

THE IMPACT OF STOCHASTIC VOLATILITY ON PRICING, HEDGING,
AND HEDGE EFFICIENCY OF WITHDRAWAL BENEFIT GUARANTEES
IN VARIABLE ANNUITIES

BY

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ABSTRACT

We analyze different types of guaranteed withdrawal benefits for life, the latest guarantee feature within variable annuities. Besides an analysis of the impact of different product features on the clients' payoff profile, we focus on pricing and hedging of the guarantees. In particular, we investigate the impact of stochastic equity volatility on pricing and hedging. We consider different dynamic hedging strategies for delta and vega risks and compare their performance. We also examine the effects if the hedging model (with deterministic volatility) differs from the data-generating model (with stochastic volatility). This is an indication for the model risk an insurer takes by assuming constant equity volatilities for risk management purposes, whereas in the real world volatilities are stochastic.

KEYWORDS

Variable Annuities, Guaranteed Minimum Benefits, Pricing, Hedging, Hedge Performance, Stochastic Volatility, Model Risk

1. INTRODUCTION

Variable annuities are fund-linked annuities where typically the policyholder pays a single premium, which is then invested in one or several mutual funds. Usually the policyholder may choose from a variety of different mutual funds. Such products were introduced in the 1970s in the United States. Two decades later, in the 1990s, insurers started to offer certain guarantee riders on top of the basic structure of variable annuity policies, leading to a significant increase in popularity and success of this type of annuity. Variable annuities including such guarantee riders were also successfully introduced in several Asian markets, and finally made their way to Europe. In the course of the recent financial crisis, however, the guarantees within Variable Annuities caused serious problems to some providers, forcing many insurers to redesign their products and some even to completely withdraw their offerings from certain markets.

There are several types of guarantee riders that come with variable annuities, including so-called guaranteed minimum death benefits (GMDB) as well as guaranteed minimum living benefits, which can be categorized in three main subcategories: guaranteed minimum accumulation benefits (GMAB), guaranteed minimum income benefits (GMIB) and guaranteed minimum withdrawal benefits (GMWB). A GMAB guarantee provides the policyholder with some guaranteed value at the maturity of the contract, while the GMIB guarantee provides a guaranteed annuity benefit, starting after a certain deferral period. However, the currently most popular type of guaranteed minimum living benefits is the GMWB rider. Under certain conditions, the policyholders may withdraw money from their account, even if the value of the account has dropped to zero. Such withdrawals are guaranteed as long as both, the amount that is withdrawn within each policy year and the total amount that is withdrawn over the term of the policy, stay within certain limits.

Recently, insurers started to include additional features in GMWB products. The most prominent is called “GMWB for Life” (also known as guaranteed lifetime withdrawal benefits, GLWB). With this guarantee type, the total amount of withdrawals is unlimited. However, the annual amount that may be withdrawn while the insured is still alive may not exceed some maximum value; otherwise the guarantee will be affected. The withdrawals made by the policyholders are deducted from their account value, as long as this value is positive. Afterwards, the insurer has to provide the guaranteed withdrawals for the rest of the insured’s life. In return for this guarantee, the insurer receives guarantee charges, which are usually deducted as a fixed annual percentage from the policyholder’s account value (as long as this value is positive). In contrast to a conventional annuity, in which the assets covering the liabilities are owned by the pool of insured, in a GLWB policy, the fund units of the contract are owned by the individual policyholder and remain accessible to the policyholder even in the payout phase. The policyholder may access the remaining fund assets at any time by (partially) surrendering the contract. In case of death of the insured, any remaining fund value is paid out to the insured’s beneficiary.

Therefore, from an insurer’s point of view, these products contain a combination of several risks resulting from policyholder behavior (e.g. surrender and withdrawal), financial markets, and longevity, which makes these guarantees difficult to hedge. Moreover, in practice, the insurer faces a variety of additional risks including operational risk, reputational risk, basis risk, etc., which are not in the focus of this paper.

To deal with the significant financial risks resulting from the guarantees, in general risk management strategies such as dynamic hedging are applied. However, during the recent financial crisis, insurers suffered from inefficient hedging strategies¹. Among other effects, the financial crisis led to a significant

¹ Cf. e.g. different articles and papers in “Life and Pensions”: “A challenging environment” (June 2008), “Variable Annuities – Flawed product design costs Old Mutual £150m” (September 2008), “Variable annuities – Milliman denies culpability for clients’ hedging losses” (October 2008), “Variable Annuities – Axa injects \$3bn into US arm” (January 2009).

increase in actual and implied equity volatility, and thus to a tremendous increase in the value of most standard and non-standard equity-linked options, including the value of typical variable annuity guarantees. Insurers who chose not to hedge certain risks or used insufficient hedging programs (in particular with respect to increasing volatility levels) suffered from losses due to the increase in the option's economic value.

There already exists some literature on the pricing of different guaranteed minimum benefits and in particular on the pricing of GMWB rider options: Valuation methods have been proposed by e.g., Milevsky and Posner (2001) for the GMDB option, Milevsky and Salisbury (2006) for the GMWB option, Bacinello et al. (2009) for life insurance contracts with surrender guarantees, and Holz et al. (2008) for GMWB for Life riders.

Bauer et al. (2008) introduced a general model framework that allows for the simultaneous and consistent pricing and analysis of different variable annuity guarantees. They also give a comprehensive analysis over non-pricing related literature on variable annuities. To our knowledge, there exists little literature on the performance of different strategies for hedging the market risk within variable annuity guarantees. Coleman et al. (2006 and 2007) provide such analyses for death benefit guarantees under different hedging and data-generating models. However, to our knowledge, the performance of different hedging strategies for GLWB contracts under stochastic equity volatility has not yet been analyzed. The present paper fills this gap.

The remainder of this paper is organized as follows. First, we give a high-level description of the guaranteed lifetime withdrawal benefits (GLWB) rider options and explain their general functionality in Section 2, where we also present the GLWB product designs that we will analyze in the numerical section of this paper. We also describe the model framework for insurance liabilities that is used for our analyses, which follows Bauer et al. (2008).

In Section 3, we provide the framework for the numerical analyses, starting with a description of the financial market models used within our analyses for pricing and hedging of insurance liabilities. For the sake of comparison, we use the well-known Black-Scholes (1973) model (with deterministic equity volatility) as a reference and the Heston (1993) model as a model that allows for stochastic equity volatility. We also describe the financial instruments involved in the hedging strategies considered in the numerical part of this paper, and show how we determine the fair values of these instruments under both financial market models and how we compute these values' sensitivity to certain model parameters.

The numerical results of our contract analyses are provided in Section 4, starting with the determination of the fair guaranteed withdrawal rate in Section 4.1 for different product designs of the GLWB rider and under different model assumptions regarding the financial market and policyholder behavior. We show that product design, policyholder behavior and market parameters like long-term volatility and interest rates have a significant impact on the pricing results of the considered GLWB riders. However, the choice of the financial model (Black-Scholes or Heston) does not. We proceed with an analysis of the

distribution of guaranteed withdrawal amounts in Section 4.2 and the GLWB rider's "trigger time", i.e. the specific point in time, when, for the first time, guaranteed payments from the insurer to the policyholder are due, in Section 4.3. We find vast differences between the considered product designs from both, the client's and the insurer's perspectives. Finally we calculate the distribution over time of the so-called Greeks of the GLWB options in Section 4.4, analyzing the different sensitivities of the considered product designs with regard to changes in the underlying's spot price, the underlying's volatility and interest rates.

In Section 5, we analyze the hedge efficiency of different hedging strategies that may be applied by the insurance company in order to reduce the financial risk that originates from selling GLWB riders. We first describe the different dynamic hedging strategies that we consider within our analyses. We then analyze and compare their performance under both considered financial market models and show that the risk arising from using a model with constant equity volatility for risk management purposes can be substantial if actual volatility is stochastic.

2. MODEL FRAMEWORK

In Bauer et al. (2008), a general framework for modeling and valuation of variable annuity contracts was introduced. Within this framework, any contract with one or several living benefit guarantees and/or a guaranteed minimum death benefit can be represented. In their numerical analysis however, only contracts with a rather short finite time horizon were considered. Within the same framework, Holz et al. (2008) describe how GMWB for Life products can be included in this model. In what follows, we introduce this model focusing on the peculiarities of the contracts considered within our numerical analyses. We refer to Bauer et al. (2008) as well as Holz et al. (2008) for the explanation of other types of guarantees and more details on the model.

2.1. High-Level Description of the considered Insurance Contracts

Variable annuities are fund-linked products. The single premium P is invested in one or several mutual funds. We call the value of the insured's individual portfolio the *account value* and denote its value at time t by AV_t . All charges are taken from the account value by cancellation of fund units. Furthermore, the insured has the possibility to surrender the contract or to withdraw a portion of the account value.

