H}^{\mathrm{in}}_\alpha$. Because of that, the hamiltonian is not diagonalizable; it is equal to $H_{\mathrm{naive}} = \mathcal{L}_v$ plus off-diagonal terms mixing the spaces $\mathcal{H}^{\mathrm{in}}_\alpha$ with $\mathcal{H}^{\mathrm{in}}_\beta$ corresponding to the strata X_β of lower dimension which are in the closure of X_α. However, this mixing occurs only within the subspaces of differential forms of a fixed degree and fixed eigenvalue with respect to $\mathcal{L}_v$. Because these subspaces are finite dimensional, all Jordan blocks are finite and their length is bounded by $\dim_{\mathbb{C}} X + 1$.*

Likewise, we will see that the supercharge Q is equal to d plus correction terms mixing $\mathcal{H}^{\mathrm{in}}_\alpha$ with $\mathcal{H}^{\mathrm{in}}_\beta$ corresponding to the strata X_β in the closure of X_α. However, we will show in § 4.9 that these correction terms do not change the cohomology of Q.

* We will see in § 5.5 that the maximal length of the Jordan blocks may well be less than $\dim_{\mathbb{C}} X + 1$.

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The space of 'out' states has a similar structure, but with respect to the stratification of X by the descending manifolds X^α. For each stratum X^α we have the space $\mathcal{H}_\alpha^{\text{out}}$ of *delta-forms supported on* X_α. In particular, it contains the ground state Δ^α constructed in § 3.6. Moreover, the space $\mathcal{H}_\alpha^{\text{out}}$ is generated from Δ^α under the action of differential operators defined in the neighbourhood of X_α. Similarly to the 'in' spaces, it exhibits *holomorphic factorization*

$$\mathcal{H}_\alpha^{\text{out}} = \mathcal{F}_\alpha^{\text{out}} \otimes \bar{\mathcal{F}}_\alpha^{\text{out}},$$

where $\mathcal{F}_\alpha^{\text{out}}$ (respectively, $\bar{\mathcal{F}}_\alpha^{\text{out}}$) is the space of holomorphic (respectively, anti-holomorphic) delta-forms supported on X^α. Finally, the space of 'out' states is isomorphic to

$$\mathcal{H}^{\text{out}} \simeq \bigoplus_{\alpha\in A} \mathcal{H}_\alpha^{\text{out}} = \bigoplus_{\alpha\in A} \mathcal{F}_\alpha^{\text{out}} \otimes \bar{\mathcal{F}}_\alpha^{\text{out}}.$$

In reality, this direct sum decomposition is not canonical. Instead, there is a canonical filtration with the spaces $\mathcal{H}_\alpha^{\text{out}}$ appearing as consecutive quotients, and the hamiltonian is $-\mathcal{L}_v$ plus non-diagonal terms, as for $\mathcal{H}^{\text{in}}$. Nevertheless, there is a canonical pairing between $\mathcal{H}^{\text{in}}$ and $\mathcal{H}_\alpha^{\text{out}}$, as expected on general grounds. We will discuss all this in the next section.

4. The structure of the space of states

In the previous section we have determined, in the first approximation, the spaces $\mathcal{H}^{\text{in}}$ and $\mathcal{H}^{\text{out}}$ of 'in' and 'out' states of our quantum mechanical model in the limit $\lambda = \infty$. In this section we will give a more precise description of these spaces. We will show that states are naturally interpreted as *distributions* (or *currents*) on our manifold X. Because some of these distributions require regularization (reminiscent of the Epstein–Glaser regularization [**13**] familiar in quantum field theory), the action of the Hamiltonian on them becomes non-diagonalizable. We compute this action, as well as the action of the supercharges, in terms of the so-called *Grothendieck–Cousin operators* associated to the stratification of our manifold by the ascending and descending manifolds. We also compute the cohomology of the supercharges using the Grothendieck–Cousin complex [**26**].

In § 5 we will realize the evaluation observables of our model as linear operators acting on the spaces of states. We will then be able to obtain the correlation functions as matrix elements of these operators and to test our predictions by comparing these matrix elements with the integrals over the moduli of the gradient trajectories which were obtained in the path integral approach, as explained in § 2.4.

4.1. States as distributions

The answer we have given for the space of states $\mathcal{H}^{\text{in}}$ in formula (3.35) requires some explanation. Consider for example the case of $X = \mathbb{CP}^1$. We have claimed that

$$\mathcal{H}^{\text{in}} \simeq \mathcal{H}_{\mathbb{C}_0} \oplus \mathcal{H}_\infty, \tag{4.1}$$

where $\mathcal{H}_{\mathbb{C}_0}$ is the space of differential forms on the one-dimensional cell $\mathbb{C}_0 = \mathbb{CP}^1\backslash\infty$ and $\mathcal{H}_\infty$ is the space of delta-forms supported at ∞. These delta-forms are naturally

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functionals, i.e. distributions (more precisely, currents) on the space of differential forms on $\mathbb{CP}^1$. For example, the ground state $\delta^{(2)}(w, \bar{w})\, \mathrm{d}w\, \mathrm{d}\bar{w}$ is the functional whose value on a function f on $\mathbb{CP}^1$ is equal to $f(\infty)$, and it is equal to zero on all differential forms of positive degree.

Thus, the subspace $\mathcal{H}_{\mathbb{C}_0}$ appears to be realized in the space of differential forms, while the subspace $\mathcal{H}_\infty$ is realized in the space of distributions or currents, that is functionals on the space of differential forms. This is puzzling because the two spaces appear to be of different nature. The second puzzle is that elements of $\mathcal{H}_{\mathbb{C}_0}$ are well defined only on the subset $\mathbb{C}_0 \subset \mathbb{CP}^1$ and, with the exception of the constant functions, have poles at ∞. Therefore, the integrals of the products of such elements with an observable of our theory, which is *a priori* an arbitrary smooth differential form on X, is not well defined. But integrals of this type naturally appear as the one-point correlation functions of our theory at finite values of λ.

For example, let $\hat{\omega}$ be the observable of our theory which corresponds to a smooth 2-form ω on $\mathbb{CP}^1$, and consider the one-point function represented by the matrix element

$$\langle {}_\infty\Psi_{\mathrm{vac}} | \mathrm{e}^{(t-t_{\mathrm{f}})H} \hat{\omega} \mathrm{e}^{(t_{\mathrm{i}}-t)H} | {}_0\Psi_{n,\bar{n},0,0} \rangle = q^{E_{n,\bar{n}}} \int_{\mathbb{CP}^1} {}_\infty\Psi^{(0)}_{0,0}\, \omega\, {}_0\Psi^{(0)}_{n,\bar{n},0,0},$$

where $q = \mathrm{e}^{t_{\mathrm{i}}-t}$ and $E_{n,\bar{n}}$ is the eigenvalue of H on ${}_0\Psi_{n,\bar{n},0,0}$. We have argued that in the limit $\lambda \to \infty$ the 'out' state corresponding to ${}_\infty\Psi_{\mathrm{vac}}$ becomes equal to 1, while the 'in' state corresponding to ${}_0\Psi_{n,\bar{n},0,0}$ becomes equal to $z^n \bar{z}^{\bar{n}}$. Therefore, to make sense of the theory at $\lambda = \infty$ we should be able to compute integrals of the form

$$\int_{\mathbb{CP}^1} z^n \bar{z}^{\bar{n}} \omega, \tag{4.2}$$

for a general smooth 2-form ω on $\mathbb{CP}^1$.

Unfortunately, these integrals generally diverge for $n, \bar{n} > 0$. But this discussion leads us to an important idea: it suggests that a proper definition of the state corresponding to ${}_0\Psi_{n,\bar{n},0,0}$ in the limit $\lambda \to \infty$ assumes that we can evaluate the integrals of the form (4.2). Therefore, it is only natural to view these states not as functions on $\mathbb{CP}^1$, but as distributions! This at least allows us to treat $\mathcal{H}_{\mathbb{C}_0}$ and $\mathcal{H}_\infty$ on equal footing. It now becomes clear that our space of states $\mathcal{H}^{\mathrm{in}}$ with its decomposition (4.1) should be considered as a subspace of the space of distributions (or currents) on the space of smooth differential forms on $\mathbb{CP}^1$.

Viewing states as distributions is most natural from the point of view of the path integral. Recall formula (3.7) describing states as path integrals. Now, given a differential form ω on X, set

$$\langle \omega | \mathcal{O}_n(t_n) \cdots \mathcal{O}_1(t_1) | 0 \rangle = \int_X \omega \wedge \int_{x(t)\colon (-\infty,0] \to X;\, x(0)=x} \mathcal{O}_1(t_1) \cdots \mathcal{O}_n(t_n) \mathrm{e}^{-S}.$$

Thus, the state $\mathcal{O}_n(t_n) \cdots \mathcal{O}_1(t_1)|0\rangle$ is naturally interpreted as a linear functional on differential forms, i.e. a distribution.

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The delta-forms supported at ∞ are legitimate distributions. But what kind of distribution can we associate to the function $z^n\bar{z}^{\bar{n}}$ which has a pole at $\infty \in \mathbb{CP}^1$? This question has a well-known answer in the theory of generalized functions, as we now explain following [**23**, § 3.2] and [**20**, § B1].

First of all, let us recall that by definition a *distribution* on $\mathbb{CP}^1$ is a continuous linear functional on the space of smooth functions on $\mathbb{CP}^1$, equipped with the topology induced by the norm $\|f\| = \sup|f(x)|$.* In what follows we will use the term 'distribution' in more general sense, as a continuous linear functional on the space of differential forms on $\mathbb{CP}^1$ (a more common term for such an object is 'current'). We denote the space of such distributions by $D(\mathbb{CP}^1)$.

Next, we define the space $\mathcal{S}(\mathbb{C}_0)$ of *Schwartz functions* on $\mathbb{C}_0 = \mathbb{CP}^1\backslash\infty \subset \mathbb{CP}^1$: its elements are smooth functions f on $\mathbb{C}_0$ such that $z^n\bar{z}^{\bar{n}}\partial_z^m\partial_{\bar{z}}^{\bar{m}}f$ is bounded on C_0 (and hence well defined at ∞) for all $n, \bar{n}, m, \bar{m} \in \mathbb{Z}_{\geqslant 0}$. Such a function therefore extends to a smooth function on $\mathbb{CP}^1$; moreover, it necessarily decays as $z, \bar{z} \to \infty$. Define a topology on the space $\mathcal{S}(\mathbb{C}_0)$ induced by the family of semi-norms

$$f \mapsto |z^n\bar{z}^{\bar{n}}\partial_z^m\partial_{\bar{z}}^{\bar{m}}f|.$$

A *tempered distribution* on $\mathbb{C}_0$ is by definition a continuous linear functional on $\mathcal{S}(\mathbb{C}_0)$.

We define in the same way the space $\mathcal{S}\Omega(\mathbb{C}_0)$ of Schwartz differential forms on $\mathbb{C}_0$. We will call continuous linear functionals on $\mathcal{S}\Omega(\mathbb{C}_0)$ 'tempered distributions on differential forms on $\mathbb{C}_0$'.

Now observe that for all $n, \bar{n} \in \mathbb{Z}_{\geqslant 0}$ the monomial $z^n\bar{z}^{\bar{n}}$ defines a continuous linear functional $\varphi_{n,\bar{n}}$, hence a tempered distribution on differential forms on $\mathbb{C}_0$ by the formula

$$\varphi_{n,\bar{n}}(\omega) = \int_{\mathbb{C}_0} z^n\bar{z}^{\bar{n}}\omega, \quad \omega \in \mathcal{S}\Omega(\mathbb{C}_0). \tag{4.3}$$

The integral converges because of the condition imposed on elements of $\mathcal{S}\Omega(\mathbb{C}_0)$ (note that it is non-zero only if ω is a 2-form).

Thus, we are now in the following situation: we have the subspace $\mathcal{S}\Omega(\mathbb{C}_0) \subset \Omega(\mathbb{CP}^1)$ and a continuous linear functional $\varphi_{n,\bar{n}}$ on $\mathcal{S}\Omega(\mathbb{C}_0)$ defined by formula (4.3). Can we extend this functional to the larger space $\Omega(\mathbb{CP}^1)$?

It turns out that we can, but there are many possible extensions and there is no canonical choice among them, unless $n = 0$ or $\bar{n} = 0$. The good news, however, is that any two possible extensions differ by a distribution supported at ∞. Therefore, even though the span of all functionals $\varphi_{n,\bar{n}}, n, \bar{n} \in \mathbb{Z}_{\geqslant 0}$, is not canonically defined as a subspace of $D(\mathbb{CP}^1)$, the span of these functionals together with the functionals $\partial_w^m\partial_{\bar{w}}^{\bar{m}}\delta^{(2)}(w, \bar{w})$ is well defined.

We have an analogous statement for the i-form versions of these spaces, where $i = 1, 2$. Thus, we obtain that the sum $\mathcal{H}_{\mathbb{C}_0} + \mathcal{H}_\infty$ is a well-defined subspace of the space of all distributions on $\mathbb{CP}^1$.

* For a general manifold X, distributions are continuous linear functionals on the space of smooth functions on X with compact support, but on a compact manifold X the 'compact support' condition is vacuous.

We now *define* the space $\mathcal{H}^{\text{in}}$ of 'in' states of the $\mathbb{CP}^1$ model in the $\lambda = \infty$ limit as this subspace of $D(\mathbb{CP}^1)$. This way we resolve the first puzzle pointed out at the beginning of this section (the fact that $\mathcal{H}_{\mathbb{C}_0}$ and $\mathcal{H}_\infty$ seem to be objects of different nature). But we have not yet explained how to extend the linear functionals $\varphi_{n,\bar{n}}$ to $D(\mathbb{CP}^1)$ and make sense of the integrals (4.2) for arbitrary smooth differential forms $\omega \in \Omega(\mathbb{CP}^1)$. We will explain that in the next section.

4.2. Regularization of the integrals in the case of $\mathbb{CP}^1$

A particular extension of the tempered distribution $\varphi_{n,\bar{n}}$ to a distribution on $\mathbb{CP}^1$ is constructed by introducing a 'cutoff': for $\omega \in \Omega(\mathbb{CP}^1)$, consider the integral

$$\int_{|z|<\epsilon^{-1}} z^n \bar{z}^{\bar{n}} \omega, \tag{4.4}$$

which is well defined for any positive real ϵ. One can show that as a function in ϵ it may be uniquely represented in the form

$$C_0 + \sum_{i>0} C_i \epsilon^{-i} + C_{\log} \log \epsilon + o(1), \tag{4.5}$$

where the C_i and $C_{\log}$ are some numbers (see [**23**, pp. 70–71]). Therefore, one defines, following Hadamard, the *partie finie* of the above integral as the constant coefficient C_0 obtained after discarding the terms with negative powers of ϵ and $\log \epsilon$ in the integral (4.4) and taking the limit $\epsilon \to 0$. We denote it by

$$⨍_{|z|<\epsilon^{-1}} z^n \bar{z}^{\bar{n}} \omega = C_0.$$

A similar regularization has also been used in quantum field theory, in particular, in the works of Epstein and Glaser [**13**].

It is clear that if $\omega \in \mathcal{S}\Omega(\mathbb{C}_0)$, then

$$⨍_{|z|<\epsilon^{-1}} z^n \bar{z}^{\bar{n}} \omega = \int_{\mathbb{CP}^1} z^n \bar{z}^{\bar{n}} \omega,$$

so we indeed obtain an extension of $\varphi_{n,\bar{n}}$ to a distribution on $\mathbb{CP}^1$. We will denote it by $\tilde{\varphi}_{n,\bar{n}}$. Note that the distribution we obtain takes non-zero values only on 2-forms on $\mathbb{CP}^1$.

The problem with this definition is that it is not canonical. Indeed, we do not have a canonical coordinate on $\mathbb{CP}^1$, because we are only given points 0 and ∞, so our coordinate z is only defined up to multiplication by a non-zero scalar. If we rescale our coordinate $z \mapsto az$, then the functional $\tilde{\varphi}_{n,\bar{n}}$ will change as we will now integrate over the region $|z| < a\epsilon^{-1}$, and the integral will be different due to the presence of the logarithmic term in (4.5). However, the resulting change will amount to a distribution supported at $\infty \in \mathbb{CP}^1$, so that the span of these distributions and the distributions supported at ∞ is a canonically defined subspace of $D(\mathbb{CP}^1)$.

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Let us compute the values of the distributions $\tilde{\varphi}_{n,\bar{n}}$ in some examples. Let

$$\omega = \sum_\alpha \frac{\omega_\alpha}{z\bar{z} + R_\alpha}\, \mathrm{d}z\, \mathrm{d}\bar{z}, \quad R_\alpha \in \mathbb{C}\backslash\mathbb{R}_{\leqslant 0},$$

where the numbers ω_α satisfy the condition

$$\sum_\alpha \omega_\alpha = 0,$$

which ensures that ω is well defined at ∞. We recall our convention (3.16).

Writing $z = \sqrt{x}\mathrm{e}^{\mathrm{i}\theta}$, we find that

$$\begin{aligned}\tilde{\varphi}_{n,\bar{n}}(\omega) &= \fint_{|z|<\epsilon^{-1}} z^n \bar{z}^{\bar{n}} \omega \\ &= \sum_\alpha \int_0^{2\pi} \mathrm{d}\theta\, \mathrm{e}^{\mathrm{i}\theta(n-\bar{n})} \left[\int_0^{\epsilon^{-2}} \frac{\omega_\alpha x^{(n+\bar{n})/2}}{x + R_\alpha}\, \mathrm{d}x \right]_{\epsilon^0} \\ &= 2\pi(-1)^{n+1}\delta_{n,\bar{n}} \sum_\alpha \omega_\alpha R_\alpha^n \log R_\alpha. \end{aligned} \tag{4.6}$$

Here we use the subscript ϵ^0 to denote the constant term in the ϵ-expansion (this is the C_0 of formula (4.5)).

The distributions $\tilde{\varphi}_{n,\bar{n},p,\bar{p}}$ corresponding to the basis elements $z^n\bar{z}^{\bar{n}}(\mathrm{d}z)^p(\mathrm{d}\bar{z})^{\bar{p}}$ of $\mathcal{H}_{\mathbb{C}_0}$ are defined in the same way. These distributions take non-zero values on differential forms of degree $(1-p, 1-\bar{p})$. Again, they are not canonically defined, but their span together with the span of $(p,\bar{p})$-forms in $\mathcal{H}_\infty$ will be well defined. This span is the $(p,\bar{p})$-form part of our space of 'in' states $\mathcal{H}^{\mathrm{in}}$.

4.3. Action of the hamiltonian

Having defined the space $\mathcal{H}^{\mathrm{in}}$ as a particular subspace of the space of distributions on $\mathbb{CP}^1$, we can now find explicitly the action of the Hamiltonian $H = \mathcal{L}_v$, where $v = z\partial_z + \bar{z}\partial_{\bar{z}}$. Actually, we will compute separately the action of $\mathcal{L}_\xi$ and $\mathcal{L}_{\bar{\xi}}$, where $\xi = z\partial_z$, $\bar{\xi} = \bar{z}\partial_{\bar{z}}$. We will see that due to the non-canonical nature of the decomposition (4.1) these operators act non-diagonally, with Jordan blocks.

Consider first the action of $\mathcal{L}_\xi$ and $\mathcal{L}_{\bar{\xi}}$ on the subspace $\mathcal{H}_\infty \subset \mathcal{H}^{\mathrm{in}}$ (which is a canonical subspace of $\mathcal{H}^{\mathrm{in}}$). This subspace has the following basis:

$$|n,\bar{n},p,\bar{p}\rangle_\infty := \frac{(-1)^{n+\bar{n}}}{n!\bar{n}!} \partial_w^n \partial_{\bar{w}}^{\bar{n}} \delta^{(2)}(w,\bar{w})(\mathrm{d}w)^p(\mathrm{d}\bar{w})^{\bar{p}}.$$

By definition,

$$|n,\bar{n},p,\bar{p}\rangle_\infty(\omega_{w\bar{w}}(\mathrm{d}w)^r(\mathrm{d}\bar{w})^{\bar{r}}) = (-\mathrm{i})^p \mathrm{i}^{\bar{p}} \delta_{r,1-p}\delta_{\bar{r},1-\bar{p}} \partial_w^n \partial_{\bar{w}}^{\bar{n}} \omega_{w\bar{w}}|_{w=0}. \tag{4.7}$$

We find that

$$\begin{aligned}\mathcal{L}_\xi \cdot |n,\bar{n},p,\bar{p}\rangle_\infty &= (n+1-p)|n,\bar{n},p,\bar{p}\rangle_\infty, \\ \mathcal{L}_{\bar{\xi}} \cdot |n,\bar{n},p,\bar{p}\rangle_\infty &= (\bar{n}+1-\bar{p})|n,\bar{n},p,\bar{p}\rangle_\infty.\end{aligned}$$

Next, we consider the subspace $\mathcal{H}_{\mathbb{C}_0}$. It has the following basis:

$$|n,\bar{n},p,\bar{p}\rangle_{\mathbb{C}_0} := \tilde{\varphi}_{n,\bar{n},p,\bar{p}},$$

where $\tilde{\varphi}_{n,\bar{n},p,\bar{p}}$ is the distribution defined at the end of the previous section:

$$\tilde{\varphi}_{n,\bar{n},p,\bar{p}}(\omega) = ⨍_{|z|<\epsilon^{-1}} z^n \bar{z}^{\bar{n}} (\mathrm{d}z)^p (\mathrm{d}\bar{z})^{\bar{p}} \wedge \omega.$$

These elements, and their span, are not canonically defined, but depend on a particular 'partie finie' regularization of the above integral defined above.

Let us compute $\mathcal{L}_\xi \cdot |n,\bar{n},0,0\rangle_{\mathbb{C}_0}$. By definition, this is the distribution, whose value on a 2-form ω on $\mathbb{CP}^1$ is equal to

$$\tilde{\varphi}_{n,\bar{n}}(-\mathcal{L}_\xi\omega) = -⨍_{|z|<\epsilon^{-1}} z^n \bar{z}^{\bar{n}} \mathcal{L}_\xi \omega.$$

Writing $\mathcal{L}_\xi$ as $\{\mathrm{d}, \imath_\xi\}$ and using the Stokes formula, we find that in addition to the differentiation of z^n, which results in multiplication by n, there is also a boundary term, which is the ϵ^0-coefficient in the expansion of the integral

$$\int_{|w|=\epsilon} w^{-n} w^{-\bar{n}} \imath_\xi \omega$$

in power series in $\epsilon^{\pm 1}$ and $\log\epsilon$ (the change of sign here is due to the change of orientation of the circle $|z| = \epsilon^{-1}$ under the change of variables $z \mapsto w = z^{-1}$).

Writing $\xi = -w\partial_w$, $\omega = \omega_{w\bar{w}}\,\mathrm{d}w\,\mathrm{d}\bar{w} = \mathrm{i}\omega_{w\bar{w}}\,\mathrm{d}w \wedge \mathrm{d}\bar{w}$ and $w = \epsilon \mathrm{e}^{\mathrm{i}\theta}$, we find that this boundary term is equal to

$$\left[\int_0^{2\pi} w^{1-n}\bar{w}^{1-n}\omega_{w\bar{w}}\,\mathrm{d}\theta\right]_{\epsilon^0} = \begin{cases} -\dfrac{2\pi}{(n-1)!(\bar{n}-1)!}\partial_w^{n-1}\partial_{\bar{w}}^{\bar{n}-1}\omega_{w\bar{w}}\Big|_{w=0} & \text{if } n,\bar{n}>0, \\ 0 & \text{if } n=0 \text{ or } \bar{n}=0. \end{cases}$$

Thus, we obtain the following formula

$$\mathcal{L}_\xi \cdot |n,\bar{n},0,0\rangle_{\mathbb{C}_0} = n|n,\bar{n},0,0\rangle_{\mathbb{C}_0} - 2\pi|n-1,\bar{n}-1,0,0\rangle_\infty.$$

Likewise, we obtain

$$\mathcal{L}_\xi \cdot |n,\bar{n},p,\bar{p}\rangle_{\mathbb{C}_0} = (n+p)|n,\bar{n},p,\bar{p}\rangle_{\mathbb{C}_0} - 2\pi|n+2p-1,\bar{n}+2\bar{p}-1,p,\bar{p}\rangle_\infty, \quad (4.8)$$

$$\mathcal{L}_{\bar{\xi}} \cdot |n,\bar{n},p,\bar{p}\rangle_{\mathbb{C}_0} = (\bar{n}+\bar{p})|n,\bar{n},p,\bar{p}\rangle_{\mathbb{C}_0} - 2\pi|n+2p-1,\bar{n}+2\bar{p}-1,p,\bar{p}\rangle_\infty. \quad (4.9)$$

Here we use the convention

$$|n,\bar{n},p,\bar{p}\rangle_\infty \equiv 0 \quad \text{if } n<0 \text{ or } \bar{n}<0.$$

Thus, we find that the operators $\mathcal{L}_\xi$ and $\mathcal{L}_{\bar{\xi}}$, and hence the Hamiltonian $\mathcal{L}_v$, have Jordan blocks of length two. The generalized eigenspace of the operators $\mathcal{L}_\xi$ and $\mathcal{L}_{\bar{\xi}}$ corresponding to the eigenvalues $n+p \geqslant 0$ and $\bar{n}+\bar{p} \geqslant 0$ on the space of $(p,\bar{p})$-forms in $\mathcal{H}^{\mathrm{in}}$ is two dimensional, spanned by the vectors $|n,\bar{n},p,\bar{p}\rangle_{\mathbb{C}_0}$ and $|n+2p-1,\bar{n}+2p-1,p,\bar{p}\rangle_\infty$. The former is an eigenvector, and the second is a generalized eigenvector which is adjoint to it.

