
On the Richter–Thomassen Conjecture about Pairwise Intersecting Closed Curves[†]

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A long-standing conjecture of Richter and Thomassen states that the total number of intersection points between any n simple closed Jordan curves in the plane, so that any pair of them intersect and no three curves pass through the same point, is at least $(1 - o(1))n^2$.

We confirm the above conjecture in several important cases, including the case (1) when all curves are convex, and (2) when the family of curves can be partitioned into two equal classes such that each curve from the first class touches every curve from the second class. (Two closed or open curves are said to be touching if they have precisely one point in common and at this point the two curves do not properly cross.)

An important ingredient of our proofs is the following statement. Let S be a family of n open curves in \mathbb{R}^2 , so that each curve is the graph of a continuous real function defined on \mathbb{R} , and no three of them pass through the same point. If there are nt pairs of touching curves in S , then the number of crossing points is $\Omega(nt\sqrt{\log t/\log \log t})$.

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1. Introduction

Studying combinatorial properties of arrangements of curves in the plane and surfaces in higher dimensions is one of the central themes of discrete and computational geometry, and has numerous applications in motion planning, ray shooting, computer graphics, pattern recognition, combinatorial optimization, *etc.* [8, 16, 20]. Analysis of the complexity of many geometric algorithms depends crucially on extremal results on arrangements. In the past 40 years, powerful probabilistic, algebraic and algorithmic techniques have been developed to deal with such questions. The classical open problem of Erdős [9] on the maximum number of times the unit distance can occur among n points in the plane can also be paraphrased as a problem on arrangements, in two different ways.

- (1) What is the maximum number of incidences between n points and n unit circles in the plane, where a point p and a circle c are called *incident* if p belongs to c ?
- (2) What is the maximum number of *touchings* among n circles of radius $1/2$ in the plane? Two such circles are tangent to each other if and only if their centres are at unit distance.

A seminal result in this area is the Szemerédi–Trotter theorem [23, 24], which states that the number of incidences between n points and m lines in the plane is $O(n^{2/3}m^{2/3} + m + n)$; for many important generalizations and applications; see [4, 21, 26]. Nearly 20 years ago, Székely [22] found an elegant argument using crossing numbers of graphs, which shows that the same result holds for *pseudo-segments*, that is, for systems of curves, every pair of which intersect at most once. This observation opened the door to generalizations to systems of more general curves: circles, *pseudo-circles*, that is, curves with two (or a bounded number of) pairwise intersections. The first substantial step in this direction was taken by Tamaki and Tokuyama [25]. They defined a *lens* to be the union of two arcs connecting the same pair of points along different curves belonging to a family S , and proved that for pseudo-circles one can cut the curves at a number of points proportional to the maximum number $\nu(S)$ of non-overlapping lenses, so that the curves break into pieces, any two of which intersect at most once. Then one can apply the Szemerédi–Trotter bound on pseudo-segments to upper-bound the number of incidences between the curves and the points. An interesting instance of pseudo-circles, whose study is motivated by robotic motion planning, is the family of homothetic copies of a fixed convex set in the plane [13]. The line of research initiated by Tamaki and Tokuyama was continued in [2, 3, 5, 6, 7, 14]. Note that any point of tangency (*i.e.* a *locally* non-transversal intersection) between the curves of S can be viewed as a degenerate lens. Thus, the parameter $\nu(S)$ is at least as large as the number of tangencies (‘degenerate lenses’) between the curves in S , and in many cases it can be bounded in terms of this quantity.

Other geometric problems that boil down to estimating the maximum number of tangencies between a set of curves in the plane are discussed by Agarwal, Nevo, Pach, Pinchasi, Sharir and Smorodinsky [1]. In particular, they showed that any family of n pairwise intersecting pseudo-circles admits at most $O(n)$ tangencies. As a consequence of the spectacular success of techniques from algebraic geometry in this area, some of these problems have recently been revisited from an algebraic perspective [12].

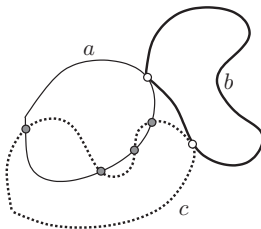


Figure 1. In the depicted family of three pairwise-intersecting curves, b touches each of the remaining two curves a and c , which cross one another.

In this paper we address an old conjecture of Richter and Thomassen [18]. To formulate this conjecture, we need to introduce some terminology. Let S be a collection of curves in the plane in *general position*, in the sense that no three curves pass through the same point and no two share infinitely many points. Two curves $s_1, s_2 \in S$ are said to *touch* each other if they have *precisely one* point in common, at which the curves do not properly cross.

The unique point that two touching curves have in common is called a *touching point* or, in short, a *touching*. All other points shared by two curves of the collection are called *crossing points* of S , or simply *crossings*, and the corresponding curves of S are said to *cross* each other. The term *intersection* refers to both touchings and crossings. In what follows, we may assume that any point of tangency is necessarily a touching. Indeed, we can apply an arbitrarily small perturbation to any pair $s_1, s_2 \in S$ so as to remove all of their tangencies, with the possible exception of one last tangency, which then becomes a touching.

Richter–Thomassen conjecture 18. The total number of intersection points between n pairwise intersecting simple closed curves in the plane which are in general position is at least $(1 - o(1))n^2$.

Richter and Thomassen established the weaker lower bound $(3/4 - o(1))n^2$, which was later improved by Mubayi [15] to $(4/5 - o(1))n^2$. Observe that if every pair of curves intersect at least twice, then the number of intersection points is at least $2\binom{n}{2} = (1 - o(1))n^2$. The Richter–Thomassen conjecture states that we cannot substantially decrease the number of intersection points by allowing touching pairs. Most likely, the presence of many such pairs significantly increases the total number of intersection points.