Products with a GMWB option give the policyholder the possibility of guaranteed withdrawals. In this paper, we focus on the case where such withdrawals are guaranteed lifelong ("GMWB for Life" or "Guaranteed Lifetime Withdrawal Benefits", GLWB). The initially guaranteed withdrawal amount is usually a certain percentage x_{WL} of the single premium P . Any remaining

account value at the time of death is paid to the beneficiary as death benefit². If, however, the account value of the policy drops to zero while the insured is still alive, the policyholder can still continue to withdraw the guaranteed amount each year until death. The insurer charges a fee for this guarantee, which is usually a pre-specified annual percentage of the account value.

Often, GLWB products contain certain features that lead to an increase of the guaranteed withdrawal amount if the underlying funds perform well. Typically, on every policy anniversary, the current account value is compared to a certain reference value, which is referred to as withdrawal benefit base. Whenever the account value exceeds that withdrawal benefit base either the guaranteed annual withdrawal amount is increased (step-up or ratchet) or (a part of) the difference is paid out to the client (surplus distribution). In our numerical analyses in Sections 4 and 5, we have a closer look on four different product designs that can be observed in the market:

- **No Ratchet:** The first and simplest alternative is one where no ratchets or surplus exist at all. In this case, the guaranteed annual withdrawal amount is constant and does not depend on market movements.
- **Lookback Ratchet:** The second alternative is a ratchet mechanism where a withdrawal benefit base at outset is given by the single premium paid. During the contract term, on each policy anniversary date the withdrawal benefit base is increased to the account value, if the account value exceeds the previous withdrawal benefit base. The guaranteed annual withdrawal is increased accordingly to x_{WL} multiplied by the new withdrawal benefit base. This effectively means that the fund performance needs to compensate for policy charges and annual withdrawals in order to cause an increase of the guaranteed annual withdrawals. With this product design, increases in the guaranteed withdrawal amount are permanent, i.e. over time, the guaranteed withdrawal amount may only increase, never decrease.
- **Remaining WBB Ratchet:** With the third ratchet mechanism, the withdrawal benefit base at outset is also given by the single premium paid. The withdrawal benefit base is however reduced by every guaranteed withdrawal. On each policy anniversary where the current account value exceeds the current withdrawal benefit base, the withdrawal benefit base is increased to the account value. The guaranteed annual withdrawal is increased by x_{WL} multiplied by the difference between the account value and the previous withdrawal benefit base. This effectively means that, in order to cause an increase of guaranteed annual withdrawals, the fund performance needs to compensate for policy charges, but not for annual withdrawals. This ratchet mechanism is therefore c.p. somewhat “richer” than the Lookback Ratchet. Therefore, typically the initially guaranteed withdrawal amount should c.p. be lower than with a product offering a Lookback Ratchet.

² Some products also contain guaranteed minimum death benefits. However, we do not consider this feature in this paper.

As with the Lookback Ratchet design, increases in the guaranteed amount are permanent.

- **Performance Bonus:** In this version of the product, the withdrawal benefit base is defined similarly as in the Remaining WBB ratchet, but with the difference that in this design the withdrawal benefit base is never increased. Instead of permanently increasing the guaranteed withdrawal amount, on each policy anniversary date where the account value exceeds the withdrawal benefit base, 50% of the difference is added to this year's guaranteed amount as a "performance bonus". In contrast to the previous two designs, the guaranteed withdrawal amounts remain unchanged in the Performance Bonus design. For the calculation of the withdrawal benefit base only guaranteed annual withdrawals are deducted from the benefit base, not the performance bonus payments.

2.2. Liability Model

Throughout the paper, we assume that administration charges and guarantee charges are deducted at the end of each policy year as a percentage ϕ^{adm} and ϕ^{guar} of the account value. Additionally, we allow for upfront acquisition charges ϕ^{acq} as a percentage of the single premium P . This leads to $AV_0 = P \cdot (1 - \phi^{acq})$.

We denote the guaranteed withdrawal amount at time t by W_t^{guar} and the withdrawal benefit base by WBB_t . At inception, for each of the considered products, the initial guaranteed withdrawal amount is given by $W_0^{guar} = x_{WL} \cdot WBB_0 = x_{WL} \cdot P$. The amount actually withdrawn by the client is denoted by W_t^3 . Thus, the state vector $y_t = (AV_t, WBB_t, W_t, W_t^{guar})$ at time t contains all information about the contract at that point in time.

Since we restrict our analyses to single premium contracts, policyholder actions during the life of the contract are limited to withdrawals, (partial) surrender and death.

During the year, all processes are subject to capital market movements. For the sake of simplicity, we allow for withdrawals at policy anniversaries only. Also, we assume that death benefits are paid out at policy anniversaries if the insured person has died during the previous year. Thus, at each policy anniversary $t = 1, 2, \dots, T$, we have to distinguish between the value of a variable in the state vector $(\cdot)_t^-$ immediately before and the value $(\cdot)_t^+$ after withdrawals, (partial) surrender, and death benefit payments.

In what follows, we first describe the development between two policy anniversaries and then the transition at policy anniversaries for different contract designs. From these, we are finally able to determine all benefits for any given policy holder strategy and any capital market path. This allows for an analysis of such contracts in a Monte-Carlo framework.

³ Note that the client can chose to withdraw less than the guaranteed amount, thereby increasing the probability of future ratchets/bonuses. If the client wants to withdraw more than the guaranteed amount, any exceeding withdrawal would be considered a partial surrender.

2.2.1. *Development between two Policy Anniversaries*

We assume that the annual fees ϕ^{adm} and ϕ^{guar} are deducted from the policyholder’s account value at the end of each policy year. Thus, the development of the account value between two policy anniversaries is given by the development of the value S_t of the variable annuity’s underlying (typically a fund or a basket of mutual funds) after deduction of the guarantee fee, i.e.

$$AV_{t+1}^- = AV_t^+ \cdot \frac{S_{t+1}}{S_t} \cdot \exp(-\phi^{adm} - \phi^{guar}). \tag{1}$$

At the end of each year, the different ratchet mechanisms or the performance bonus are applied after charges are deducted and before any other actions are taken. Thus W_t^{guar} develops as follows:

- **No Ratchet:** $WBB_{t+1}^- = WBB_t^+ = P$ and $W_{t+1}^{guar-} = W_t^{guar+} = x_{WL} \cdot P$.
- **Lookback Ratchet:** $WBB_{t+1}^- = \max\{WBB_t^+, AV_{t+1}^-\}$
and $W_{t+1}^{guar-} = x_{WL} \cdot WBB_{t+1}^- = \max\{W_t^{guar+}, x_{WL} \cdot AV_{t+1}^-\}$.
- **Remaining WBB Ratchet:** Since withdrawals are only possible on policy anniversaries, the withdrawal benefit base during the year develops like in the Lookback Ratchet case. Thus, we have $WBB_{t+1}^- = \max\{WBB_t^+, AV_{t+1}^-\}$ and $W_{t+1}^{guar-} = W_t^{guar+} + x_{WL} \cdot \max\{AV_{t+1}^- - WBB_t^+, 0\}$.
- **Performance Bonus:** For this alternative a withdrawal benefit base is defined, which is similar to the one in the Remaining WBB ratchet, but is not increased at policy anniversaries, i.e. $WBB_{t+1}^- = WBB_t^+$. Additionally to the constant guaranteed withdrawal amount $x_{WL} \cdot P$, 50% of the difference between the account value and the withdrawal benefit base is added to this year’s guaranteed amount as a “performance bonus”. Thus, we have $W_{t+1}^{guar-} = x_{WL} \cdot P + 0.5 \cdot \max\{AV_{t+1}^- - WBB_t^+, 0\}$. Note that in this case, the state variable W_{t+1}^{guar-} can be decreasing in t .

2.2.2. *Transition at a Policy Anniversary t*

At the policy anniversaries, we have to distinguish the following four cases:

a) The insured has died within the previous year ($t - 1, t$]

If the insured has died within the previous policy year, the account value is paid out as death benefit. With the payment of the death benefit, the insurance contract matures. Thus, $AV_t^+ = 0$, $WBB_t^+ = 0$, $W_t^+ = 0$, and $W_t^{guar+} = 0$.

b) The insured has survived the previous policy year and does not withdraw any money from the account at time t

If no death benefit is paid out to the policyholder and no withdrawals are made from the contract, i.e. $W_t^+ = 0$, we get $AV_t^+ = AV_t^-$, $WBB_t^+ = WBB_t^-$,

and $W_t^{guar+} = W_t^{guar-}$. In the Performance Bonus product, the guaranteed annual withdrawal amount is reset to its original level since W_t^{guar-} might have contained performance bonus payments. Thus, for this alternative we have $W_t^{guar+} = x_{WL} \cdot P$.

c) The insured has survived the previous policy year and at the policy anniversary withdraws an amount within the limits of the withdrawal guarantee

If the insured has survived the past year, no death benefits are paid. Any withdrawal W_t below the guaranteed annual withdrawal amount W_t^{guar-} reduces the account value by the withdrawn amount. Of course, we do not allow for negative policyholder account values and thus get $AV_t^+ = \max\{0, AV_t^- - W_t\}$.