In particular, the indeterminacy of the vector $|n,\bar{n},p,\bar{p}\rangle_{\mathbb{C}_0}$ with $n+2p-1 \geqslant 0$ and $\bar{n}+2\bar{p}-1 \geqslant 0$ is contained in the two-dimensional subspace of $\mathcal{H}^{\mathrm{in}}$ spanned by it and $|n+2p-1,\bar{n}+2\bar{p}-1,p,\bar{p}\rangle_\infty$.

This could actually be seen from the outset. As we have explained, the indeterminacy comes from the fact that in the definition of the distribution $\tilde{\phi}_{n,\bar{n}}$ as the 'partie finie' of the integral (4.4) we use the 'cutoff' $|z| < \epsilon^{-1}$, and so if we replace ϵ by $\tilde{\epsilon} = a^{-1}\epsilon$, then the 'partie finie' of the integral will get shifted by the term $-C_{\log}\log a$, where $C_{\log}$ is defined in formula (4.5). But the term $C_{\log}$ is easy to evaluate explicitly. Indeed, we may split the integral (4.4) into the sum of two: over $|z| < 1$ and $1 < |z| < \epsilon^{-1}$. The former converges and hence cannot contribute to $C_{\log}$. On the other hand, writing ω in the form $\omega_{w\bar{w}}\,\mathrm{d}w\,\mathrm{d}\bar{w}$, as before, we rewrite the latter as

$$\int_{\epsilon<|w|<1} w^{-n}\bar{w}^{-\bar{n}}\omega_{w\bar{w}}\,\mathrm{d}w\,\mathrm{d}\bar{w}.$$

The $\log\epsilon$ contribution to this integral is equal to

$$4\pi\log\epsilon\frac{1}{(n-1)!(\bar{n}-1)!}\partial_w^{n-1}\partial_{\bar{w}}^{\bar{n}-1}\omega_{w\bar{w}}\Big|_{w=0}.$$

Thus, by changing the 'cutoff' in the definition of $\tilde{\phi}_{n,\bar{n}}$, we replace $\tilde{\phi}_{n,\bar{n}}$ by its linear combination with

$$\frac{1}{(n-1)!(\bar{n}-1)!}\partial_w^{n-1}\partial_{\bar{w}}^{\bar{n}-1}\delta^{(2)}(w,\bar{w})\,\mathrm{d}w\bar{w},$$

for $n,\bar{n} > 0$.

Note, however, that the extra term is equal to 0 if $n=0$ or $\bar{n}=0$, that is for purely holomorphic or anti-holomorphic functions on $\mathbb{C}_0$. This shows that the distributions $\tilde{\varphi}_{n,0}$ and $\tilde{\varphi}_{0,\bar{n}}$ corresponding to the states $|n,0,0,0\rangle_{\mathbb{C}_0}$ and $|0,\bar{n},0,0\rangle_{\mathbb{C}_0}$, respectively, are actually well defined. In fact, it is easy to see that they have canonical regularizations which are eigenvectors of $\mathcal{L}_\xi$ and $\mathcal{L}_{\bar{\xi}}$. The vectors $|n,0,0,1\rangle_{\mathbb{C}_0}$ and $|0,\bar{n},1,0\rangle_{\mathbb{C}_0}$ are also eigenvectors.

One can analyse the indeterminacy of $(p,\bar{p})$-forms in a similar fashion, reproducing the above result that it is confined to the two-dimensional subspaces of $\mathcal{H}^{\mathrm{in}}$ spanned by $|n,\bar{n},p,\bar{p}\rangle_{\mathbb{C}_0}$ and $|n+2p-1,\bar{n}+2p-1,p,\bar{p}\rangle_\infty$. We will denote this subspace by $\mathcal{H}^{\mathrm{in}}_{n,\bar{n},p,\bar{p}}$.

4.4. Action of the supercharges

Next, we analyse the action of the supercharges. Recall that we have two of them: $Q = \mathrm{d}$, the de Rham differential, and $Q^* = 2\imath_v$, the contraction with the vector field $2v = 2(z\partial_z + \bar{z}\partial_{\bar{z}})$. As in the case of the Hamiltonian, we split each of them into the sum of holomorphic and anti-holomorphic terms: $Q = \partial + \bar{\partial}$ and $\imath_v = \imath_\xi + \imath_{\bar{\xi}}$. These operators naturally act on the space of distributions on $\mathbb{CP}^1$. For instance, we have

$$\langle \partial\varphi, \omega \rangle = -\langle \varphi, \partial\omega \rangle,$$

and so on. As we will see, these operators preserve the subspace $\mathcal{H}^{\text{in}}$. We wish to compute the corresponding action of these operators on $\mathcal{H}^{\text{in}}$.

The operators $\imath_\xi$ and $\imath_{\bar{\xi}}$ are the easiest to compute. Their action is given by the standard formulae on both $\mathcal{H}^{\text{in}}_\infty$ (considered as the space of delta-forms supported at $\infty \in \mathbb{CP}^1$) and $\mathcal{H}^{\text{in}}_{\mathbb{C}_0}$ (considered as the space of polynomial differential forms on $\mathbb{C}_0$).

The action of ∂ and $\bar{\partial}$ on $\mathcal{H}^{\text{in}}_\infty$ is also the obvious one. However, due to boundary terms similar to the ones arising in the above calculation of the Hamiltonian, the action of ∂ and $\bar{\partial}$ on the subspace $\mathcal{H}^{\text{in}}_{\mathbb{C}_0}$ has correction terms which belong to $\mathcal{H}^{\text{in}}_\infty$.

To see how this works, let us compute explicitly $\partial \cdot |n, \bar{n}, 0, 0\rangle_{\mathbb{C}_0}$. It follows from the definition that this is the distribution, whose value on a 1-form $\omega = \omega_{\bar{w}}\, \mathrm{d}\bar{w}$ on $\mathbb{CP}^1$ is equal to

$$\tilde{\varphi}_{n,\bar{n}}(-\partial\omega) = -\int\!\!\!\!\!\!\diagup_{|z|<\epsilon^{-1}} z^n \bar{z}^{\bar{n}} \partial\omega.$$

Using the Stokes formula, we find that this integral has two terms: one corresponds to the obvious action of ∂ on $z^n \bar{z}^{\bar{n}}$, sending it to $n z^{n-1} \bar{z}^{\bar{n}}\, \mathrm{d}z$, and the other is the boundary term

$$\left[\int_{|w|=\epsilon} w^{-n} \bar{w}^{-\bar{n}} \omega_{\bar{w}}\, \mathrm{d}\bar{w} \right]_{\epsilon^0} = -\frac{2\pi \mathrm{i}}{n!(\bar{n}-1)!} \partial_w^n \partial_{\bar{w}}^{\bar{n}-1} \omega_{w\bar{w}} \bigg|_{w=0}.$$

Therefore, we obtain that

$$\partial |n, \bar{n}, 0, 0\rangle_{\mathbb{C}_0} = n|n-1, \bar{n}, 1, 0\rangle_{\mathbb{C}_0} + 2\pi |n, \bar{n}-1, 1, 0\rangle_\infty$$

(see formula (4.7)).

Similarly, we find that

$$\bar{\partial} |n, \bar{n}, 0, 0\rangle_{\mathbb{C}_0} = n|n, \bar{n}-1, 0, 1\rangle_{\mathbb{C}_0} + 2\pi |n-1, \bar{n}, 0, 1\rangle_\infty,$$

and obtain analogous formulae for differential forms of higher degrees. These formulae may be summarized as follows.

Let us use holomorphic factorization and realize the spaces $\mathcal{H}^{\text{in}}_{\mathbb{C}_0}$ and $\mathcal{H}_\infty$ as the tensor products

$$\mathcal{H}^{\text{in}}_{\mathbb{C}_0} = \mathbb{C}[z] \otimes \Lambda[\mathrm{d}z] \otimes \mathbb{C}[\bar{z}] \otimes \Lambda[\mathrm{d}\bar{z}],$$
$$\mathcal{H}^{\text{in}}_\infty = (\mathbb{C}[w, w^{-1}]/\mathbb{C}[w]) \otimes \Lambda[\mathrm{d}w] \otimes (\mathbb{C}[\bar{w}, \bar{w}^{-1}]/\mathbb{C}[\bar{w}]) \otimes \Lambda[\mathrm{d}\bar{w}].$$

In this realization our basis elements of $\mathcal{H}^{\text{in}}_\infty$ correspond to

$$|n, \bar{n}, p, \bar{p}\rangle_\infty = \frac{(-1)^{n+\bar{n}}}{n!\bar{n}!} \partial_w^n \partial_{\bar{w}}^{\bar{n}} \delta^{(2)} (\mathrm{d}w)^p (\mathrm{d}\bar{w})^{\bar{p}} = w^{-n-1} (\mathrm{d}w)^p \otimes \bar{w}^{-\bar{n}-1} (\mathrm{d}\bar{w})^{\bar{p}}.$$

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This realization is convenient, because we can use the natural linear maps

$$\delta\colon \mathbb{C}[z]\otimes\Lambda[\mathrm{d}z]\to(\mathbb{C}[w,w^{-1}]/\mathbb{C}[w])\otimes\Lambda[\mathrm{d}w],$$
$$\bar\delta\colon \mathbb{C}[\bar z]\otimes\Lambda[\mathrm{d}\bar z]\to(\mathbb{C}[\bar w,\bar w^{-1}]/\mathbb{C}[\bar w])\otimes\Lambda[\mathrm{d}\bar w],$$

obtained using the composition

$$\mathbb{C}[z]\to\mathbb{C}[z,z^{-1}]=\mathbb{C}[w,w^{-1}]\to\mathbb{C}[w,w^{-1}]/\mathbb{C}[w]$$

(we recall that $w=z^{-1}$). As we will see in § 4.8, these are the simplest examples of the Grothendieck–Cousin operators.

Then the formulae for ∂ and $\bar\partial$ are expressed in terms of these operators as follows:

$$\partial(\Psi\otimes\bar\Psi)=\partial_{\text{naive}}(\Psi\otimes\bar\Psi)+2\pi\delta\biggl(\frac{\mathrm{d}w}{w}\wedge\Psi\biggr)\otimes\bar\delta(\bar\Psi),$$
$$\bar\partial(\Psi\otimes\bar\Psi)=\bar\partial_{\text{naive}}(\Psi\otimes\bar\Psi)+2\pi\delta(\Psi)\otimes\bar\delta\biggl(\frac{\mathrm{d}\bar w}{\bar w}\wedge\bar\Psi\biggr)$$

(here we use the convention that $\delta(\mathrm{d}w/w)=\delta(\mathrm{d}\bar w/\bar w)=0$).

Using this formula and the Cartan formula $\mathcal{L}_v=\{\mathrm{d},\imath_v\}$, we obtain the following formula for the chiral and anti-chiral components of the Hamiltonian:

$$\mathcal{L}_\xi=\mathcal{L}_{\xi,\text{naive}}-2\pi\delta\otimes\bar\delta,$$
$$\mathcal{L}_{\bar\xi}=\mathcal{L}_{\bar\xi,\text{naive}}-2\pi\delta\otimes\bar\delta,$$

so that the Hamiltonian is given by the formula

$$H=\tfrac{1}{2}\{Q,Q^*\}=H_{\text{naive}}-4\pi\delta\otimes\bar\delta.$$

This agrees with formulae (4.8) and (4.9).

4.5. The space of states as a $\boldsymbol{\lambda\to\infty}$ limit

The two-dimensional subspaces $\mathcal{H}^{\text{in}}_{n,\bar n,p,\bar p}$ are thus the building blocks of the space of 'in' states of our model at the point $\lambda=\infty$. Using these spaces, we can now clarify the result of the semi-classical analysis of the low-lying eigenvectors of the Hamiltonian $\tilde H_\lambda=\mathcal{L}_v-(1/(2\lambda))\Delta$. This Hamiltonian commutes with the operator $P=\mathcal{L}_{z\partial_z-\bar z\partial_{\bar z}}$ and we will consider their joint eigenstates. For simplicity, we will restrict ourselves to the 0-forms.

According to the semi-classical analysis, for each $n,\bar n\geqslant 0$, there is a two-dimensional space of eigenstates of $\tilde H_\lambda$ and P in the space of functions on $\mathbb{CP}^1$ with the eigenvalues $n+\bar n+O(\lambda^{-1})$ and $n-\bar n$, respectively. These eigenstates are obtained by multiplying by $\mathrm{e}^{\lambda f}$ the eigenstates of the conjugated hermitian operator H_λ, given by formula (3.23), with the same eigenvalues. Let $\mathcal{H}^\lambda_{n,\bar n}$ be the corresponding two-dimensional space of functions on $\mathbb{CP}^1$. Now observe that each smooth function Ψ on $\mathbb{CP}^1$ defines a distribution by the formula

$$\Psi(\omega)=\int_{\mathbb{CP}^1}\Psi\omega,\quad \omega\in\Omega^2(\mathbb{CP}^1).$$

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Therefore, $\mathcal{H}^{\lambda}_{n,\bar{n}}$ defines a two-dimensional subspace of the space $D(\mathbb{CP}^1)$ of distributions on differential forms on $\mathbb{CP}^1$, depending on λ.

We conjecture that the limit of the subspace $\mathcal{H}^{\lambda}_{n,\bar{n}} \subset D(\mathbb{CP}^1)$ as $\lambda \to \infty$ is the subspace $\mathcal{H}^{\mathrm{in}}_{n,\bar{n},0,0}$ introduced above.

We have a similar conjecture for the $(p, \bar{p})$-form subspaces $\mathcal{H}^{\mathrm{in}}_{n,\bar{n},p,\bar{p}}$.

Thus, we conjecture that our subspace $\mathcal{H}^{\mathrm{in}} \subset D(\mathbb{CP}^1)$ naturally appears as the $\lambda \to \infty$ limit of the span of the low-lying eigenstates of the hamiltonian $\tilde{H}_\lambda$, viewed as distributions. Note that the eigenstates of $\tilde{H}_\lambda$ are obtained by multiplying the eigenstates of the hermitian operator H_λ by the function $\mathrm{e}^{\lambda f}$ (which are differential forms on $\mathbb{CP}^1$ viewed as distributions).

It is natural to ask whether we can develop a perturbation theory for the eigenstates at finite λ around the eigenstates $|n, \bar{n}, p, \bar{p}\rangle_{\mathbb{C}_0}$ and $|n + 2p - 1, \bar{n} + 2p - 1, p, \bar{p}\rangle_\infty$ of the theory at $\lambda = \infty$. This will be discussed in § 6.1.

4.6. Definition of the 'out' space and the pairing

We now briefly describe the space of 'out' states in the same way. We have previously said that it is isomorphic to the direct sum

$$\mathcal{H}^{\mathrm{out}} \simeq \mathcal{H}^{\mathrm{out}}_{\mathbb{C}_\infty} \oplus \mathcal{H}^{\mathrm{out}}_0.$$

Now we realize $\mathcal{H}^{\mathrm{out}}$ as a subspace of the space $D(\mathbb{CP}^1)$ of distributions on $\mathbb{CP}^1$, following §§ 4.1 and 4.2. The subspace $\mathcal{H}^{\mathrm{out}}_0$ is by definition the space of distributions on the differential forms on $\mathbb{CP}^1$ supported at the point $0 \in \mathbb{CP}^1$. It is spanned by the eigenstates

$${}_0\langle n, \bar{n}, p, \bar{p}| = \frac{(-1)^{n+\bar{n}}}{n!\bar{n}!} \partial_z^n \partial_{\bar{z}}^{\bar{n}} \delta^{(2)}(z, \bar{z})(\mathrm{d}z)^p(\mathrm{d}\bar{z})^{\bar{p}}.$$

The subspace $\mathcal{H}^{\mathrm{out}}_{\mathbb{C}_\infty}$ is not canonical, but we choose as its basis vectors

$${}_{\mathbb{C}_\infty}\langle n, \bar{n}, p, \bar{p}|$$

the distributions defined by the formula

$${}_{\mathbb{C}_\infty}\langle n, \bar{n}, p, \bar{p}|\omega\rangle = \int\!\!\!\!\!\!\!\!\;\; -_{|w|<\epsilon^{-1}} w^n \bar{w}^{\bar{n}} (\mathrm{d}w)^p (\mathrm{d}\bar{w})^{\bar{p}} \wedge \omega.$$

The above integral needs to be regularized because the differential form has a pole at $w = \infty$ (or $z = 0$). We use the 'partie finie' regularization introduced in § 4.2. If we change the regularization, then the corresponding distribution will get shifted by a distribution supported at 0.

Therefore, the sum of the spaces $\mathcal{H}^{\mathrm{out}}_{\mathbb{C}_\infty}$ and $\mathcal{H}^{\mathrm{out}}_0$ is a well-defined subspace of the space $D(\mathbb{CP}^1)$ of distributions on $\mathbb{CP}^1$, and this is by definition the space $\mathcal{H}^{\mathrm{out}}$ of 'out' states of our model.

The Hamiltonian is $-\mathcal{L}_v = -\mathcal{L}_\xi - \mathcal{L}_{\bar{\xi}}$. The states ${}_0\langle n, \bar{n}, p, \bar{p}|$ are eigenvectors:

$$\begin{aligned} {}_0\langle n, \bar{n}, p, \bar{p}| \cdot -\mathcal{L}_\xi &= (n + 1 - p)\,{}_0\langle n, \bar{n}, p, \bar{p}|, \\ {}_0\langle n, \bar{n}, p, \bar{p}| \cdot -\mathcal{L}_{\bar{\xi}} &= (\bar{n} + 1 - \bar{p})\,{}_0\langle n, \bar{n}, p, \bar{p}|. \end{aligned}$$

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The states ${}_{\mathbb{C}_\infty}\langle n, \bar{n}, p, \bar{p}|$ are generalized eigenvectors (unless $n = 0$, $p = 0$ or $\bar{n} = 0$, $\bar{p} = 0$) which satisfy

$$ {}_{\mathbb{C}_\infty}\langle n,\bar{n},p,\bar{p}| \cdot -\mathcal{L}_\xi = (n+p)_{\mathbb{C}_\infty}\langle n,\bar{n},p,\bar{p}| + 2\pi_0\langle n+2p-1,\bar{n}+2\bar{p}-1,p,\bar{p}|, $$
$$ {}_{\mathbb{C}_\infty}\langle n,\bar{n},p,\bar{p}| \cdot -\mathcal{L}_{\bar{\xi}} = (\bar{n}+\bar{p})_{\mathbb{C}_\infty}\langle n,\bar{n},p,\bar{p}| + 2\pi_0\langle n+2p-1,\bar{n}+2\bar{p}-1,p,\bar{p}|. $$

In particular, we see that, just as in the case of 'in' states, the 'mixing' between $\mathcal{H}^{\mathrm{out}}_{\mathbb{C}_\infty}$ and $\mathcal{H}^{\mathrm{out}}_0$ is confined to the two-dimensional generalized eigenspaces of $\mathcal{L}_\xi$ and $\mathcal{L}_{\bar{\xi}}$.

As in the case of 'in' states, we conjecture that the subspace $\mathcal{H}^{\mathrm{out}} \subset D(\mathbb{CP}^1)$ naturally appears as the $\lambda \to \infty$ limit of the span of the low-lying eigenstates of the hermitian operator H_λ by the function $\mathrm{e}^{-\lambda f}$ (which are differential forms on $\mathbb{CP}^1$, viewed as distributions).

On general grounds (see § 3.2) we expect that there is a canonical pairing

$$ \mathcal{H}^{\mathrm{in}} \times \mathcal{H}^{\mathrm{out}} \to \mathbb{C}. $$

Explicitly, it is defined as follows: let $\Psi \in \mathcal{H}^{\mathrm{in}}$ and $\Phi \in \mathcal{H}^{\mathrm{out}}$. Then

$$ \langle \Phi, \Psi \rangle = \int_{\mathbb{CP}^1} \Phi \wedge \Psi. $$

Some comments are necessary here, because *a priori* Ψ and Φ are distributions and so it is not clear what the above integral means. However, Ψ and Φ are distributions of a very special kind, and for those this integral is well defined. Intuitively, this is because Ψ may be viewed a distribution in a small neighbourhood of ∞ and a function elsewhere, while Φ is a distribution in a small neighbourhood of 0 and a function elsewhere.

More precisely, we write $\Phi = \Phi_0 + \Phi_\infty$, $\Psi = \Psi_0 + \Psi_\infty$, where $\Phi_0 = c_{|z|\leqslant 1}\Phi$, $\Phi_\infty = c_{|z|\geqslant 1}\Phi$ and $c_{|z|\leqslant 1}$, $c_{|z|\geqslant 1}$ are the characteristic functions of the discs $|z| \leqslant 1$ and $|z| \geqslant 1$. We then split the above integral into the sum

$$ \int_{|z|\leqslant 1} \Phi_0 \wedge \Psi_0 + \int_{|z|\geqslant 1} \Phi_\infty \wedge \Psi_\infty. $$

In the first summand Φ_0 is a smooth function, whereas Ψ_0 is a distribution. Therefore, their pairing is well defined. The second summand is well defined for the same reason, with the roles of Φ and Ψ reversed.

This pairing is especially easy to describe when either $\Psi \in \mathcal{H}^{\mathrm{in}}_\infty$ or $\Phi \in \mathcal{H}^{\mathrm{out}}_0$. First of all, if both inclusions are satisfied, then they are distributions supported at two different points on $\mathbb{CP}^1$: ∞ and 0, respectively. Therefore, the pairing between them is equal to 0.

If $\Phi \in \mathcal{H}^{\mathrm{out}}_0$, then

$$ \langle \Phi, \Psi \rangle = \int \Phi \wedge \mathrm{pr}(\Psi) = \Phi(\mathrm{pr}(\Psi)), $$

the evaluation of the distribution Φ on the projection $\mathrm{pr}(\Psi)$ of Ψ onto $\mathcal{H}^{\mathrm{in}}_{\mathbb{C}_0} \simeq \Omega(\mathbb{C}_0)$, which is a differential form well defined in the neighbourhood of $0 \in \mathbb{CP}^1$.

Likewise, if $\Psi \in \mathcal{H}^{\mathrm{out}}_\infty$, then

$$ \langle \Phi, \Psi \rangle = \int \mathrm{pr}(\Phi) \wedge \Psi = \Psi(\mathrm{pr}(\Phi)), $$

the evaluation of the distribution Ψ on the projection $\mathrm{pr}(\Phi)$ of Φ onto $\mathcal{H}^{\mathrm{in}}_{\mathbb{C}_\infty} \simeq \Omega(\mathbb{C}_\infty)$, which is a differential form well defined in the neighbourhood of $\infty \in \mathbb{CP}^1$.

We find from this description that

$$\begin{aligned} {}_0\langle m, \bar{m}, r, \bar{r} | n, \bar{n}, p, \bar{p} \rangle_{\mathbb{C}_0} &= {}_{\mathbb{C}_\infty}\langle m, \bar{m}, r, \bar{r} | n, \bar{n}, p, \bar{p} \rangle_\infty \\ &= (-\mathrm{i})^p \mathrm{i}^{\bar{p}} (-1)^{p\bar{r}} \delta_{n,m} \delta_{\bar{n},\bar{m}} \delta_{p,1-r} \delta_{\bar{p},1-\bar{r}}. \end{aligned} \tag{4.10}$$

Finally, we compute the pairing between the states $|n, \bar{n}, p, \bar{p}\rangle_{\mathbb{C}_0}$ and ${}_{\mathbb{C}_\infty}\langle m, \bar{m}, r, \bar{r}|$. According to the above definition, it is equal to the 'partie finie' of the integral

$$(-1)^{p\bar{r}} \int_{\epsilon_1 < |z| < \epsilon_2^{-1}} z^{n-m-2} \bar{z}^{\bar{n}-\bar{m}-2} (\mathrm{d}z)^{p+r} (\mathrm{d}\bar{z})^{\bar{p}+\bar{r}},$$

which is obtained by discarding all terms that contain $(\epsilon_1)^{-1}, \log \epsilon_1, (\epsilon_2)^{-1}, \log \epsilon_2$ and setting $\epsilon_1, \epsilon_2 = 0$ in the remainder. It is easy to see that the result is always zero.