The conjecture was confirmed by Salazar [19] in the case when every pair of curves have at most a bounded number of points in common. However, the problem has remained open for general families of simple closed curves, with perhaps the most intriguing special case involving n *convex curves*.

The aim of this paper is to prove the Richter–Thomassen conjecture in this special case. More generally, we settle the question for collections of closed curves that can be decomposed into a constant number of x -monotone pieces. An open curve is called *x -monotone* if no two points of it have the same x -coordinate; see Figure 2(a).

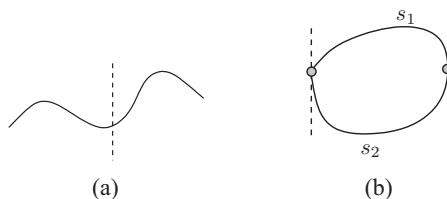


Figure 2. (a) A curve is x -monotone if no two points have the same x -coordinate, so any vertical line can intersect it at most once. (b) Any convex curve can be decomposed into a pair of x -monotone curves s_1 and s_2 up to some small perturbation of the axes.

Theorem 1.1. *Let k be a fixed positive integer. The total number of intersection points between n pairwise intersecting closed curves in general position in the plane, each of which can be decomposed into at most k x -monotone curves, is at least $(1 - o_k(1))n^2$.*

Given a finite collection of closed convex curves in the plane, by slightly changing the direction of the x -axis, if necessary, we can assume that each element can be split into two x -monotone pieces; see Figure 2(b). Applying Theorem 1.1 with $k = 2$, we establish the conjecture for all families of closed convex curves (that is, boundaries of convex regions).

Corollary 1.2. *The total number of intersection points between n pairwise intersecting closed convex curves in general position in the plane is at least $(1 - o(1))n^2$.*

Theorem 1.1 can be deduced from a general lower bound on the number of crossings between n x -monotone curves in terms of the number of touchings between them. An x -monotone curve is called *bi-infinite* or *two-way infinite* if it is the graph of a continuous real function defined over the entire real line.

Theorem 1.3. *Let S_1 and S_2 be two collections of bi-infinite x -monotone curves in general position. Suppose that $|S_1| = |S_2| = n$ and that the number of touching pairs (s_1, s_2) with $s_1 \in S_1, s_2 \in S_2$ is tn , where t is larger than an absolute constant C .*

Then the number of crossing points between the elements of S_1 or the number of crossing points between the elements of S_2 is at least $\Omega(nt\sqrt{\log t / \log \log t})$.¹

Let S be a collection of n bi-infinite x -monotone curves with tn touching pairs. Partition S randomly into two sets, S_1 and S_2 , of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$, respectively. The expected number of touchings between the curves in S_1 and the curves in S_2 will be more than $tn/2$. Fix a partition $S = S_1 \cup S_2$ for which this number exceeds $tn/2$, and apply Theorem 1.3 to S_1 and S_2 . We obtain the following.

¹ Below we use $\log x$ to denote the binary logarithm $\log_2 x$.

Corollary 1.4. *Let S be a collection of n bi-infinite x -monotone curves in general position with tn touching pairs, where t is bigger than an absolute constant. Then the number of crossings between the elements of S is $\Omega(nt\sqrt{\log t/\log \log t})$.*

At the beginning of Section 2 we will show that Corollary 1.4 implies the following statement for not necessarily bi-infinite curves.

Corollary 1.5. *Let S be a collection of n x -monotone curves in general position, with at least ϵn^2 touching pairs. If $\epsilon > C\sqrt{\log \log n/\log n}$ for a suitable absolute constant C , then S determines $\Omega(\epsilon n^2\sqrt{\log n/\log \log n})$ crossing points.*

Corollary 1.5 easily extends to the following more general result, which immediately implies Theorem 1.1.

Theorem 1.6. *Let $k > 0$ be a fixed positive integer. Let S be a collection of n closed curves in general position and with at least ϵn^2 touching pairs, so that each curve in S can be decomposed into k x -monotone curves. Suppose that $\epsilon > C_k\sqrt{\log \log n/\log n}$ for a suitable constant $C_k > 0$ which may depend on k . Then S determines $\Omega_k(\epsilon n^2\sqrt{\log n/\log \log n})$ crossing points.*

It is instructive to compare Theorems 1.1 and 1.6 in the case when S consists of pairwise intersecting curves. Note that the statement of Theorem 1.1 can still hold even if both the number of crossings and the number of touchings determined by S are quadratic in n (for arbitrary large n). In contrast, Theorem 1.6 guarantees that, in the above setup, the overall number of crossing points must asymptotically *exceed* the number of touching pairs. In particular, given a sufficiently large number n of curves, each decomposable into k x -monotone curves, and $\Omega(n^2)$ touchings among them, then some pair of curves must cross *super-constantly* many times. For this conclusion, we can trade the assumption of subdivisibility to a bounded number of x -monotone curves to a different assumption: that each pair of curves intersect (see the remark after Theorem 1.7).

We prove Theorem 1.3, which is the main technical result of this paper, in Section 3. The proof is based on a double counting argument. We define a ‘charging scheme’, that is, a weighted graph G whose vertices correspond to the touchings and crossings between the curves in $S_1 \cup S_2$. Theorem 1.3 follows by comparing our upper and lower bounds on the total weight of the edges.