For the alternatives “No Ratchet” and “Lookback Ratchet”, the withdrawal benefit base and the guaranteed annual withdrawal amount remain unchanged, i.e. $WBB_t^+ = WBB_t^-$, and $W_t^{guar+} = W_t^{guar-}$. For the alternative “Remaining WBB Ratchet”, the withdrawal benefit base is reduced by the withdrawal taken, i.e. $WBB_t^+ = \max\{0, WBB_t^- - W_t\}$ and the guaranteed annual withdrawal amount remains unchanged, i.e. $W_t^{guar+} = W_t^{guar-}$. For the “Performance Bonus”, the withdrawal benefit base is at a maximum reduced by the initially guaranteed withdrawal amount (without performance bonus), i.e. $WBB_t^+ = \max\{0, WBB_t^- - \min\{W_t, x_{WL} \cdot P\}\}$ and the guaranteed annual withdrawal amount is set back to its original level, i.e. $W_t^{guar+} = x_{WL} \cdot P$.

d) The insured has survived the previous policy year and at the policy anniversary withdraws an amount exceeding the limits of the withdrawal guarantee

In this case again, no death benefits are paid. For the sake of brevity, we only give the formulae for the case of full surrender, since partial surrender is not analyzed in what follows⁴. In case of full surrender, the complete account value is withdrawn, we then set $AV_t^+ = 0$, $WBB_t^+ = 0$, $W_t^+ = AV_t^-$, and $W_t^{guar+} = 0$ and the contract terminates.

2.3. Contract Valuation

The valuation framework in this section follows in some parts the one used in Bacinello et al. (2009) and in others Bauer et al. (2008). We take as given a filtered probability space (Ω, Σ, F, P) , in which P is the real-world (or physical) probability measure and $F \doteq (F_t)_{t \geq 0}$ is a filtration with $F_0 = \{\emptyset, \Omega\}$ and $F_t \subset \Sigma \forall t \geq 0$. We assume that trading takes place continuously over time and without any transaction costs or spreads. Furthermore, we assume that the price processes of the traded assets in the market are adapted and of bounded variation. Assuming the absence of arbitrage opportunities in the financial market, there exists a probability measure Q that is equivalent to P and under which the gain from holding a traded asset is a Q -martingale after discounting

⁴ For details on partial surrender, we refer the reader to Bauer et al. (2008).

with the chosen numéraire process, in our case the money-market account. Q is called equivalent martingale measure (EMM). Details on the derivation and existence of an EMM, subject to the used financial market model, are given in section 3.2.1. Assuming independence between financial markets, policyholder behavior and mortality, as well as risk-neutrality of the insurer with respect to mortality and behavioral risk, we are able to use the product measure of Q and the measures for mortality and policyholder behavior. In what follows, we denote this product measure by \widehat{Q} and use the enlargement $(\widehat{\Omega}, \widehat{\Sigma}, \widehat{F}, \widehat{P})$ of the filtered probability space (Ω, Σ, F, P) , with $\widehat{F} \doteq (\widehat{F}_t)_{t \geq 0}$ being the enlargement of the filtration F in order to include the filtrations associated to the mortality and the behavior processes.

As already mentioned, for the contracts considered within our analysis, policyholder actions are limited to withdrawals and (partial) surrender. In our numerical analyses in sections 4 and 5, we do not consider partial surrender. To keep notation simple, we only give formulae for the considered cases (cf. Bauer et al. (2008) for formulae for the other cases). Additionally, we only consider annual policy calculation dates and assume that surrender is only possible at these annual dates.

We denote by x_0 the insured’s age at the start of the contract, ${}_t p_{x_0}$ the probability under \widehat{Q} for a x_0 -year old to survive the next t years, q_{x_0+t} the probability under \widehat{Q} for a $(x_0 + t)$ -year old to die within the next year, and let ω be the limiting age of the mortality table, i.e. the age beyond which survival is impossible. The probability under \widehat{Q} that an insured aged x_0 at inception passes away in the year $(t, t + 1]$ is thus given by ${}_t p_{x_0} \cdot q_{x_0+t}$. The limiting age ω allows for a finite time horizon $T = \omega - x_0 + 1$. We denote by $\tau_D \in \{1, 2, \dots, T\}$ the policy anniversary date following the death of the insured.

Further, we denote by $\tau_S \in \{1, 2, \dots, T\}$ the point in time at which the policyholder surrenders the contract. Since it is only possible to surrender the contract while the insured is still alive, we will interpret any value $\tau_S \geq \tau_D$ as if the policyholder does not surrender during the contract’s lifetime; the same applies for $\tau_S = T$. For any given value of τ_S and τ_D , all contractual cash flows and thus all guarantee payments (i.e. payments made by the insurer after the account value has dropped to zero) at times $i \in \{1, 2, \dots, T\}$, denoted by $G_i^P(\tau_D, \tau_S)$, and all guarantee fee payments $G_i^F(\tau_D, \tau_S)$, again at times $i \in \{1, 2, \dots, T\}$, are specified for each capital market scenario. For given τ_S and τ_D , the time- t value $V_t^G(\tau_D, \tau_S)$ of the GLWB rider (from the policyholder’s perspective) is given by the expected present value of all future guarantee payments $G_i^P(\tau_D, \tau_S)$, $i \in \{1, 2, \dots, T\}$, minus future guarantee fees $G_i^F(\tau_D, \tau_S)$, $i \in \{1, 2, \dots, T\}$:

$$V_t^G(\tau_D, \tau_S) = E_{\widehat{Q}} \left[\sum_{i=t+1}^{\tau_D} e^{-r(i-t)} (G_i^P(\tau_D, \tau_S) - G_i^F(\tau_D, \tau_S)) \middle| \widehat{F}_t \right]. \tag{2}$$

Thus, the time- t value of the option for a given time of surrender τ_S , using the mortality probabilities as defined above, is given by

$$V_t^G(\tau_S) = \sum_{u=t+1}^T q_{x_0+u-1} \cdot p_{x_0+t} \cdot V_t^G(u, \tau_S). \tag{3}$$

Finally, we consider the case that the policyholder surrenders the contract at each anniversary t with a certain (state-independent) probability p_t^S , conditional to the insured being alive, the contract being still in force and an account value exceeding zero. The time- t probability that the policyholder surrenders at time $\tau_S > t$ is therefore given by $(\prod_{i=t+1}^{\tau_S-1} (1 - p_i^S)) p_{\tau_S}^S$. We denote this probability by \hat{p}_{t,τ_S}^S . Then, the time- t value of the GLWB rider is given by

$$V_t^G = \sum_{s=t+1}^T \hat{p}_{t,s}^S \cdot V_t^G(s). \tag{4}$$

Note that this approach assumes policyholders to behave completely path-independently, i.e. after inception of the contract, policyholders are assumed to disregard the actual option value of the GLWB rider and any market parameters when deciding whether or not to surrender their contract. Risks arising from rational policyholder behavior or, in general, any path-dependent behavior are not considered within this modeling approach and are not part of the analyses undertaken in this paper.

3. NUMERICAL ANALYSIS FRAMEWORK

3.1. Models of the Financial Market

In all of the following, we take as given the same filtered probability space and assumptions that were introduced in section 2.3. For our analyses we assume two primary tradable assets: the underlying fund (or basket of funds⁵), whose spot price we will denote by $S(\cdot)$, and the money-market account, denoted by $B(\cdot)$. The focus of our analyses lies on the risk arising if the stochasticity of equity volatility is ignored. In order to separate the volatility-related effects from other influences, like e.g. stochastic interest rates, we limit the considered market models and only allow for interest rates that are deterministic and constant. Furthermore, we assume the interest spread to be zero and the money-market account to evolve at a constant risk-free rate of interest r :

$$\begin{aligned} dB(t) &= rB(t)dt \\ \Rightarrow B(t) &= B(0)\exp(rt) \end{aligned} \tag{5}$$

⁵ In this case, the modeled volatility (or volatility process) of the underlying can be interpreted as the resulting volatility of the whole basket, which may not only vary if the volatility of a specific asset changes, but also if the basket's composition is changed.