Note, however, that if we chose different regularization schemes for defining these states (which would shift them by the 'delta-like' states), then the pairing between them would change.

Thus, we find that the bases

$$\{|n, \bar{n}, p, \bar{p}\rangle_{\mathbb{C}_0}, |n, \bar{n}, p, \bar{p}\rangle_\infty\} \quad \text{and} \quad \{{}_0\langle n, \bar{n}, p, \bar{p}|, {}_{\mathbb{C}_\infty}\langle n, \bar{n}, p, \bar{p}|\}$$

are dual to each other in $\mathcal{H}^{\mathrm{in}}$ and $\mathcal{H}^{\mathrm{out}}$, up to a power of i (see formula (4.10)).

4.7. The general case

We now briefly discuss how to generalize the above results to the case of a general Kähler manifold X and a holomorphic vector field ξ satisfying the conditions of § 3.6. Recall that we have two stratifications of X, defined in formula (3.28). In the first approximation we defined the space of 'in' states as the direct sum given by formula (3.35). Each summand $\mathcal{H}^{\mathrm{in}}_\alpha$ consists of delta-forms supported on the stratum X_α. However, this decomposition is not canonical. We now apply the same reasonings as in the case of $X = \mathbb{CP}^1$ to define the space $\mathcal{H}^{\mathrm{in}}$ of 'in' states as a canonical subspace of the space of distributions on differential forms on X.

This space has a canonical filtration $\mathcal{H}^{\mathrm{in}}_{\leqslant i}$, $i = 0, \ldots, \dim_{\mathbb{C}} X$, such that each consecutive quotient $\mathcal{H}^{\mathrm{in}}_{\leqslant i}/\mathcal{H}^{\mathrm{in}}_{\leqslant (i-1)}$ is isomorphic to the direct sum

$$\bigoplus_{\dim_{\mathbb{C}} X_\alpha = i} \mathcal{H}^{\mathrm{in}}_\alpha. \tag{4.11}$$

We construct $\mathcal{H}^{\mathrm{in}}_{\leqslant i}$ by induction on i. The space $\mathcal{H}^{\mathrm{in}}_{\leqslant 0}$ is by definition the space $\mathcal{H}^{\mathrm{in}}_{\alpha_{\max}}$ for the critical point $x_{\alpha_{\max}}$ such that the corresponding stratum $X_{\alpha_{\max}}$ consists of a single point, $x_{\alpha_{\max}}$, which happens when it is the absolute maximum of the Morse function. Thus, $\mathcal{H}^{\mathrm{in}}_{\leqslant 0}$ is the space of distributions on X supported at this point.*

* Here, as above, we use the term 'distribution' to mean functionals on the space of all smooth differential forms, and not just smooth functions

Now suppose that we have already constructed the subspace $\mathcal{H}^{\rm in}_{\leqslant(i-1)}$. Let X_α be a stratum of complex dimension i. We construct a new space $\mathcal{H}^{\rm in}_{\leqslant(i-1),\alpha}$ which is an extension

$$0 \to \mathcal{H}^{\rm in}_{\leqslant(i-1)} \to \mathcal{H}^{\rm in}_{\leqslant(i-1),\alpha} \to \mathcal{H}^{\rm in}_{\alpha} \to 0 \tag{4.12}$$

as follows.

Let $B_\alpha \subset \bar{X}_\alpha \backslash X_\alpha$ be the union of the strata X_β of complex dimension $(i-1)$ that belong to the closure $\bar{X}_\alpha$ of X_α. We denote the set of βs appearing in its decomposition by A_α. We define $\mathcal{S}_\alpha \Omega(X)$ as the space of smooth differential forms on X which decay very fast along B_α. More precisely, we define it as the intersection of the spaces $\mathcal{S}_{\alpha\beta}\Omega(X), \beta \in A_\alpha$, which are constructed as follows.

Consider the union $X_\alpha \bigsqcup X_\beta$, where $\beta \in A_\alpha$. It is isomorphic to a fibration over $\mathbb{CP}^1$, with fibres being vector spaces isomorphic to X_β. The stratum X_β is embedded as the fibre at $\infty \in \mathbb{CP}^1$ and the stratum X_α as its complement, the preimage of $\mathbb{C}_0 = \mathbb{CP}^1 \backslash \infty$. There is a section $\mathbb{CP}^1 \to X_\alpha \bigsqcup X_\beta$ such that the image of $\infty \in \mathbb{CP}^1$ is the critical point x_β, the image of $0 \in \mathbb{CP}^1$ is the critical point x_α, and the image of $\mathbb{C}^\times \subset \mathbb{CP}^1$ is the intersection of X_α with the descending manifold X^β corresponding to x_β. Moreover, the $\mathbb{C}^\times$-action on X restricted to $X_\alpha \bigsqcup X_\beta$ lifts the standard $\mathbb{C}^\times$-action on $\mathbb{CP}^1$.

The function w on $\mathbb{CP}^1$ which has a zero of order one at ∞ and a pole of order one at 0 pulls back to a function on $X_\alpha \bigsqcup X_\beta$ which we denote by $w_{\alpha\beta}$. Note that X_β is the zero set of $w_{\alpha\beta}$ and that $\xi \cdot w_{\alpha\beta} = a_{\alpha\beta} w_{\alpha\beta}$, where $a_{\alpha\beta}$ is a negative real number.

Now the condition on $\omega \in \Omega(X)$ to belong to $\mathcal{S}_{\alpha\beta}\Omega(X)$ is that after we apply to ω any sequence of Lie derivatives with respect to vector fields defined in a neighbourhood of $X_\alpha \bigsqcup X_\beta$, the restriction of the resulting form to $X_\alpha \bigsqcup X_\beta$ will tend to zero as $w_{\alpha\beta} \to 0$ faster than any polynomial in $w_{\alpha\beta}$.

Now we define $\mathcal{S}_\alpha \Omega(X)$ as the intersection of the spaces $\mathcal{S}_{\alpha\beta}\Omega(X), \beta \in A_\alpha$.

Any vector Ψ in the space $\mathcal{H}^{\rm in}_\alpha$ of delta-forms on X_α gives rise to a linear functional on $\mathcal{S}_\alpha \Omega(X)$ whose value on $\omega \in \mathcal{S}_\alpha \Omega(X)$ is given by the integral

$$\langle \Psi, \omega \rangle = \int_X \Psi \wedge \omega.$$

The integral converges because of the conditions we imposed on ω.

Now we wish to extend this functional to a distribution on all smooth differential forms on X. Such an extension is constructed by introducing a 'cutoff', following the example of $\mathbb{CP}^1$ discussed in § 4.2. Namely, to define its value on $\omega \in \Omega(X)$ we take the integral of $\Psi \wedge \omega$ over X minus the union of subsets in X_α defined by the inequalities $|w_{\alpha\beta}| < \epsilon$, for all $\beta \in A_\alpha$. Then we take the 'partie finie' of the corresponding integral, i.e. discard all terms involving negative powers of ϵ and $\log \epsilon$ and set $\epsilon = 0$ in the remainder.

The resulting functional is regularization dependent because it depends on the choice of the functions $w_{\alpha\beta}$. But the difference of two possible regularizations is a distribution which belongs to the previously constructed space $\mathcal{H}^{\rm in}_{\leqslant(i-1)}$. Therefore, we obtain an extension $\mathcal{H}^{\rm in}_{\leqslant(i-1),\alpha}$ as in (4.12).

Finally, we define $\mathcal{H}^{\rm in}_{\leqslant i}$ as the sum of the extensions (4.12) over all $\alpha \in A$ such that $\dim_{\mathbb{C}} X_\alpha = i$. This completes the inductive construction of the space of 'in' states as

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a well-defined canonical subspace of the space $D(X)$ of all distributions (on differential forms) on X.

Recall the hamiltonian $\tilde{H}_\lambda = \mathcal{L}_v - (1/(2\lambda))\Delta$ which is the regularized version of our Hamiltonian $\mathcal{L}_v$. Its 'low-lying' eigenfunctions are by definition the eigenfunctions whose eigenvalues are equal to $C + O(\lambda^{-1})$, where C is a constant. We recall that each of these eigenfunctions is equal to $\mathrm{e}^{\lambda f}$ times an eigenfunction of the hermitian Hamiltonian H_λ given by formula (2.4).

We conjecture that the space $\mathcal{H}^{\mathrm{in}}$ appears as the limit of the span of the low-lying eigenfunctions of the Hamiltonian $\tilde{H}_\lambda$ (considered as distributions).

The space $\mathcal{H}^{\mathrm{out}}$ of 'out' states is defined in the same way, by using the opposite stratification by the submanifolds $X^\alpha, \alpha \in A$.

Finally, we define a pairing

$$\mathcal{H}^{\mathrm{in}} \times \mathcal{H}^{\mathrm{out}} \to \mathbb{C},$$
$$\langle \Phi, \Psi \rangle \mapsto \int_X \Phi \wedge \Psi.$$

To see that this integral makes sense, we argue in the same way as in the case of $\mathbb{CP}^1$ (see § 4.6).

4.8. Action of the supercharges and the Hamiltonian

The supercharges of our theory: $Q = \mathrm{d}$, the de Rham differential, and $Q^* = 2\imath_v$, and the Hamiltonian, $H = \mathcal{L}_v = \mathcal{L}_\xi + \mathcal{L}_{\bar{\xi}}$, act naturally on the space $\mathcal{H}^{\mathrm{in}}$, and their adjoints act on $\mathcal{H}^{\mathrm{out}}$. However, this action is rather complicated because elements of the space $\mathcal{H}^{\mathrm{in}}$ are constructed by a non-trivial regularization procedure. This procedure distorts the action of these operators and as the result they acquire correction terms. We have analysed this in the case of $X = \mathbb{CP}^1$ in § 4.3 and we have seen that as the result of these correction terms the Hamiltonian is non-diagonalizable. The same happens for general X and ξ satisfying the assumptions of § 3.6. From now on we will assume in addition that X is a projective algebraic variety.

We will now describe a model for the space of 'in' states in which the action of the supersymmetry charges and the Hamiltonian are given by very transparent formulae. This generalizes the formulae in the case of $X = \mathbb{CP}^1$ presented in § 4.4. The key ingredients are the GC boundary operators which act between the spaces of delta-forms supported on the strata X_α and X_β of our decomposition, with X_β being a codimension one stratum in the closure of X_α. We will write $X_\alpha \succ X_\beta$ if this is the case. The construction of these operators is explained in detail in [**26**, § 7].

The GC operators act between the spaces of *local cohomology* of a sheaf $\mathcal{F}$ on X with support on locally closed submanifolds of X. Let us recall the definition of the functor of local cohomology. Let Z be a closed submanifold of X. We will say that a section s of a sheaf $\mathcal{F}$ on X is supported on Z if its restriction to $X \backslash Z$ is equal to 0. Let $\Gamma_Z(X, \mathcal{F})$ be the space of sections of $\mathcal{F}$ supported on Z. Thus, we obtain a functor $\mathcal{F} \mapsto \Gamma_Z(X, \mathcal{F})$. It is left exact, but not right exact. We will denote its higher derived functors by $H^i_Z(\mathcal{F})$.

More generally, let Y be a locally closed subset of X such that $Y = Z'\backslash Z$, where $Z \subset Z'$ are two closed subsets of X. Then we denote by $H^i_Y(\mathcal{F})$ the higher derived functors of $\Gamma_Y(X\backslash Z, \mathcal{F}|_{X\backslash Z})$.

Using standard technique of homological algebra, we then obtain boundary maps

$$H^i_Y(\mathcal{F}) \to H^{i+1}_Z(\mathcal{F})$$

for any sheaf $\mathcal{F}$ on X and a pair of closed subsets $Z \subset Z'$ of X and $Y = Z'\backslash Z$ (see [**26**, § 7] for the precise definition). These are the GC operators that we will need.

Consider the special case when Y is an open subset of X and Z is a divisor. In this case the GC operator may be described in very concrete terms. For simplicity suppose that Z is a smooth divisor in a smooth (possibly non-compact) algebraic variety X such that Z and $Y = X\backslash Z$ are affine algebraic varieties. Suppose that our sheaf $\mathcal{F}$ is a holomorphic vector bundle $\mathcal{E}$ on X. Since Y is open and dense in X, we have $H^i_Y(\mathcal{E}) = H^i(Y, \mathcal{E}|_Y)$. From now on we will simply write $\mathcal{E}$ for $\mathcal{E}|_Y$. Since Y is assumed to be affine, $H^i(Y, \mathcal{E}) = 0$ for $i > 0$ and $H^0(Y, \mathcal{E})$ is the space of (regular) holomorphic sections of $\mathcal{E}$ on Y. On the other hand, consider $H^1_Z(\mathcal{E})$, the first local cohomology of $\mathcal{E}$ with support on Z. To define it, choose another smooth divisor Z_1 in X such that $Z \cap Z_1 = \emptyset$ and $X\backslash Z_1$ is affine. Then we have the following exact sequence

$$0 \to H^0(X\backslash Z_1, \mathcal{E}) \to H^0(X\backslash(Z \sqcup Z_1), \mathcal{E}) \to H^1_Z(\mathcal{E}) \to 0,$$

which allows us to define $H^1_Z(\mathcal{E})$ as the quotient

$$H^1_Z(\mathcal{E}) \simeq H^0(X\backslash(Z \sqcup Z_1), \mathcal{E})/H^0(X\backslash Z_1, \mathcal{E}).$$

This definition is independent of the choice of Z_1 satisfying the above conditions.

Now we define the GC operator corresponding to the pair (Y, Z),

$$H^0(Y, \mathcal{E}) \to H^1_Z(\mathcal{E}),$$

as the composition

$$H^0(Y, \mathcal{E}) \to H^0(X\backslash(Z \sqcup Z_1), \mathcal{E}) \to H^1_Z(\mathcal{E}). \tag{4.13}$$

Informally, it corresponds to taking the polar part along Z of meromorphic sections of $\mathcal{E}$ defined on a small neighbourhood of Z, which are allowed to have poles only along Z.

For example, if $X = \mathbb{CP}^1$, $Z = \infty$, $Z_1 = 0$, so that $Y = \mathbb{C}_0 = \mathbb{CP}^1\backslash\infty$, and $\mathcal{E} = \mathcal{O}$ is the trivial line bundle, then

$$H^0(Y, \mathcal{E}) = \mathbb{C}[w^{-1}], \qquad H^1_Z(\mathcal{O}) = \mathbb{C}[w, w^{-1}]/\mathbb{C}[w], \tag{4.14}$$

where w is a function on $\mathbb{CP}^1$ which vanishes at ∞ to order one and has a pole of order one at 0.* The corresponding GC operator (4.13) is just the natural map

$$\mathbb{C}[w^{-1}] \to \mathbb{C}[w, w^{-1}]/\mathbb{C}[w], \tag{4.15}$$

* Note that we have already encountered this space in our discussion of 'holomorphic delta-functions' in § 3.7.

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obtained by composing the maps

$$\mathbb{C}[w^{-1}] \to \mathbb{C}[w, w^{-1}] \quad \text{and} \quad \mathbb{C}[w, w^{-1}] \to \mathbb{C}[w, w^{-1}]/\mathbb{C}[w].$$

Now let us return to our situation. So we have a projective algebraic variety X of complex dimension n with a stratification by smooth locally closed strata $X_\alpha, \alpha \in A$, isomorphic to $\mathbb{C}^{n_\alpha}$. Let $\mathcal{E}$ be a holomorphic vector bundle. Then for each stratum X_α of complex dimension i one defines the local cohomology groups $H^{n-i}_{X_\alpha}(\mathcal{E})$ of $\mathcal{E}$ with support on X_α. One can show that

$$H^{n-i}_{X_\alpha}(\mathcal{E}) = 0, \quad i \neq n_\alpha,$$

so the local cohomology is non-trivial only in dimension $n - n_\alpha$, which is the codimension of X_α.

How to relate this discussion to our space of 'in' states? Recall from § 3.8 that we have holomorphic factorization

$$\mathcal{H}^{\text{in}}_\alpha = \mathcal{F}^{\text{in}}_\alpha \otimes \bar{\mathcal{F}}^{\text{in}}_\alpha,$$

where $\mathcal{F}^{\text{in}}_\alpha$ and $\bar{\mathcal{F}}^{\text{in}}_\alpha$ are the spaces of holomorphic and anti-holomorphic delta-forms supported on X_α, respectively. The point is that $\mathcal{F}^{\text{in}}_\alpha$ is precisely the local cohomology $H^{n-n_\alpha}_{X_\alpha}(\mathcal{E})$, for $\mathcal{E} = \Omega_{X,\text{hol}}$, the sheaf of holomorphic differential forms on X (and similarly for $\bar{\mathcal{F}}^{\text{in}}_\alpha$):

$$\mathcal{F}^{\text{in}}_\alpha = H^{n-n_\alpha}_{X_\alpha}(\Omega_{X,\text{hol}}). \tag{4.16}$$

Now for each pair of strata such that $X_\alpha \succ X_\beta$ (which means that X_β is a codimension one stratum in the closure of X_α) there is a canonical GC operator

$$\delta_{\alpha\beta}\colon H^{n-n_\alpha}_{X_\alpha}(\mathcal{E}) \to H^{n-n_\beta}_{X_\beta}(\mathcal{E}).$$

Therefore, we obtain canonical maps

$$\delta_{\alpha\beta}\colon \mathcal{F}^{\text{in}}_\alpha \to \mathcal{F}^{\text{in}}_\beta$$

for all $\alpha, \beta \in A$ such that $X_\alpha \succ X_\beta$.

Likewise, we have anti-holomorphic analogues of these maps:

$$\bar{\delta}_{\alpha\beta}\colon \bar{\mathcal{F}}^{\text{in}}_\alpha \to \bar{\mathcal{F}}^{\text{in}}_\beta$$

for $X_\alpha \succ X_\beta$.

Now we use these maps to write formulae for Q, Q^* and H. Actually, all of these operators decompose into sums of holomorphic and anti-holomorphic parts:

$$Q = \mathrm{d} = \partial + \bar{\partial}, \qquad Q^* = 2\imath_v = 2(\imath_\xi + \imath_{\bar{\xi}}),$$
$$H = \mathcal{L}_v = \mathcal{L}_\xi + \mathcal{L}_{\bar{\xi}},$$

and we will write separate formulae for these parts.

Let us choose, as in § 4.7, a function $w_{\alpha\beta}$ on $X_\alpha \bigsqcup X_\beta$ such that X_β is the divisor of zeros of $w_{\alpha\beta}$. Once we choose these coordinates, we obtain particular regularizations of

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our integrals, as explained in the previous section, and hence we may identify $\mathcal{H}^{\text{in}}$ with the direct sum $\bigoplus_{\alpha\in A}\mathcal{H}^{\text{in}}_\alpha$. This gives us a concrete realization of the space of 'in' states, which is more convenient for computations than its more abstract definition as a subspace of the space of distributions on X. We now describe the action of the supersymmetry charges and the Hamiltonian on the space of states in this realization.

Let

$$\Psi = (\Psi_\alpha \otimes \bar{\Psi}_\alpha) \in \bigoplus_{\alpha\in A} \mathcal{F}^{\text{in}}_\alpha \otimes \bar{\mathcal{F}}^{\text{in}}_\alpha \simeq \mathcal{H}^{\text{in}}.$$

In the same way as in the case of $\mathbb{CP}^1$ (see § 4.4) we obtain the following formulae for the action of these operators:

$$\partial\Psi = \partial_{\text{naive}}\Psi + 2\pi \sum_{\beta;X_\alpha\succ X_\beta} \delta_{\alpha\beta}\left(\frac{\mathrm{d}w_{\alpha\beta}}{w_{\alpha\beta}} \wedge \Psi_\alpha\right) \otimes \bar{\delta}_{\alpha\beta}(\bar{\Psi}_\alpha), \tag{4.17}$$

$$\bar{\partial}\Psi = \bar{\partial}_{\text{naive}}\Psi + 2\pi \sum_{\beta;X_\alpha\succ X_\beta} \delta_{\alpha\beta}(\Psi_\alpha) \otimes \bar{\delta}_{\alpha\beta}\left(\frac{\mathrm{d}\bar{w}_{\alpha\beta}}{\bar{w}_{\alpha\beta}} \wedge \bar{\Psi}_\alpha\right), \tag{4.18}$$

$$\mathcal{L}_\xi = \{\partial, \imath_\xi\} = \mathcal{L}_{\xi,\text{naive}} + 2\pi \sum_{\beta;X_\alpha\succ X_\beta} a_{\alpha\beta}\delta_{\alpha\beta} \otimes \bar{\delta}_{\alpha\beta}, \tag{4.19}$$

$$\mathcal{L}_{\bar{\xi}} = \{\bar{\partial}, \imath_{\bar{\xi}}\} = \mathcal{L}_{\bar{\xi},\text{naive}} + 2\pi \sum_{\beta;X_\alpha\succ X_\beta} a_{\alpha\beta}\delta_{\alpha\beta} \otimes \bar{\delta}_{\alpha\beta}, \tag{4.20}$$

where the numbers $a_{\alpha\beta}$ are defined by the formula $\xi \cdot w_{\alpha\beta} = a_{\alpha\beta}w_{\alpha\beta}$, so that

$$\imath_\xi \cdot \frac{\mathrm{d}w_{\alpha\beta}}{w_{\alpha\beta}} = a_{\alpha\beta}$$

(see § 5.5 below, where we discuss explicitly the example of $X = \mathbb{CP}^2$). Therefore, we find that

$$H = \{\mathrm{d}, \imath_v\} = H_{\text{naive}} + 4\pi \sum_{X_\alpha\succ X_\beta} a_{\alpha\beta}\delta_{\alpha\beta} \otimes \bar{\delta}_{\alpha\beta}.$$

The operators $\imath_\xi$ and $\imath_{\bar{\xi}}$ have no correction terms.

The fact that $\partial^2 = 0$ is the consequence of a non-trivial property of the GC operators: suppose we have four strata X_α, $X_{\beta'}$, $X_{\beta''}$, X_γ, such that $X_\alpha \succ X_{\beta'} \succ X_\gamma$ and $X_\alpha \succ X_{\beta''} \succ X_\gamma$. Then we have

$$\delta_{\beta'\gamma} \circ \delta_{\alpha\beta'} = \delta_{\beta''\gamma} \circ \delta_{\alpha\beta''} \tag{4.21}$$

(see [**26**]). The fact that $\bar{\partial}^2 = 0$ is proved in the same way.

While the identification of $\mathcal{H}^{\text{in}}$ with the direct sum $\bigoplus_{\alpha\in A}\mathcal{H}^{\text{in}}_\alpha$ depends on the choice of the coordinates $w_{\alpha\beta}$, we see that the formulae for the operators ∂ and $\bar{\partial}$ depend on their logarithmic derivatives, and the formulae for the Hamiltonian and its holomorphic and anti-holomorphic parts only depend on the eigenvalues of ξ on these coordinates.

Thus, we find that, as in the $\mathbb{CP}^1$ model, the hamiltonian is non-diagonalizable. It has off-diagonal terms which are given by the GC operators.

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Remark. We want to stress that the appearance of Jordan blocks in the Hamiltonian is tied up with the assumptions we have made in § 3.6 about the manifold X being Kähler and the gradient vector field of the Morse function f coming from a $\mathbb{C}^\times$-action. If we allow more general Morse functions, then the spectrum of the Hamiltonian in our model may (and generically will) be non-degenerate, and hence the Hamiltonian will be diagonalizable. This is related to the well-known property of the 'partie finie' regularization: the functions $x^{-\alpha}$, where α is not a positive integer, have canonical extensions to homogeneous distributions on the line, unlike the functions with $\alpha \in \mathbb{Z}_{>0}$ which we discussed above (see [**20**, **23**]).

4.9. Cohomology of the supercharges

A natural application of the above formula for the supercharge Q is to use it to compute its cohomology on the space $\mathcal{H}^{\text{in}}$ of states of our theory and check that it coincides with $H^\bullet(X, \mathbb{C})$.

We compute this cohomology by using a spectral sequence. Consider the filtration $\mathcal{H}^{\text{in}}_{\leqslant i}$ introduced in § 4.7. According to formulae (4.17) and (4.18), the supercharge Q preserves this filtration. Therefore, we may compute the cohomology of Q by using the spectral sequence associated to this filtration. The 0th term of this spectral sequence is

$$\bigoplus_{i\geqslant 0} \mathcal{H}^{\text{in}}_{\leqslant i}/\mathcal{H}^{\text{in}}_{\leqslant (i-1)} = \bigoplus_{\alpha\in A} \mathcal{H}^{\text{in}}_{\alpha}. \tag{4.22}$$

Let us compute the 0th differential. We find that the second terms in formulae (4.17) and (4.18) map $\mathcal{H}^{\text{in}}_{\leqslant i}$ to $\mathcal{H}^{\text{in}}_{\leqslant (i-1)}$. Therefore, the corresponding differential on $\bigoplus_{\alpha\in A} \mathcal{H}^{\text{in}}_{\alpha}$ is just the de Rham differential.

