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ON THE GLOBAL GAUSSIAN LIPSCHITZ SPACE

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Abstract It is well known that the standard Lipschitz space in Euclidean space, with exponent $\alpha \in (0,1)$, can be characterized by means of the inequality $|\partial \mathcal{P}_t f/\partial t| \lesssim t^{\alpha-1}$, where $\mathcal{P}_t f$ is the Poisson integral of the function f. There are two cases: one can either assume that the functions in the space are bounded, or one can not make such an assumption. In the setting of the Ornstein–Uhlenbeck semigroup in $\mathbb{R}^n$, Gatto and Urbina defined a Lipschitz space by means of a similar inequality for the Ornstein–Uhlenbeck Poisson integral, considering bounded functions. In a preceding paper, the authors characterized that space by means of a Lipschitz-type continuity condition. The present paper defines a Lipschitz space in the same setting in a similar way, but now without the boundedness condition. Our main result says that this space can also be described by a continuity condition. The functions in this space turn out to have at most logarithmic growth at infinity.

Keywords: Gauss measure space; Lipschitz space; Ornstein–Uhlenbeck Poisson kernel

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1. Introduction and main result

Consider the Euclidean space $\mathbb{R}^n$ endowed with the Gaussian measure γ, given by

$$\mathrm{d}\gamma(x) = \pi^{-n/2} \mathrm{e}^{-|x|^2}.$$

The Gaussian analogue of the Euclidean Laplacian is the *Ornstein–Uhlenbeck operator*

$$\mathcal{L} = -\tfrac{1}{2}\Delta + x \cdot \nabla,$$

where $\nabla = (\partial_{x_1}, \dots, \partial_{x_n})$. The heat semigroup generated by $\mathcal{L}$ and defined in $L^2(\gamma)$ is the so-called *Ornstein–Uhlenbeck semigroup*

$$T_t = \mathrm{e}^{-t\mathcal{L}}, \quad t \geqslant 0.$$

The *Ornstein–Uhlenbeck Poisson semigroup* $P_t = \mathrm{e}^{-t\sqrt{\mathcal{L}}}$, $t \geqslant 0$, can be defined from $\{T_t\}_{t\geqslant 0}$ by subordination as

$$P_t f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{\mathrm{e}^{-u}}{\sqrt{u}} T_{t^2/4u} f(x)\,\mathrm{d}u, \quad x \in \mathbb{R}^n,$$

for $f \in L^2(\gamma)$. As explained in § 2, $P_t f$ is given by integration against a kernel $P_t(x,y)$.

Via $\{P_t\}_{t\geqslant 0}$, Gatto and Urbina [**3**] introduced the *Gaussian Lipschitz space* GLip_α for all $\alpha > 0$. We will always have $\alpha \in (0,1)$. Then the definition says that a function f in $\mathbb{R}^n$ is in GLip_α if it is bounded and satisfies

$$\|t\partial_t P_t f\|_{L^\infty} \leqslant A t^\alpha, \quad t > 0, \tag{1.1}$$

for some $A > 0$.

In [**4**] the authors characterized GLip_α, $0 < \alpha < 1$, in terms of a Lipschitz-type continuity condition. Indeed, Theorem 1.1 of [**4**] says that $f \in \mathrm{GLip}_\alpha$ if and only if there exists a positive constant K such that

$$|f(x) - f(y)| \leqslant K \min\left\{|x-y|^\alpha, \left(\frac{|x-y_x|}{1+|x|}\right)^{\alpha/2} + |y'_x|^\alpha\right\}, \quad x, y \in \mathbb{R}^n. \tag{1.2}$$

Here and in what follows we use a decomposition of y as $y = y_x + y'_x$, where y_x is parallel to x and y'_x is orthogonal to x; however, if $x = 0$ or $n = 1$, we let $y_x = y$ and $y'_x = 0$.

As is well known, a condition analogous to (1.1) for the standard Poisson integral characterizes the ordinary Lipschitz space (see [**6**, § V.4]). If only bounded functions are considered, one obtains the inhomogeneous Lipschitz space, and without the boundedness assumption one obtains the larger, homogeneous Lipschitz space.

In our setting we will see that the condition (1.1) without the boundedness condition defines a Gaussian analogue of the homogeneous Lipschitz space. Since here no homogeneity is involved, we will call it the *global Gaussian Lipschitz space*.

In (1.1), an *a priori* assumption is needed to ensure that $P_t f$ exists. Here we apply a recent result by Garrigós *et al*. [**1**]. Clearly, a measurable function f in $\mathbb{R}^n$ has a well-defined Gaussian Poisson integral if

$$\int_{\mathbb{R}^n} P_t(x,y)|f(y)|\,\mathrm{d}y < \infty$$

for all $x \in \mathbb{R}^n$ and $t > 0$. Theorem 1.1 of [**1**] says that this is equivalent to the growth condition

$$\int_{\mathbb{R}^n} \frac{\mathrm{e}^{-|y|^2}}{\sqrt{\ln(\mathrm{e}+|y|)}}|f(y)|\,\mathrm{d}y < \infty. \tag{1.3}$$

Moreover, (1.3) ensures that $P_t f(x) \to f(x)$ as $t \to 0$ for almost all $x \in \mathbb{R}^n$.

We can now define the global Gaussian Lipschitz space.