For the dynamics of $S(\cdot)$, we will use two different models: first we will assume the equity volatility to be deterministic and constant over time, and hence use the Black-Scholes model for our simulations. To allow for a more realistic equity volatility model, we will also use the Heston model, in which both, the underlying and its (instantaneous) variance, are modeled by stochastic processes. There are many approaches to volatility modeling, including approaches that use the assumption of volatility to be uncertain – instead of being either deterministic or stochastic as in the two models that we consider. For an overview of volatility modeling, we refer the reader to Wilmott (2006).

3.1.1. Black-Scholes Model

In the Black-Scholes (1973) model, the underlying's spot price $S(\cdot)$ follows a geometric Brownian motion whose dynamics under the real-world measure P are given by the following stochastic differential equation (SDE)

$$dS(t) = \mu S(t)dt + \sigma_{BS} S(t) dW(t), \quad S(0) \geq 0, \quad (6)$$

where μ is the (constant) drift of the underlying, σ_{BS} its constant volatility and $W(\cdot)$ denotes a P -Brownian motion. By Itô's lemma, $S(\cdot)$ has the solution (cf. eg. Bingham and Kiesel (2004))

$$S(t) = S(0) \exp\left(\left(\mu - \frac{\sigma_{BS}^2}{2}\right)t + \sigma_{BS} W(t)\right), \quad S(0) \geq 0. \quad (7)$$

3.1.2. Heston Model

There are various extensions to the Black-Scholes model that allow for a more realistic modeling of the underlying's volatility. We use the Heston (1993) model in our analyses where the instantaneous (or local) volatility of the asset is stochastic. Under the Heston model, the market is assumed to be driven by two stochastic processes: the underlying's price $S(\cdot)$, and its instantaneous variance $V(\cdot)$, which is assumed to follow a one-factor square-root process identical to the one used in the Cox-Ingersoll-Ross (1985) interest rate model. The dynamics of the two processes under the real-world measure P are given by the following system of stochastic differential equations:

$$dS(t) = \mu S(t)dt + \sqrt{V(t)} S(t) (\rho dW_1(t) + \sqrt{1 - \rho^2} dW_2(t)), \quad S(0) \geq 0 \quad (8)$$

$$dV(t) = \kappa(\theta - V(t))dt + \sigma_v \sqrt{V(t)} dW_1(t), \quad V(0) \geq 0, \quad (9)$$

where μ again is the drift of the underlying, $V(t)$ is the local variance at time t , κ is the speed of mean reversion, θ is the long-term average variance, σ_v is the so-called "vol of vol", or (more precisely) the volatility of the variance process,

ρ denotes the correlation between the underlying and the variance process, and $W_{1/2}$ are P -Wiener processes. The condition $2\kappa\theta \geq \sigma_v^2$ ensures that the variance process will remain strictly positive almost surely (cf. Cox et al. (1985)).

There is no analytical solution for $S(\cdot)$ available, thus numerical methods must be used for simulation. For our analyses, we use the quadratic exponential discretization scheme proposed by Andersen (2007).

3.2. Valuation

3.2.1. Equivalent Martingale Measure

In this section, we explain how the equivalent martingale measure Q that was introduced in section 2.3 is derived. Within our model, in order to determine the fair values of the assets used in the hedging strategies and of the guarantees to be hedged, we have to transform the real-world measure P into an equivalent (local) martingale measure Q , i.e. into a measure under which the process of the discounted spot price of the underlying is a (local) martingale. While – under the usual assumptions – the transformation to such a measure is unique under the Black-Scholes model (cf. e.g. Bingham and Kiesel (2004)), it is not under the Heston model. In the Heston model, since there are two sources of risk, there are also two market-price-of-risk processes, denoted by γ_1 and γ_2 (corresponding to W_1 and W_2). Heston (1993) proposed the following restriction on the market-price-of-volatility-risk process, assuming it to be linear in volatility,

$$\gamma_1(t) = \lambda\sqrt{V(t)}. \tag{10}$$

Provided both measures, P and Q , exist, the Q -dynamics of $S(t)$ and $V(t)$, again under the assumption that no dividends are paid, are then given by

$$dS(t) = rS(t)dt + \sqrt{V(t)}S(t)(\rho dW_1^Q(t) + \sqrt{1 - \rho^2} dW_2^Q(t)), \quad S(0) \geq 0 \tag{11}$$

$$dV(t) = \kappa^*(\theta^* - V(t))dt + \sigma_v\sqrt{V(t)}dW_1^Q(t), \quad V(0) \geq 0 \tag{12}$$

where $W_1^Q(\cdot)$ and $W_2^Q(\cdot)$ are two independent Q -Wiener processes and where

$$\kappa^* = (\kappa + \lambda\sigma_v), \quad \theta^* = \frac{\kappa\theta}{(\kappa + \lambda\sigma_v)} \tag{13}$$

are the risk-neutral counterparts to κ and θ (cf., for instance, Wong and Heyde (2006)).

Wong and Heyde (2006) also show that the equivalent local martingale measure that corresponds to the market price of volatility risk, $\lambda\sqrt{V(t)}$, exists if the inequality $-\kappa/\sigma_v \leq \lambda < \infty$ is fulfilled. They further show that, if an

equivalent local martingale measure Q exists and $\kappa + \lambda\sigma_v \geq \sigma_v\rho$, the discounted stock price $\frac{S(t)}{B(t)}$ is a Q -martingale.

3.2.2. Valuation of the GLWB Rider

For both equity models, we use Monte-Carlo simulations to compute the value of the GLWB rider V_t^G defined in Section 2.3, i.e. the difference between expected future guarantee payments made by the insurer and expected future guarantee fees deducted from the policyholders' fund assets. At inception, we call the contract "fair" (in an actuarial sense), if $V_0^G = 0$.

3.2.3. Standard Option Valuation

In some of the hedging strategies considered in Section 5, European standard (or "plain vanilla") put and call options are used. Under the Black-Scholes model, closed form solutions for the price of European call and put options exist (cf. Black (1976)). For the Heston stochastic volatility model, Heston (1993) found a semi-analytical solution for pricing European call and put options using Fourier inversion techniques. In our analyses, we use the numerical scheme proposed by Kahl and Jäckel (2006).

3.3. Computation of Sensitivities (Greeks)

Where no analytical solutions for the sensitivity of the options' or guarantees' values to changes in model parameters (the so-called Greeks, cf. e.g. Hull (2008)) exist, we use Monte-Carlo methods to compute the respective sensitivities numerically. We use finite differences (cf. Glasserman (2003)) as approximations of the partial derivatives, where the direction of the shift is chosen accordingly to the direction of the risk, i.e. for delta we shift the stock downwards in order to compute the backward finite difference, and shift the volatility upwards for vega, this time to compute a forward finite difference.

4. CONTRACT ANALYSIS

4.1. Determination of the Fair Guaranteed Withdrawal Rate

In this section, we first calculate the guaranteed withdrawal rate x_{WL} that makes a contract fair at inception, all other parameters given. In order to calculate x_{WL} , we perform a root search with x_{WL} as argument and the value of the option as function value. For all of the analyses we use the fee structure given in Table 1.

We further assume the policy holder to be a 65 years old male. For pricing purposes, we use best-estimate mortality probabilities given in the DAV 2004R table published by the German Actuarial Society (DAV).

TABLE 1

ASSUMED FEE STRUCTURE FOR ALL REGARDED CONTRACTS.

Acquisition charges	4.00% of lump sum
Management charges	1.50% p.a. of NAV
Guarantee charges	1.50% p.a. of NAV

4.1.1. Results for the Black-Scholes model

Table 3 displays the fair guaranteed withdrawal rates for different ratchet mechanisms, different volatilities, different interest rate levels, and different policyholder behavior assumptions: We assume that – as long as their contracts are still in force – policyholders withdraw each year exactly the guaranteed withdrawal amount⁶. Further, we look at the scenarios no surrender (*no surr*), surrender according to Table 2 (*surr 1*) and surrender with twice the probabilities as given in Table 2 (*surr 2*).

TABLE 2

ASSUMED DETERMINISTIC SURRENDER RATES.

Year	Surrender rate p_t^S
1	6%
2	5%
3	4%
4	3%
5	2%
≥ 6	1%

A comparison of the different product designs shows that, obviously, the highest annual guarantee can be provided if no ratchet or performance bonus is provided at all. If no surrender and a volatility of 20% is assumed, the guarantee is almost 5%. Including a Lookback Ratchet would require a reduction of the initial annual guarantee by 66 basis points to 4.32%. If a richer ratchet mechanism is provided such as the Remaining WBB Ratchet, the guarantee needs to be reduced to 4.01%. About the same annual guarantee (4.00%) can be provided if no ratchet is provided but a performance bonus is paid out annually.