It is easy to see from the description of $\mathcal{H}^{\text{in}}_{\alpha}$ given in § 3.8 that the cohomology of the de Rham differential acting on $\mathcal{H}^{\text{in}}_{\alpha}$ is one dimensional, occurring in cohomological degree $(n - n_\alpha, n - n_\alpha)$ and spanned by the delta-form Δ_α. Therefore, the first term of our spectral sequence is spanned by the delta-forms Δ_α, $\alpha \in A$. We have

$$\mathcal{H}^{\text{in}}_{\alpha} \simeq \mathcal{F}^{\text{in}}_{\alpha} \otimes \bar{\mathcal{F}}^{\text{in}}_{\alpha},$$

where $\mathcal{F}^{\text{in}}_{\alpha}$ and $\bar{\mathcal{F}}^{\text{in}}_{\alpha}$ are the spaces of holomorphic and anti-holomorphic delta-forms on X_α, respectively (see § 3.8). Using local cohomology, we may express them as follows:

$$\mathcal{F}^{\text{in}}_{\alpha} = H^{n-n_\alpha}_{X_\alpha}(\Omega_{X,\text{hol}}), \qquad \bar{\mathcal{F}}^{\text{in}}_{\alpha} = H^{n-n_\alpha}_{X_\alpha}(\Omega_{X,\text{anti-hol}}).$$

With respect to this tensor product decomposition, we have

$$\Delta_\alpha = \Delta^{\text{hol}}_\alpha \otimes \Delta^{\text{anti-hol}}_\alpha,$$

where $\Delta^{\text{hol}}_\alpha$ and $\Delta^{\text{anti-hol}}_\alpha$ are generating vectors of $\mathcal{F}^{\text{in}}_{\alpha}$ and $\bar{\mathcal{F}}^{\text{in}}_{\alpha}$, respectively, considered as $\mathcal{D}$-modules.

According to formulae (4.17), (4.18), the differential d_1 of the first term of our spectral sequence is given by a linear combination of the GC operators:

$$\mathrm{d}_1(\Delta_\alpha) = 2\pi \sum_{\beta; X_\alpha \succ X_\beta} \left(\delta_{\alpha\beta}\left(\frac{\mathrm{d}w_{\alpha\beta}}{w_{\alpha\beta}} \wedge \Delta_\alpha^{\mathrm{hol}} \right) \otimes \bar{\delta}_{\alpha\beta}(\Delta^{\text{anti-hol}}) + \delta_{\alpha\beta}(\Delta_\alpha^{\mathrm{hol}}) \otimes \bar{\delta}_{\alpha\beta}\left(\frac{\mathrm{d}\bar{w}_{\alpha\beta}}{\bar{w}_{\alpha\beta}} \wedge \Delta^{\text{anti-hol}} \right) \right).$$

It follows from the definition that $\Delta_\alpha^{\mathrm{hol}}$ and $\Delta_\alpha^{\text{anti-hol}}$ extend to all strata X_β of codimension 1 in the closure of X_α. Therefore, $\delta_{\alpha\beta}(\Delta_\alpha^{\mathrm{hol}}) = 0$ and $\bar{\delta}_{\alpha\beta}(\Delta_\alpha^{\text{anti-hol}}) = 0$, and so $\mathrm{d}_1(\Delta_\alpha) = 0$. Hence we conclude that d_1, as well as all higher differentials of our spectral sequence, are all equal to zero. Therefore, the cohomology is spanned by the delta-forms Δ_α. These form the dual basis to the homology basis represented by the even-dimensional cycles X_α.

Therefore, we conclude that the cohomology of Q acting on $\mathcal{H}_\alpha^{\mathrm{in}}$ coincides with the cohomology of the de Rham differential and is isomorphic to the cohomology of X, $H^\bullet(X, \mathbb{C})$, as expected.

Let us recall that our supercharge $Q = \mathrm{d}$ has a canonical decomposition $Q = \partial + \bar{\partial}$. Therefore, it is also interesting to compute the cohomology of the differentials ∂ and $\bar{\partial}$ separately. The operators ∂ and $\bar{\partial}$ are quantum mechanical analogues of the left and right moving supercharges in two-dimensional sigma models. According to [**24**, **44**] (see also [**18**, **37**]), the cohomology of the right moving supercharge is a chiral algebra which is closely related to the chiral de Rham complex [**31**] of X. The cohomology of $\bar{\partial}$ may be thought of as a 'baby version' of this chiral algebra. The explicit formulae of the previous section give us an effective tool for computing this cohomology.

This tool is the *Grothendieck–Cousin resolution* (GC resolution for short). This is a complex $C^\bullet(\mathcal{E}) = \bigoplus_{i \geqslant 0} C^i(\mathcal{E})$, defined for a holomorphic vector bundle $\mathcal{E}$ on X, whose cohomology coincides with the cohomology of $\mathcal{E}$, considered as a coherent sheaf on X, $H^\bullet(X, \mathcal{E})$. The ith term $C^i(\mathcal{E})$ of the complex is equal to

$$C^i(\mathcal{E}) = \bigoplus_{\dim X_\alpha = i} H^{n-i}_{X_\alpha}(\mathcal{E}),$$

where $H^{n-i}_{X_\alpha}(\mathcal{E})$ is the local cohomology of $\mathcal{E}$ with support on X_α that was introduced in § 4.8.

The differential $\delta^i \colon C^i(\mathcal{E}) \to C^{i+1}(\mathcal{E})$ is given by the alternating sum of the GC operators $\delta_{\alpha\beta} = \delta^{\mathcal{E}}_{\alpha\beta} \colon H^{n-n_\alpha}_{X_\alpha}(\mathcal{E}) \to H^{n-n_\alpha+1}_{X_\beta}(\mathcal{E})$ introduced in § 4.8:

$$\delta^i = \sum_{\beta; X_\alpha \succ X_\beta} \epsilon_{\alpha\beta}\delta_{\alpha\beta}, \tag{4.23}$$

where $\epsilon_{\alpha\beta} = \pm 1$ are signs chosen so as to ensure that $\delta^{i+1} \circ \delta^i = 0$. The existence of such signs follows from the fact that the GC operators satisfy the identity (4.21) if we have $X_\alpha \succ X_{\beta'} \succ X_\gamma$ and $X_\alpha \succ X_{\beta''} \succ X_\gamma$.

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Under our assumptions on the stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ the jth cohomology of the complex $C^\bullet(\mathcal{E})$ coincides with $H^j(X, \mathcal{E})$ (see [**26**]).

Let us see how this works in the simplest example of $X = \mathbb{CP}^1$ and the stratification $\mathbb{CP}^1 = \mathbb{C}_0 \bigsqcup \infty$.

Consider first the case when $\mathcal{E}$ is the trivial line bundle on $\mathbb{CP}^1$. The corresponding complex $C^\bullet(\mathcal{O})$ looks as follows:

$$H^0(\mathbb{C}_0, \mathcal{O}) \to H^1_\infty(\mathcal{O}).$$

We have already determined these spaces in formula (4.14). The resulting complex is (4.15), where the differential is obtained by composing the maps

$$\mathbb{C}[w^{-1}] \to \mathbb{C}[w, w^{-1}] \quad \text{and} \quad \mathbb{C}[w, w^{-1}] \to \mathbb{C}[w, w^{-1}]/\mathbb{C}[w].$$

It is easy to see that the differential is surjective, and its kernel is one-dimensional and spanned by the constants in $\mathbb{C}[w^{-1}]$. This coincides with the cohomology of $\mathcal{O}$ on $\mathbb{CP}^1$.

More generally, suppose that $\mathcal{E}$ is the line bundle $\mathcal{O}(n)$, $n \in \mathbb{Z}$. Then we can trivialize this line bundle on $\mathbb{C}_0$ and $\mathbb{C}_\infty$, and the transition function is w^n. We still have the identifications (4.14), but now the GC operator is the map (4.15) obtained by composing the embedding $\mathbb{C}[w^{-1}] \to \mathbb{C}[w, w^{-1}]$, multiplication by w^n, and the projection $\mathbb{C}[w, w^{-1}] \to \mathbb{C}[w, w^{-1}]/\mathbb{C}[w]$. As the result, the kernel and the cokernel of the GC operator change: if $n \geqslant 0$, then the kernel is $(n+1)$-dimensional, spanned by $1, w^{-1}, \ldots, w^{-n}$, and the cokernel is zero. If $n < -1$, then the kernel is zero, and the cokernel is spanned by $w^{-1}, \ldots, w^{n+1}$. If $n = -1$, then both kernel and cokernel are zero. Again, we find the agreement with the cohomology $H^i(\mathbb{CP}^1, \mathcal{O}(n))$.

Now we use the GC complex to compute the cohomology of the anti-chiral supercharge $\bar{\partial}$. We compute this cohomology using the spectral sequence associated to the same filtration that we used in the above computation of the cohomology of Q. The 0th term of the spectral sequence is again (4.22) and the 0th differential is the operator $\bar{\partial}$ acting on this space. Now recall that each of the summands in the direct sum (4.22) factorizes into the tensor product

$$\mathcal{H}^{\mathrm{in}}_\alpha = \mathcal{F}^{\mathrm{in}}_\alpha \otimes \bar{\mathcal{F}}^{\mathrm{in}}_\alpha.$$

The operator $\bar{\partial}$ acts along the second factor $\bar{\mathcal{F}}^{\mathrm{in}}_\alpha$. It is easy to see that its cohomology is one dimensional, spanned by the generator $\Delta^{\text{anti-hol}}_\alpha$ of $\bar{\mathcal{F}}^{\mathrm{in}}_\alpha$, which is in cohomological dimension $n - n_\alpha$. Therefore, we obtain that the first term of the spectral sequence is concentrated in the 0th row, and the terms in this row are isomorphic to

$$E_1^{i,0} \simeq \bigoplus_{\alpha \in A;\, i = n - n_\alpha} \mathcal{F}^{\mathrm{in}}_\alpha \otimes \Delta^{\text{anti-hol}}_\alpha.$$

According to formula (4.18), the action of the differential $\bar{\partial}_1$ of the first term of the spectral sequence is given by the formula

$$\bar{\partial}_1(\Psi_\alpha \otimes \Delta^{\text{anti-hol}}_\alpha) = 2\pi \sum_{\beta;\, X_\alpha \succ X_\beta} \delta_{\alpha\beta}(\Psi_\alpha) \otimes \bar{\delta}_{\alpha\beta}\biggl(\frac{\mathrm{d}\bar{w}_{\alpha\beta}}{\bar{w}_{\alpha\beta}} \wedge \Delta^{\text{anti-hol}}_\alpha\biggr).$$

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However, it follows from the definitions that we can normalize the states $\Delta_\alpha^{\text{anti-hol}}$ in such a way that

$$\bar{\delta}_{\alpha\beta}\left(\frac{\mathrm{d}\bar{w}_{\alpha\beta}}{\bar{w}_{\alpha\beta}} \wedge \Delta_\alpha^{\text{anti-hol}}\right) = \epsilon_{\alpha\beta}\Delta_\beta^{\text{anti-hol}},$$

where the sign $\epsilon_{\alpha\beta} = \pm 1$ is due to the fact that we obtain $\Delta_\beta^{\text{anti-hol}}$ by multiplying $\Delta_\alpha^{\text{anti-hol}}$ with $\mathrm{d}\bar{w}_{\alpha\beta}$, and so for fixed β and varying α we obtain different signs in general.

Let us now identify the $(i,0)$ group of the first term of the spectral sequence with

$$E_1^{i,0} = \bigoplus_{\alpha\in A;\, i=n-n_\alpha} \mathcal{F}_\alpha^{\text{in}}$$

by $\Psi_\alpha \otimes \Delta_\alpha^{\text{anti-hol}} \mapsto \Psi_\alpha$. Recall that according to formula (4.16) we have

$$\mathcal{F}_\alpha^{\text{in}} = H_{X_\alpha}^{n-n_\alpha}(\Omega_{X,\text{hol}}),$$

where $\Omega_{X,\text{hol}} = \bigoplus_{j\geqslant 0} \Omega^j_{X,\text{hol}}$ is the sheaf of holomorphic differential forms. Thus,

$$E_1^{i,0} = \bigoplus_{\alpha\in A;\, i=n-n_\alpha} H_{X_\alpha}^{n-n_\alpha}(\Omega_{X,\text{hol}})$$

is precisely the ith term of the GC complex associated with $\Omega_{X,\text{hol}}$. Moreover, we find that the first differential $\bar{\partial}_1^i \colon E_1^{i,0} \to E_1^{i+1,0}$ of our spectral sequence is given by the following formula: for $\Psi = (\Psi_\alpha) \in E_1^{i,0}$ we have (up to the inessential factor of 2π)

$$\bar{\partial}_1(\Psi) = \sum_{\beta;\, X_\alpha \succ X_\beta} \epsilon_{\alpha\beta}\delta_{\alpha\beta}(\Psi_\alpha).$$

This is precisely the differential (4.23) of the GC complex $C^\bullet(\Omega_{X,\text{hol}})$ associated with the sheaf $\Omega_{X,\text{hol}}$. Since the GC complex computes the cohomology of this sheaf, we find that the cohomology of $\bar{\partial}_1$ is equal to

$$H^i(X, \Omega_{X,\text{hol}}).$$

Recall that the first term of our spectral sequence has only one row. Therefore, the spectral sequence collapses in the first term, and we find that this is in fact the answer for the cohomology of $\bar{\partial}$ on $\mathcal{H}^{\text{in}}$.

Since X is Kähler, we have

$$H^k(X,\mathbb{C}) \simeq \bigoplus_{i,j\geqslant 0} H^{j,i}, \qquad H^{j,i} = H^i(X, \Omega^j_{X,\text{hol}}).$$

Thus, the cohomology of the space $\mathcal{H}^{\text{in}}$ of 'in' states with respect to the anti-chiral supercharge $\bar{\partial}$ is isomorphic to the cohomology of X, but with the cohomological grading coming from the anti-holomorphic cohomological degrees of differential forms. However, the holomorphic cohomological degree is preserved by $\bar{\partial}$, so we could consider it as an extra grading on our complex. Thus, we find that the cohomology of the anti-chiral supercharge $\bar{\partial}$ coincides with the cohomology of the full supercharge Q. In § 6.3 we will generalize this result to a class of non-supersymmetric models.

5. Action of observables on the space of states

In the previous section we described the space of states of our quantum mechanical model in the limit $\lambda = \infty$. We have seen that there are actually two spaces, $\mathcal{H}^{\text{in}}$ and $\mathcal{H}^{\text{out}}$, and that their structure is dramatically different from the usual structure in hermitian quantum mechanics. The reason is that before we pass to the limit $\lambda \to \infty$ we make a violent transformation of the states: the 'in' states are multiplied by $\mathrm{e}^{\lambda f}$, and the 'out' states are multiplied by $\mathrm{e}^{-\lambda f}$. In contrast, the *observables* of our theory are getting conjugated by the function $\mathrm{e}^{\lambda f}$. There is a large class of observables, namely, all evaluation observables introduced in § 2.4 (corresponding to smooth differential forms on X), which commute with $\mathrm{e}^{\lambda f}$. Those remain intact in the limit $\lambda \to \infty$. This immediately leads to the question: how do these observables act on the spaces of states, $\mathcal{H}^{\text{in}}$ and $\mathcal{H}^{\text{out}}$?

This is the question that we take up in this section, first in the case of $\mathbb{CP}^1$ and then in the general case. We will see that analytic properties of the observables play an important role in the limit $\lambda \to \infty$. We will also see that factorization of the correlation functions over intermediate states leads to some non-trivial identities on analytic differential forms.

5.1. The case of $\mathbb{CP}^1$

The spaces of states in this case have been described in great detail in the previous sections. The space $\mathcal{H}^{\text{in}}$ has a basis consisting of the states

$$|n, \bar{n}, p, \bar{p}\rangle_{\mathbb{C}_0} \quad \text{and} \quad |n, \bar{n}, p, \bar{p}\rangle_{\infty}, \tag{5.1}$$

and the space $\mathcal{H}^{\text{out}}$ has a basis consisting of the states

$${}_0\langle n, \bar{n}, p, \bar{p}| \quad \text{and} \quad {}_{\mathbb{C}_\infty}\langle n, \bar{n}, p, \bar{p}|. \tag{5.2}$$

Recall that we have defined the vectors in $\mathcal{H}^{\text{in}}_{\mathbb{C}_0}$ and $\mathcal{H}^{\text{out}}_{\mathbb{C}_\infty}$ by using particular regularizations of the integrals of the differential forms $z^n \bar{z}^{\bar{n}} (\mathrm{d}z)^p (\mathrm{d}\bar{z})^{\bar{p}}$ and $w^n \bar{w}^{\bar{n}} (\mathrm{d}w)^p (\mathrm{d}\bar{w})^{\bar{p}}$, respectively, as explained in § 4.2 (recall that z is a coordinate at $0 \in \mathbb{CP}^1$, and $w = z^{-1}$ is a coordinate at ∞). We have seen in § 4.6 that if we choose the 'cutoffs' appearing in these regularized integrals in a compatible way ($|z| < \epsilon^{-1}$ in the first case, $|w| < \epsilon^{-1}$ in the second case), then we have

$${}_{\mathbb{C}_\infty}\langle n, \bar{n}, p, \bar{p}|n, \bar{n}, p, \bar{p}\rangle_{\mathbb{C}_0} = 0,$$

and so the bases (5.1) and (5.2) are dual to each other (up to powers of i). We will use this property in what follows. If one were to choose regularizations of 'in' and 'out' states independently, then some of our formulae below would need to be modified to account for that.

The action of the evaluation observable $\hat{\omega}$ corresponding to a smooth differential form on $\mathbb{CP}^1$ on $\mathcal{H}^{\text{in}}$ and $\mathcal{H}^{\text{out}}$ may be found from the matrix coefficients

$$\langle \Psi^{\text{out}}|\hat{\omega}|\Psi^{\text{in}}\rangle.$$

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It follows from our construction that this matrix coefficient is equal to the integral

$$\int \Psi^{\text{out}} \wedge \omega \wedge \Psi^{\text{in}},$$

understood in the same way as in § 4.6.

This allows us to compute explicitly the action of the evaluation observables. For example, consider the case when ω is a smooth $(r, \bar{r})$-differential form on $\mathbb{CP}^1$, and write

$$\omega = \omega_{z\bar{z}}(z, \bar{z})(\mathrm{d}z)^r(\mathrm{d}\bar{z})^{\bar{r}} = \omega_{w\bar{w}}(w, \bar{w})(\mathrm{d}w)^r(\mathrm{d}\bar{w})^{\bar{r}},$$

so that

$$\omega_{w\bar{w}}(w, \bar{w}) = (-1)^{r+\bar{r}}\omega_{z\bar{z}}(w^{-1}, \bar{w}^{-1})w^{-2r}\bar{w}^{-2\bar{r}}$$

(we recall our convention (3.16)).

Then we find from the above formula that

$$\begin{aligned}
&\hat{\omega}|n, \bar{n}, p, \bar{p}\rangle_{\mathbb{C}_0} \\
&\quad = (-1)^{p\bar{r}} \sum_{m,\bar{m}=0}^{\infty} \frac{1}{m!\bar{m}!} \partial_z^m \partial_{\bar{z}}^{\bar{m}} (\omega_{z\bar{z}} z^n \bar{z}^{\bar{n}}) \Big|_{z=0} |m, \bar{m}, p + r, \bar{p} + \bar{r}\rangle_{\mathbb{C}_0} \\
&\qquad + (-1)^{p+\bar{p}+p\bar{r}} \sum_{m,\bar{m}=0}^{\infty} \int_{|w|<\epsilon^{-1}} \omega_{w\bar{w}} w^{m-n-2p} \bar{w}^{\bar{m}-\bar{n}-2\bar{p}} \,\mathrm{d}w\,\mathrm{d}\bar{w} |m, \bar{m}, p + r, \bar{p} + \bar{r}\rangle_{\infty}
\end{aligned}$$

and

$$\hat{\omega}|n, \bar{n}, p, \bar{p}\rangle_{\infty} = (-1)^{p\bar{r}} \frac{1}{n!\bar{n}!} \sum_{m=0}^{n} \sum_{\bar{m}=0}^{\bar{n}} \partial_w^n \partial_{\bar{w}}^{\bar{n}} (\omega_{w\bar{w}} w^m \bar{w}^{\bar{m}}) \Big|_{w=0} |m, \bar{m}, p + r, \bar{p} + \bar{r}\rangle_{\infty}.$$

Here we use the convention that a state is equal to zero if at least one of its indices takes a value that is not allowed.

Likewise, we find that

$$\begin{aligned}
&{}_{\mathbb{C}_\infty}\langle n, \bar{n}, p, \bar{p}|\hat{\omega} \\
&\quad = (-1)^{p\bar{r}} \sum_{m,\bar{m}=0}^{\infty} \frac{1}{m!\bar{m}!} \partial_w^m \partial_{\bar{w}}^{\bar{m}} (\omega_{w\bar{w}} w^n \bar{w}^{\bar{n}}) \Big|_{w=0} {}_{\mathbb{C}_\infty}\langle m, \bar{m}, p + r, \bar{p} + \bar{r}| \\
&\qquad + (-1)^{p+\bar{p}+p\bar{r}} \sum_{m,\bar{m}=0}^{\infty} \int_{|z|<\epsilon^{-1}} \omega_{z\bar{z}} z^{m-n-2p} \bar{z}^{\bar{m}-\bar{n}-2\bar{p}} \,\mathrm{d}z\,\mathrm{d}\bar{z}\, {}_0\langle m, \bar{m}, p + r, \bar{p} + \bar{r}|
\end{aligned}$$

and

$${}_0\langle n, \bar{n}, p, \bar{p}|\hat{\omega} = (-1)^{p\bar{r}} \frac{1}{n!\bar{n}!} \sum_{m=0}^{n} \sum_{\bar{m}=0}^{\bar{n}} \partial_z^n \partial_{\bar{z}}^{\bar{n}} (\omega_{z\bar{z}} z^m \bar{z}^{\bar{m}}) \Big|_{z=0} {}_0\langle m, \bar{m}, p + r, \bar{p} + \bar{r}|.$$

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The right-hand sides of these formulae, as well as the other formulae that appear below, are in general infinite linear combinations of our states. This means that they are really vectors in the completions of the spaces $\mathcal{H}^{\mathrm{in}}$ and $\mathcal{H}^{\mathrm{out}}$, which are the direct products of the (finite-dimensional) generalized eigenspaces with respect to $\mathcal{L}_\xi$ and $\mathcal{L}_{\bar{\xi}}$. However, these naive completions are too big, and we would like to define some more reasonable subspaces whose elements possess some analytic properties. Here is a possible definition of such a completion $\tilde{\mathcal{H}}^{\mathrm{in}}_{\mathbb{C}_0}$ of $\mathcal{H}^{\mathrm{in}}_{\mathbb{C}_0}$: take the space of all analytic differential forms on C_0 which grow not faster than a polynomial at ∞. It contains the monomials $z^n \bar{z}^{\bar{n}}\,\mathrm{d}z^p\,\mathrm{d}\bar{z}^{\bar{p}}$ that we have considered previously and the products of $z^n \bar{z}^{\bar{n}}$ with analytic differential forms on $\mathbb{CP}^1$. The expansion of such a form at $z = 0$ gives rise to a (possibly infinite) linear combination of our monomial states. The completion of $\mathcal{H}^{\mathrm{in}}$ is then defined as the sum of $\mathcal{H}^{\mathrm{in}}_\infty$ and the subspace of the space of distributions on $\mathbb{CP}^1$ spanned by all possible regularizations of elements of $\tilde{\mathcal{H}}^{\mathrm{in}}_{\mathbb{C}_0}$. This completion of $\mathcal{H}^{\mathrm{in}}$ contains, in particular, all finite linear combinations of the derivatives of the delta-forms supported at ∞ as well as some of their infinite linear combinations.

The completion of $\mathcal{H}^{\mathrm{out}}$ is defined in a similar way. However, we note that the pairing between $\mathcal{H}^{\mathrm{in}}$ and $\mathcal{H}^{\mathrm{out}}$ does not extend to a pairing between their completions. Instead, we have a weaker property. For example, for a 2-form ω in the completion of $\mathcal{H}^{\mathrm{out}}$ and a 0-form f in the completion of $\mathcal{H}^{\mathrm{in}}$ the pairing $\langle \omega, f\rangle$ will in general be a divergent infinite sum. But the pairing $\langle \omega, \phi^*(q)(f)\rangle$, where $\phi^*(q)(f)(z,\bar{z}) = f(qz, \bar{q}\bar{z})$ should converge if $0 < |q| < \delta$ for sufficiently small $\delta > 0$ (which depends on ω and f). We will observe a similar phenomenon in the next section when we discuss factorization of the correlation functions.