Definition 1.1. Let $\alpha \in (0,1)$. A measurable function f defined in $\mathbb{R}^n$ and satisfying (1.3) belongs to the global Gaussian Lipschitz space GGLip_α if (1.1) holds. The corresponding norm is

$$\|f\|_{\mathrm{GGLip}_\alpha} = \inf\{A > 0\colon A \text{ satisfies } (1.1)\}.$$

Strictly speaking, this space consists of functions modulo constants. A natural question is now what continuity condition characterizes this space. To state the answer we start in one dimension and introduce a distance by

$$d(x,y) = \left| \int_x^y \frac{\mathrm{d}\xi}{1+|\xi|} \right|, \quad x, y \in \mathbb{R}. \tag{1.4}$$

Then

$$d(x,y) = |\ln(1+|x|) - \operatorname{sgn} xy \ln(1+|y|)|$$

for all $x, y \in \mathbb{R}$, provided we define $\operatorname{sgn} 0 = 1$. In several dimensions, we use this distance on the line spanned by x, defining

$$d(x, y_x) = |\ln(1+|x|) - \operatorname{sgn}\langle x, y\rangle \ln(1+|y_x|)|, \quad x, y \in \mathbb{R}^n,$$

with y_x as before.

Our result reads as follows.

Theorem 1.2. *Let $\alpha \in (0,1)$ and let f be a measurable function in $\mathbb{R}^n$. The following are equivalent:*

(i) *f satisfies (1.3) and $f \in \mathrm{GGLip}_\alpha$;*

(ii) *there exists a positive constant K such that*

$$|f(x) - f(y)| \leqslant K \min\{|x-y|^\alpha, d(x,y_x)^{\alpha/2} + |y'_x|^\alpha\}, \quad x, y \in \mathbb{R}^n, \tag{1.5}$$

after correction of f on a null set.

Moreover,

$$\|f\|_{\mathrm{GGLip}_\alpha} \simeq \inf\{K \colon K \text{ satisfies } (1.5)\}. \tag{1.6}$$

The meaning of the symbol $\simeq$ is explained below.

Remark 1.3. Comparing (1.5) and (1.2), one easily verifies that if $\langle x, y\rangle > 0$ and $\frac{1}{2} < |x|/|y_x| < 2$, then

$$d(x, y_x) \simeq \frac{|x - y_x|}{1+|x|}. \tag{1.7}$$

Moreover, the space GLip_α can be described in terms of the distance function d. Indeed, since (1.2) implies that f is bounded (see [**4**, Lemma 2.1]), it is easy to check that (1.2) holds if and only if there exists a constant $K' > 0$ such that

$$|f(x) - f(y)| \leqslant K' \min\{1, |x-y|^\alpha, d(x,y_x)^{\alpha/2} + |y'_x|^\alpha\}$$

for all $x, y \in \mathbb{R}^n$. This also tells us that for bounded functions, (1.2) is equivalent to (1.5). But (1.5) implies only that

$$f(x) = O((\ln|x|)^{\alpha/2}) \quad \text{as } |x| \to \infty.$$

This condition is sharp, as shown by an example in § 5; observe that it is much stronger than (1.3).

Comments

This paragraph and the next contain some general comments on our result. First of all, like the main result of [**4**] it answers a very natural question: namely, whether a space that is called 'Lipschitz' can be described by an explicit continuity condition. It also sheds new light on initial-value problems related to the Gaussian setting and the Ornstein–Uhlenbeck and other semigroups.

Bessel and Triebel–Lizorkin spaces related to GLip_α were introduced and studied in [**5**] and [**2**]. These spaces must have global versions related to GGLip_α and corresponding to homogeneous spaces in the Euclidean setting, which it would be interesting to explore. Our result also suggests the possibility of characterizing these spaces in terms of integrals involving differences of the function, as in the standard Euclidean case. Another issue for further work is the extension to $\alpha \geqslant 1$, which should lead to higher-order derivatives and differences and, for $\alpha = 1$, analogues of the Zygmund space.

This paper is organized as follows. Section 2 contains a needed improvement of the estimate for $P_t(x, y)$ and its derivatives in [**4**]. Some properties of the Gaussian Poisson integral are obtained in § 3. Theorem 1.2 is then proved in § 4. Finally, in § 5 we give an example of a function in GGLip_α with logarithmic growth.

Notation

Throughout the paper, we will write C for various positive constants that depend only on n and α, unless otherwise explicitly stated. Given any two non-negative quantities A and B, the notation $A \lesssim B$ stands for $A \leqslant CB$ (we say that A is controlled by B), and $A \gtrsim B$ means $B \lesssim A$. If $B \lesssim A \lesssim B$, we write $A \simeq B$.

For positive quantities X, we will write $\exp^*(-X)$, meaning $\exp(-cX)$ for some constant $c = c(n, \alpha) > 0$.

2. The Ornstein–Uhlenbeck Poisson kernel

It is known that for $f \in L^2(\gamma)$,

$$T_t f(x) = \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} M_{\mathrm{e}^{-t}}(x, y) f(y) \, \mathrm{d}y, \quad x \in \mathbb{R}^n, \ t > 0,$$

where $M_{\mathrm{e}^{-t}}$ is the *Mehler kernel* defined by

$$M_r(x, y) = \frac{\exp(-|y - rx|^2/(1 - r^2))}{(1 - r^2)^{n/2}}, \quad x, y \in \mathbb{R}^n, \ 0 < r < 1.$$

The Gaussian Poisson integral $P_t f$ is given by an integral kernel called the *Ornstein–Uhlenbeck Poisson kernel* and denoted by $P_t(x, y)$; thus

$$P_t f(x) = \int_{\mathbb{R}^n} P_t(x, y) f(y) \, \mathrm{d}y, \quad x \in \mathbb{R}^n, \ t > 0.$$

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Because of the subordination formula, $P_t(x,y)$ is given by

$$\begin{aligned} P_t(x,y) &= \frac{1}{\pi^{(n+1)/2}} \int_0^\infty \frac{\mathrm{e}^{-u}}{\sqrt{u}} M_{\mathrm{e}^{-t^2/4u}}(x,y)\,\mathrm{d}u \\ &= \frac{1}{2\pi^{(n+1)/2}} \int_0^\infty \frac{t}{s^{3/2}} \mathrm{e}^{-t^2/4s} \frac{\exp(-|y-\mathrm{e}^{-s}x|^2/(1-\mathrm{e}^{-2s}))}{(1-\mathrm{e}^{-2s})^{n/2}}\,\mathrm{d}s. \end{aligned} \tag{2.1}$$

Here, we inserted the expression for the Mehler kernel and transformed the variable.