⁶ Especially for younger insured, the right to not withdraw money in order to potentially increase future withdrawal guarantees can be of value for the policyholder. Since the main focus of this paper is on the analysis of the risk resulting from stochastic volatility, we ignore this effect.

TABLE 3

FAIR GUARANTEED WITHDRAWAL RATES FOR DIFFERENT RATCHET MECHANISMS, DIFFERENT POLICYHOLDER BEHAVIOR ASSUMPTIONS, DIFFERENT VOLATILITIES, AND DIFFERENT INTEREST RATE LEVELS.

Volatility		Ratchet mechanism	I (No Ratchet)	II (Lookback Ratchet)	III (Remaining WBB Ratchet)	IV (Performance Bonus)
Surrender						
$\sigma_{BS} = 15\%$ $r = 4\%$	<i>No surr</i>		5.26%	4.80%	4.43%	4.37%
	<i>Surr 1</i>		5.45%	5.00%	4.62%	4.57%
	<i>Surr 2</i>		5.66%	5.22%	4.83%	4.79%
$\sigma_{BS} = 20\%$ $r = 4\%$	<i>No surr</i>		4.98%	4.32%	4.01%	4.00%
	<i>Surr 1</i>		5.16%	4.50%	4.18%	4.19%
	<i>Surr 2</i>		5.35%	4.71%	4.38%	4.40%
$\sigma_{BS} = 22\%$ $r = 4\%$	<i>No surr</i>		4.87%	4.13%	3.85%	3.85%
	<i>Surr 1</i>		5.04%	4.30%	4.01%	4.03%
	<i>Surr 2</i>		5.23%	4.50%	4.20%	4.24%
$\sigma_{BS} = 25\%$ $r = 4\%$	<i>No surr</i>		4.70%	3.85%	3.61%	3.62%
	<i>Surr 1</i>		4.86%	4.01%	3.76%	3.81%
	<i>Surr 2</i>		5.04%	4.20%	3.94%	4.01%
$\sigma_{BS} = 22\%$ $r = 3\%$	<i>No surr</i>		4.51%	3.88%	3.66%	3.67%
	<i>Surr 1</i>		4.68%	4.06%	3.83%	3.86%
	<i>Surr 2</i>		4.88%	4.26%	4.02%	4.07%
$\sigma_{BS} = 22\%$ $r = 5\%$	<i>No surr</i>		5.29%	4.41%	4.06%	4.04%
	<i>Surr 1</i>		5.45%	4.59%	4.22%	4.22%
	<i>Surr 2</i>		5.63%	4.78%	4.41%	4.44%

Throughout our analyses, the Remaining WBB Ratchet and the Performance Bonus designs allow for about the same initial annual guarantee. However, for lower volatilities, the Remaining WBB Ratchet seems to be less valuable than the Performance Bonus and therefore allows for higher guarantees while for higher volatilities the Performance Bonus allows for higher guarantees. Thus, the relative impact of volatility on the price of a GLWB depends on the chosen product design and appears to be particularly high for ratchet type products (II and III). This can also be observed comparing the No Ratchet case with the Lookback Ratchet. Whereas – if volatility is increased from 15% to 25% – for the No Ratchet case, the fair guaranteed withdrawal decreases by just over half a percentage point from 5.26% to 4.7%, it decreases by almost a full percentage point from 4.8% to 3.85% in the Lookback Ratchet case (if no surrender is assumed). The reason for this is that for the products with ratchet, high volatility leads to a possible lock-in of high positive returns in some years and thus is a rather valuable feature if volatilities are high.

As expected, the fair guaranteed withdrawal rate is decreasing with decreasing interest rates since with lower interest rates, the value of the corresponding guarantee is increasing. For a volatility of 22% and a scenario where no surrender is assumed, e.g., the fair withdrawal rate in the No Ratchet case decreases from 4.87% to 4.51% (increases from 4.87% to 5.29%) if the interest rate is reduced to 3% (increased to 5%). Thus, there is quite a significant impact of a change in interest rates on the fair withdrawal rate.

If the insurance company assumes deterministic surrender probabilities, the guaranteed rates always increase. The increase of the annual guarantee is rather similar over all product types and volatilities. The annual guarantee increases by around 15-20 basis points if the surrender assumption from Table 2 is used and increases by about another 20 basis points if this surrender assumption is doubled.

4.12.2. Results for the Heston model

We use the model parameters given in Table 4, where the Heston parameters are those derived by Eraker (2004), and stated in annualized form for instance by Ewald et al. (2009).

TABLE 4
BENCHMARK PARAMETERS FOR THE HESTON MODEL.

Parameter	Numerical value
θ	0.220 ²
κ	4.75
σ_v	0.55
ρ	-0.569
$V(0)$	θ

TABLE 5
Q-PARAMETERS FOR DIFFERENT CHOICES OF THE MARKET PRICE OF VOLATILITY RISK FACTOR.

Market price of volatility risk	Speed of mean reversion κ^*	Long-run local variance θ^*
$\lambda = 3$	6.40	0.190 ²
$\lambda = 2$	5.85	0.198 ²
$\lambda = 1$	5.30	0.208 ²
$\lambda = 0$	4.75	0.220 ²
$\lambda = -1$	4.20	0.234 ²
$\lambda = -2$	3.65	0.251 ²
$\lambda = -3$	3.10	0.272 ²

One of the key parameters in the Heston model is the market price of volatility risk λ . Since absolute λ -values are hard to interpret, in Table 5 we show the values of the long-run local variance and the speed of mean reversion for different values of λ .

Higher values of λ correspond to a lower volatility and a higher mean-reversion speed, while lower (and negative) values of λ correspond to high volatilities and lower speed of mean reversion. $\lambda = 2$ implies a long-term volatility of 19.8% and $\lambda = -2$ implies a long-term volatility of 25.1%.

In the following table, we show the fair annual guaranteed withdrawal rate under the Heston model for all different product designs, different interest rate levels, the same assumptions regarding policyholder behavior as for the Black-Scholes model, and values of λ between -2 and 2 .

TABLE 6

FAIR GUARANTEED WITHDRAWAL RATES UNDER THE HESTON MODEL FOR DIFFERENT RATCHET MECHANISMS, DIFFERENT ASSUMPTIONS REGARDING POLICYHOLDER BEHAVIOR, DIFFERENT VALUES OF THE MARKET PRICE OF VOLATILITY RISK PARAMETER λ , AND DIFFERENT INTEREST RATE LEVELS.

Market price of volatility risk	Ratchet mechanism	I (No Ratchet)	II (Lookback Ratchet)	III (Remaining WBB Ratchet)	IV (Performance Bonus)
	Surrender				
$\lambda = 2$ $r = 4\%$	<i>No surr</i>	4.99%	4.36%	4.03%	4.00%
	<i>Surr 1</i>	5.18%	4.56%	4.21%	4.22%
	<i>Surr 2</i>	5.38%	4.76%	4.40%	4.43%
$\lambda = 1$ $r = 4\%$	<i>No surr</i>	4.93%	4.27%	3.95%	3.93%
	<i>Surr 1</i>	5.12%	4.46%	4.13%	4.14%
	<i>Surr 2</i>	5.31%	4.66%	4.32%	4.35%
$\lambda = 0$ $r = 4\%$	<i>No surr</i>	4.87%	4.17%	3.86%	3.84%
	<i>Surr 1</i>	5.05%	4.35%	4.03%	4.06%
	<i>Surr 2</i>	5.24%	4.55%	4.22%	4.27%
$\lambda = -1$ $r = 4\%$	<i>No surr</i>	4.79%	4.05%	3.75%	3.74%
	<i>Surr 1</i>	4.97%	4.23%	3.92%	3.95%
	<i>Surr 2</i>	5.16%	4.42%	4.10%	4.16%
$\lambda = -2$ $r = 4\%$	<i>No surr</i>	4.70%	3.90%	3.62%	3.62%
	<i>Surr 1</i>	4.87%	4.08%	3.79%	3.82%
	<i>Surr 2</i>	5.05%	4.26%	3.97%	4.04%
$\lambda = 0$ $r = 3\%$	<i>No surr</i>	4.52%	3.93%	3.68%	3.68%
	<i>Surr 1</i>	4.69%	4.10%	3.85%	3.86%
	<i>Surr 2</i>	4.89%	4.30%	4.03%	4.07%
$\lambda = 0$ $r = 5\%$	<i>No surr</i>	5.31%	4.46%	4.06%	4.02%
	<i>Surr 1</i>	5.46%	4.63%	4.23%	4.20%
	<i>Surr 2</i>	5.64%	4.82%	4.41%	4.41%

Under the Heston model, the fair annual guaranteed withdrawal is very similar as under the Black-Scholes model with a comparable constant volatility. E.g. for $\lambda = 0$, which corresponds to a long-term volatility of 22%, the fair annual guaranteed withdrawal rate for a contract without ratchet is given by 4.87%, exactly the same number as under the Black-Scholes model. In the Lookback Ratchet case, the Heston model leads to a fair guaranteed withdrawal rate of 4.17%, the Black-Scholes model of 4.13%. For the other two product designs, again, both asset models almost exactly lead to the same withdrawal rates. This also holds for different interest rate levels. Hence, interest rate sensitivities under the Heston model lead to the same effects as observed under the Black-Scholes model.