5.2. Correlation functions and their factorization over intermediate states

We now compare the above formulae for the matrix elements of evaluation observables attached to differential forms on $\mathbb{CP}^1$ with the exact expression (2.21) for the correlation functions of these observables.

We recall that in general correlation functions are labelled by pairs of critical points x_-, x_+ corresponding to the choice of the 'in' and 'out' vacua. The corresponding path integral is given by the integral over the moduli space $\mathcal{M}_{x_-,x_+}$ of gradient trajectories of the differential forms on X, pulled back to $\mathcal{M}_{x_-,x_+}$ via the evaluation maps.

In our case $X = \mathbb{CP}^1$, there are two critical points: 0 and ∞, and there are three moduli spaces. Two of them, $\mathcal{M}_{0,0}$ and $\mathcal{M}_{\infty,\infty}$, consist of a single point. They correspond to constant maps (taking the value 0 and ∞ in $\mathbb{CP}^1$, respectively). The only non-trivial moduli space is $\mathcal{M}_{0,\infty}$ which corresponds to the only possible instanton transition: gradient trajectories going from $x_- = 0$ to $x_+ = \infty$. This moduli space is isomorphic to $\mathbb{C}^\times$, which is naturally isomorphic to the subset $\mathbb{CP}^1\backslash\{0,\infty\}$ under the evaluation at $t = 0$ map ev_0. Thus, it has a natural compactification isomorphic to $X = \mathbb{CP}^1$.

Consider the two-point function of the evaluation observables $\hat{\omega}$ and $\hat{F}$, where ω is a smooth 2-form on $\mathbb{CP}^1$ and F is a smooth function on $\mathbb{CP}^1$. Let us insert $\hat{\omega}$ at the time $t_1 \in \mathbb{R}$ and $\hat{\omega}$ at the time $t_2 < t_1$. Then the corresponding correlation function is equal to

$$ {}_{\infty}\langle \hat{\omega}(t_1)\hat{F}(t_2)\rangle_0 = \int_{\mathbb{CP}^1} \omega\phi(\mathrm{e}^{-t})^*(F), \tag{5.3} $$

where $(\phi(\mathrm{e}^{-t})^*(F))(z,\bar{z}) = F(qz, q\bar{z})$ with $q = \mathrm{e}^{-t} \in \mathbb{R}^{\times}$ and $t = t_1 - t_2$.

Actually, it will be convenient to break $H = \mathcal{L}_v$ into the sum $H = \mathcal{L}_{\xi} + \mathcal{L}_{\bar{\xi}}$ where $\xi = z\partial_z$ and $\bar{\xi} = \bar{z}\partial_{\bar{z}}$ and to allow t_1, t_2 and $t = t_1 - t_2$ to be complex. Then we will have the same formula as (5.3), except that

$$ (\phi(\mathrm{e}^{-t})^*(F))(z,\bar{z}) = F(qz, \bar{q}\bar{z}), $$

where $q = \mathrm{e}^{-t}$, $\bar{q} = \mathrm{e}^{-\bar{t}}$.

The right-hand side of formula (5.3) is the answer that we obtain from the Lagrangian, or path integral, formulation of the model. On the other hand, we may compute the same correlation function from our Hamiltonian formulation. In the hamiltonian realization the vacuum 'in' state corresponding to the critical point 0 is

$$ |0\rangle_{\mathbb{C}_0} = |0,0,0,0\rangle_{\mathbb{C}_0}, $$

and the covacuum 'out' state corresponding to the critical point ∞ is

$$ {}_{\mathbb{C}_\infty}\langle 0| = {}_{\mathbb{C}_\infty}\langle 0,0,0,0|. $$

Therefore, the same correlation function should be equal to the matrix element

$$ {}_{\infty}\langle \hat{\omega}(t)\hat{F}(0)\rangle_0 = {}_{\mathbb{C}_\infty}\langle 0|\hat{\omega}\mathrm{e}^{-t\mathcal{L}_{\xi} - \bar{t}\mathcal{L}_{\bar{\xi}}}\hat{F}|0\rangle_{\mathbb{C}_0}. \tag{5.4} $$

We evaluate this matrix element using the formulae for the action of the observables on the states obtained in the previous section for the action of $\mathcal{L}_{\xi}$ and $\mathcal{L}_{\bar{\xi}}$ on the states obtained in § 4.3.

We have

$$ \hat{F}|0\rangle_{\mathbb{C}_0} = \sum_{m,\bar{m}=0}^{\infty} \frac{1}{m!\bar{m}!}\partial_z^m \partial_{\bar{z}}^{\bar{m}} F\bigg|_{z=0} |m,\bar{m},0,0\rangle_{\mathbb{C}_0} \\ + \sum_{m,\bar{m}=0}^{\infty} \int\!\!\!\!\!\!\diagup_{|w|<\epsilon^{-1}} F w^m \bar{w}^{\bar{m}}\, \mathrm{d}w \wedge \mathrm{d}\bar{w} |m,\bar{m},0,0\rangle_{\infty}. $$

Next, we find using formulae (4.8) and (4.9) that

$$ \mathrm{e}^{-t\mathcal{L}_{\xi} - \bar{t}\mathcal{L}_{\bar{\xi}}}\hat{F}|0\rangle_{\mathbb{C}_0} = \sum_{m,\bar{m}=0}^{\infty} \frac{q^m \bar{q}^{\bar{m}}}{m!\bar{m}!}\partial_z^m \partial_{\bar{z}}^{\bar{m}} F\bigg|_{z=0} |m,\bar{m},0,0\rangle_{\mathbb{C}_0} \\ - 2\pi\log(q\bar{q}) \sum_{m,\bar{m}=1}^{\infty} \frac{q^m \bar{q}^{\bar{m}}}{m!\bar{m}!}\partial_z^m \partial_{\bar{z}}^{\bar{m}} F\bigg|_{z=0} |m-1,\bar{m}-1,0,0\rangle_{\infty} \\ + \sum_{m,\bar{m}=0}^{\infty} q^{m+1}\bar{q}^{\bar{m}+1} \int\!\!\!\!\!\!\diagup_{|w|<\epsilon^{-1}} F w^m \bar{w}^{\bar{m}}\, \mathrm{d}w \wedge \mathrm{d}\bar{w} |m,\bar{m},0,0\rangle_{\infty}. $$

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On the other hand, writing $\omega = \omega_{w\bar{w}}\,\mathrm{d}w\,\mathrm{d}\bar{w} = \omega_{z\bar{z}}\,\mathrm{d}z\,\mathrm{d}\bar{z}$, we obtain that

$$
{}_{\mathbb{C}_\infty}\langle 0|\hat{\omega} = \sum_{m,\bar{m}=0}^{\infty} \frac{1}{m!\bar{m}!}\partial_w^m \partial_{\bar{w}}^{\bar{m}} \omega_{w\bar{w}}\Big|_{w=0} {}_{\mathbb{C}_\infty}\langle m,\bar{m},1,1| + \sum_{m,\bar{m}=0}^{\infty} \fint_{|z|<\epsilon^{-1}} \omega_{z\bar{z}} z^m \bar{z}^{\bar{m}}\,\mathrm{d}z\,\mathrm{d}\bar{z}_0\langle m,\bar{m},1,1|.
$$

Therefore, the right-hand side of (5.4) is equal to

$$
\begin{aligned}
&{}_{\mathbb{C}_\infty}\langle 0|\hat{\omega}\mathrm{e}^{-t\mathcal{L}_\xi - \bar{t}\mathcal{L}_{\bar{\xi}}}\hat{F}|0\rangle_{\mathbb{C}_0} \\
&\quad= \sum_{m,\bar{m}=0}^{\infty} \fint_{|z|<\epsilon^{-1}} \omega_{z\bar{z}} z^m \bar{z}^{\bar{m}}\,\mathrm{d}z\,\mathrm{d}\bar{z} \cdot \frac{q^m \bar{q}^{\bar{m}}}{m!\bar{m}!}\partial_z^m \partial_{\bar{z}}^{\bar{m}} F\Big|_{z=0} \\
&\qquad + q\bar{q} \sum_{m,\bar{m}=0}^{\infty} \frac{q^m \bar{q}^{\bar{m}}}{m!\bar{m}!}\partial_w^m \partial_{\bar{w}}^{\bar{m}} \omega_{w\bar{w}}\Big|_{w=0} \cdot \fint_{|w|<\epsilon^{-1}} F w^m \bar{w}^{\bar{m}}\,\mathrm{d}w\,\mathrm{d}\bar{w} \\
&\qquad - 2\pi\log(q\bar{q}) \sum_{m,\bar{m}=1}^{\infty} \frac{1}{(m-1)!(\bar{m}-1)!}\partial_w^{m-1} \partial_{\bar{w}}^{\bar{m}-1} \omega_{w\bar{w}}\Big|_{w=0} \cdot \frac{q^m \bar{q}^{\bar{m}}}{m!\bar{m}!}\partial_z^m \partial_{\bar{z}}^{\bar{m}} F\Big|_{z=0}.
\end{aligned}
$$

Combining this formulae with formula (5.3) obtained using the path integral,

$$
{}_{\mathbb{C}_\infty}\langle 0|\hat{\omega}\mathrm{e}^{-t\mathcal{L}_\xi - \bar{t}\mathcal{L}_{\bar{\xi}}}\hat{F}|0\rangle_{\mathbb{C}_0} = \int_{\mathbb{CP}^1} \omega F(qz, \bar{q}\bar{z}),
$$

we arrive at the following identity:

$$
\begin{aligned}
&\int_{\mathbb{CP}^1} \omega F(qz, \bar{q}\bar{z}) \\
&\quad= \sum_{m,\bar{m}=0}^{\infty} \fint_{|z|<\epsilon^{-1}} \omega_{z\bar{z}} z^m \bar{z}^{\bar{m}}\,\mathrm{d}z\,\mathrm{d}\bar{z} \cdot \frac{q^m \bar{q}^{\bar{m}}}{m!\bar{m}!}\partial_z^m \partial_{\bar{z}}^{\bar{m}} F\Big|_{z=0} \\
&\qquad + q\bar{q} \sum_{m,\bar{m}=0}^{\infty} \frac{q^m \bar{q}^{\bar{m}}}{m!\bar{m}!}\partial_w^m \partial_{\bar{w}}^{\bar{m}} \omega_{w\bar{w}}\Big|_{w=0} \cdot \fint_{|w|<\epsilon^{-1}} F w^m \bar{w}^{\bar{m}}\,\mathrm{d}w\,\mathrm{d}\bar{w} \\
&\qquad - 2\pi\log(q\bar{q}) \sum_{m,\bar{m}=1}^{\infty} \frac{1}{(m-1)!(\bar{m}-1)!}\partial_w^{m-1} \partial_{\bar{w}}^{\bar{m}-1} \omega_{w\bar{w}}\Big|_{w=0} \cdot \frac{q^m \bar{q}^{\bar{m}}}{m!\bar{m}!}\partial_z^m \partial_{\bar{z}}^{\bar{m}} F\Big|_{z=0}.
\end{aligned}
\tag{5.5}
$$

Note that the identity (5.5) may also be rewritten in the following way:*

$$
\begin{aligned}
\int_{\mathbb{CP}^1} \omega F(qz, \bar{q}\bar{z}) &= \sum_{m,\bar{m}=0}^{\infty} \fint_{|z|<\epsilon^{-1}} \omega_{z\bar{z}} z^m \bar{z}^{\bar{m}}\,\mathrm{d}z\,\mathrm{d}\bar{z} \cdot \frac{q^m \bar{q}^{\bar{m}}}{m!\bar{m}!}\partial_z^m \partial_{\bar{z}}^{\bar{m}} F\Big|_{z=0} \\
&\quad + q\bar{q} \sum_{m,\bar{m}=0}^{\infty} \frac{q^m \bar{q}^{\bar{m}}}{m!\bar{m}!}\partial_w^m \partial_{\bar{w}}^{\bar{m}} \omega_{w\bar{w}}\Big|_{w=0} \cdot \fint_{|w|<q^{-1}\epsilon^{-1}} F w^m \bar{w}^{\bar{m}}\,\mathrm{d}w\,\mathrm{d}\bar{w}.
\end{aligned}
\tag{5.6}
$$

* We note that this formula played the role of 'formula of love' in the film *Rites of Love and Math* by Reine Graves and Edward Frenkel (see http://ritesofloveandmath.com for more details).

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The terms with $\log |q|^2$, which appeared in the third term of (5.5) are now hidden in the definition of the 'partie finie' regularization of the integral in the second term: instead of the 'cutoff' $|w| < \epsilon^{-1}$ as in (5.5) we are now using the 'cutoff' $|w| < q^{-1}\epsilon^{-1}$.

These identities express the factorization of the two point correlation functions over intermediate states. Indeed, assuming for simplicity that t is real, we rewrite it as follows:

$$\langle 0|\hat{\omega}\mathrm{e}^{-tH}\hat{F}|0\rangle = \sum_{\nu}\langle 0|\hat{\omega}|\Psi_\nu\rangle\langle\Psi_\nu^*|\mathrm{e}^{-tH}\hat{F}|0\rangle = \sum_{\mu,\nu}\langle 0|\hat{\omega}|\Psi_\nu\rangle\langle\Psi_\nu^*|\mathrm{e}^{-tH}|\Psi_\mu\rangle\langle\Psi_\mu^*\hat{F}|0\rangle, \quad (5.7)$$

where $\{\Psi_\mu\}$ and $\{\Psi_\mu^*\}$ are dual bases of the spaces of 'in' and 'out' states, respectively. Identities of this type are taken for granted in conventional (i.e. CPT-invariant) quantum mechanics, where the space of states is a Hilbert space, so that we may take as $\{\Psi_\mu\}$ and $\{\Psi_\mu^*\}$ a complete orthonormal basis. However, our model is not CPT-invariant, and so we do not have the structure of Hilbert space on the space of states. Instead, we have two distinct spaces of 'in' and 'out' states and a pairing between them. Because of that, identity (5.5) is more subtle, as we will see in the next section. It requires that F and ω be analytic, and also the equality should be understood in the sense that the right-hand side is the $q, \bar{q}$-power series expansion of the left-hand side in the domain $0 < |q| < \delta$ for some δ which depends on F and ω. Before discussing these subtleties, we point out some salient features of this identity.

The most important feature of the identity (5.5) is the appearance of the logarithms of q and $\bar{q}$ in the right-hand side. If the operators $\mathcal{L}_\xi$ and $\mathcal{L}_{\bar{\xi}}$ and the Hamiltonian were diagonalizable, then the right-hand side would be a power series in $q = \mathrm{e}^{-t\mathcal{L}_\xi}$ and $\bar{q} = \mathrm{e}^{-t\mathcal{L}_{\bar{\xi}}}$. But identity (5.5) shows that there are also the terms involving $\log q$ and $\log \bar{q}$. This means that the operators $\mathcal{L}_\xi$ and $\mathcal{L}_{\bar{\xi}}$, and the Hamiltonian $H = \mathcal{L}_\xi + \mathcal{L}_{\bar{\xi}}$, are not diagonalizable, but have Jordan blocks.

This leads us to the following conclusion: the logarithmic nature of our model, which manifests itself in the fact that the Hamiltonian has Jordan block structure, can be detected from, and is in fact *dictated* by the correlation functions of evaluation observables. These correlation functions are given by a simple and explicit formula (2.21). Applying this formula to various observables of our model, we find logarithmic terms in q and $\bar{q}$, which implies that the Hamiltonian is not diagonalizable.

However, it is important to stress that in order to observe these 'logarithmic effects' at least one of our observables should be a *non-BPS* observable, i.e. not be annihilated by the supercharge Q. If both $\hat{F}$ and $\hat{\omega}$ were BPS observables, i.e. Q-closed, then the one-point functions appearing in the right-hand side of formula (5.7) would be non-zero only when the intermediate states Ψ_μ, Ψ_ν are the BPS states (i.e. the vacuum states). On such states the Hamiltonian is diagonalizable, so we would not be able to observe the logarithmic terms. Since Q acts as the de Rham differential and ω is a 2-form, we find that $\hat{\omega}$ is Q-closed. However, $\hat{F}$ is not Q-closed if F is non-constant (if it were constant, then the logarithmic terms in (5.5) would indeed disappear).

Thus, we can discover the structure of the space of states of the theory, and in particular, the existence of the Jordan blocks of the Hamiltonian, *only* if we consider correlation functions of non-BPS observables. This is one more reason for considering correlation functions beyond the topological sector of the theory: one simply cannot observe these important features of the model within the topological sector.

One can write similar identities for n-point correlation functions of evaluation observables with $n > 2$.

5.3. Analytic aspects of the identity

We now analyse identity (5.5) in more detail. There are two important aspects that we notice right away. The first one is that it does not hold for differential forms/functions that are not analytic at the points 0 and ∞. Indeed, suppose that all derivatives of F vanish at the point 0 (i.e. at $z = 0$), and all derivatives of $\omega_{w\bar{w}}$ vanish at the point ∞ (i.e. when $w = 0$). Then the right-hand side of (5.5) is equal to 0, but the left-hand side may well be non-zero.

Thus, from now on we will assume that F and ω are *analytic*. For the function F this means that for each point $x \in \mathbb{CP}^1$ there is a small neighbourhood of x in which F is equal to its Taylor series expansion in z_x, $\bar{z}_x$ (where z_x is a holomorphic coordinate at x). In other words, this means that the real and imaginary components of the function F are real-analytic (this does not mean that F is holomorphic!). For the 2-form ω this means that the functions $\omega_{w\bar{w}}$ and $\omega_{z\bar{z}}$ are analytic on $\mathbb{C}_\infty$ and $\mathbb{C}_0$, respectively.

We conjecture that the identity (5.5) holds whenever F and ω are analytic on $\mathbb{CP}^1$, in the sense that the right-hand side is a $q, \bar{q}$-series expansion of the left-hand side, which converges in the punctured disc $0 < |q| < \delta$ for some $\delta > 0$ depending on F and ω.

We also expect that the left-hand side is an analytic function in $q, \bar{q}$ on $\mathbb{C}^\times$.

In what follows we present some evidence for this conjecture. We start by proving it for a large class of analytic functions and differential forms which may be represented as follows:

$$\omega = \sum_\alpha \frac{\varpi_\alpha}{z\bar{z} + R_\alpha}\, \mathrm{d}z\, \mathrm{d}\bar{z}, \qquad F = \sum_\beta \frac{f_\beta}{z\bar{z} + Q_\beta},$$

where $R_\alpha, Q_\beta \in \mathbb{C}\backslash\mathbb{R}_{\leqslant 0}$, and the condition

$$\sum_\alpha \varpi_\alpha = 0 \tag{5.8}$$

is satisfied to ensure that ω is well defined at $z = \infty$.

Let us compute the left-hand side of the identity (5.5). To this end we need to compute the integral of the form

$$\int_{\mathbb{CP}^1} \frac{1}{z\bar{z} + R_\alpha} \frac{1}{|q|^2 z\bar{z} + Q_\beta}\, \mathrm{d}z\, \mathrm{d}\bar{z}.$$

By making a charge of variables $z = \sqrt{x}\mathrm{e}^{\mathrm{i}\theta}$, we rewrite this as

$$2\pi \int_0^\infty \frac{1}{(x + R_\alpha)(|q|^2 x + Q_\beta)}\, \mathrm{d}x.$$

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Next, we represent it as a contour integral of

$$-\mathrm{i}\log(-x)\frac{1}{(x+R_\alpha)(|q|^2x+Q_\beta)}\,\mathrm{d}x,$$

over the contour that goes along the real axis from $+\infty$ to ϵ, then goes counterclockwise around 0 and then along the real axis from ϵ to ∞, in the limit when $\epsilon\to 0$. The latter is evaluated as the sum of residues of the integrand in the complex plane, and we obtain the following answer:

$$\int_{\mathbb{CP}^1}\omega F(qz,\bar{q}\bar{z}) = -2\pi\sum_{\alpha,\beta}\varpi_\alpha f_\beta\frac{\log(|q|^2R_\alpha/Q_\beta)}{Q_\beta-|q|^2R_\alpha}. \tag{5.9}$$

Now we compute the right-hand side. Using formula (4.6), we find that

$$⨍_{|z|<\epsilon^{-1}}\omega_{z\bar{z}}z^m\bar{z}^{\bar{m}}\,\mathrm{d}z\,\mathrm{d}\bar{z} = 2\pi(-1)^{m+1}\delta_{m,\bar{m}}\sum_\alpha\varpi_\alpha R_\alpha^m\log R_\alpha.$$

Similarly, we obtain that

$$⨍_{|w|<\epsilon^{-1}}Fw^m\bar{w}^{\bar{m}}\,\mathrm{d}w\,\mathrm{d}\bar{w} = 2\pi(-1)^{m+1}\delta_{m,\bar{m}}\sum_\beta f_\beta Q_\beta^{-m-2}\log Q_\beta.$$

On the other hand, we have

$$\frac{1}{m!\bar{m}!}\partial_w^m\partial_{\bar{w}}^{\bar{m}}\omega_{w\bar{w}}\bigg|_{w=0} = (-1)^{m+1}\delta_{m,\bar{m}}\sum_\alpha\varpi_\alpha R_\alpha^{m+1},$$
$$\frac{1}{m!\bar{m}!}\partial_z^m\partial_{\bar{z}}^{\bar{m}}F\bigg|_{z=0} = (-1)^{m}\delta_{m,\bar{m}}\sum_\beta f_\beta Q_\beta^{-m-1}.$$

Substituting these expressions into the right-hand side of formula (5.5) and using the condition (5.8), we obtain the following series

$$-2\pi\sum_{\alpha,\beta}\varpi_\alpha f_\beta\sum_{m=0}^{\infty}Q_\beta^{-m-1}R_\alpha^m|q|^{2m}(\log R_\alpha-\log Q_\beta+\log|q|^2).$$

But this is precisely the $|q|$-series expansion of (5.9), which converges to (5.9) for all non-zero q inside the disc of radius $\min_{\alpha,\beta}\{|Q_\beta|/|R_\alpha|\}$. Note that this is true if and only if the condition (5.8) holds which is needed to make ω well defined at $z=\infty$. So it appears that the identity (5.5) somehow 'knows' about this condition.

Thus, we discover an interesting phenomenon: the factorization identity (5.5) should be understood in the analytic continuation sense. Namely, the right-hand side is the expansion of the left-hand side in powers of q, $\bar{q}$ (note, however, that it also includes terms with $\log|q|^2$), which converges in the domain $0<|q|<\delta$ for some δ which depends on the choice of F and ω. Moreover, we expect that the left-hand side of (5.5) is an analytic function in q for all $q\in\mathbb{C}^\times$.

In the same way as above we prove the identity (5.5) in the more general case of F and ω of the form

$$\omega_{n,\bar{n}} = \sum_{\alpha} \varpi_\alpha \frac{z^n \bar{z}^{\bar{n}}}{z\bar{z} + R_\alpha} \, dz \, d\bar{z}, \tag{5.10}$$

$$f_{m,\bar{m}} = \sum_{\beta} f_\beta \frac{z^m \bar{z}^{\bar{m}}}{z\bar{z} + Q_\beta}, \tag{5.11}$$

where the R_α and the Q_β are in $\mathbb{C}\backslash\mathbb{R}_{\leqslant 0}$. The numbers ϖ_α and R_α (respectively, f_β and Q_β) should also satisfy some conditions which ensure that ω (respectively, F) is well defined at $z = \infty$. In the same way as above we obtain that the left-hand side of (5.5) is given by the formula

$$2\pi\delta_{n+m,\bar{n}+\bar{m}} q^m \bar{q}^{\bar{m}} (-1)^{n+m+1} \sum_{\alpha,\beta} \varpi_\alpha f_\beta \frac{R_\alpha^{n+m} \log(R_\alpha) - (Q_\beta |q|^{-2})^{n+m} \log(Q_\beta |q|^{-2})}{Q_\beta - |q|^2 R_\alpha},$$

and the right-hand side of (5.5) is equal to the $q, \bar{q}$-series expansion of the left-hand side.

Thus, we have now checked the validity of the identity (5.5) for a large class of functions and 2-forms.