The following estimate for P_t and its first derivatives is established in [**4**, Theorems 1.2 and 1.3].

Proposition 2.1. *For all $t>0$, $x,y\in\mathbb{R}^n$ and $i\in\{1,2,\dots,n\}$, the kernel P_t satisfies*

$$\begin{aligned} P_t(x,y) + |t\partial_t P_t(x,y)| + |t\partial_{x_i} P_t(x,y)| \\ \leqslant C[K_1(t,x,y)+K_2(t,x,y)+K_3(t,x,y)+K_4(t,x,y)], \end{aligned}$$

where

$$K_1(t,x,y) = \frac{t}{(t^2+|x-y|^2)^{(n+1)/2}} \exp^*(-t(1+|x|));$$

$$\begin{aligned} K_2(t,x,y) &= \frac{t}{|x|}\left(t^2 + \frac{|x-y_x|}{|x|} + |y'_x|^2\right)^{-(n+2)/2} \\ &\qquad \times \exp^*\left(-\frac{(t^2+|y'_x|^2)|x|}{|x-y_x|}\right)\chi_{\{|x|>1,\,x\cdot y>0,\,|x|/2\leqslant|y_x|<|x|\}}; \end{aligned}$$

$$K_3(t,x,y) = \min(1,t)\exp^*(-|y|^2);$$

$$K_4(t,x,y) = \frac{t}{|y_x|}\left(\ln\frac{|x|}{|y_x|}\right)^{-3/2} \exp^*\left(-\frac{t^2}{\ln(|x|/|y_x|)}\right)\exp^*(-|y'_x|^2)\chi_{\{x\cdot y>0,\,1<|y_x|<|x|/2\}}.$$

We need a slight sharpening of this lemma. The term K_3 will be modified to decay for large x.

Lemma 2.2. *The estimate of Proposition 2.1 remains valid if the kernel $K_3(t,x,y)$ is replaced by*

$$\tilde{K}_3(t,x,y) = \min\left\{1, \frac{t}{[\ln(\mathrm{e}+|x|)]^{1/2}}\right\}\exp^*(-|y|^2).$$

Proof. From the proof of [**4**, Theorem 1.3], we see that $|t\partial_t P_t(x,y)|$ and $|t\partial_{x_i} P_t(x,y)|$ can be controlled by an integral similar to the right-hand side of (2.1) (only with exp in (2.1) replaced by $\exp^*$). Thus, we only need to consider $P_t(x,y)$.

When $|x|\leqslant 4+2|y|$, we have $\exp^*(-|y|^2)\lesssim \exp^*(-|y|^2)\exp^*(-|x|^2)$, and hence $K_3(t,x,y)\lesssim \tilde{K}_3(t,x,y)$.

We therefore assume from now on that $|x|>4+2|y|$. We will sharpen a few arguments in the proof of [**4**, Proposition 4.1]. By the rotation invariance of $P_t(x,y)$ and $\tilde{K}_3(t,x,y)$, we may assume that $x=(x_1,0,\dots,0)$ with $x_1>0$. The decomposition of y will then be written $y=(y_1,0,\dots,0)+(0,y')$, and we will have $x_1>4$ and $|y_1|<x_1/2$.

Case 1 ($-x_1/2 < y_1 \leqslant 0$). Using the notation from the proof of Proposition 4.1 (i) in [**4**], we see that we only need to verify that $J_2 \lesssim \tilde{K}_3$. By [**4**, (4.9)] and the fact that $y_1 \leqslant 0 < x_1$, we have

$$J_2 \simeq \exp^*(-|y'|^2) \int_{\ln 2}^{\infty} \frac{t}{s^{3/2}} \exp^*\left(-\frac{t^2}{s}\right) \exp^*(-|y_1 - \mathrm{e}^{-s} x_1|^2)\, \mathrm{d}s$$
$$\lesssim \exp^*(-|y|^2) \int_{\ln 2}^{\infty} \frac{t}{s^{3/2}} \exp^*\left(-\frac{t^2}{s}\right) \exp^*(-\mathrm{e}^{-2s} x_1^2)\, \mathrm{d}s. \tag{2.2}$$

Note that

$$\int_{\frac{1}{2}\ln x_1}^{\infty} \frac{t}{s^{3/2}} \exp^*\left(-\frac{t^2}{s}\right) \exp^*(-\mathrm{e}^{-2s} x_1^2)\, \mathrm{d}s \simeq \int_{\frac{1}{2}\ln x_1}^{\infty} \frac{t}{s^{3/2}} \exp^*\left(-\frac{t^2}{s}\right) \mathrm{d}s$$
$$\lesssim \min\{1, t(\ln x_1)^{-1/2}\}$$

and

$$\int_{\ln 2}^{\frac{1}{2}\ln x_1} \frac{t}{s^{3/2}} \exp^*\left(-\frac{t^2}{s}\right) \exp^*(-\mathrm{e}^{-2s} x_1^2)\, \mathrm{d}s \leqslant \exp^*(-x_1) \int_{\ln 2}^{\frac{1}{2}\ln x_1} \frac{t}{s^{3/2}} \exp^*\left(-\frac{t^2}{s}\right) \mathrm{d}s$$
$$\lesssim \exp^*(-x_1) \min\{1, t\},$$

from which the required estimate follows.