Thus, for the pricing (as opposed to hedging, see Section 5) of the GLWB riders considered in our numerical analyses, the long-term volatility assumption is much more crucial than the question whether volatility should be modeled stochastic or deterministic.

4.2. Distribution of Withdrawals

In this subsection, we compare the (real-world) distributions of the guaranteed withdrawal benefits (given the policyholder is still alive, has not surrendered the contract yet and has always withdrawn exactly the guaranteed amount) for each policy year and for all four different ratchet mechanisms that were presented in Section 2.

For the analyses in this subsection and the following subsections 4.3 and 4.4, we use the Black-Scholes model assuming a constant risk-free rate of interest $r = 4\%$, an underlying's drift $\mu = 7\%$ and a constant equity volatility of $\sigma_{BS} = 22\%$. For all four ratchet types, we use the guaranteed withdrawal rates derived in Section 4.1.1 (assuming no surrender). In Figure 1, for each product design we show the development of arithmetic average (diamonds), median (dashes), 10th-90th percentile (outlined area), and 25th-75th percentile (solid black area) of the guaranteed annual withdrawal amount over time. If two percentiles coincide, this means that the corresponding probability mass is concentrated on one point. In the case of no ratchet, the guaranteed withdrawal amount is deterministic and thus all percentiles coincide.

Obviously, the different considered product designs lead to significantly different risk/return-profiles for the policyholder. While the No Ratchet case provides deterministic cash flows over time, the payoffs under the other product designs are uncertain with distributions that differ quite considerably. Both ratchet products have potentially increasing benefits. For the Lookback Ratchet, however, the 25th percentile (lower end of the solid black area; in this case coinciding with the 10th percentile) remains constant at the level of the first withdrawal amount. Thus, the probability that a ratchet never happens exceeds 25%. The median increases for the first 10 years and then reaches some constant level implying that with a probability of more than 50% no withdrawal increments will take place thereafter.

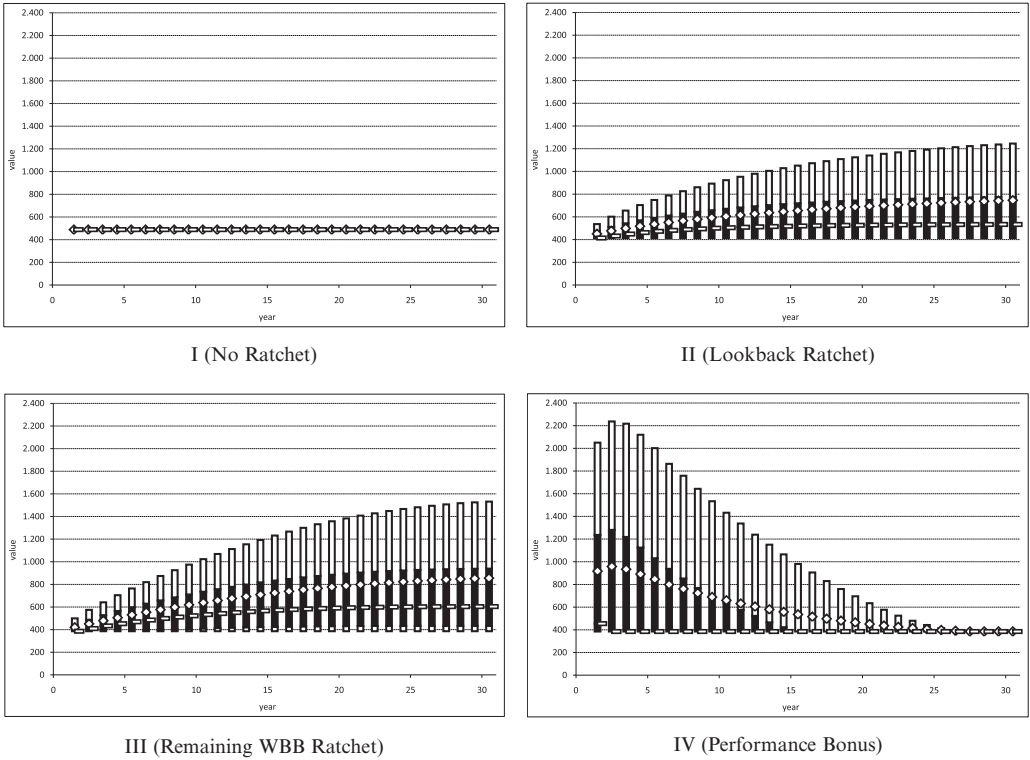


FIGURE 1: Development of percentiles, median and mean of the guaranteed withdrawal amount over policy years 0 to 30 for each ratchet type.

Product III (Remaining WBB Ratchet) provides more potential for increasing withdrawals: For this product, the 25th percentile increases over the first few years and the median is increasing for around 20 years. In the 90th percentile, the guaranteed annual withdrawal amount reaches 1,500 after slightly more than 25 years while the Lookback Ratchet hardly reaches 1,200. On average, the annual guaranteed withdrawal amount more than doubles over time while the Lookback Ratchet doesn't; of course this is only possible since the guaranteed withdrawal at $t = 0$ is lower.

A completely different profile is achieved by the fourth product design, the product with Performance Bonus. Here, annual withdrawal amounts are rather high in the first years and are decreasing later. After 15 years, with a 75% probability no more performance bonus is paid, after 25 years, with a probability of 90% no more performance bonus is paid.

For all three product designs with some kind of bonus, the probability distribution of the annual withdrawal amount is rather skewed: the arithmetic average is significantly above the median. For the product with Performance Bonus, the median exceeds the guarantee only in the first year. Thus, the probability

of receiving a performance bonus in later years is less than 50%. The expected value, however, is more than twice as high.

4.3. Distribution of Trigger Times

In Figure 2, for each of the products, we show the probability distribution of trigger times, i.e. the distribution of the point in time when the account value drops to zero and the guarantee is triggered, given the insured is still alive at this point in time. Any probability mass at $t = 57$ (the limiting age of the mortality table used is 121), refers to scenarios where the guarantee is not triggered.

For the No Ratchet product, trigger times vary from 7 to over 55 years. With a probability of 17%, there is still some account value available at the end of the simulation period, when the limiting age is reached. For this product, on the one hand, the insurance company’s uncertainty with respect to if and when guarantee payments have to be paid is very high; on the other hand, there is a significant chance that the guarantee is not triggered at all.

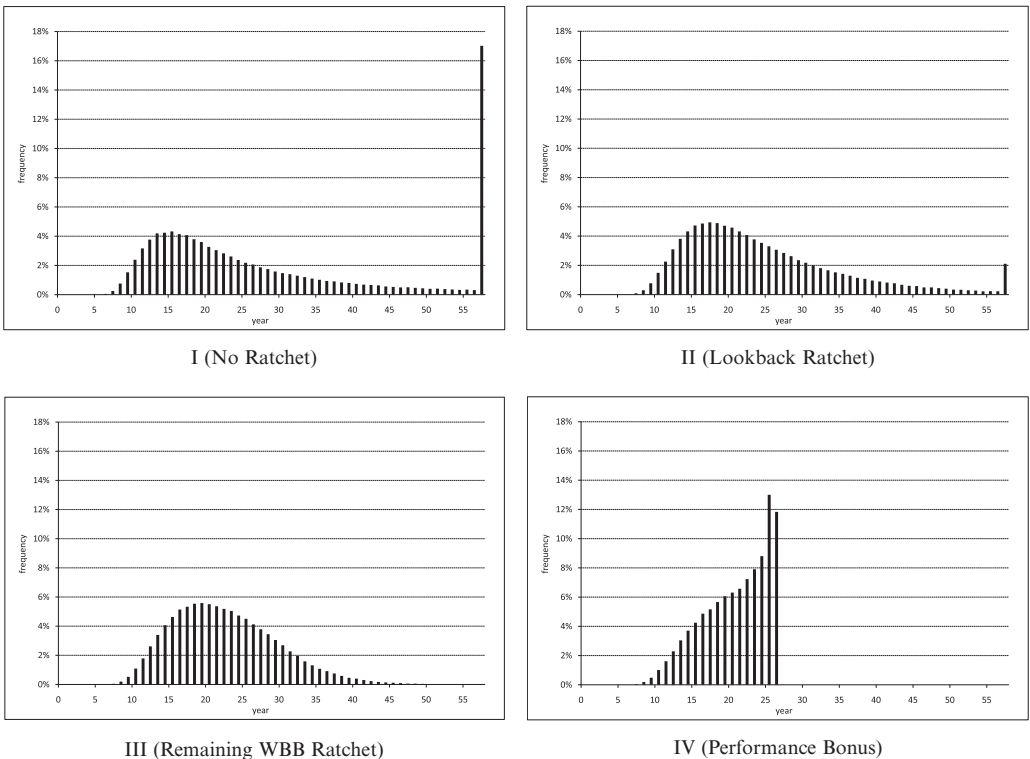


FIGURE 2: Distribution of trigger times for each of the product designs.