Consider the setup of Theorem 1.3 in the special case when, for every pair $s_1 \in S_1, s_2 \in S_2$, the curve s_1 lies entirely below s_2 (with a possible tangency). Fox, Frati, Pach and Pinchasi [10] established a tight bound for this case. Namely, they proved that the number of crossing points between the elements of S_1 or the number of crossing points between the elements of S_2 is at least $\Omega(tn \log(tn))$, where, as above, tn is the overall number of tangent pairs $s_1 \in S_1$ and $s_2 \in S_2$. In particular, if every element of S_1 touches all elements of S_2 , we have $t = n$, so that this bound becomes $\Omega(n^2 \log n)$. In Theorem 1.3, we prove a slightly weaker bound, but we drop the condition on the bipartite structure of the family of curves.

In Section 4, we insist on a complete bipartite pattern of tangencies, but we drop the condition of x -monotonicity. We prove the following.

Theorem 1.7. *Let S_1 and S_2 be collections of closed curves in general position. If $|S_1| = |S_2| = n$, any two elements of $S_1 \cup S_2$ intersect and every element of S_1 touches all elements of S_2 , then the number of crossing points between the elements of S_1 or the number of crossing points between the elements of S_2 is $\Omega(n^2 \sqrt{\log n / \log \log n})$.*

Note that this theorem also implies the statement formulated in the paragraph after Theorem 1.6. Namely, if S is a collection of n closed curves in general position such that all pairs intersect and ϵn^2 pairs touch, then some pair intersects super-constantly many times. Indeed, by the Kővári–Sós–Turán theorem, we find disjoint subsets S_1 and S_2 of S with $|S_1| = |S_2| = \Omega(\log n)$ such that every curve in S_1 touches all curves in S_2 . Then, by Theorem 1.7, the curves in S_1 or S_2 have many crossings and thus there are two curves that intersect $\Omega(\sqrt{\log \log n / \log \log \log n})$ times.

2. Proofs of Corollary 1.5, Theorems 1.1 and 1.6 using Theorem 1.3

We have seen in the Introduction that Corollary 1.4 is a direct consequence of Theorem 1.3. In this section, we first deduce Corollary 1.5 from Corollary 1.4. Then we show how Theorem 1.1 and 1.6 follow from Corollary 1.5.

Proof of Corollary 1.5 (using Corollary 1.4). Partition S into two parts, S_1 and S_2 , such that at least $\epsilon n^2/4$ touchings are *good* in the sense that they are formed by a curve from S_1 touching a curve from S_2 from above. This is possible, as the expected number of good touchings for a uniform random partition of S into two parts is $\epsilon n^2/4$.

Perturb the curves in S a little bit so that (1) they become x -monotone polygonal paths consisting of finitely many straight-line segments, (2) all crossing points and good touching points remain unchanged, and (3) all other touchings are eliminated. Extend the resulting curves to bi-infinite polygonal paths, as follows (see Figure 3). Let z be a sufficiently large positive number. Extend each curve in S_1 beyond its right endpoint and each curve in S_2 beyond its left endpoint by a ray of slope z . Extend each curve in S_1 beyond its left endpoint and each curve in S_2 beyond its right endpoint by a ray of slope $-z$. By choosing z large enough, we can make sure that this last transformation preserves all good touching pairs and creates only at most two new crossings for any pair of curves.² This means that we kept at least $\epsilon n^2/4$ touchings and created fewer than n^2 new crossing points.

² For instance, choosing z larger than the absolute slopes of all straight-line segments that constitute the curves of S will do.

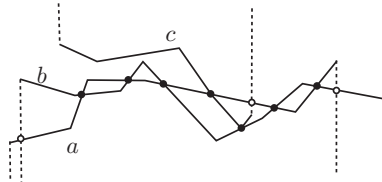


Figure 3. We extend each (polygonal) curve of $S_1 \cup S_2$ to a bi-infinite polygonal path by sufficiently steep terminal rays. (Among the three depicted curves, a and b belong to S_2 , whereas c belongs to S_1 .) The extension introduces at most two additional crossings for any pair of curves.

In view of the condition on ϵ , we can apply Corollary 1.4 to the modified bi-infinite curves with $t = \epsilon n/4$ to conclude that the number of crossing points is at least

$$\Omega\left(nt\sqrt{\frac{\log t}{\log \log t}}\right) = \Omega\left(\frac{\epsilon n^2}{4}\sqrt{\frac{\log(\epsilon n/4)}{\log \log(\epsilon n/4)}}\right) = \Omega\left(\epsilon n^2\sqrt{\frac{\log n}{\log \log n}}\right).$$

The lower bound on ϵ guarantees that over half of these points were already present in the original collection. □

Proof of Theorem 1.6 (using Corollary 1.5). Let S be a collection of n closed curves with at least ϵn^2 touching pairs as in Theorem 1.6, so that each curve in S can be decomposed into k x -monotone curves. Clearly, dividing the closed curves of S into x -monotone pieces can introduce altogether at most kn cutpoints. The resulting collection of curves, S' , still has at least $|T| - kn$ touching points, after losing all touchings that coincide with locally x -extremal cutpoints of the curves. Assuming $\epsilon > C_k\sqrt{\log \log n / \log n}$ for a suitable constant C_k guarantees that the new collection S' satisfies the hypothesis of Corollary 1.5 with $|S'| = n' = O_k(n)$ and $\epsilon' = \Omega_k(\epsilon)$. □

Proof of Theorem 1.1 (using Theorem 1.6). Two intersecting closed curves have at least two points in common, unless they touch each other. Thus, the total number of intersection points between the n curves is at least $2\binom{n}{2} - |T|$, where T denotes the set of touching points. If $|T| = o(n^2)$, we are done. If this is not the case, we can apply Theorem 1.6 and conclude that the total number of crossings is $\Omega(n^2\sqrt{\log n / \log \log n})$. □

3. Proof of Theorem 1.3

We say that a point p is to the right of point q and q is to the left of p in the plane if the x -coordinate of p is larger than the x -coordinate of q .