Case 2 ($0 < y_1 < x_1/2$). Considering now the proof of [**4**, Proposition 4.1 (iii)], we only need to estimate the terms $J_{2,1}^{(2)}$ and $J_{2,3}$, and also $J_{2,2}$ when $y_1 \in (0, 1]$.

From [**4**, (4.16)], for $y_1 \in (0, 1]$ we obtain

$$J_{2,2} \simeq \frac{t}{(\ln(x_1/y_1))^{3/2}} \exp^*\left(-\frac{t^2}{\ln(x_1/y_1)}\right) \exp^*(-|y'|^2)$$
$$\lesssim \min\left\{\frac{t}{(\ln(x_1/y_1))^{3/2}}, \frac{1}{\ln(x_1/y_1)}\right\} \exp^*(-|y|^2)$$
$$\lesssim \tilde{K}_3(t, x, y),$$

since here $\ln(x_1/y_1) \gtrsim \ln(\mathrm{e} + |x|)$. Furthermore,

$$J_{2,1}^{(2)} + J_{2,3} \leqslant \exp^*(-|y'|^2) \int \frac{t}{s^{3/2}} \exp^*\left(-\frac{t^2}{s}\right) \exp^*(-|y_1 - \mathrm{e}^{-s} x_1|^2)\, \mathrm{d}s, \tag{2.3}$$

where the integral is taken over the set $\{s > \ln 2 \colon |s - \ln(x_1/y_1)| > c_0\}$ for some $c_0 > 0$. Thus the quotient $\mathrm{e}^{-s} x_1/y_1$ stays away from 1 in this integral, so that $|y_1 - \mathrm{e}^{-s} x_1| \simeq \max\{\mathrm{e}^{-s} x_1, y_1\} \simeq \mathrm{e}^{-s} x_1 + y_1$. This implies that the right-hand side of (2.3) is controlled by the expression in (2.2) and thus by $\tilde{K}_3$.

Lemma 2.2 is proved. □

3. Auxiliary lemmas

Lemma 3.1. *There exists a constant $C > 0$ such that for all $x, y \in \mathbb{R}^n$ and $t > 0$,*

$$|\partial_t P_t(x,y)| \leqslant C\frac{1}{t}P_{t/2}(x,y).$$

Proof. Differentiating (2.1), we get

$$\partial_t P_t(x,y) = \frac{1}{2\pi^{(n+1)/2}}\frac{1}{t}\int_0^\infty \frac{t}{s^{3/2}}\mathrm{e}^{-t^2/4s}\left(1-\frac{t^2}{2s}\right)\frac{\exp(-|y-\mathrm{e}^{-s}x|^2/(1-\mathrm{e}^{-2s}))}{(1-\mathrm{e}^{-2s})^{n/2}}\,\mathrm{d}s.$$

It is now enough to observe that

$$\mathrm{e}^{-t^2/4s}\left|1-\frac{t^2}{2s}\right| \lesssim \mathrm{e}^{-(t/2)^2/4s}$$

and compare with (2.1). □

Lemma 3.2. *Fix $i \in \{1, 2, \dots, n\}$ and let $R > 0$. Then there exists a constant $C > 0$, depending only on n and R, such that for all $x, y \in \mathbb{R}^n$ with $|x| < R$,*

$$|\partial_{x_i} P_t(x,y)| \leqslant C(1+t^{-4-n})P_{t/2}(x,y), \quad t > 0, \tag{3.1}$$

and

$$|\partial_{x_i} P_t(x,y)| \leqslant Ct^{-1/2}\mathrm{e}^{-|y|^2}[\ln(\mathrm{e}+|y|)]^{-3/4}, \quad t > 1. \tag{3.2}$$

Proof. In this proof all constants denoted by C will depend only on n and R, and the same applies to the implicit constants in the $\lesssim$ and $\simeq$ symbols. We let $|x| < R$, and we can clearly assume that $R > 1$.

Differentiating (2.1), we get

$$\partial_{x_i} P_t(x,y) = \frac{1}{\pi^{(n+1)/2}}\int_0^\infty \frac{t}{s^{3/2}}\mathrm{e}^{-t^2/4s}\frac{\mathrm{e}^{-s}(y_i-\mathrm{e}^{-s}x_i)}{1-\mathrm{e}^{-2s}}\frac{\exp(-|y-\mathrm{e}^{-s}x|^2/(1-\mathrm{e}^{-2s}))}{(1-\mathrm{e}^{-2s})^{n/2}}\,\mathrm{d}s. \tag{3.3}$$

Compared with (2.1), the integral now has an extra factor: $\mathrm{e}^{-s}(y_i - \mathrm{e}^{-s}x_i)/(1-\mathrm{e}^{-2s})$.

With $\gamma > 0$, we will repeatedly use the simple inequality

$$\mathrm{e}^{-t^2/4s} \leqslant C_\gamma\left(\frac{s}{t^2}\right)^\gamma \mathrm{e}^{-(t/2)^2/4s} \tag{3.4}$$

for some $C_\gamma > 0$, and here we sometimes drop the last factor.