For the products with ratchet features, very late or even no triggering of the guarantee appears to be less likely. The more upside potential a ratchet mechanism provides for the client, the earlier the guarantee tends to be triggered. While for the Lookback Ratchet still 2% of the contracts do not trigger at all, the Remaining WBB Ratchet almost certainly triggers within the first 40 years. However, the mode of the trigger time is around 20 years, which is rather late.

The least uncertainty in the trigger time appears to be in the product with Performance Bonus. While the probability distribution looks very similar to that of the Remaining WBB Ratchet for the first 15 years, trigger probabilities then increase rapidly and reach a maximum at $t = 25$ and 26 years. Later triggers did not occur at all within our simulation. The reason for this is quite obvious: The Performance Bonus is given by 50% of the difference between the current account value and the remaining withdrawal benefit base. However, this benefit base is annually reduced by the initially guaranteed withdrawal amount and therefore reaches 0 after 26 years (1 / 3.85%). Thus, after around 20 years, almost half of the account value is paid out as bonus every year. This, of course, leads to a tremendously decreasing account value in later years. Therefore, there is less uncertainty with respect to the trigger time on the insurance company's side.

Whenever the guarantee is triggered, the insurance company must pay an annual lifelong annuity equal to the guaranteed annual withdrawal amount. This is the guarantee that needs to be hedged by the insurer. Thus, in the following section, we have a closer look on the so-called "Greeks" of the guarantees of the different product designs.

4.4. Greeks of the GLWB Rider

Within our Monte-Carlo simulation, we can calculate different sensitivities of the option value as defined in Section 2.3 with respect to changes in model parameters. These so-called Greeks (cf. Hull (2008)) of the option value are calculated for a pool of identical policies with a total single premium volume of US\$ 100m under certain assumptions of future mortality and future surrender. All the results shown in this section are calculated under the same mortality assumptions as in Section 4.1 and the assumption that the policyholders do not surrender.

In Figure 3, we show different percentiles (10th, 25th, 75th and 90th, solid lines) as well as the arithmetic average (dotted line) and the median (dashed line) of the so-called delta, i.e. the sensitivity of the option value with respect to changes in the price of the underlying.

Similarly, Figure 4 and Figure 5 show the same percentiles as well as the arithmetic average of the so-called rho, i.e. the sensitivity of the option value with respect to changes in the interest rate, and vega, i.e. the sensitivity of the option value with respect to changes in the volatility parameter.

For the sake of comparability, at each t in each simulation path, we multiplied the delta with the then-current spot price of the underlying.

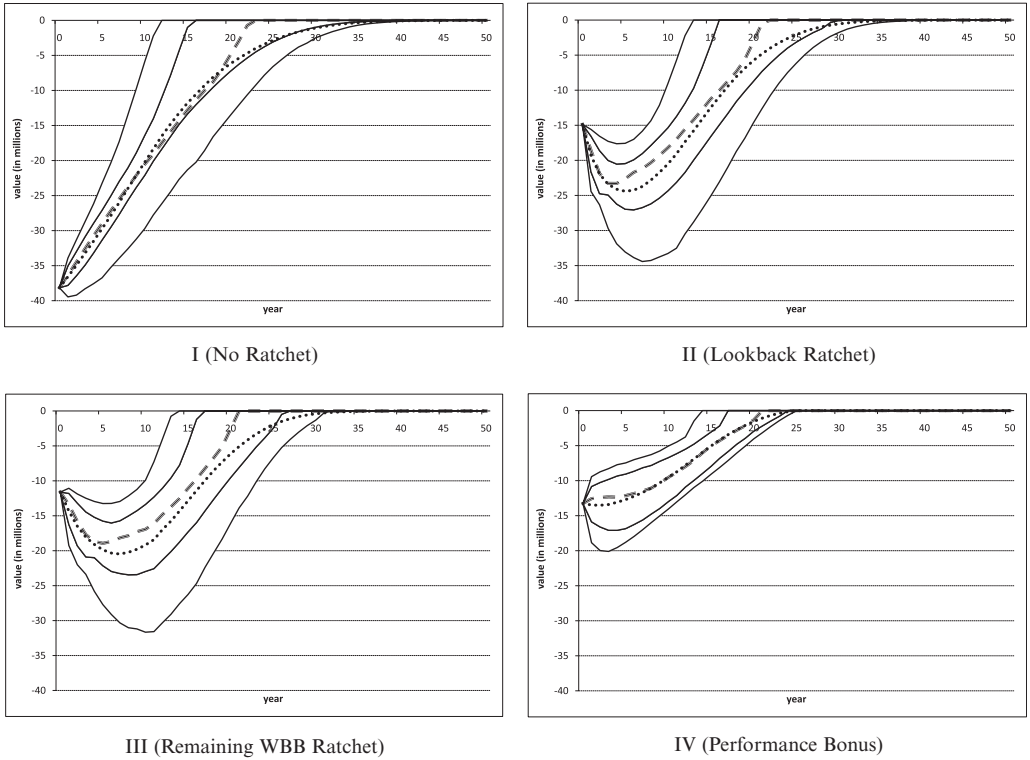


FIGURE 3: Development over time of the percentiles of the GLWB rider's delta for a pool of policies multiplied by the current spot price of the underlying.

Once the guarantee is triggered, no more account value is available, thus the GLWB rider is independent of the underlying and therefore, from this point on, its delta and vega are zero.

It is evident that (until the guarantee is triggered) all products have negative deltas, negative rhos and positive vegas at any point in time. The negative deltas (and rhos) result from the fact that the value of the guarantee increases with falling stock markets (or interest rates, resp.) and vice versa. At the same time, the value of the guarantee increases with increasing volatility leading to a positive vega.

In what follows, we call the greeks “large” whenever their absolute value is large.

At outset, the product without any ratchet or bonus has the largest delta and thus the highest sensitivity with respect to changes in the underlying's price. The reason for this is mainly the fact that the guarantee is not adjusted when fund prices rise. In this case, the value of the guarantee decreases much stronger than with any product where either a ratchet leads to an increasing guarantee or a performance bonus leads to a reduction of the account value.

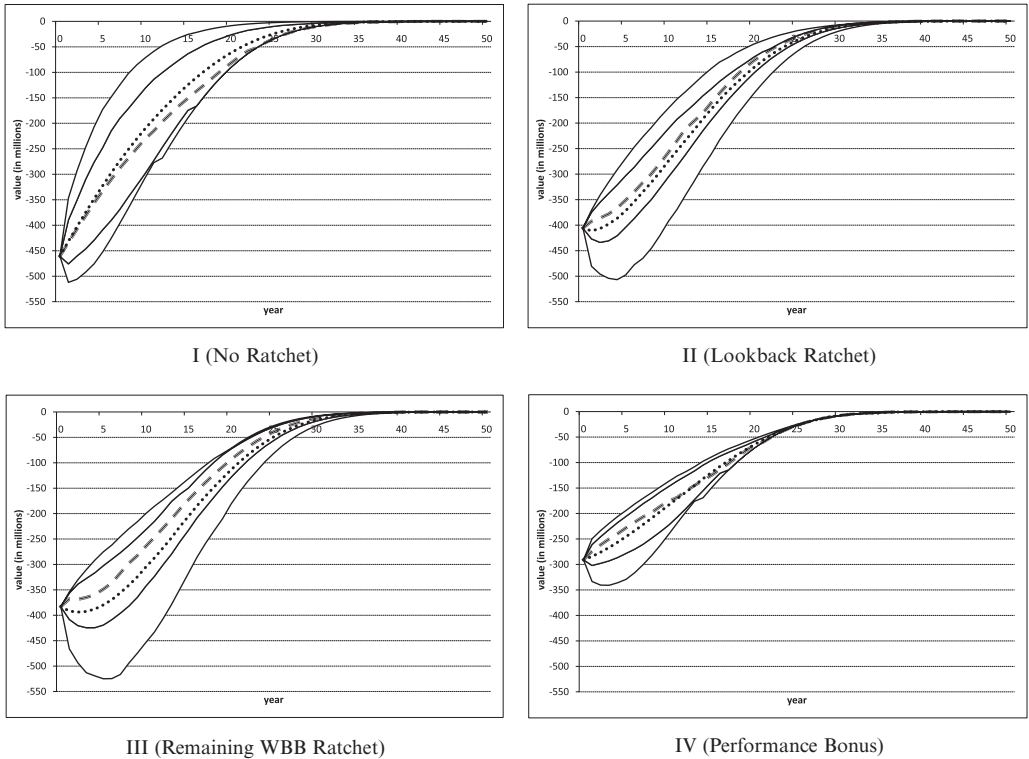


FIGURE 4: Development over time of the percentiles of the GLWB rider's rho for a pool of policies.