It is important to realize that we could reverse our calculation. Namely, after computing the integral $\int_{\mathbb{CP}^1} \omega F(qz, \bar{q}\bar{z})$ we could expand it in a power series in q, $\bar{q}$ and interpret the result as the formula for the factorization of the two-point correlation function of ω and F over intermediate states, as in (5.7). This determines completely the matrix elements of F between the vacuum $|0\rangle_{\mathbb{C}_0}$ and generic 'out' states, and matrix elements of ω between the covacuum ${}_{\mathbb{C}_\infty}\langle 0|$ and generic 'in' states. Thus, we could start with the two-point, and more generally, n-point correlation functions of evaluation observables, which are given by explicit integrals over moduli spaces of instantons, and use them to *derive* the matrix elements of these observables acting on the space of states. In particular, this way we find that the Hamiltonian of our model is non-diagonalizable. Moreover, we can estimate the maximal size of the Jordan blocks (the maximal power of the logarithm of q plus 1). This points to an effective strategy for determining matrix elements from the correlation functions, which can be applied to more general models, such as two-dimensional sigma models and four-dimensional Yang–Mills theory that we will discuss in Part II. Again, we stress that in order to obtain non-trivial results we must consider *non-BPS observables.*

To conclude this section, we point out another case when the identity (5.5) obviously holds. Namely, suppose that F is analytic, but $\omega_{z,\bar{z}}$ has compact support on the complex plane $\mathbb{C}_0 = \mathbb{CP}^1\backslash\infty$, i.e. away from the point $z = \infty$ (thus, ω is not analytic, and so this is not a special case of our conjecture). Then there exist positive numbers R and r such that $\omega_{z\bar{z}} \equiv 0$ for all z such that $|z| > R$, while F is equal to its Taylor series expansion in the disc $|z| < r$. In this case for all $0 < |q| < r/R$ we have

$$\begin{aligned}\int_{\mathbb{CP}^1} \omega F(qz, \bar{q}\bar{z}) &= \int_{|z|<R} F(qz, \bar{q}\bar{z}) \omega_{z\bar{z}} \, dz \, d\bar{z} \\ &= \sum_{m,\bar{m}=0}^{\infty} \frac{q^m \bar{q}^{\bar{m}}}{m! \bar{m}!} \partial_z^m \partial_{\bar{z}}^{\bar{m}} F \bigg|_{z=0} \cdot \int_{\mathbb{CP}^1} \omega_{z\bar{z}} z^m \bar{z}^{\bar{m}} \, dz \, d\bar{z}.\end{aligned}$$

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Thus, the left-hand side of (5.5) is equal to the $q, \bar{q}$-series expansion of the first term in the right-hand side, while the second and the third terms vanish. Therefore, the identity (5.5) holds in this case.

In the same way we prove the identity with the roles of F and ω reversed, i.e. assuming that F has compact support away from $z = 0$ while ω is analytic at $z = \infty$.

Remark. It is interesting to investigate what will happen if we allow smooth but non-analytic observables in the spectral decomposition of the correlation functions. For example, consider the correlation functions of the evaluation observables whose support does not contain the critical points of the Morse function.

Let us pass to the coordinates in which the gradient vector field looks like a translation in one of them, say,

$$v = \partial_t.$$

For simplicity, suppose that the observables are independent of the remaining coordinates. Then the correlation function reduces to a one-dimensional integral over the t-line. For example, if we have two observables, giving rise to a 1-form $\omega = \omega(t)\,\mathrm{d}t$ and a function $f(t)$ on the t-line, then the correlation function will look like this:

$$\mathcal{C}(q) = \int_{-\infty}^{+\infty} \omega(t) f(t + \log(q\bar{q}))\,\mathrm{d}t.$$

This integral converges because we have assumed that f and ω have compact support on the complement to the set of critical points (which in the model example consists of the points $t = \pm\infty$). Clearly, in this case the decomposition of $\mathcal{C}(q)$ as a sum of the contributions of the eigenstates of the Hamiltonian looks as the integral

$$\mathcal{C}(q) = \int \frac{\mathrm{d}k}{2\pi} \mathrm{e}^{\mathrm{i}k\log(q\bar{q})} \hat{\mathcal{C}}_{f,\omega}(k), \tag{5.12}$$

with

$$\hat{\mathcal{C}}_{f,\omega}(k) = \hat{f}(-k)\hat{\omega}(k)$$

being the product of the Fourier transforms of f and ω. Formula (5.12) implies that the spectrum of the Hamiltonian contains a continuous part, with the eigenvalues given by

$$E_k = \mathrm{i}k,$$

i.e. purely imaginary!

Thus, we are facing a dilemma: either these compactly supported functions and forms require a new, infinite-dimensional, sector in the space of states, or, by some sort of resummation, they are already included in the space of states that we have constructed.

It is instructive to reconsider from this point of view the example of the harmonic oscillator, i.e. the quantum mechanical model on $\mathbb{C}$, with the quadratic Morse function. We have analysed in detail the $\lambda \to \infty$ limit of the full set of the eigenstates of the Hamiltonian in § 3.3, and we did not see any need for the continuous imaginary spectrum. How could it be that the functions with compact support not containing zero are included in the space of states built from polynomials? Physically, the explanation is the following. For $\lambda = \infty$ the evolution looks like a constant velocity motion in the logarithmic coordinate t. However, once λ becomes finite, there is an admixture of diffusion, caused by the term $(1/(2\lambda))\Delta$ in the Hamiltonian $\widetilde{H}_\lambda$. What is more important, this diffusion takes place in the linear, as opposed to the logarithmic, coordinates. Roughly speaking, the evolution during some time T spreads the initially localized object as $\sim \sqrt{T/\lambda} \gg \mathrm{e}^{-T}$, for large T. Thus, even if we start with a distribution with compact support not containing zero, the critical point, the diffusion will spread it so it will contain zero. Once this has happened, the resulting distribution can be well-approximated by the Taylor series at zero, i.e. by the wave functions from our space of states.

Another point worth mentioning is that the wave functions of the Hamiltonian $\mathcal{L}_v$ corresponding to the imaginary eigenvalues are not smooth at the critical points. For instance, in the case of the $\mathbb{CP}^1$ model, where v is the Euler vector field, these eigenfunctions have the form $|z|^{\mathrm{i}k}$. When we deform away from the point $\lambda = \infty$, we add the term $\lambda^{-1}\Delta$ to the Hamiltonian. If these eigenfunctions were present in the spectrum, then we would be able to deform them to eigenfunctions of the deformed Hamiltonian. But applying Δ to $|z|^{\mathrm{i}k}$, we obtain a function which has poles at the critical points, and this shows that it cannot be deformed to a smooth eigenfunction of the Hamiltonian in perturbation theory in λ^{-1}.

Let us mention, however, that in Part II, when we discuss the quantum mechanical models on non-simply connected manifolds, we shall see some version of the 'imaginary' space of states. Its appearance (in a much more tame form, with discrete spectrum) will be related to the existence of gradient trajectories which go from 'nowhere to nowhere', i.e. never terminate. But this is only possible for Morse–Novikov, i.e. multivalued, functions.

5.4. Interpretation as an expansion of the delta-form on the q-shifted diagonal

The identity (5.5) expresses the factorization over intermediate states of the two-point correlation function of evaluation observables, one of which is a function and the other is a 2-form. But we could consider instead the correlation functions of two 1-forms, or to switch F and ω (so that $\mathbb{C}^\times$ acts on the 2-form rather than the function). In each case we obtain a similar identity.

It is instructive to think of all of these identities as expressing the delta-form supported on the 'q-shifted diagonal' in $\mathbb{CP}^1 \times \mathbb{CP}^1$ in terms of distributions along the first and the second factors. More precisely, consider the submanifold

$$\mathrm{Diag}_q = \{(x, y) \in \mathbb{CP}^1 \times \mathbb{CP}^1 \mid y = qx\} \subset \mathbb{CP}^1 \times \mathbb{CP}^1.$$

Note that qx simply means the point obtained by acting on $x \in \mathbb{CP}^1$ with $q \in \mathbb{C}^\times$. This is the q-shifted diagonal. Now let Δ_q be the delta-form (of degree 2) supported on Diag_q. Note that Δ_q is precisely the kernel of the evolution operator in our quantum mechanical model on $\mathbb{CP}^1$.

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Observe that

$$\int_{\mathbb{CP}^1\times\mathbb{CP}^1} \Delta_q \wedge (\omega \boxtimes F) = \int_{\mathbb{CP}^1} \omega\phi(q)^*(F), \tag{5.13}$$

where $(\phi(q)^*(F))(z,\bar{z}) = F(qz,\bar{q}\bar{z})$, is precisely the two-point function appearing in the left-hand side of the identity (5.5). Likewise, if we take two 1-forms η_1 and η_2 on $\mathbb{CP}^1$, then we have

$$\int_{\mathbb{CP}^1\times\mathbb{CP}^1} \Delta_q \wedge (\eta_1 \boxtimes \eta_2) = \int_{\mathbb{CP}^1} \eta_1 \wedge \phi(q)^*(\eta_2), \tag{5.14}$$

and similarly for a function and a 2-form switched:

$$\int_{\mathbb{CP}^1\times\mathbb{CP}^1} \Delta_q \wedge (F \boxtimes \omega) = \int_{\mathbb{CP}^1} F\phi(q)^*(\omega). \tag{5.15}$$

In these formulae, given differential forms ω_1 and ω_2 on $\mathbb{CP}^1$, we denote by $\omega_1 \boxtimes \omega_2$ the corresponding differential form on $\mathbb{CP}^1 \times \mathbb{CP}^1$.

The identities discussed above correspond to an expansion of different components of the distribution Δ_q. More precisely, Δ_q is the sum of three components which are differential forms of degrees $(0,2)$, $(1,1)$, and $(2,0)$ on $\mathbb{CP}^1 \times \mathbb{CP}^1$. The expansion of each of them gives rise to the three identities considered above.

For example, consider the $(0,2)$ part of Δ_q which we will denote by $\Delta_q^{(0,2)}$. This is the part which contributes to the integral (5.13). Let ${}_z\tilde{\varphi}_{n,\bar{n}}$ and ${}_w\tilde{\varphi}_{n,\bar{n}}$ be the distributions on $\mathbb{CP}^1$ defined by the formulae

$$\begin{aligned} {}_z\tilde{\varphi}^{\epsilon}_{m,\bar{m}}(\omega) &= \fint_{|z|<\epsilon^{-1}} \omega_{z\bar{z}} z^m \bar{z}^{\bar{m}}\, \mathrm{d}z\, \mathrm{d}\bar{z}, \qquad \omega \in \Omega^2(\mathbb{CP}^1),\\ {}_w\tilde{\varphi}^{\epsilon}_{m,\bar{m}}(F) &= \fint_{|w|<\epsilon^{-1}} F w^m \bar{w}^{\bar{m}}\, \mathrm{d}w\, \mathrm{d}\bar{w}, \quad F \in \Omega^0(\mathbb{CP}^1). \end{aligned}$$

Given two distributions φ, ϕ on $\mathbb{CP}^1$, we will denote by $\varphi \boxtimes \phi$ the corresponding distribution on $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Then the identity (5.5) may be rewritten as follows:

$$\begin{aligned} \Delta_q^{(0,2)} &= \sum_{m,\bar{m}=0}^{\infty} q^m \bar{q}^{\bar{m}}\, {}_z\tilde{\varphi}^{\epsilon}_{m,\bar{m}} \boxtimes \frac{1}{m!\bar{m}!}(-\partial_z)^m(-\partial_{\bar{z}})^{\bar{m}}\delta^{(2)}(z,\bar{z})\, \mathrm{d}z\, \mathrm{d}\bar{z}\\ &\quad + q\bar{q} \sum_{m,\bar{m}=0}^{\infty} q^m \bar{q}^{\bar{m}} \frac{1}{m!\bar{m}!}(-\partial_w)^m(-\partial_{\bar{w}})^{\bar{m}}\delta^{(2)}(w,\bar{w}) \boxtimes {}_w\tilde{\varphi}^{\epsilon}_{m,\bar{m}}\\ &\quad - 2\pi \log(q\bar{q}) \sum_{m,\bar{m}=1}^{\infty} q^m \bar{q}^{\bar{m}} \frac{\partial_w^{m-1}\partial_{\bar{w}}^{\bar{m}-1}}{(m-1)!(\bar{m}-1)!}\delta^{(2)}(w,\bar{w}) \boxtimes \frac{\partial_z^m \partial_{\bar{z}}^{\bar{m}}}{m!\bar{m}!}\delta^{(2)}(z,\bar{z})\, \mathrm{d}z\, \mathrm{d}\bar{z}. \end{aligned} \tag{5.16}$$

One needs to be careful in interpreting this identity. Since we expect the identity (5.5) to be true only for analytic functions, it is natural to consider (5.16) as an identity in the space of *hyperfunctions* on $\mathbb{CP}^1 \times \mathbb{CP}^1$.* We recall that the space of hyperfunctions is the dual space to the space of analytic functions, equipped with an appropriate topology. We will use the term 'hyperfunction' in the broader sense as an element of the dual space to the space of analytic differential forms. The right-hand side of (5.16) should be understood as a $q, \bar{q}$-expansion of the left-hand side, as in the case of the identity (5.5), which converges for $0 < |q| < \delta$ for some real $\delta > 0$. It is clear that applying (5.16) to the differential form $\omega \boxtimes F$ on $\mathbb{CP}^1 \times \mathbb{CP}^1$ we obtain (5.5).

Likewise, there are similar identities on the components of degrees $(1,1)$ and $(2,0)$ in Δ_q. Those contribute to the integrals (5.14) and (5.15), respectively, and give rise to the corresponding identities.

One may wonder whether there is a simplified analogue of the identity (5.16) in the case of the target manifold $X = \mathbb{C}$. There is indeed such an analogue, which is however more restrictive due to the non-compactness of $\mathbb{C}$. Nevertheless, it is instructive to look at this identity.

The analogue of the q-shifted diagonal in this case is just the line $y = xq$ in $\mathbb{C} \times \mathbb{C}$, and the corresponding delta-form is just

$$\Delta_q = \delta^{(2)}(y - xq, \bar{y} - \bar{q}\bar{x})\, \mathrm{d}(y - xq) \wedge \mathrm{d}(\bar{y} - \bar{q}\bar{x}),$$

which is the kernel of the evolution operator of our model that we discussed in § 3.4. It appears as the $\lambda \to \infty$ limit of the kernel $K_{t,\bar{t}}$ of the model at finite λ (see formula (3.21)). Let us look at its $(0,2)$ component, which reads

$$\delta^{(2)}(y - xq, \bar{y} - \bar{q}\bar{x})\, \mathrm{d}y \wedge \mathrm{d}\bar{y}.$$

The naive Taylor series expansion of this distribution looks as follows:

$$\sum_{m,\bar{m}=0}^{\infty} q^m \bar{q}^{\bar{m}} x^m \bar{x}^{\bar{m}} \boxtimes \frac{1}{m!\bar{m}!} (-\partial_y)^m (-\partial_{\bar{y}})^{\bar{m}} \delta^{(2)}(y, \bar{y})\, \mathrm{d}y \wedge \mathrm{d}\bar{y}. \tag{5.17}$$

This formula may be interpreted in the following way. Let ω be a 2-form on $\mathbb{C}$ with compact support and F a function that is analytic at $0 \in \mathbb{C}$. Then for sufficiently small q the series

$$\sum_{m,\bar{m}=0}^{\infty} q^m \bar{q}^{\bar{m}} \int x^m \bar{x}^{\bar{m}} \omega \cdot \frac{1}{m!\bar{m}!} \partial_y^m \partial_{\bar{y}}^{\bar{m}} F \bigg|_{y=0},$$

obtained by applying (5.17) to $\omega \boxtimes F$, converges to the integral

$$\int_{\mathbb{C}\times\mathbb{C}} \Delta_q \wedge (\omega \boxtimes F) = \int_{\mathbb{C}} \omega \phi(q)^*(F).$$

The $(1,1)$ and $(2,0)$ parts of the decomposition of Δ_q have a similar structure and interpretation. The decomposition of Δ_q obtained this way may be viewed as the $\lambda \to \infty$ limit of the decomposition of the kernel of the evolution operator $K_{t,\bar{t}}$ in terms of the

* We thank P. Schapira and K. Vilonen for discussions of our identity in the context of hyperfunctions.

orthonormal basis of eigenstates of the Hamiltonian, which is given in formula (3.20). However, because we do not have the structure of Hilbert space on our space of states at $\lambda = \infty$, this decomposition becomes more subtle: it has to be understood in the sense of analytic continuation and the observables are required to have some analytic properties.

Thus, we see that an analogue of the identity (5.5) in the case of $X = \mathbb{C}$ may be obtained simply by applying a Taylor series expansion to the delta-form supported on the q-shifted diagonal. While it seems very easy to derive it, its validity is very limited because $\mathbb{C}$ is not compact. Indeed, for the integral to converge we need to make an additional assumption of compactness of support of at least one of the objects involved, F and ω. On the other hand, in order to make sense of this identity we need them to be analytic. Thus, we have a clash between two seemingly irreconcilable properties of differential forms on $\mathbb{C}$: analyticity and compactness of support. The best we can do is to assume that F is analytic and ω has compact support. The resulting identity is not very useful, but it is still instructive to consider it as a toy model for the corresponding identity in the case of $\mathbb{CP}^1$.

In the case of $X = \mathbb{CP}^1$ the structure of the identity is more complicated, but it is applicable to a larger class of differential forms. Now instead of one infinite sum we have three infinite sums. The first two have the structure similar to that of the sum appearing in the identity for $\mathbb{C}$. They correspond to the two critical points of the Morse function on $\mathbb{CP}^1$ (or equivalently, the fixed points of the $\mathbb{C}^\times$-action): 0 and ∞. The third term has to do with the non-diagonalizable nature of the Hamiltonian. It would be interesting to understand its meaning from the analytic point of view. On the positive side, all differential forms on $\mathbb{CP}^1$ have compact support, so the convergence of the left-hand side of (5.5) is not an issue. Hence it makes sense to impose the condition of analyticity on both F and ω.

In order to understand the generalization of these identities to other Kähler manifolds, it is more convenient to work with the other version (5.6) of our identity. This version also has an interpretation in terms of the decomposition of the delta-form on the q-shifted diagonal:

$$\Delta_q^{(0,2)} = \sum_{m,\bar{m}=0}^{\infty} q^m \bar{q}^{\bar{m}} {}_z\tilde{\varphi}^{\epsilon}_{m,\bar{m}} \boxtimes \frac{1}{m!\bar{m}!}(-\partial_z)^m(-\partial_{\bar{z}})^{\bar{m}} \delta^{(2)}(z,\bar{z})\,\mathrm{d}z\,\mathrm{d}\bar{z} + q\bar{q} \sum_{m,\bar{m}=0}^{\infty} q^m \bar{q}^{\bar{m}} \frac{1}{m!\bar{m}!}(-\partial_w)^m(-\partial_{\bar{w}})^{\bar{m}} \delta^{(2)}(w,\bar{w}) \boxtimes {}_w\tilde{\varphi}^{q\epsilon}_{m,\bar{m}}. \tag{5.18}$$

The terms with $\log|q|^2$ have now disappeared at the cost of changing the regularization in the second term: we now use ${}_w\tilde{\varphi}^{q\epsilon}_{m,\bar{m}}$ instead of ${}_w\tilde{\varphi}^{\epsilon}_{m,\bar{m}}$. The resulting identity has two terms, corresponding to the fixed points 0 and ∞ of the $\mathbb{C}^\times$-action on $\mathbb{CP}^1$. There are also similar identities for the $(1,1)$ and $(2,0)$ components of Δ_q.

5.5. Generalization to other Kähler manifolds

We now briefly discuss how to generalize the results of the previous section to more general Kähler manifolds, using $\mathbb{CP}^2$ as the main example. Points of $\mathbb{CP}^2$ will be represented by triples $(z_1 : z_2 : z_3)$ of non-zero complex numbers, up to an overall scalar multiple. Consider the $\mathbb{C}^\times$-action on $\mathbb{CP}^2$ generated by the vector field $v = \xi + \bar{\xi}$, where

$$\xi = z_1 \partial_{z_1} + \gamma z_2 \partial_{z_2},$$

where γ is a rational number such that $0 < \gamma < 1$. If γ does not satisfy these inequalities, then the structure of the ascending manifolds described below will be different. Rationality of γ is needed to ensure that ξ comes from a $\mathbb{C}^\times$-action.

The vector field v is the gradient of the Morse function

$$f = \frac{1}{2} \frac{2|z_1|^2 + (1+\gamma)|z_2|^2}{|z_1|^2 + |z_2|^2 + |z_3|^2}.$$

Its critical points are $(0:0:1)$ of index 0, $(0:1:0)$ of index 2 and $(1:0:0)$ of index 4. The corresponding ascending manifolds X_α are $X_{(0:0:1)} = \{(z_1 : z_2 : 1)\} \simeq \mathbb{C}^2$, $X_{(0:1:0)} = \{(w_1 : 1 : 0)\} \simeq \mathbb{C}$, and $X_{(1:0:0)}$, which is a point.

The coordinates z_1, z_2 are local coordinates at $(0:0:1)$ representing points $(z_1 : z_2 : 1)$. Let us introduce local coordinates w_1, w_2 at $(0:1:0)$ by representing nearby points as $(w_1 : 1 : w_2)$. Then we have

$$w_1 = \frac{z_1}{z_2}, \qquad w_2 = \frac{1}{z_2}. \tag{5.19}$$

We also choose local coordinates u_1, u_2 at $(1:0:0)$ by representing nearby points as $(1 : u_1 : u_2)$, so that

$$u_1 = \frac{z_2}{z_1}, \qquad u_2 = \frac{1}{z_1}.$$

The space $\mathcal{H}^{\text{in}}$ of 'in' states is isomorphic to

$$\mathcal{H}^{\text{in}} \simeq \mathcal{H}^{\text{in}}_{(0:0:1)} \oplus \mathcal{H}^{\text{in}}_{(0:1:0)} \oplus \mathcal{H}^{\text{in}}_{(1:0:0)},$$

where

$$\begin{aligned}
\mathcal{H}^{\text{in}}_{(0:0:1)} &= \mathbb{C}[z_1, z_2, \bar{z}_1, \bar{z}_2] \otimes \Lambda[\mathrm{d}z_1, \mathrm{d}z_2, \mathrm{d}\bar{z}_1, \mathrm{d}\bar{z}_2],\\
\mathcal{H}^{\text{in}}_{(0:1:0)} &= \mathbb{C}[w_1, \partial_{w_2}, \bar{w}_1, \partial_{\bar{w}_2}] \otimes \Lambda[\mathrm{d}w_1, \imath_{\partial_{w_2}}, \mathrm{d}\bar{w}_1, \imath_{\partial_{\bar{w}_2}}] \cdot \delta^{(2)}(w_2, \bar{w}_2)\, \mathrm{d}^2 w_2,\\
\mathcal{H}^{\text{in}}_{(1:0:0)} &= \mathbb{C}[\partial_{u_1}, \partial_{u_2}, \partial_{\bar{u}_1}, \partial_{\bar{u}_2}] \otimes \Lambda[\imath_{\partial_{u_1}}, \imath_{\partial_{u_2}}, \imath_{\partial_{\bar{u}_1}}, \imath_{\partial_{\bar{u}_2}}] \cdot \delta^{(4)}(u_1, u_2, \bar{u}_1, \bar{u}_2)\, \mathrm{d}^2 u_1 \wedge \mathrm{d}^2 u_2.
\end{aligned}$$

The ground states are

$$\Delta_{(0:0:1)} = 1, \qquad \Delta_{(0:1:0)} = \delta^{(2)}(w_2, \bar{w}_2)\, \mathrm{d}^2 w_2,$$
$$\Delta_{(1:0:0)} = \delta^{(4)}(u_1, u_2, \bar{u}_1, \bar{u}_2)\, \mathrm{d}^2 u_1 \wedge \mathrm{d}^2 u_2.$$

Note that each space exhibits holomorphic factorization $\mathcal{H}^{\text{in}}_\alpha = \mathcal{F}^{\text{in}}_\alpha \otimes \bar{\mathcal{F}}_\alpha$.

We realize $\mathcal{H}^{\text{in}}$ as a subspace in the space of distributions on differential forms on $\mathbb{CP}^2$. The subspace $\mathcal{H}^{\text{in}}_{(1:0:0)}$ is a canonical subspace, which consists of distributions supported at the point $(1:0:0)$. The subspaces $\mathcal{H}^{\text{in}}_{(0:1:0)}$ and $\mathcal{H}^{\text{in}}_{(0:0:1)}$ are not canonical. The definition of elements of these subspaces depends on a particular choice of regularization of divergent integrals, as in the case of $\mathbb{CP}^1$ which we have studied in great detail earlier. Changing regularization would add to elements of $\mathcal{H}^{\text{in}}_{(0:1:0)}$ correction terms that lie in $\mathcal{H}^{\text{in}}_{(1:0:0)}$, and to elements of $\mathcal{H}^{\text{in}}_{(0:0:1)}$ correction terms in $\mathcal{H}^{\text{in}}_{(0:1:0)}$ and $\mathcal{H}^{\text{in}}_{(1:0:0)}$. Because of that, we only have a canonical filtration with associated graded spaces being $\mathcal{H}^{\text{in}}_{(1:0:0)}$, $\mathcal{H}^{\text{in}}_{(0:1:0)}$ and $\mathcal{H}^{\text{in}}_{(0:0:1)}$.