We start with the simple case of bounded y; more precisely, we assume that $|y| \leqslant \mathrm{e}^{12}R$. Then the extra factor is no larger than $C\mathrm{e}^{-s}/(1-\mathrm{e}^{-2s})$. An application of (3.4) with $\gamma = 1 + n/2$ yields

$$|\partial_{x_i} P_t(x,y)| \lesssim t^{-2-n}\int_0^\infty \frac{t}{s^{3/2}}\mathrm{e}^{-(t/2)^2/4s}\frac{\mathrm{e}^{-s}s^{1+n/2}}{(1-\mathrm{e}^{-2s})^{1+n/2}}\exp\left(-\frac{|y-\mathrm{e}^{-s}x|^2}{1-\mathrm{e}^{-2s}}\right)\mathrm{d}s.$$

Comparing this with (2.1), one sees that this estimate implies (3.1). If we choose $\gamma = 2 + n/2$ instead, (3.2) will also follow, since y stays bounded.

From now on we assume that $|y| > \mathrm{e}^{12}R$. Then (3.3) implies

$$|\partial_{x_i} P_t(x,y)| \lesssim \int_0^\infty \frac{t}{s^{3/2}} \mathrm{e}^{-t^2/4s} \frac{\mathrm{e}^{-s}|y|}{1-\mathrm{e}^{-2s}} \frac{\exp(-|y-\mathrm{e}^{-s}x|^2/(1-\mathrm{e}^{-2s}))}{(1-\mathrm{e}^{-2s})^{n/2}}\,\mathrm{d}s. \tag{3.5}$$

First we estimate the exponent

$$E(s,x,y) = -\frac{|y-\mathrm{e}^{-s}x|^2}{1-\mathrm{e}^{-2s}}$$

from (3.5). It satisfies

$$E(s,x,y) \leqslant \frac{-|y|^2 + 2\mathrm{e}^{-s} y\cdot x}{1-\mathrm{e}^{-2s}} \leqslant \frac{-|y|^2 + \frac{1}{2}\mathrm{e}^{-2s}|y|^2 + 2|x|^2}{1-\mathrm{e}^{-2s}},$$

where we applied the inequality between the geometric and arithmetic means. If $\mathrm{e}^{-s} < \frac{1}{2}$, then

$$E(s,x,y) \leqslant \frac{-|y|^2 + \frac{1}{2}\mathrm{e}^{-2s}|y|^2}{1-\mathrm{e}^{-2s}} + C.$$

If, instead, $\mathrm{e}^{-s} \geqslant \frac{1}{2}$, we have $2|x|^2 < \mathrm{e}^{-2s}|y|^2/4$ since $|y| > \mathrm{e}^{12}|x|$, and thus

$$E(s,x,y) \leqslant \frac{-|y|^2 + \frac{3}{4}\mathrm{e}^{-2s}|y|^2}{1-\mathrm{e}^{-2s}}.$$

In both cases,

$$E(s,x,y) \leqslant -|y|^2 \frac{1-\frac{3}{4}\mathrm{e}^{-2s}}{1-\mathrm{e}^{-2s}} + C \leqslant -|y|^2(1+\tfrac{1}{4}\mathrm{e}^{-2s}) + C,$$

and this implies that

$$\mathrm{e}^{E(s,x,y)} \lesssim \mathrm{e}^{-|y|^2} \min\left(1, \frac{\mathrm{e}^{2s}}{|y|^2}\right). \tag{3.6}$$

We also need a converse inequality, under the assumption that $s > \ln|y|$. Then

$$E(s,x,y) \geqslant \frac{-|y|^2 - 2\mathrm{e}^{-s}|y||x| - \mathrm{e}^{-2s}|x|^2}{1-\mathrm{e}^{-2s}} \geqslant \frac{-|y|^2}{1-|y|^{-2}} - C \geqslant -|y|^2 - C. \tag{3.7}$$

Now split the integral in (3.5) as, say,

$$\left(\int_0^3 + \int_3^{\ln|y|} + \int_{\ln|y|}^\infty\right) \frac{t}{s^{3/2}} \mathrm{e}^{-t^2/4s} \frac{\mathrm{e}^{-s}|y|}{1-\mathrm{e}^{-2s}} \frac{\exp(-|y-\mathrm{e}^{-s}x|^2/(1-\mathrm{e}^{-2s}))}{(1-\mathrm{e}^{-2s})^{n/2}}\,\mathrm{d}s$$
$$= I_1 + I_2 + I_3;$$

observe that $\ln|y| > 12$. We shall prove that these three integrals satisfy the bounds in (3.1) and (3.2).

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In I_3 we have $\mathrm{e}^{-s}|y|/(1-\mathrm{e}^{-2s}) \lesssim 1$. Comparing this with (2.1), we conclude that

$$I_3 \lesssim P_t(x,y) \lesssim P_{t/2}(x,y),$$

which is part of (3.1). Aiming at (3.2), we apply (3.4) with $\gamma = \frac{3}{4}$ and (3.6), where the minimum is 1, to conclude that

$$I_3 \lesssim \int_{\ln|y|}^{\infty} t^{-1/2} s^{-3/4} \mathrm{e}^{-s} |y| \mathrm{e}^{-|y|^2} \,\mathrm{d}s \lesssim t^{-1/2} (\ln|y|)^{-3/4} \mathrm{e}^{-|y|^2},$$

as desired.