Consider the collections S_1 and S_2 of bi-infinite x -monotone curves, as in the statement of Theorem 1.3, and let $S = S_1 \cup S_2$. By slightly perturbing the curves in S , if necessary, eliminate all touchings between them, except those at which a curve in S_1 touches a curve in S_2 from above. This operation does not change the set of crossing points, and we may assume without loss of generality that the number of remaining touchings is at least $tn/2$. Indeed, otherwise we can swap the roles of S_1 and S_2 and the assumption will hold. By a slight abuse of notation, we continue to denote the collections of perturbed curves S_1

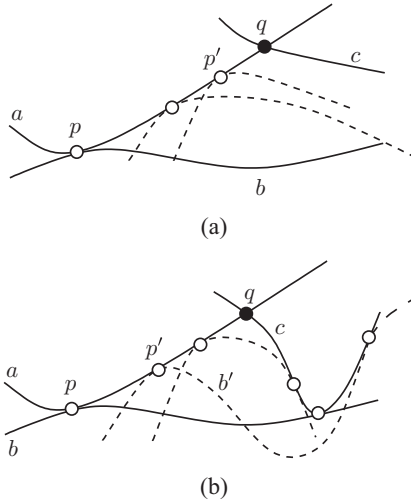


Figure 4. The construction of A_k and B_k (resp. (a) and (b)). Each edge (p, q) connects a touching $p \in T$ between a and b to a crossing $q \in X$ between a and some third curve c .

and S_2 , and their union by S . Henceforth we will use the property that each curve in S is touched only from one side.

Next, we construct a weighted bipartite multigraph G such that its vertex classes are T , the set of touching points and X , the set of crossing points between pairs of curves coming from the same collection S_1 or S_2 . That is, X does not contain crossings between a curve in S_1 and a curve in S_2 .

Next, we define some sets of weighted edges, A_k, B_k and C_k , running between T and X . The edge set of G is defined as the disjoint union of the sets A_k, B_k and C_k , for all $1 \leq k \leq t/2$ that are powers of 2. Every edge between a touching $p \in T$ and a crossing $q \in X$ has the property that p and q belong to the same curve in S and p lies to the left of q . In case p and q are connected in several of these sets, we consider these distinct edges and G contains several parallel edges connecting p and q , typically with different weights.

The definitions of A_k, B_k and C_k are illustrated in Figures 4 and 5. Namely, let $p \in T$ be the touching between two curves, $a, b \in S$, and let $q \in X$ be a crossing to the right of p , where a crosses $c \in S$. Here we must have $a, c \in S_1$ and $b \in S_2$, or alternately $a, c \in S_2$ and $b \in S_1$. Let $\alpha > 1$ be a parameter to be specified later.

- A. All edges of A_k have weight $1/k$. The edge connecting a touching p to a crossing q is present in A_k if there are fewer than k touching points p' on a between p and q ; see Figure 4(a).
- B. All edges of B_k have weight α/k . The edge connecting p and q is present in B_k if the following conditions are satisfied (refer to Figure 4(b)):
 - B1. the curves b and c touch to the right of q and
 - B2. there are fewer than k/α touching points p' on a between p and q with the property that the curve $b' \in S \setminus \{a\}$ containing p' touches c .
- C. All edges of C_k have weight α^2/k . The edge connecting p and q is present in C_k if:

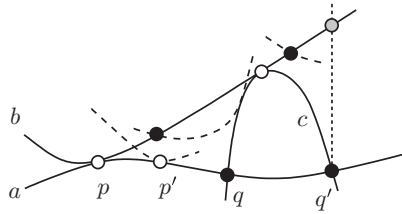


Figure 5. The construction of C_k . The arc of $(p, q) \in C_k$ is the part of b to the right of p and left of q' .

- C1. the curves a and c cross to the right of q with q' being the next such crossing,
- C2. the curve b touches c between q and q' ,
- C3. there are fewer than k touching points p' on a between p and q , and
- C4. there are fewer than αk points of X on b to the right of p and to the left of q' .

We call the part of b to the right of p and left of q' the *arc* of the corresponding edge in C_k (see Figure 5).

Our proof relies on a double counting of the total weight of the edges in G . We prove an upper bound for the total weight of edges incident to crossing points and a lower bound for the total weight of edges incident to most touching points. This proof technique is commonly referred to as the charging method.

3.1. Upper bound on total weight

We start by considering the edges incident to a fixed crossing point $q \in X$ of two curves of S . Since p can lie on either of these two curves, there are at most $2k$ edges $(p, q) \in A_k$ incident to q , for a total weight of at most 2. Summing over all possible values of k we get a total weight of at most $2 \log t$.

Similarly, there are at most $2\lceil k/\alpha \rceil$ edges in B_k incident to q for a total weight of at most $2 + 2\alpha/k$. Summing for all k we get that the total weight of these edges is less than $2 \log t + 4\alpha$.

For edges in C_k incident to q , only one³ of the two curves passing through q may play the role of a ; the other curve must play the role of c . Considering condition C3 in the definition of C_k (and ignoring the arcs of the edges for a moment), we see that q is incident to at most k edges of C_k for a total weight of at most α^2 .

Proposition 3.1. *For any pair of edges $(p, q) \in C_k$ and $(p', q) \in C_{k'}$ incident to the same point $q \in X$, their respective arcs either coincide or cross. In case of a crossing, all the crossing points must belong to X . □*

Proof. Assume with no loss of generality that q lies on $a \in S_2$, and let $b, b' \in S_1$ be the respective second curves that are incident to p and p' . If $p = p'$, then the arcs of the two edges coincide. Otherwise we must have $b \neq b'$. Let p_1 and p_2 be the respective points at which b and b' touch c (as prescribed in condition C2). The proposition now follows

³ This distinction depends on whether these two curves crossing at q belong to S_1 or to S_2 , and uses that the curves of S_1 are allowed to touch the curves of S_2 only from above.