This is reflected in the non-diagonalizability of the operators $\mathcal{L}_\xi$ and $\mathcal{L}_{\bar\xi}$. According to our general formulae (4.19) and (4.20), we have

$$\mathcal{L}_\xi = \mathcal{L}_{\xi,\text{naive}} + 2\pi(-\gamma\delta_{12}\otimes\bar\delta_{12} + (\gamma-1)\delta_{23}\otimes\bar\delta_{23}),$$
$$\mathcal{L}_{\bar\xi} = \mathcal{L}_{\bar\xi,\text{naive}} + 2\pi(-\gamma\delta_{12}\otimes\bar\delta_{12} + (\gamma-1)\delta_{23}\otimes\bar\delta_{23}).$$

where $\delta_{12}\colon \mathcal{H}^{\text{in}}_{(0:0:1)} \to \mathcal{H}^{\text{in}}_{(0:1:0)}$ and $\delta_{23}\colon \mathcal{H}^{\text{in}}_{(0:1:0)} \to \mathcal{H}^{\text{in}}_{(1:0:0)}$ are the GC operators, and $\bar\delta_{12}$ and $\bar\delta_{23}$ are their complex conjugates. Note that (in the notation of § 4.8) we have $w_{12} = w_2$, $w_{23} = u_1$ and so $a_{12} = -\gamma$, $a_{23} = \gamma - 1$.

As an example, we describe the action of δ_{12} on the subspace $\mathbb{C}[z_1, z_2]$ of 0-forms in $\mathcal{F}^{\text{in}}_{(0:0:1)}$. Given an element of $\mathbb{C}[z_1, z_2]$, we rewrite it as a polynomial in w_1, $w_2^{\pm1}$, using the substitution (5.19). Then we project it onto the quotient $\mathbb{C}[w_1, w_2^{\pm1}]/\mathbb{C}[w_1, w_2^{-1}]$, which we identify with the space $\mathbb{C}[w_1, \partial_{w_2}]$ by the formula

$$w_1^n w_2^{-m} \mapsto w_1^n \frac{1}{(m-1)!}(-\partial_{w_2})^{m-1}.$$

One defines the action of δ_{12} on differential forms of degree greater than 0 in the same way. The operator δ_{23} is defined similarly.

What is the maximal length of the Jordan blocks of the operators $\mathcal{L}_\xi$, $\mathcal{L}_{\bar\xi}$ and the Hamiltonian? In the case of $\mathbb{CP}^1$ the space $\mathcal{H}^{\text{in}}$ was an extension of two subspaces, $\mathcal{H}^{\text{in}}_{\mathbb{C}_0}$ and $\mathcal{H}^{\text{in}}_\infty$, and the off-diagonal parts of these operators were acting from one of them to the other. Therefore, the Jordan blocks had maximal length 2. Now the space $\mathcal{H}^{\text{in}}$ is an extension of three spaces and the off-diagonal parts of these operators act from the first to the second and from the second to the third. Therefore, *a priori* one could expect Jordan blocks of length 3. It comes as a bit of a surprise when we learn that in fact the maximal Jordan blocks have length 2.

There are two ways to see that. The first is to find the spectra of the diagonal parts of $\mathcal{L}_\xi$ and $\mathcal{L}_{\bar\xi}$ on each of the three subspaces of $\mathcal{H}^{\text{in}}$. According to the above description of $\mathcal{H}^{\text{in}}_{(0:0:1)}$, the eigenvalues of $\mathcal{L}_\xi$ on it (which are the same as the eigenvalues of $\mathcal{L}_{\xi,\text{naive}}$) have the form $n_1 + \gamma n_2$, where $n_1, n_2 \in \mathbb{Z}_{\geqslant 0}$. We also find that the eigenvalues on $\mathcal{H}^{\text{in}}_{(0:1:0)}$ have the form $m_1(1-\gamma) + (m_2+1)\gamma$, where $m_1, \mu_2 \in \mathbb{Z}_{\geqslant 0}$, and the eigenvalues on $\mathcal{H}^{\text{in}}_{(1:0:0)}$ have the form $(l_1+1)(1-\gamma) + (l_2+1)$, where $l_1, l_2 \in \mathbb{Z}_{\geqslant 0}$. We have similar formulae for the eigenvalues of $\mathcal{L}_{\bar\xi}$. By inspection of these formulae we find that for irrational values of γ there is an overlap of the spectra between $\mathcal{H}^{\text{in}}_{(0:0:1)}$ and $\mathcal{H}^{\text{in}}_{(0:1:0)}$, and between $\mathcal{H}^{\text{in}}_{(0:1:0)}$ and $\mathcal{H}^{\text{in}}_{(1:0:0)}$, but none between $\mathcal{H}^{\text{in}}_{(0:0:1)}$ and $\mathcal{H}^{\text{in}}_{(1:0:0)}$. Even though we are only allowed to take rational values of γ, we expect the operators $\mathcal{L}_\xi$ and $\mathcal{L}_{\bar{\xi}}$ to depend continuously on γ. Hence we find that they cannot have off-diagonal terms acting from $\mathcal{H}^{\mathrm{in}}_{(0:0:1)}$ to $\mathcal{H}^{\mathrm{in}}_{(1:0:0)}$, and so the maximal size of their Jordan blocks cannot be greater than 2.

Another way to see that is to use the identity (4.21), which in our case reads

$$\delta_{23} \circ \delta_{12} = 0, \qquad \bar{\delta}_{23} \circ \bar{\delta}_{12} = 0. \tag{5.20}$$

Therefore,

$$(\delta_{12} \otimes \bar{\delta}_{12}) \circ (\delta_{23} \otimes \bar{\delta}_{23}) = 0.$$

It implies that the square of the off-diagonal parts of $\mathcal{L}_\xi$ and $\mathcal{L}_{\bar{\xi}}$ (which are equal to each other) is zero. This means that the Jordan blocks have length at most 2.

This should be contrasted to the case of the model defined on the manifold $\mathbb{CP}^1 \times \mathbb{CP}^1$, with the vector field $\xi = z_1 \partial_{z_1} + z_2 \partial_{z_2}$. In that case we have four ascending manifolds: one two dimensional, X_1, two one dimensional, X_2 and X_3, and one zero dimensional, X_4. The off-diagonal part of $\mathcal{L}_\xi$ now has four terms:

$$\delta_{12} \otimes \bar{\delta}_{12} + \delta_{13} \otimes \bar{\delta}_{13} + \delta_{24} \otimes \bar{\delta}_{24} + \delta_{34} \otimes \bar{\delta}_{34}. \tag{5.21}$$

The analogue of the identity (5.20) reads as follows:

$$\delta_{24} \circ \delta_{12} = \delta_{34} \circ \delta_{13}, \qquad \bar{\delta}_{24} \circ \bar{\delta}_{12} = \bar{\delta}_{34} \circ \bar{\delta}_{13}.$$

However, it does not mean that the square of the sum (5.21) is equal to zero. It would be zero only if we would change one of the four signs to minus (this is a good example of how the signs $\epsilon_{\alpha\beta}$ discussed in § 4.8 are chosen).* Thus, because we now have two 'channels' from the two-dimensional stratum to the zero-dimensional stratum (via two intermediate one-dimensional strata) we find that the square of the off-diagonal term of $\mathcal{L}_\xi$ (and, likewise, $\mathcal{L}_{\bar{\xi}}$) is non-zero. Therefore, there are Jordan blocks of length three in the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$.

For $\mathbb{CP}^2$ we can go from the two-dimensional stratum to the zero-dimensional stratum in only one way, because there is only one one-dimensional stratum. Therefore, the identity (5.20) on the GC operators now implies that the squares of the off-diagonal terms of the chiral and anti-chiral components of the Hamiltonian are equal to zero. This analysis leads us to a non-trivial prediction: the two-point correlation functions of evaluation observables of the $\mathbb{CP}^2$ model can only contain $\log q$ and $\log \bar{q}$, but not $(\log q)^2$ or $(\log \bar{q})^2$. (Those can appear if and only if the Hamiltonian has Jordan blocks of length three, as formula (5.7) makes clear.) This prediction turns out to be in perfect agreement with experiment.

Indeed, consider the largest moduli space of gradient trajectories, corresponding to the trajectories going from $(0:0:1)$ to $(1:0:0)$. This moduli space is naturally identified with an open dense subset of $\mathbb{CP}^2$ and it is naturally compactified by $\mathbb{CP}^2$. The correlation function of evaluation observables corresponding to an analytic function F on

* Note also that the squares of the supersymmetry charges ∂ and $\bar{\partial}$ are equal to zero, but this is because of the presence of the differentials $dw_{\alpha\beta}$ and $d\bar{w}_{\alpha\beta}$ in formulae (4.17) and (4.18).

$\mathbb{CP}^2$ (inserted at the time 0) and an analytic four-form ω on $\mathbb{CP}^2$ (inserted at the time t, which we allow to be complex, as before) is then given by the integral

$$\int_{\mathbb{CP}^2} \omega\phi(q)^*(F), \tag{5.22}$$

where $q = \mathrm{e}^{-t}$. At first glance, it is easy to produce examples of ω and F for which this integral contains $(\log q)^2$, for example,

$$\omega = \frac{\mathrm{d}^2 z_1 \wedge \mathrm{d}^2 z_2}{(z_1\bar{z}_1 + z_2\bar{z}_2 + R)^3}, \qquad F = \frac{1}{(z_1\bar{z}_1 + Q)(z_2\bar{z}_2 + P)}, \tag{5.23}$$

where P, Q, R are positive real numbers. However, the problem is that the function F is not smooth at the point $(0 : 1 : 0)$, because in terms of the local coordinates w_1, w_2 around this point it reads

$$F = \frac{w_2\bar{w}_2}{((w_1\bar{w}_1/w_2\bar{w}_2) + Q)(1 + Pw_2\bar{w}_2)},$$

and so it is clear that its second derivative with respect to w_1 and $\bar{w}_1$ is not continuous. Likewise, F is not smooth at the point $(1 : 0 : 0)$. To make it smooth, we would need to pull it back to the blow-up of $\mathbb{CP}^2$ at the points $(0 : 1 : 0)$ and $(1 : 0 : 0)$. The resulting manifold is a del Pezzo surface, and the pullback of F is a legitimate observable on it. But the instanton picture is different on this del Pezzo surface than on $\mathbb{CP}^2$, and so it is not surprising that $(\log q)^2$ appears in the correlation functions on this del Pezzo surface, even though it does not appear on $\mathbb{CP}^2$.

The forms (5.23) are also legitimate observables in the $\mathbb{CP}^1 \times \mathbb{CP}^1$ model, where the appearance of $(\log q)^2$ in the correlation functions of this model is to be expected due to the existence of Jordan blocks of length 3 in the action of the Hamiltonian.

But the forms (5.23) are *not* legitimate observables in the $\mathbb{CP}^2$ model, because F is not smooth, let alone analytic. Therefore, this calculation is not a valid counterexample to our claim (based on the analysis of the off-diagonal terms in the Hamiltonian action) that there are no terms with $(\log q)^2$. In fact, in all examples of correlation functions given by the integrals (5.22), where ω and F are truly analytic on $\mathbb{CP}^2$, that we have computed, we have observed the appearance of $\log q$, but not of $(\log q)^2$. Thus, the correlation functions really distinguish between $\mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$ (or del Pezzo surface) instantons, in agreement with our predictions.

Note that we expect the same phenomenon for the $\mathbb{CP}^n$ model (with a generic $\mathbb{C}^\times$ action) as well. Here we again have a single stratum in each dimension, so the same argument as above again applies to show that the square of the off-diagonal part of the Hamiltonian is equal to zero. This means that the maximal size of the Jordan blocks of the Hamiltonian (and its chiral and anti-chiral components) is 2 (and not n, as one might have thought). Therefore, we expect the appearance of $\log q$ and $\log \bar{q}$ in the correlation functions, but not their higher powers.

The action of evaluation observables on the spaces of 'in' and 'out' states of the $\mathbb{CP}^2$ model is obtained in the same way as in the $\mathbb{CP}^1$ model. It is not difficult to write

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analogues of the identity (5.6), corresponding to factorization over intermediate states in the $\mathbb{CP}^2$ model. In the most interesting case of instantons propagating from $(0:0:1)$ to $(1:0:0)$, the left-hand side of the identity is given by the integral (5.22). The right-hand side is the sum of three terms, each corresponding to one of the fixed points of the $\mathbb{C}^\times$-action. It is a $q, \bar{q}$-series that should converge to the left-hand side of the identity inside a sufficiently small disc on the q-plane.

For a general Kähler manifold X, the compactification of the largest instanton moduli space of gradient trajectories is X itself, and we have a similar identity in which the right-hand side has terms corresponding to the fixed points of the $\mathbb{C}^\times$. This identity may be interpreted as a decomposition of the delta-form on $X \times X$ supported on the q-shifted diagonal, as we explained in the case of $\mathbb{CP}^1$ in § 5.4. There are also similar identities for other instanton moduli spaces. It would be very interesting to find a general proof of these identities for all analytic evaluation observables.

We conclude our discussion of the observables with the following remark. So far we have considered the evaluation observables of our theory which correspond to differential forms on X. As we pointed out in § 2.7, there are other observables in the theory which correspond to differential operators on X. In particular, those include global holomorphic and anti-holomorphic differential operators. The algebras of such operators should be viewed as the precursors of the chiral (and anti-chiral) de Rham complex of the two-dimensional sigma model that we will discuss in Part II. Differential operators on X naturally act on the spaces of states $\mathcal{H}^{\text{in}}$ and $\mathcal{H}^{\text{out}}$ of our theory (it follows from the definition that these spaces are $\mathcal{D}$-modules).

These operators are particularly important for computing the perturbation theory expansion of the correlation functions of our theory away from the point $\lambda = \infty$. This is because perturbation to finite values of λ is achieved by adding to the action the term $\lambda^{-1} g^{a\bar{b}} p_a p_{\bar{b}}$, which corresponds, in the Hamiltonian formalism, to a differential operator on X. Therefore, in order to compute the correlation functions in the theory defined for finite values of λ by perturbation theory in λ^{-1} we need to insert these operators in the correlation functions of the theory at $\lambda = \infty$. We will discuss this in § 6.1.

6. Various generalizations

In this section we comment on possible generalizations of the results obtained above. We begin by discussing the perturbation theory around the point $\lambda = \infty$. In the previous chapters we have exhibited the structure of the space of states of the theory in the limit $\lambda = \infty$ and discussed various methods for computing the correlation functions. It would be highly desirable to use these results to obtain information about the theories defined at finite values of λ. In particular, we consider the question of how the space of states of the model changes when we move away from the point $\lambda = \infty$, first in the case when $X = \mathbb{C}$ and then for $X = \mathbb{CP}^1$. Because the Hamiltonian is non-diagonalizable we cannot apply the standard tools of quantum mechanics and the perturbation theory turns out to be a more challenging task. We then discuss the computation of the correlation functions in λ^{-1} perturbation theory. We present some evidence that the quantum mechanical models for finite values of λ may be studied by using this perturbation theory.

Next, we consider some non-supersymmetric analogues of our models. We discuss in particular the computation of the cohomology of the anti-chiral supercharge $\bar{\partial}$ (in those models in which it exists; they are one-dimensional analogues of the $(0,2)$ supersymmetric two-dimensional sigma models). We make contact to the GC complexes of arbitrary vector bundles on Kähler manifolds and results of Witten [**39**] and Wu [**46**] on holomorphic Morse theory.

We also discuss briefly the generalization in which a Morse functions is replaced by a Morse–Bott function having non-isolated critical points. More precisely, we will consider the situation where the Morse function comes from a $\mathbb{C}^{\times}$-action on our Kähler manifold X with non-isolated fixed points. We show how some of the features of the models with Morse functions change in this more general situation.

Finally, we comment on the Morse–Novikov functions, which are multivalued analogues of the Morse functions. They are particularly important for applications to two-dimensional and four-dimensional models.

6.1. Perturbation theory around the point $\lambda = \infty$

We have described above the structure of the spaces of 'in' and 'out' states in the limit $\lambda \to \infty$ of our quantum mechanical model. This structure is very different from the structure observed at finite values of λ. It is natural to ask whether one can relate the two pictures by some kind of perturbation theory. This question is important because we would like to understand our models at finite values of λ using the results obtained at $\lambda = \infty$, where the theory simplifies dramatically.

Note that here we have to deal with a somewhat unfamiliar situation, where the problem does not have a Hilbert space formulation, so we cannot use the hermitian inner product, as is customary in the quantum mechanics, at least in all of its textbook examples.

We start with the case of the flat space $\mathbb{C}$. We recall from § 3.3 that in this case the space of 'in' states is the space of differential forms on $\mathbb{C}$ if $\omega > 0$, and the space of distributions supported at $0 \in \mathbb{C}$ if $\omega < 0$. In what follows we will restrict ourselves to the subspace of 0-forms.

In the former case the perturbation theory is very simple and finite. Indeed, the Hamiltonian $\tilde{H}_\lambda$ at finite λ is obtained from the Hamiltonian $\mathcal{L}_v$, $v = z\partial_z + \bar{z}\partial_{\bar{z}}$ at $\lambda = \infty$ by adding the term $-(2/\lambda)\partial_z\partial_{\bar{z}}$. This extra term lowers the degree of a polynomial by 1 in z and by 1 in $\bar{z}$. Therefore, starting with a monomial $z^n\bar{z}^{\bar{n}}$, which is an eigenvector of $\mathcal{L}_v$, we can obtain an eigenvector of $\tilde{H}_\lambda$ by adding monomials of lower degrees. The resulting polynomial is closely related to the Hermite polynomials.

The perturbation theory in the second case is more subtle. In this case we are trying to reproduce the eigenfunctions of $\tilde{H}_\lambda$, which look like $\mathrm{e}^{-\lambda|\omega|z\bar{z}}$ times a polynomial in z, $\bar{z}$, as linear combinations of the derivatives of the delta-function $\delta^{(2)}(z,\bar{z})$. These linear combinations are sums of infinitely many terms, which may be thought of as asymptotic expansions of the eigenfunctions at $\lambda^{-1} = 0$. That there are infinitely many terms in the expansion is easy to see from the fact that now the additional term $-(2/\lambda)\partial_z\partial_{\bar{z}}$ appearing in the Hamiltonian increases the number of derivatives of the delta-function.

The exact formulae for the eigenstates look as follows (in the notation of § 3.3, where for simplicity we set $\omega = -1$):

$$\tilde{\Psi}^{\mathrm{in}}_{n,\bar{n}} = \frac{\lambda}{2\pi}\frac{1}{n!\bar{n}!}\partial_z^n \partial_{\bar{z}}^{\bar{n}} \mathrm{e}^{-\lambda z\bar{z}} \sim \sum_{k=0}^{\infty} \frac{1}{n!\bar{n}!k!}\lambda^{-k}\partial_z^{n+k}\partial_{\bar{z}}^{\bar{n}+k}\delta^{(2)}(z,\bar{z})$$
$$\sim \sum_{k=0}^{\infty} \frac{(n+k)!(\bar{n}+k)!}{n!\bar{n}!k!}\lambda^{-k}|\bar{n}+k, n+k\rangle,$$

where

$$|m,\bar{m}\rangle = \frac{1}{m!\bar{m}!}\partial_z^m \partial_{\bar{z}}^{\bar{m}}\delta^{(2)}(z,\bar{z})$$

and the right-hand side is understood as the asymptotic expansion of the left-hand side (viewed as a distribution on $\mathbb{C}$) at $\lambda = \infty$. The Borel summation of this series gives the left-hand side.

We shall now sketch some aspects of the perturbation theory in λ^{-1} in the $\mathbb{CP}^1$ model. We start with the following simple remark. Suppose we want to solve the following eigenvalue problem:

$$H\Psi = E\Psi, \qquad H = H_0 + \frac{1}{\lambda}H_1,$$
$$\Psi = \Psi^{[0]} + \sum_{k=1}^{\infty}\frac{1}{\lambda^k}\Psi^{[k]}, \qquad E = E^{[0]} + \sum_{k=1}^{\infty}\frac{1}{\lambda^k}E^{[k]}.$$

Then we have the following simple relations:

$$\left.\begin{aligned} H_0\Psi^{[0]} &= E^{[0]}\Psi^{[0]}, \\ (H_1 - E^{[1]})\Psi^{[0]} &= -(H_0 - E^{[0]})\Psi^{[1]}, \\ (-E^{[2]})\Psi^{[0]} + (H_1 - E^{[1]})\Psi^{[1]} &= -(H_0 - E^{[0]})\Psi^{[2]}, \\ &\vdots \end{aligned}\right\} \tag{6.1}$$

In our case we have the additional subtlety of 'almost' degenerate perturbation theory.

Consider the Hamiltonian acting on functions, i.e. on 0-forms. To simplify our notation, we will write $|n,\bar{n}\rangle_\infty$ and $|n,\bar{n}\rangle_{\mathbb{C}_0}$ for $|n,\bar{n},0,0\rangle_\infty$ and $|n,\bar{n},0,0\rangle_{\mathbb{C}_0}$. We have

$$H_0 = z\partial_z + \bar{z}\partial_{\bar{z}} = -(w\partial_w + \bar{w}\partial_{\bar{w}}), \tag{6.2}$$
$$H_1 = -2(1+z\bar{z})^2\partial_z\partial_{\bar{z}} = -2(1+w\bar{w})^2\partial_w\partial_{\bar{w}}. \tag{6.3}$$

The subspace $\mathcal{H}^{\mathrm{in}}_\infty$ of the space of states is preserved by H_0 and H_1:

$$H_0|n,\bar{n}\rangle_\infty = (n+\bar{n}+2)|n,\bar{n}\rangle_\infty,$$
$$H_1|n,\bar{n}\rangle_\infty = -2(n+1)(\bar{n}+1)(|n-1,\bar{n}-1\rangle_\infty + 2|n,\bar{n},\rangle_\infty + |n+1,\bar{n}+1\rangle_\infty).$$

The equations (6.1) can be solved explicitly:

$$\begin{aligned}
\Psi^{[0]} &= |n,\bar{n}\rangle_\infty = \frac{1}{n!\bar{n}!}\partial_w^n \partial_{\bar{w}}^{\bar{n}} \delta^{(2)}(w,\bar{w}),\\
\Psi^{[1]} &= (n+1)(\bar{n}+1)(|n+1,\bar{n}+1\rangle_\infty - |n-1,\bar{n}-1\rangle_\infty),\\
\Psi^{[2]} &= \tfrac{1}{2}(n+1)(\bar{n}+1)[(n+2)(\bar{n}+2)(|n+2,\bar{n}+2\rangle_\infty + 4|n+1,\bar{n}+1\rangle_\infty)\\
&\qquad + n\bar{n}(|n-2,\bar{n}-2\rangle_\infty + 4|n-1,\bar{n}-1\rangle_\infty)]
\end{aligned}$$

and

$$\begin{aligned}
E_\infty^{[0]} &= n+\bar{n}+2,\\
E_\infty^{[1]} &= -4(n+1)(\bar{n}+1),\\
E_\infty^{[2]} &= 4(n+1)(\bar{n}+1)(n+\bar{n}+2).
\end{aligned}$$

We conjecture that

$$E_\infty^{[k]} = (E^{[0]})^{k+1} O(1).$$

This would imply that the radius of convergence of the corresponding perturbative expansion is approximately $\lambda^{-1} < (E^{[0]})^{-1}$, which means that we understand well the eigenfunctions whose eigenvalues are less than λ. This agrees with the qualitative picture suggested by the semi-classical analysis.

The determination of the perturbation series for the 'in' states that belong to the subspace $\mathcal{H}_{\mathbb{C}_0}^{\text{in}}$ is more complicated because of the Jordan block structure of the Hamiltonian. Indeed, according to the calculations of § 4.3, almost all of the states $|n,\bar{n}\rangle_{\mathbb{C}_0}$ in $\mathcal{H}_{\mathbb{C}_0}^{\text{in}}$ are generalized eigenvectors of the Hamiltonian:

$$H_0|n,\bar{n}\rangle_{\mathbb{C}_0} = (n+\bar{n})|n,\bar{n}\rangle_{\mathbb{C}_0} - 2\pi|n-1,\bar{n}-1\rangle_\infty, \tag{6.4}$$

as we have seen in the previous section (the exception is the states $|n,0\rangle_{\mathbb{C}_0}$ and $|0,n\rangle_{\mathbb{C}_0}$). We also find the following formula for the action of H_1:

$$\begin{aligned}
H_1|n,\bar{n}\rangle_{\mathbb{C}_0} &= -2n\bar{n}(|n-1,\bar{n}-1\rangle_{\mathbb{C}_0} + 2|n,\bar{n}\rangle_{\mathbb{C}_0} + |n+1,\bar{n}+1\rangle_{\mathbb{C}_0})\\
&\qquad + 4\pi(n+\bar{n})(|n-1,\bar{n}-1\rangle_\infty + 2|n,\bar{n}\rangle_\infty + |n+1,\bar{n}+1\rangle_\infty).
\end{aligned}$$

The corresponding perturbation theory is unusual, because normally one considers hermitian Hamiltonians which cannot have Jordan blocks. The first question is whether the degeneracy of the eigenvalues is removed and the Jordan block structure is broken in the λ^{-1} perturbation theory. At first glance, it appears that the degeneracy and the Jordan block structure should remain, because we know that the difference between the two eigenvalues for finite λ is of the order $e^{-\lambda}$, which appears to be out of reach of the perturbation theory. However, here we could in principle obtain the asymptotic expansion of this difference. It would be very interesting to analyse this perturbative expansion explicitly.