To deal with I_2, we apply (3.6), now with the second quantity in the minimum, and obtain

$$I_2 \lesssim \int_3^{\ln|y|} \frac{t}{s^{3/2}} \mathrm{e}^{-t^2/4s} \frac{\mathrm{e}^s}{|y|} \mathrm{e}^{-|y|^2} \,\mathrm{d}s. \tag{3.8}$$

Using (3.4), again with $\gamma = \frac{3}{4}$, we can estimate this integral by

$$t^{-1/2} \mathrm{e}^{-|y|^2} \int_3^{\ln|y|} s^{-3/4} \frac{\mathrm{e}^s}{|y|} \,\mathrm{d}s,$$

which gives the bound in (3.2) for I_2. Thinking of (3.1), we write the integral in (3.8) as

$$t\mathrm{e}^{-|y|^2} |y|^{-1} \int_3^{\ln|y|} \phi(s) \mathrm{e}^{s/2} \,\mathrm{d}s,$$

where

$$\phi(s) = \frac{\mathrm{e}^{s/2}}{s^{3/2}} \mathrm{e}^{-t^2/4s}.$$

Here, both the factors are increasing functions of s in $(3,\infty)$, and so is ϕ. Thus, for any $\eta \in (0,1)$,

$$\sup_{(3,\ln|y|)} \phi(s) \leqslant \phi(\eta + \ln|y|),$$

and so

$$I_2 \lesssim t\mathrm{e}^{-|y|^2} |y|^{-1} \phi(\eta + \ln|y|) \int_3^{\ln|y|} \mathrm{e}^{s/2} \,\mathrm{d}s \simeq t\mathrm{e}^{-|y|^2} \frac{1}{(\eta + \ln|y|)^{3/2}} \mathrm{e}^{-t^2/4(\eta + \ln|y|)}.$$

Integrating in η we see that

$$I_2 \lesssim \int_{\ln|y|}^{1+\ln|y|} \frac{t}{s^{3/2}} \mathrm{e}^{-t^2/4s} \mathrm{e}^{-|y|^2} \,\mathrm{d}s. \tag{3.9}$$

Because of (3.7), this integral is dominated by the one defining $P_t(x,y)$ in (2.1). Since $P_t(x,y) \lesssim P_{t/2}(x,y)$, it follows that $I_2 \lesssim P_{t/2}(x,y)$.

Finally, we estimate I_1 by means of (3.6). Since here $1 - \mathrm{e}^{-2s} \simeq s$, we get

$$I_1 \lesssim \int_0^3 \frac{t}{s^{3/2}} \mathrm{e}^{-t^2/4s} \frac{1}{|y| s^{1+n/2}} \mathrm{e}^{-|y|^2} \,\mathrm{d}s.$$

Using (3.4) with $\gamma = 2 + n/2$, we conclude that

$$I_1 \lesssim t^{-3-n} \int_0^3 s^{-1/2} \mathrm{e}^{-(t/2)^2/4s} \frac{1}{|y|} \mathrm{e}^{-|y|^2} \,\mathrm{d}s. \tag{3.10}$$

This leads immediately to the bound in (3.2). For (3.1), we can estimate the right-hand side in (3.10) by

$$t^{-3-n} \frac{1}{|y|} \mathrm{e}^{-t^2/48} \mathrm{e}^{-|y|^2} \lesssim t^{-4-n} \frac{t}{(\eta + \ln|y|)^{3/2}} \mathrm{e}^{-t^2/4(\eta+\ln|y|)} \mathrm{e}^{-|y|^2}$$

with $\eta \in (0,1)$ as before, since $\ln|y| > 12$. As a result, we get a bound for I_1 similar to (3.9) but with an extra factor t^{-4-n}, and thus also the bound in (3.1).

Lemma 3.2 is proved. □

Proposition 3.3. *Let f be a measurable function on $\mathbb{R}^n$ satisfying (1.3). Then, for all $i \in \{1, 2, \dots, n\}$ and $x \in \mathbb{R}^n$,*

$$\partial_{x_i} \partial_t P_{s+t} f(x) = \int_{\mathbb{R}^n} \partial_{x_i} P_s(x,y) \partial_t P_t f(y) \,\mathrm{d}y, \quad s, t > 0, \tag{3.11}$$

and

$$\lim_{t \to \infty} \partial_{x_i} P_t f(x) = 0. \tag{3.12}$$

Proof. We can assume that $|x| < R$ for some $R > 0$ and thus apply the estimates from Lemma 3.2. First we verify the absolute convergence of the integral in (3.11) by showing that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\partial_{x_i} P_s(x,y)| \, |\partial_t P_t(y,z)| \, |f(z)| \,\mathrm{d}y \,\mathrm{d}z < \infty.$$

Lemmas 3.2 and 3.1 imply that this integral is, up to a factor $C(n,R)$, no larger than

$$\frac{1 + s^{-4-n}}{t} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} P_{s/2}(x,y) P_{t/2}(y,z) |f(z)| \,\mathrm{d}y \,\mathrm{d}z = \frac{1 + s^{-4-n}}{t} \int_{\mathbb{R}^n} P_{(s+t)/2}(x,z) |f(z)| \,\mathrm{d}z < \infty,$$

where the equality comes from the semigroup property. The last integral here is finite because of (1.3); indeed, [**1**, (6.4)] says that $P_t(x,y)$ is controlled by $\mathrm{e}^{-|y|^2}/\sqrt{\ln(\mathrm{e} + |y|)}$, locally uniformly in x and t.