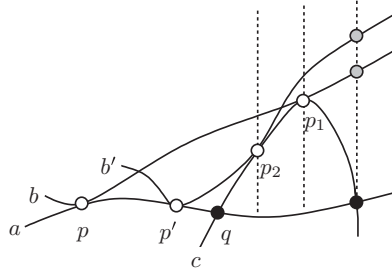


Figure 6. Proof of Proposition 3.1. The curves b and b' are tangent to c at the respective points p_1 and p_2 , so they must cross between the vertical lines through p_1 and p_2 .

because the vertical lines through p_1 and p_2 intersect the curves b and b' in a different order, each time within the respective arc of (p, q) or (p', q) ; see Figure 6. □

Let k_0 be the smallest index such that an edge in C_{k_0} is incident to q . Consider the arc of any such edge. By condition C4, there are fewer than αk_0 crossing points on this arc that belong to X . The crossing point with the arc of an edge in C_k incident to q determines the edge, and so does the fact that the two arcs coincide. Therefore, there are at most $\lceil \alpha k_0 \rceil$ edges in C_k incident to q . This means that the total weight is at most $\alpha^3 k_0/k + \alpha^2/k$. Summing over all values of k and always using the better of the two estimates proved, we obtain that the total weight of all edges in the sets C_k incident to q is less than $\alpha^2(\log \alpha + 2)$.

The total weight of all edges in G incident to a fixed vertex $q \in X$ is, therefore, at most $4 \log t + \alpha^2 \log \alpha + 6\alpha^2$. The total weight of all edges in G is at most $|X|(4 \log t + \alpha^2 \log \alpha + 6\alpha^2)$.

3.2. Lower bound on total weight

Now we fix a touching point $p \in T$. Let a and b be the two curves in S that touch at p and assume that both a and b contain at least k touching points to the right of p . This assumption holds for all but at most $2nk$ touching points.

Fix $1 \leq k \leq nt/2$. Let the next k touching points along a to the right of p be r_1, \dots, r_k . Let b_0 be the segment of b to the right of p and to the left of r_k . We assume that there are fewer than k touching points on b_0 . We can assume this without loss of generality, as if this condition is violated, we can switch the roles of a and b .

For $i = 1, \dots, k$, let $c_i \in S$ be the curve different from a containing r_i . Observe that c_i must cross b_0 to the left of r_i at least once; see Figure 7. We are using here that all touchings are on the same side of any curve. Let q_i be any one of these crossing points.

We claim that the total weight of the edges in A_k, B_k and C_k incident to p is at least α . We prove this claim by case analysis.

Case 1. b_0 contains at least αk points of X . Note that all these points are connected to p by an edge in A_k . These edges have a total weight of at least α , as claimed.

Case 2. q_i is connected to p by an edge in B_k for $i = 1, \dots, k$. Observe that these edges alone provide the total weight α , as claimed.

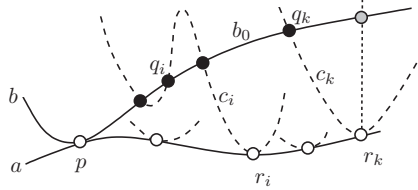


Figure 7. Obtaining a lower bound on the total weight received by a touching p between a and b . We consider the first k touching points r_i , for $1 \leq i \leq k$, along a and to the right of p . The other curve c_i must cross b to the right of p and to the left of r_i .

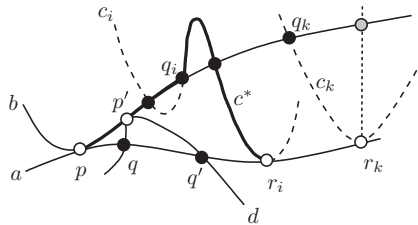


Figure 8. Case 3. We encounter at least k/α curves d , each touching b at a point p' between p and q_i . For each d of this kind, we set q (resp. q') to be the first crossing between a and d to the left (resp. right) of p' , and argue that the edge (p, q) belongs to C_k .

Case 3. Neither Case 1 nor Case 2 applies. We fix an index $1 \leq i \leq k$ such that the crossing point q_i is not connected to p by an edge in B_k . Note that condition B1 for this edge to be present in B_k is satisfied, thus condition B2 must be violated, that is, there are at least k/α distinct touching points p' on b between p and q_i with the property that the curve $d \in \mathcal{S} \setminus \{b\}$ containing p' touches c_i . Consider the triangle-like region bounded by the segment of a^* of a between p and r_i , the segment b^* of b between p and q_i and the segment c^* of c_i between q_i and r_i . Close to p' , the curve d is inside this region, as depicted in Figure 8. (We are again using that the curves are touched from one side only.) As d touches both b and c_i , it must leave the region through a^* . Let q be the first point where d and a^* cross to the left of p' , and let q' be the first point where they cross to the right of p' . Note that q is connected to p by an edge in C_k . Indeed, conditions C1 and C2 are easy to verify. Condition C3 holds, as the segment of a between p and q is contained in a^* , which has exactly $i - 1$ touching points in its interior. Condition C4 holds, as the segment of b to the right of p and left of q' is contained in b_0 , which is assumed to contain fewer than αk points of X . Note that there were at least k/α distinct choices for the touching point p' , each yielding a distinct edge in C_k incident to p , so the claim is also proved in this final case.