6.2. Perturbative expansion for correlation functions

It is important to understand to what extent the correlation functions of the quantum mechanical models at finite values of λ may be reconstructed from the correlation functions at $\lambda = \infty$ by perturbation theory. Here we consider the simplest non-trivial example, which suggests that this may indeed be done successfully.

Let $X = \mathbb{CP}^1$ and consider the two-point functions of the form ${}_\infty\langle \hat{\omega}(t_1)\hat{F}(t_2)\rangle_0$ as in § 5.2. According to formula (5.3), the value of this correlation function at $\lambda = \infty$ is equal to the integral $\int_{\mathbb{CP}^1} \omega\phi(\mathrm{e}^{-t})^*(F)$ and to the matrix element

$$ {}_{\mathbb{C}_\infty}\langle 0|\hat{\omega}\mathrm{e}^{-tH_0}\hat{F}|0\rangle_{\mathbb{C}_0}. $$

For finite values of λ this correlation function is given by the formula

$$ {}_{\mathbb{C}_\infty}\langle 0|\hat{\omega}\mathrm{e}^{-t(H_0+\lambda^{-1}H_1)}\hat{F}|0\rangle_{\mathbb{C}_0}, $$

where H_0 and H_1 (acting on functions) are given by formulae (6.2) and (6.3). Now we use the expansion formula

$$ \begin{aligned} &{}_{\mathbb{C}_\infty}\langle 0|\hat{\omega}\mathrm{e}^{-t(H_0+\lambda^{-1}H_1)}\hat{F}|0\rangle_{\mathbb{C}_0} \\ &\qquad = {}_{\mathbb{C}_\infty}\langle 0|\hat{\omega}\mathrm{e}^{-tH_0}\hat{F}|0\rangle_{\mathbb{C}_0} - \lambda^{-1}\int_0^t \mathrm{d}s\, {}_{\mathbb{C}_\infty}\langle 0|\hat{\omega}\mathrm{e}^{-sH_0}H_1\mathrm{e}^{-(t-s)H_0}\hat{F}|0\rangle_{\mathbb{C}_0} + \cdots . \end{aligned} $$

Together with formulae for $\hat{F}|0\rangle_{\mathbb{C}_0}$ and ${}_{\mathbb{C}_\infty}\langle 0|\hat{\omega}$ found in § 5.2 and the formulae for the action of H_0 and H_1 on $\mathcal{H}^{\mathrm{in}}_{\mathbb{C}_0}$ found in the previous section, this gives us an explicit perturbative λ^{-1}-expansion for the two-point correlation function ${}_\infty\langle \hat{\omega}(t_1)\hat{F}(t_2)\rangle_0$.

The same analysis may be applied to n-point correlation functions of evaluation observables. We find that each term of the corresponding λ^{-1}-expansion is given by a finite integral of a matrix element of these operators acting on the space of states at $\lambda = \infty$. Thus, in principle all of the corresponding λ^{-1} perturbation series are computable. It would be interesting to relate these perturbation series to the actual correlation functions in the theories at finite values of λ computed by other methods.

6.3. Comments on the non-supersymmetric case

Up to now we have considered the limit $\lambda = \infty$ of the supersymmetric model of quantum mechanics defined by the action (2.16). There are also analogous non-supersymmetric models, and many of our results may be applied to those models as well.

The simplest way to break supersymmetry is to consider fermions taking values in vector bundles on X that are different from the tangent and cotangent bundles. Since we have assumed X to be Kähler, we have chiral fermions: ψ^a, π_a, taking values in the holomorphic cotangent and tangent bundles on X, respectively, and anti-chiral fermions, $\psi^{\bar{a}}$, $\pi_{\bar{a}}$, taking values in the holomorphic cotangent and tangent bundles on X, respectively. We may now stipulate that ψ^a, π_a take values in vector bundles $\mathcal{E}$ and $\mathcal{E}^*$, respectively, whereas $\psi^{\bar{a}}$, $\pi_{\bar{a}}$ take values in vector bundles $\bar{\mathcal{E}}$ and $\bar{\mathcal{E}}^*$, respectively. The vector bundle $\bar{\mathcal{E}}$ need not be complex conjugate to $\mathcal{E}$, thus allowing the possibility of 'heterotic' quantum mechanical models, which are the precursors of the sigma models appearing in heterotic string theory.

The bosonic part of the action remains the same as before:

$$-\mathrm{i}\int_I \left(p_a\left(\frac{\mathrm{d}X^a}{\mathrm{d}t}-v^a\right)+p_{\bar{a}}\left(\frac{\mathrm{d}X^{\bar{a}}}{\mathrm{d}t}-\bar{v}^a\right)\right)\mathrm{d}t,$$

where $\xi = v^a\partial_{X^a}$ is a holomorphic vector field. We will again assume that $\xi+\bar{\xi}$ is the gradient vector field of a Morse function f on X and that ξ comes from a holomorphic $\mathbb{C}^\times$-action on X.

To write down the fermionic part of the action, we need to assume in addition that the $\mathbb{C}^\times$-action on X may be lifted to $\mathcal{E}$ and $\bar{\mathcal{E}}$, i.e. that $\mathcal{E}$ and $\bar{\mathcal{E}}$ are $\mathbb{C}^\times$-equivariant vector bundles on X. Then if we choose local trivialization of $\mathcal{E}$ by local sections ϕ^i and a local trivialization of $\bar{\mathcal{E}}$ by local sections $\phi^{\bar{i}}$, the generator of the one-dimensional Lie algebra of $\mathbb{C}^\times$ will act by the formula

$$\phi^i \mapsto M^i_j\phi^j, \qquad \phi^{\bar{i}} \mapsto \bar{M}^{\bar{i}}_{\bar{j}}\phi^{\bar{j}}.$$

For example, in the case when $\mathcal{E}$ and $\bar{\mathcal{E}}$ are the holomorphic and anti-holomorphic cotangent bundles of X, we have bases of sections $\mathrm{d}x^a$ and $\mathrm{d}x^{\bar{a}}$, and so

$$M^a_b = \frac{\partial v^a}{\partial X^b}, \qquad \bar{M}^{\bar{a}}_{\bar{b}} = \frac{\partial \bar{v}^a}{\partial X^{\bar{b}}}.$$

Now the fermionic part of the action is

$$\mathrm{i}\int_I \left(\pi_i\left(\frac{\mathrm{D}\psi^i}{\mathrm{D}t}-M^i_j\psi^j\right)-\pi_{\bar{i}}\left(\frac{\bar{\mathrm{D}}\psi^{\bar{i}}}{\bar{\mathrm{D}}t}-\bar{M}^{\bar{i}}_{\bar{j}}\psi^{\bar{j}}\right)\right)\mathrm{d}t.$$

Here $\mathrm{D}/\mathrm{D}t$ and $\bar{\mathrm{D}}/\bar{\mathrm{D}}t$ are the covariant derivatives corresponding to chosen connections on $\mathcal{E}$ and $\bar{\mathcal{E}}$, which may however be absorbed into the momenta p_a and $p_{\bar{a}}$ in the same way as in the supersymmetric model (see § 2.3).

The definition of the corresponding path integral for general vector bundles $\mathcal{E}$ and $\bar{\mathcal{E}}$ requires special care because the space of fields does not carry a canonical integration measure as in the supersymmetric case. We will discuss this question in Part III. Here we will only point out that our results on the Hamiltonian description of the supersymmetric model have obvious generalizations to the non-supersymmetric case.

We again have spaces of 'in' and 'out' states, $\mathcal{H}^{\mathrm{in}}$ and $\mathcal{H}^{\mathrm{out}}$. The space $\mathcal{H}^{\mathrm{in}}$ is isomorphic to the direct sum of spaces $\mathcal{H}^{\mathrm{in}}_\alpha$ labelled by the fixed points x_α of the $\mathbb{C}^\times$-action, as in the supersymmetric case. Each $\mathcal{H}^{\mathrm{in}}_\alpha$ exhibits holomorphic factorization: $\mathcal{H}^{\mathrm{in}}_\alpha = \mathcal{F}^{\mathrm{in}}_\alpha \otimes \bar{\mathcal{F}}^{\mathrm{in}}_\alpha$, where

$$\mathcal{F}^{\mathrm{in}}_\alpha = H^{n-n_\alpha}_{X_\alpha}(\wedge^\bullet\mathcal{E}), \qquad \bar{\mathcal{F}}^{\mathrm{in}}_\alpha = H^{n-n_\alpha}_{X_\alpha}(\wedge^\bullet\bar{\mathcal{E}})$$

(compare with formula (4.16)). The Hamiltonian is the vector field $v=\xi+\bar{\xi}$. The action of ξ and $\bar{\xi}$ is given by formulae similar to (4.19) and (4.20), in which the GC operators are

$$\delta^{\mathcal{E}}_{\alpha\beta}\colon \mathcal{F}^{\mathrm{in}}_\alpha \to \mathcal{F}^{\mathrm{in}}_\beta \quad \text{and} \quad \bar{\delta}^{\bar{\mathcal{E}}}_{\alpha\beta}\colon \bar{\mathcal{F}}^{\mathrm{in}}_\alpha \to \bar{\mathcal{F}}^{\mathrm{in}}_\beta$$

for $X_\alpha \succ X_\beta$.

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The space of 'out' states has a similar structure, with the ascending manifolds X_α replaced by the descending manifolds X^α. In addition, we need to replace $\mathcal{E}$ by $\Omega^{\text{top}} \otimes \mathcal{E}^*$ and $\bar{\mathcal{E}}$ by $\bar{\Omega}^{\text{top}} \otimes \bar{\mathcal{E}}^*$, where Ω^{top} and $\bar{\Omega}^{\text{top}}$ are the line bundles of holomorphic and anti-holomorphic top forms, respectively.

6.4. Cohomology of the supercharge in 'half-supersymmetric' models

An interesting special class of models arises if we let $\bar{\mathcal{E}}$ be the anti-holomorphic cotangent bundle on X, as in the supersymmetric model. Then we retain the anti-chiral supercharge $\bar{\partial}$. The corresponding 'half-supersymmetric' models may therefore be viewed as quantum mechanical analogues of the $(0,2)$ supersymmetric sigma models (just like the fully supersymmetric model may be viewed as an analogue of the $(2,2)$ supersymmetric sigma model). In such models it is interesting to compute the cohomology of the supercharge $\bar{\partial}$, which may be viewed as a 'baby version' of the cohomology of the right moving supercharge in the $(0,2)$ sigma models. This cohomology has been studied in [**44**], where it was shown that it is closely related to the chiral differential operators.

In the 'half-supersymmetric' quantum mechanical model the supercharge $\bar{\partial}$ is given by the same formula (4.18) as in the supersymmetric case, except that we need to replace the GC operators $\delta_{\alpha\beta}$ and $\bar{\delta}_{\alpha\beta}$ by $\delta^{\mathcal{E}}_{\alpha\beta}$ and $\bar{\delta}^{\bar{\mathcal{E}}}_{\alpha\beta}$, respectively.

The argument of § 4.9 for the computation of the $\bar{\partial}$-cohomology in the supersymmetric case carries over verbatim to this case, and we find that the first term of the corresponding spectral sequence is just the GC complex $C^\bullet(\wedge^\bullet\mathcal{E})$ of the vector bundle $\wedge^\bullet\mathcal{E}$ (see § 4.9 for the definition of this complex). Actually, the grading on the exterior algebra $\wedge^\bullet\mathcal{E}$ is preserved by the differential, so the GC complex decomposes into a direct sum of its subcomplexes $C^\bullet(\wedge^p\mathcal{E})$, $p = 0, \ldots, \dim\mathcal{E}$. This gives a second grading on the cohomology of $\bar{\partial}$. This cohomology is therefore equal to the direct sum of the (Dolbeault) cohomologies of the sheaves of holomorphic sections of the vector bundles $\wedge^p\mathcal{E}$, $p \geqslant 0$:

$$H^i_{\bar{\partial}} = \bigoplus_{p\geqslant 0} H^i(X, \wedge^p\mathcal{E}).$$

Thus, we find an effective way for computing the cohomology of the anti-chiral supercharge in the 'half-supersymmetric' quantum mechanical models. The result is that this cohomology is nothing but the Dolbeault cohomology of the bundle $\wedge^\bullet\mathcal{E}$, where the chiral fermions take values.

This result is closely related to the work of Witten on the holomorphic Morse theory. In [**39**] Witten has shown how to adopt his approach to Morse theory from [**38**] (discussed in § 2.1), which allows one to compute the de Rham cohomology of X in terms of an instanton complex associated to a Morse function, to the computation of the cohomology of the sheaf of holomorphic sections of a $\mathbb{C}^\times$-equivariant vector bundle $\mathcal{E}$ on X (in other words, computing Dolbeault cohomology instead of de Rham cohomology). It is assumed that X is a Kähler manifold equipped with a Morse function f whose gradient satisfies the conditions listed above. Witten has shown that this cohomology may be obtained as the cohomology of a certain complex. The groups of this complex are infinite dimensional, but they are graded by the action of $\mathbb{C}^\times$ and the corresponding graded components are finite

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dimensional. Witten has computed in [**39**] the characters of the groups of this complex. This enabled him to write down 'holomorphic Morse inequalities' giving estimates on the Dolbeault cohomology of $\mathcal{E}$ in the same way as the usual Morse inequalities give us estimates on the de Rham cohomology of X.

It was subsequently shown by Wu in [**46**] that the characters that Witten had computed were precisely the characters of the groups of the GC complex $C^\bullet(\mathcal{E})$ of $\mathcal{E}$. Therefore, it was suggested in [**46**] that Witten's holomorphic instanton complex should be interpreted as the GC complex of $\mathcal{E}$. This is in agreement with Witten's result, because we know that the cohomology of the GC complex is equal to the Dolbeault cohomology of $\mathcal{E}$. But the connection of this complex to quantum mechanics still remained a mystery.

But now we have found a natural explanation of the connection between the GC complex and quantum mechanics. Namely, we have shown that the GC complexes naturally appear in the framework of the 'half-supersymmetric' quantum mechanical models in the limit $\lambda = \infty$. We have found that the Dolbeault cohomology of a holomorphic $\mathbb{C}^\times$-equivariant line bundle on a compact Kähler manifold coincides with the cohomology of the supercharge $\bar{\partial}$ on the space of 'in' states of a particular model of this type: with left fermions living in $\mathcal{E}$ and right fermions living in the anti-holomorphic cotangent bundle. Moreover, we have shown that the computation of the cohomology of $\bar{\partial}$ naturally gives us the GC complex of $\mathcal{E}$ (and even of $\wedge^\bullet \mathcal{E}$). This explains the connection between 'holomorphic Morse theory' and quantum mechanics.

We want to point out a particularly interesting class of 'half-supersymmetric' models of this type. They are associated to a flag variety G/B, where G is a complex simple Lie group and B is its Borel subgroup. This flag variety has a natural Morse function: the hamiltonian of the vector field corresponding to a generic element of a maximal compact torus contained in B. Its critical points are parametrized by the Weyl group W of G. The corresponding ascending manifolds are called the Schubert cells, which we denote by X_w, $w \in W$. Its complex dimension is equal to the length of w, denoted by $\ell(w)$. We can choose the Morse function in such a way that they are the B-orbits on G/B. Note that if $G = \mathrm{SL}_2$, then $G/B \simeq \mathbb{CP}^1$, and the corresponding Morse function is the one we have studied extensively in the earlier sections.

Suppose that $\mathcal{E}$ is a line bundle on G/B. Then it corresponds to an integral weight μ of the group G. Let us suppose that μ is dominant, and so can be realized as the highest weight of an irreducible finite-dimensional representation V_μ of G. We denote the corresponding line bundle by $\mathcal{E}_\mu$. The GC complex of this line bundle is studied in detail in [**26**] (see also [**14, 46**]), where it is shown that this complex coincides with the dual of the so-called Bernstein–Gelfand–Gelfand (BGG) resolution of V_μ. The ith group of the GC complex $C^\bullet(\mathcal{E}_\mu)$ is equal to

$$C^i(\mathcal{E}_\mu) = \bigoplus_{\ell(w)=i} H^{\dim G/B - \ell(w)}(G/B, \mathcal{E}_\mu).$$

The group G does not act on the GC complex $C^\bullet(\mathcal{E}_\mu)$, but the Lie algebra $\mathfrak{g}$ does. Under this action

$$H^{\dim G/B - \ell(w)}(G/B, \mathcal{E}_\mu) \simeq M^*_{w(\mu+\rho)-\rho},$$

the contragredient Verma module over $\mathfrak{g}$ of highest weight $w(\mu+\rho)-\rho$, where ρ is the half-sum of positive roots of $\mathfrak{g}$. Thus, the ith group of the GC complex is

$$C^i(\mathcal{E}_\mu) = \bigoplus_{\ell(w)=i} M^*_{w(\mu+\rho)-\rho},$$

and its cohomology is equal to $H^\bullet(G/B,\mathcal{E}_\mu)$. According to the Borel–Weil–Bott theorem, $H^0(G/B,\mathcal{E}_\mu) \simeq V_\mu$ and $H^i(G/B,\mathcal{E}_\mu) = 0$, for $i>0$. Therefore, the GC complex $C^\bullet(\mathcal{E}_\mu)$ is a *resolution* of the irreducible representation V_μ. It is dual to the BGG resolution, which is well known in representation theory [**26**].*

According to the above discussion, this BGG resolution is naturally realized in the context of a 'half-supersymmetric' model on G/B in which left fermions take values in the line bundle $\mathcal{E}_\mu$ and its dual. The cohomology of the supercharge $\bar\partial$ in this model is therefore equal to

$$H^\bullet(G/B,\wedge^\bullet\mathcal{E}_\mu) = H^0(G/B,\mathcal{O}) \oplus H^0(G/B,\mathcal{E}_\mu) \simeq \mathbb{C}\oplus V_\mu.$$

Thus, we realize irreducible representations of simple Lie groups as $\bar\partial$-cohomologies of 'half-supersymmetric' models on the flag variety.

These results have interesting analogues in $(0,2)$ supersymmetric two-dimensional sigma models, as we will see in Parts II and III.

6.5. Comments on non-isolated critical points

Up to now we have considered a Morse function f on a Kähler manifold X of dimension n and the corresponding gradient vector field $v=\nabla f$ which, as we have assumed, decomposes into the sum $\xi+\bar\xi$ of a holomorphic and anti-holomorphic vector fields generating a $\mathbb{C}^\times$-action on X. The critical points of f are the fixed points of this $\mathbb{C}^\times$-action. The assumption that f is a Morse function means that these points are isolated and non-degenerate. In this section we discuss briefly what happens if we are in a situation when the fixed points are not isolated and f is a Morse–Bott function.

Let C_α, $\alpha\in A$, be the components of the fixed point set of the $\mathbb{C}^\times$-action on X (under our old assumptions, each C_α consisted of a single point). According to the results of [**6**,**8**], in this case X still has decompositions (3.28), with X_α and X^α defined in the same way as before, by formulae (3.29) and (3.30). However, in this case each X_α is a $\mathbb{C}^\times$-equivariant holomorphic fibration over C_α, whose fibres are isomorphic to $\mathbb{C}^{n_\alpha}$, where n_α is the number of positive eigenvalues of the Hessian of f at the points of C_α. Moreover, locally over C_α the bundle X_α is isomorphic to the subbundle of the normal bundle to $C_\alpha\subset X$ spanned by the eigenspaces of the Hessian of the function f with positive eigenvalues. Likewise, X^α is also a $\mathbb{C}^\times$-equivariant holomorphic bundle over C_α with fibres isomorphic to $\mathbb{C}^{n-n_\alpha-\dim C_\alpha}$. Locally over C_α the bundle X^α is isomorphic to the subbundle of the normal bundle to $C_\alpha\subset X$ spanned by the eigenspaces of the Hessian of the function f with negative eigenvalues.

* If μ is not dominant, then the cohomology is either zero or it occurs in a positive cohomological dimension; in this case the $\mathfrak{g}$-modules appearing in the complex are the twisted Verma modules (see [**14**]).

Consider, for example, the case of $X = \mathbb{CP}^2$ with the $\mathbb{C}^\times$ action $(z_1 : z_2 : z_3) \mapsto (qz_1 : z_2 : z_3)$, corresponding to the vector field $v = z_1\partial_{z_1} + \bar{z}_1\partial_{\bar{z}_1}$. Then the fixed point set has two components: the point $C_1 = (1 : 0 : 0)$ and the one-dimensional component $C_2 = \{(0 : z_2 : z_3)\}$ isomorphic to $\mathbb{CP}^1$. The corresponding strata X_1 and X_2 are the point $(1 : 0 : 0)$ and its complement, respectively. Note that X_2 is a line bundle over $\mathbb{CP}^1$ isomorphic to $\mathcal{O}(1)$, which is also isomorphic to the normal bundle of $C_2 \subset \mathbb{CP}^2$. The strata X^1 and X^2 are the plane $\{(1 : u_1 : u_2)\}$ and $C_2 = \mathbb{CP}^1$, respectively.

The description of the spaces of 'in' and 'out' states of this model is similar to the one obtained previously in the Morse function case. Namely, $\mathcal{H}^{\text{in}}$ is isomorphic to the direct sum of the spaces $\mathcal{H}^{\text{in}}_\alpha$, $\alpha \in A$. Roughly speaking, each space $\mathcal{H}^{\text{in}}_\alpha$ is the space of L_2 differential forms on C_α extended in two ways: by polynomial differential forms in the bundle directions of X_α, and then by polynomials in the derivatives in the transversal directions to X_α in X.

The ground states, on which the Hamiltonian $\mathcal{L}_v$ acts by zero, correspond to differential forms on C_α. Given such a form ω_α, let $\tilde{\omega}_\alpha$ be its pullback to X_α under the projection $X_\alpha \to C_\alpha$. Then $\tilde{\omega}_\alpha$ defines a 'delta-like' distribution supported on X_α, whose value on $\eta \in \Omega^\bullet(X)$ is equal to

$$\int_{X_\alpha} \tilde{\omega}_\alpha \wedge \eta|_{X_\alpha}$$

(compare with formula (3.31)). We will use the same notation $\tilde{\omega}_\alpha$ for these distributions. While these are the ground states of the model at $\lambda = \infty$, only those of them which correspond to harmonic differential forms $\omega_\alpha \in \Omega^\bullet(C_\alpha)$, $\alpha \in A$, may be deformed to ground states for finite values of λ.

Other elements of $\mathcal{H}_\alpha$ are distributions obtained by applying to the distributions $\tilde{\omega}_\alpha$ Lie derivatives in the transversal directions to X_α as well as multiplying them by differential forms on X_α that are polynomial along the fibres of the projection $X_\alpha \to C_\alpha$ (compare with formula (3.34)). The definition of these distributions requires a regularization similar to the one we used in the case of isolated critical points. Because of this regularization, we obtain non-trivial extensions between different spaces $\mathcal{H}^{\text{in}}_\alpha$, and the action of the Hamiltonian is not diagonalizable. However, the formula for the Hamiltonian is more complicated than in the case of isolated fixed points. Another difference with the case of isolated fixed points is that we observe holomorphic factorization only in the fibre directions of the maps $X_\alpha \to C_\alpha$, but not along the manifolds C_α themselves.

6.6. Morse–Novikov functions

In the analysis we have performed so far we worked with the single-valued Morse functions f. Morse theory has a generalization for non-simply connected manifolds, namely, the Morse–Novikov theory, in which f is multivalued and only its differential is well defined. However, according to the results of [**16**], under the assumptions that we have made: that X is a compact Kähler manifold X with a holomorphic vector field ξ such that its zeros are isolated and the set of zeros is non-empty, we have $H_1(X, \mathbb{Z}) = 0$. Therefore, all closed 1-forms on X are exact. Let β be the 1-form obtained by contraction of the vector field $v = \xi + \bar{\xi}$ and the metric on X. Then $\beta = df$, where the function f is a Morse function on X whose gradient vector field is equal to v, and whose critical points are the zeros of v and of ξ. Therefore, there is no need to consider the case of multivalued, or Morse–Novikov, functions.* However, for *infinite-dimensional* Kähler manifolds, such as the loop space LX, such functions do arise, and in fact it is necessary to study them in order to understand two-dimensional sigma models. We will study this in detail in Part II.

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* Such functions exist if we allow X to be a real manifold, for example, a circle.

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