Our next step consists of integrating the right-hand side of (3.11) along intervals in the variables x_i and t. We choose two points $x', x'' \in \mathbb{R}^n$ with $|x'|, |x''| < R$ that differ only in the ith coordinate, and also two points $t', t'' > 0$. Fubini's theorem applies because of the above estimates, and we get

$$\begin{aligned}\int_{x_i'}^{x_i''}\int_{t'}^{t''}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\partial_{x_i}P_s(x,y)\partial_t p_t(y,z)f(z)\,\mathrm{d}y\,\mathrm{d}z\right)\mathrm{d}t\,\mathrm{d}x_i \\ = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}[P_s(x'',y)-P_s(x',y)][P_{t''}(y,z)-P_{t'}(y,z)]f(z)\,\mathrm{d}y\,\mathrm{d}z \\ = P_{s+t''}f(x'')-P_{s+t''}f(x')-P_{s+t'}f(x'')+P_{s+t'}f(x').\end{aligned}$$

From this, we obtain (3.11) by differentiating with respect to x_i'' and t''.

Finally, (3.12) is a direct consequence of (3.2) and (1.3). □

Proposition 3.3 now allows us to apply the method of proof of [**4**, Proposition 3.2] and obtain the same estimates as there.

Corollary 3.4. *Let* $\alpha \in (0,1)$ *and let* $f \in \mathrm{GGLip}_\alpha$ *with norm* 1.

(i) *For all* $i \in \{1,2,\dots,n\}$, $t>0$ *and* $x \in \mathbb{R}^n$,

$$|\partial_{x_i}P_t f(x)| \leqslant Ct^{\alpha-1}.$$

(ii) *For all* $t>0$ *and* $x=(x_1,0,\dots,0)\in\mathbb{R}^n$ *with* $x_1 \geqslant 0$,

$$|\partial_{x_1}P_t f(x)| \leqslant Ct^{\alpha-2}(1+x_1)^{-1}.$$

4. Proof of Theorem 1.2

(i) $\Longrightarrow$ (ii) We assume that f satisfies (1.3) and (1.1). According to [**1**, Theorem 1.1], $P_t f(x) \to f(x)$ as $t \to 0$ for almost all $x \in \mathbb{R}^n$, and we can therefore modify f on a null set so that this convergence holds for all x.

Now fix $x, y \in \mathbb{R}^n$. For all $t>0$, we write

$$|f(x)-f(y)| \leqslant |f(x)-P_t f(x)| + |P_t f(x)-P_t f(y)| + |P_t f(y)-f(y)|. \tag{4.1}$$

Using Corollary 3.4 (i) and arguing as in the verification of [**4**, (3.7)], we get

$$|f(x)-f(y)| \lesssim |x-y|^\alpha. \tag{4.2}$$

To obtain (1.5) it is then enough to prove that

$$|f(x)-f(y)| \lesssim d(x,y_x)^{\alpha/2} + |y_x'|^\alpha.$$

By writing

$$|f(x)-f(y)| \leqslant |f(x)-f(y_x)| + |f(y_x)-f(y)|$$

and applying (4.2) to the last term here, we see that we need only verify that

$$|f(x)-f(y_x)| \lesssim d(x,y_x)^{\alpha/2}.$$

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Making a rotation, we can assume that $x = (x_1, 0, \dots, 0)$ with $x_1 \geqslant 0$ and $y_x = (y_1, 0, \dots, 0)$.

We estimate $|f(x) - f(y_x)|$ as in (4.1). Of the three terms we then get, the first and third are controlled by t^α. We apply Corollary 3.4 (ii) and the one-dimensional integral expression (1.4) for d to the second term. As a result,

$$|f(x) - f(y_x)| \lesssim t^\alpha + t^{\alpha-2} d(x, y_x),$$

and here we choose $t = d(x, y_x)^{1/2}$. This leads to (1.5), and the implication (i) $\Longrightarrow$ (ii) is proved.

(ii) $\Longrightarrow$ (i) Letting $y = 0$, we see that (1.5) implies that $f(x) = O((\ln|x|)^{\alpha/2})$ as $|x| \to \infty$ and thus also (1.3). We must verify (1.1).

Using the fact that $\int_{\mathbb{R}^n} \partial_t P_t(x, y)\,\mathrm{d}y = 0$ and Lemma 2.2, we can write

$$\begin{aligned}|t\partial_t P_t f(x)| &= \left| \int_{\mathbb{R}^n} t\partial_t P_t(x, y)[f(y) - f(x)]\,\mathrm{d}y \right| \\ &\lesssim \int_{\mathbb{R}^n} [K_1(t, x, y) + K_2(t, x, y) + \tilde{K}_3(t, x, y) + K_4(t, x, y)]|f(y) - f(x)|\,\mathrm{d}y.\end{aligned}$$

We thus get four integrals to bound by t^α. For $\int_{\mathbb{R}^n} K_1(t, x, y)|f(y) - f(x)|\,\mathrm{d}y$, we can apply the same simple argument as at the end of § 3 in [**4**], since it uses only the quantity $|x - y|^\alpha$ in (1.5).

The integral involving $K_2(t, x, y)$ can also be estimated as in [**4**], because (1.7) applies in the support of $K_2(t, x, y)$.

For the integral with $\tilde{K}_3(t, x, y)$, we apply the inequality $(a+b)^\kappa \leqslant a^\kappa + b^\kappa$ with $a, b > 0$ and $\kappa = \alpha/2 \in (0, 1)$ to the expression in (1.5) and get

$$\begin{aligned}&\int_{\mathbb{R}^n} \tilde{K}_3(t, x, y)|f(y) - f(x)|\,\mathrm{d}y \\ &\lesssim \min\left\{1, \frac{t}{\sqrt{\ln(\mathrm{e} + |x|)}}\right\} \int_{\mathbb{R}^n} ((\ln(1 + |x|))^{\alpha/2} + (\ln(1 + |y_x|))^{\alpha/2} + |y_x'|^\alpha) \exp^*(-|y|^2)\,\mathrm{d}y.\end{aligned}$$

The minimum here is no larger than $t^\alpha/[\ln(\mathrm{e} + |x|)]^{\alpha/2}$, which leads immediately to the bound t^α for the whole expression.