Summing for all the touching points in T , except the at most $2nk$ that are among the last k on some curve, we get that the total weight of the edges in A_k , B_k and C_k (k fixed) is at least $\alpha(|T| - 2nk)$. Using that $|T| \geq tn/2$ and summing over all k , we obtain that the total weight of all edges in G is at least $(\log t - 3)\alpha tn/2$. Comparing this with the

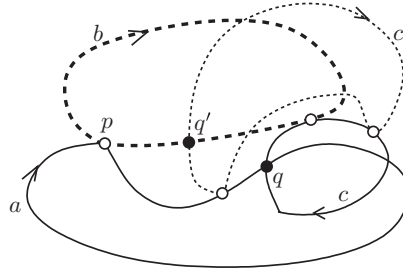


Figure 9. The construction of A_k, A'_k and A''_k . A touching $p \in T$ between $a \in S_1$ and $b \in S_2$ is connected in A_k to a crossing $q \in X_1$ between a and a third curve $c \in S_1$ if and only if the arc of a from p to q contains fewer than k points of T . Each edge $(p, q') \in A'_k \cup A''_k$ connects a touching $p \in T$ between $a \in S_1$ and $b \in S_2$ to a crossing $q' \in X_2$ between b and some third curve $c' \in S_2$.

upper bound obtained earlier, we get

$$|X| \geq \frac{(\log t - 3)\alpha t n / 2}{4 \log t + \alpha^2 \log \alpha + 6\alpha^2}.$$

Substituting $\alpha = \sqrt{\log t / \log \log t}$ finishes the proof of Theorem 1.3.

4. Proof of Theorem 1.7

This proof is also based on the charging method, but we have to modify the charging scheme used in the proof of Theorem 1.3. We build the same type of weighted bipartite graph between the vertex sets X and T , where $X = X_1 \cup X_2$ and X_i is the set of intersection points between curves in S_i , while T is the set of points where a curve of S_1 touches a curve of S_2 .

Note that each curve in $S = S_1 \cup S_2$ is touched from one side only. Indeed, a curve touching a closed curve from one side cannot intersect a curve touching the same closed curve from the other side. We orient each curve $a \in S$ so that the curves touching it lie on its left. This orientation specifies which of the two arcs of a closed curve a connecting two distinct points $p, q \in a$ is considered *the arc of a from p to q* . We consider this arc relatively open, thus p and q do not belong to the arc connecting them.

We define the sets of weighted edges, A_k, A'_k, A''_k, B_k and C_k , running between T and X , whose definition is detailed below and illustrated in Figures 9, 10 and 11. The edge set of G is defined as the disjoint union of the sets A_k, A'_k, A''_k, B_k and C_k , for all $1 \leq k < n$ that are powers of 2. Thus we have $l := \lceil \log n \rceil$ distinct values of k to consider. The parameter $\alpha > 1$ plays the same role as in the proof of Theorem 1.3 and will be set later.

- A. The edge connecting a touching $p \in T$ to a crossing $q \in X_1$ is present in A_k with weight $1/k$ if p and q lie on a common curve $a \in S_1$ and the arc of a from p to q contains fewer than k touching points.
- A'. The edge connecting $p \in T$ and $q \in X_2$ is present in A'_k with weight α/k if p and q lie on a common curve $b \in S_2$ and the arc of b from q to p contains fewer than k/α touching points.

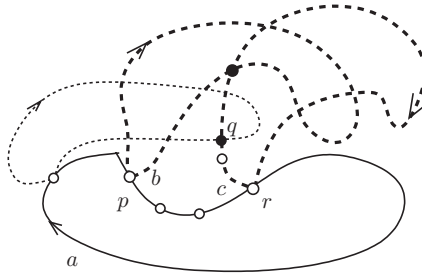


Figure 10. The construction of B_k . A touching $p \in T$ between $a \in S_1$ and $b \in S_2$ is connected to a crossing $q \in X_2$ if there is a touching r between a and some $c \in S_2$ containing q so that the arc of a from p to r contains fewer than k points of T and the arc of c from r to q contains fewer than αk points of T .

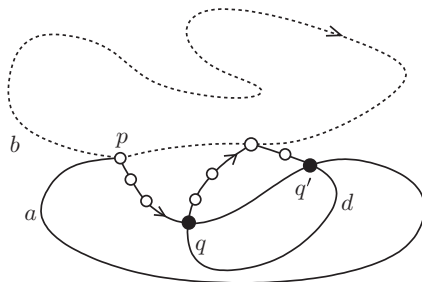


Figure 11. The construction of C_k . A touching $p \in T$ between $a \in S_1$ and $b \in S_2$ is connected to a crossing $q \in X_1$ between a and some other curve $d \in S_1$ if the arc of a from p to q contains fewer than k points of T , the arc of d between q and the next crossing q' of a and d contains less than $3\alpha^2 k$ points of T , and b touches d within its arc from q to q' .

- A''.** The edge connecting $p \in T$ and $q \in X_2$ is present in A''_k with weight $1/(\alpha k)$ if p and q lie on a common curve $b \in S_2$ and the arc of b from q to p contains fewer than αk touching points. Note that if α^2 is a power of 2, then A''_k and $A'_{\alpha^2 k}$ are identical.
- B.** The edge connecting $p \in T$ and $q \in X_2$ is present in B_k with weight $1/(\alpha k^2)$ if there is a touching $r \in T$ between some curves $a \in S_1$ and $c \in S_2$ satisfying that:
 - B1.** p is on a and the arc of a from p to r contains fewer than k touching points and
 - B2.** q is on c and the arc of c from r to q contains fewer than αk touching points.
- C.** Let q be an intersection point of two curves a and d belonging to S_1 . Let q' be the intersection of a and d coming right after q along d . That is, the arc of d from q to q' is disjoint from a (see Figure 11). The edge connecting p and q is present in C_k with weight α^2/k if:
 - C1.** p lies on a and the arc of a from p to q contains fewer than k points of T ,
 - C2.** the arc of d from q to q' contains fewer than $3\alpha^2 k$ points of T and
 - C3.** the curve $b \in S_2$ through p touches d within the arc of d from q to q' .