Finally,

$$\begin{aligned}&\int_{\mathbb{R}^n} K_4(t, x, y)|f(y) - f(x)|\,\mathrm{d}y \\ &\qquad\leqslant \int_{\substack{x\cdot y>0 \\ 1<|y_x|<|x|/2}} \frac{t}{|y_x|}\left(\ln\frac{|x|}{|y_x|}\right)^{-3/2} \exp^*\left(-\frac{t^2}{\ln(|x|/|y_x|)}\right) \\ &\qquad\qquad\times \exp^*(-|y_x'|^2)([\ln(1 + |x|) - \ln(1 + |y_x|)]^{\alpha/2} + |y_x'|^\alpha)\,\mathrm{d}y \qquad (4.3)\end{aligned}$$

When $1 < |y_x| < |x|/2$, we have

$$|\ln(1 + |x|) - \ln(1 + |y_x|)| = \ln\frac{1 + |x|}{1 + |y_x|} \simeq \ln\frac{|x|}{|y_x|}.$$

After a rotation we can assume that $x = (x_1, 0, \ldots, 0)$ with $x_1 > 0$, so that $y_x = (y_1, 0, \ldots, 0)$ and $y'_x = (0, y')$ and we have $1 < y_1 < x_1/2$. The right-hand integral in (4.3) is bounded by a constant times

$$\int_1^{x_1/2} \int_{\mathbb{R}^{n-1}} \frac{t}{y_1} \left(\ln \frac{x_1}{y_1} \right)^{-3/2} \exp^* \left(-\frac{t^2}{\ln(x_1/y_1)} \right) \\ \times \exp^*(-|y'|^2) \left(\left[\ln \frac{x_1}{y_1} \right]^{\alpha/2} + |y'|^\alpha \right) \mathrm{d}y' \, \mathrm{d}y_1.$$

Integrating in y' and noting that $\ln(x_1/y_1) \gtrsim 1$, we can bound this double integral by

$$\int_1^{x_1/2} \frac{t}{y_1} \left(\ln \frac{x_1}{y_1} \right)^{\alpha/2-3/2} \exp^* \left(-\frac{t^2}{\ln(x_1/y_1)} \right) \mathrm{d}y_1.$$

The transformation of variable $s = t^{-2}(\ln x_1 - \ln y_1)$ now gives the desired bound t^α.

Summing up, we have verified (1.1) and (i). The norm equivalence (1.6) also follows, and this ends the proof of Theorem 1.2.

5. An example of a function in $\mathbf{GGLip}_\alpha$

With $\alpha \in (0, 1)$ we consider the function

$$f(x) = [\ln(\mathrm{e} + |x|)]^{\alpha/2}, \quad x \in \mathbb{R}^n.$$

We shall verify that f belongs to GGLip_α using Theorem 1.2.

The estimate

$$|f(x) - f(y)| \lesssim |x - y|^\alpha \tag{5.1}$$

is easy and is left to the reader.

To show that

$$|f(x) - f(y)| \lesssim |\ln(\mathrm{e} + |x|) - \mathrm{sgn}\langle x, y \rangle \ln(\mathrm{e} + |y_x|)|^{\alpha/2} + |y'_x|^\alpha, \tag{5.2}$$

write

$$|f(x) - f(y)| \leqslant |f(x) - f(y_x)| + |f(y_x) - f(y)|.$$

The last term here is controlled by $|y'_x|^\alpha$, because of (5.1). We apply the inequality $|a^\kappa - b^\kappa| \leqslant |a - b|^\kappa$, $a, b > 0$, with $\kappa = \alpha/2 \in (0, 1)$, to the first term on the right, obtaining

$$|f(x) - f(y_x)| = |[\ln(\mathrm{e} + |x|)]^{\alpha/2} - [\ln(\mathrm{e} + |y_x|)]^{\alpha/2}| \leqslant |\ln(\mathrm{e} + |x|) - \ln(\mathrm{e} + |y_x|)|^{\alpha/2}.$$

This implies (5.2), and it follows that $f \in \mathrm{GGLip}_\alpha$.

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References

1. G. Garrigós, S. Harzstein, T. Signes, J. L. Torrea and B. Viviani, Pointwise convergence to intial data of heat and Laplace equations, *Trans. Am. Math. Soc.* **368**(9) (2016), 6575–6600.
2. A. E. Gatto, E. Pineda and W. Urbina, Riesz potentials, Bessel potentials, and fractional derivatives on Besov–Lipschitz spaces for the Gaussian measure, in *Recent advances in harmonic analysis and applications*, Springer Proceedings in Mathematics & Statistics, Volume 25, pp. 105–130 (Springer, 2013).
3. A. E. Gatto and W. Urbina, On Gaussian Lipschitz spaces and the boundedness of fractional integrals and fractional derivatives on them, *Quaest. Math.* **38**(1) (2015), 1–25.
4. L. Liu and P. Sjögren, A characterization of the Gaussian Lipschitz space and sharp estimates for the Ornstein–Uhlenbeck Poisson kernel, *Rev. Mat. Iber.* **32**(4) (2016), 1189–1210.
5. E. Pineda and W. Urbina, Some results on Gaussian Besov–Lipschitz spaces and Gaussian Triebel–Lizorkin spaces, *J. Approx. Theory* **161** (2009), 529–564.
6. E. M. Stein, *Singular integrals and differentiability properties of functions* (Princeton University Press, 1970).