In the case when p and q are connected in several of these sets, we consider the corresponding edges distinct and G contains several parallel edges connecting p and q , possibly with different weights.

4.1. Upper bound on total weight

We start by considering the edges incident to a fixed crossing point $q \in X$.

A vertex $q \in X_1$ is incident to at most $2k$ edges in A_k for a total weight of at most 2. Summing over all possible values of k , we get a total weight of at most $2l$.

A vertex $q \in X_2$ is incident to at most $2\lceil k/\alpha \rceil$ edges in A'_k and at most $2\lceil \alpha k \rceil$ edges of A''_k for a total weight of less than $4 + 4\alpha/k$. Summing over all k , we obtain that the total weight of these edges is less than $4l + 8\alpha$.

In a similar way, given a vertex $q \in X_2$, we can choose $r \in T$ to satisfy condition B2 in $2\lceil \alpha k \rceil$ ways and each such vertex r brings about k edges of B_k incident to q for a total of at most $2k\lceil \alpha k \rceil$ edges with a total weight less than $2 + 2/(\alpha k)$. Summing over all k , we obtain that the total weight of these edges is less than $2l + 4/\alpha$.

Finally, we consider the edges in C_k . Let us fix $q \in X_1$. This point lies on two curves in S_1 . To produce an edge in C_k incident to q , one must play the role of a and the other the role of d . By condition C3, a point of the arc of d from q to q' is shared by a curve in S_2 , so it must be on the left side of a . This also holds for the entire arc, as it is disjoint from a . This property uniquely determines which of the two curves through q must play the role of a and which one must play the role of d . Thus, q' is also determined by q and is independent of k and of the particular edge in C_k incident to q .

Condition C1 guarantees that the number of edges in C_k incident to q is at most k , for a total weight of at most α^2 .

Let k_0 be the smallest index with an edge of C_{k_0} incident to a given vertex $q \in X_1$. By condition C2, the arc of d from q to q' contains at most $3\alpha^2 k_0$ points of T . If p is adjacent to q in C_k for some k , then the curve $b \in S_2$ through p must touch d in its arc from q to q' and this touching point determines p . Thus, the number of edges in C_k (k fixed) incident to q is at most $3\alpha^2 k_0$ for a total weight of $3\alpha^4 k_0/k$.

Summing over all values of $k \geq k_0$ and always using the better of the two estimates proved, we get that the total weight of all edges in the sets C_k incident to q is less than $2\alpha^2 \log \alpha + 4\alpha^2$.

The total weight of all edges in G incident to a fixed vertex $q \in X$ is therefore at most $6l + 2\alpha^2 \log \alpha + 12\alpha^2$. The total weight of all edges in G is at most $|X|$ times this value.

4.2. Lower bound on total weight

Fix a touching point $p \in T$. Let $a \in S_1$ and $b \in S_2$ be the two curves in S that touch at p .

Let us fix k , and let the k touching points that follow p along a (in the chosen orientation) be denoted by r_1, \dots, r_k (with r_1 being closest). Note that a contains exactly n points in T , so p and r_1, \dots, r_k are all distinct.

For every $i = 1, \dots, k$, let $c_i \in S_2$ be the curve containing r_i . Let q_i be the first intersection of the curves b and c_i after r_i , with respect to the orientation of c_i . Consider the closed curve consisting of the following three arcs: the arc b_i of b from q_i to p , the arc c'_i of c_i from r_i to q_i , and the arc a_i of a from p to r_i . This (together with the points p, q_i and r_i) is a simple closed curve and, hence, partitions the plane into two connected components; see Figure 12. We denote by T_i the component that lies on the left side of the three arcs. Note that the curves a and b are disjoint from T_i , but c_i may enter T_i through the arc b_i several times.

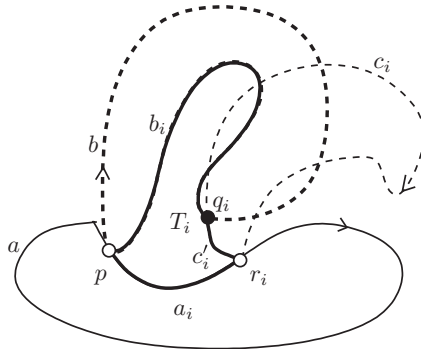


Figure 12. Lower bound for the proof of Theorem 1.7: the overall setup. q_i is the first intersection point of c_i and b after r_i along c_i . T_i is the region whose boundary is composed of the arc b_i of b from q_i to p , the arc c'_i of c_i from r_i to q_i , and the arc a_i of a from p to r_i .

We claim that the total weight of the edges in A_k, A'_k, A''_k, B_k and C_k incident to p is at least α . We prove this claim by case analysis.

Case 1. The arc a_k contains at least αk points of X . Note that all these points belong to X_1 (as $a \in S_1$), and are connected to p by an edge in A_k . These edges have a total weight of at least α , as claimed.

Note that any curve in S_1 touching b in b_i or touching c_i in c'_i must be inside T_i in a small neighbourhood of this touching point. But as it intersects a it cannot be entirely within T_i , so it must cross the boundary of T_i . In particular it must cross a_i , as it can only touch b and c_i (due to the complete bipartite touching structure of $S_1 \times S_2$); see Figure 13(a). The curve a_i is contained in a_k . Thus, if Case 1 does not hold, b_i and c'_i contain fewer than αk points from T , for any $1 \leq i \leq k$.

Case 2. Case 1 does not hold and b_i contains at least $\alpha^2 k$ points from X , for some i . As b_i has fewer than αk points of T , every point of b_i that belongs to X is connected to p by an edge in A''_k , providing a total weight α , as required.

Case 3. Case 1 does not hold and for every $i = 1, \dots, k$, either c'_i contains at least $2\alpha^2 k$ points of X or b_i contains fewer than k/α points of T . Note that a_i contains $i - 1 < k$ points of T , while c'_i contains fewer than αk points of T . Hence, every crossing point on c'_i is connected to p by an edge in B_k . This represents a total weight of at least $2\alpha/k$ if $|c'_i \cap X| \geq 2\alpha^2 k$. For values of i with $|b_i \cap T| < k/\alpha$, the vertex q_i is adjacent to p in A'_k , and this edge alone has weight α/k . If either possibility happens for all k possible values of i , these weights add up, except that some edges in B_k may be counted twice. This results in a total weight of at least α for the edges in A'_k and B_k incident to p .

Case 4. None of the Cases 1, 2 or 3 hold. We fix an index $1 \leq i \leq k$ such that c'_i contains fewer than $2\alpha^2 k$ points of X and b_i contains at least k/α points of T . This is possible, as Case 3 does not hold. Let us consider one of the curves in S_1 touching b at a point z of b_i . As was pointed out earlier, this curve d lies inside T_i in a small neighbourhood of z and leaves T_i by crossing a_i . Let q be the last crossing between d and a before z , and

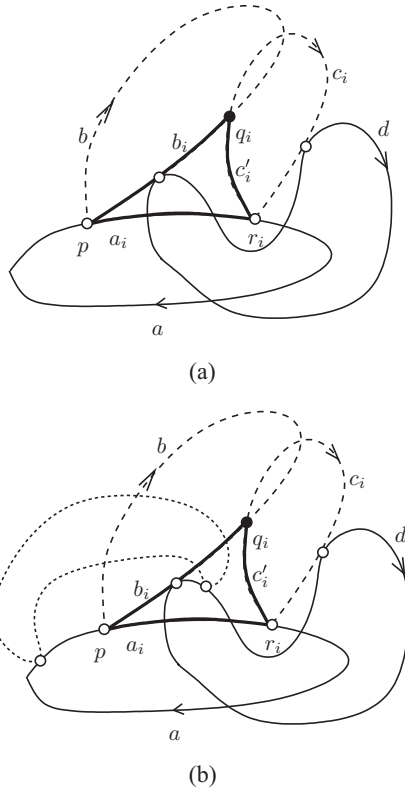


Figure 13. Lower bound for the proof of Theorem 1.7. (a) Any curve in S_1 that touches b or c_i within the respective arcs b_i or c'_i must cross a within a_i . Therefore, if Case 1 does not occur, the overall number of such curves cannot exceed αk . (b) In Case 4, the curve b is touched within b_i by at least k/α curves d of S_1 . Notice that any curve of S_2 that touches d within T_i must enter that region through b_i or c'_i . Since none of the Cases 1–3 holds, the overall number of such curves cannot exceed $3\alpha^2 k$.

let q' be the first crossing between them after z , with respect to the orientation of d . See Figure 13(b) for an illustration.

We claim that q is adjacent to p in C_k . Indeed, conditions C1 and C3 hold trivially. To verify C2, consider a curve $e \in S_2$ touching d at a point in the arc of d from q to q' . This touching point (as the whole arc of d from q to q') lies inside T_i . However, e must intersect b , so it must leave T_i . It touches a , so it must leave through either b_i or c'_i . Here c'_i has fewer than $2\alpha^2 k$ points in X , by our choice of i , while b_i contains fewer than $\alpha^2 k$ points in X , since Case 2 does not hold. Hence, e has fewer than $3\alpha^2 k$ points where it can leave T_i , and there are at most $3\alpha^2 k$ possible choices for the curve e . This means that condition C2 is satisfied and q is adjacent to p in C_k .

By our choice of i , we can select the curve d in at least k/α different ways, each giving rise to a different edge in C_k incident to p . The total weight of these edges is α . This completes the analysis of the last case, showing that the total weight of all edges in A_k, A'_k, A''_k, B_k and C_k incident to p is at least α .

Summing over all l possible values of k and over all n^2 touching points in T , we conclude that the total weight of G is at least $\alpha \lceil \log n \rceil n^2$. Comparing this lower bound with the upper bound proved in the preceding subsection and substituting $\alpha = \sqrt{\log n / \log \log n}$, Theorem 1.7 follows.

5. Latest developments

Recently, we managed to prove the Richter–Thomassen conjecture in full generality; see [17]. The main contribution of [17] is the following asymptotic result.

Theorem 5.1. *Let S be a family of n pairwise intersecting Jordan curves in general position. Then the number $|X|$ of crossing points in S is asymptotically larger the number of overall number $|T|$ of touchings. That is, $|X|/|T|$ is bounded from below by a function $f(n)$ so that $\lim_{n \rightarrow \infty} f(n) = \infty$.*

In [17], this result is proved with $f(n) = \Omega((\log \log n)^{1/8})$. Note that, although our Theorem 1.6 applies only to families of k -monotone curves for a fixed integer k , it is stronger than Theorem 5.1 in two different ways. First, it establishes the same asymptotic relation with a better function $f(n) = \Omega_k(\sqrt{\log n / \log \log n})$. More importantly, in Theorem 1.6 the condition that the curves are pairwise intersecting is replaced by the weaker assumption that there are at least ϵn^2 touching pairs for some $\epsilon > C_k \sqrt{\log \log n / \log n}$.

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