On the other hand, if fund prices decrease, the first product is deeper in the money since it does have the highest initial guaranteed withdrawal amount. Over time, all percentiles of the delta in the No Ratchet case are decreasing.

For product designs II and III, the guarantee can never be far out of the money due to the ratchet feature. Thus delta increases in the first few years. All percentiles reach a maximum after ten years and tend to be decreasing from then on.

For the product with Performance Bonus, delta and rho exposure is by far the lowest. This is in line with our results of the previous subsection, where we concluded that the uncertainty for the insurance company is the lowest in the Performance Bonus case.

Observations of the rho risk for all product types are quite similar to those of the delta risk. The No Ratchet product shows the highest rho risk at outset of the contract. Over time, the level of interest rate risk (with the exception of a few tail scenarios in the first few years) is decreasing steadily over time. Comparing the different product designs, the product without ratchet and the two products with ratchet mechanism show a rather similar level of interest rate risk.

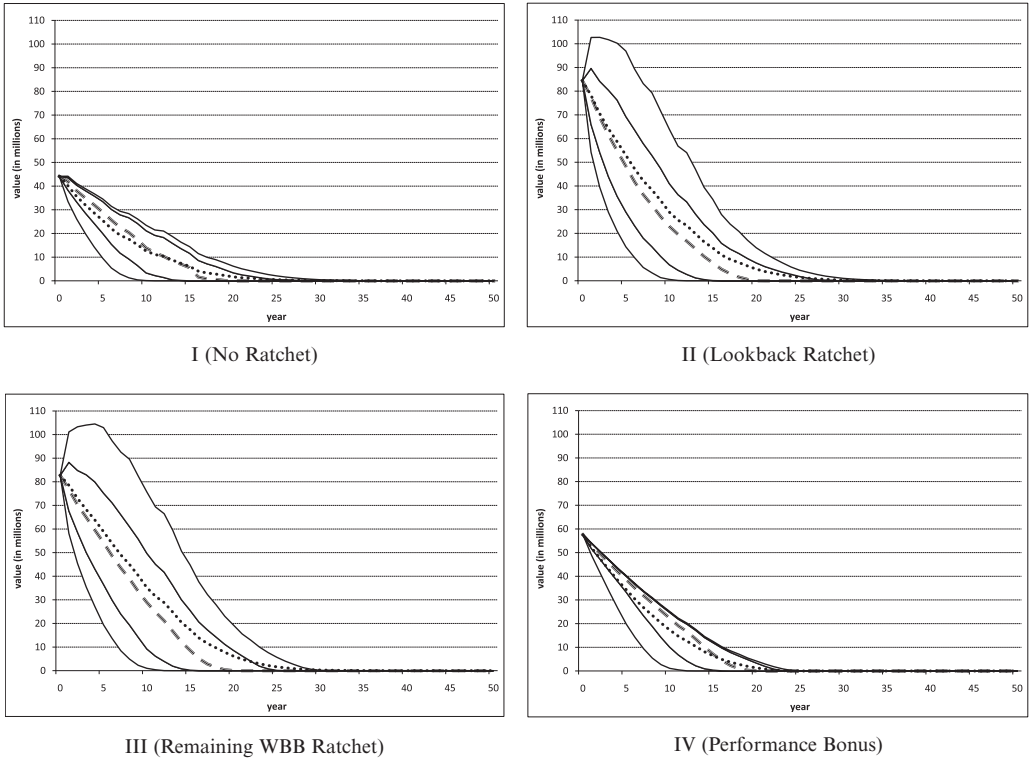


FIGURE 5: Development over time of the percentiles of the GLWB rider’s vega for a pool of policies.

Considering vega risk, the products behave differently: Vega exposure for the products including a ratchet mechanism is distinctively higher since ratchets (and also the Performance Bonus) gain in value if volatility is increased. On the other hand, the product without ratchet is the one that is the least sensitive to changes in volatilities, while the Performance Bonus leads to a vega that is similar in shape but higher. Throughout, the two product designs with ratchets face roughly twice the volatility risk of the product without ratchet.

5. ANALYSIS OF HEDGE EFFICIENCY

In this section, we analyze the performance of different (dynamic) hedging strategies, which can be applied by the insurer in order to reduce exposure to financial risk – and thereby the required economic capital – caused by selling GLWB guarantees. First, we describe the analyzed hedging strategies; we then define the risk measures that we use to compare the (simulated) hedge efficiency of the analyzed strategies, before we finally present the simulation results in the last part of this section.

5.1. Hedge Portfolio

We assume that the insurer has sold a pool of policies with GLWB guarantees. We denote by $\Psi(\cdot)$ the option value for that pool, i.e. the sum of the option values V^G defined in Section 2.3 of each policy. We assume that the insurer cannot influence the value of $\Psi(\cdot)$ by changing the underlying fund (e.g. changing the fund's exposure to risky assets or forcing the policyholder to switch to a different, e.g. less volatile, fund). We further assume that the insurer invests the guarantee fees in a hedge portfolio $\Pi^{Hedge}(\cdot)$ and applies some hedging strategy within this portfolio. In case the guarantee of a policy is triggered, the guaranteed payments due are deducted from this portfolio. Thus,

$$\Pi(t) := -\Psi(t) + \Pi^{Hedge}(t) \quad (14)$$

is the insurer's cumulative profit/loss (in what follows sometimes just denoted as insurer's profit) at time t stemming from the guarantee and the corresponding hedging strategy.

The following hedging strategies aim at reducing the insurer's risk by implementing certain investment strategies within the hedge portfolio $\Pi^{Hedge}(\cdot)$. Note that the value $\Psi(\cdot)$ of the pool of policies at time t does not only depend on the number and size of contracts and the underlying fund's current level, but also on several retrospective factors, such as the historical prices of the fund at previous withdrawal dates, and on model and parameter assumptions.

The insurer's choice of model and parameters can also have a significant impact on the hedging strategies. Therefore, we will differentiate in the following between the hedging model that is chosen and used by the insurer, and the data-generating model that we use to simulate the development of the underlying and the market prices of European call and put options. This allows us, e.g., to analyze the model risk faced by an insurer basing pricing and hedging on a simple Black-Scholes model (hedging model) with deterministic volatility, whereas in reality (data-generating model) volatility is stochastic. We assume the value of the guarantee to be marked-to-model, where the same model the insurer uses for hedging is used for the valuation of $\Psi(\cdot)$. All other assets in the insurer's portfolio are marked-to-market, which within our analysis means that their prices are determined by the (external) data-generating model.

We assume that, in addition to the underlying $S(\cdot)$ and the money-market account $B(\cdot)$, a market for European "plain vanilla" options on the underlying exists. However, we assume that only options with limited time to maturity are liquidly traded. As well as the underlying and the money-market account, we assume the option prices (i.e. the implied volatilities) to be driven by the data-generating model, and presume risk-neutrality with respect to volatility risk, i.e. the market price of volatility is set to zero in case the Heston model is used as data-generating model. Additionally, we assume the spread between bid and ask prices/volatilities to be zero.

For all considered hedging strategies we assume the hedging portfolio to consist of three assets, whose quantities are rebalanced at the beginning of each hedging period: a position of quantity $\Delta_S(\cdot)$ in the underlying, a position of quantity $\Delta_B(\cdot)$ in the money-market account and a position of $\Delta_X(\cdot)$ in a 1-year ATMF straddle (i.e. an option consisting of one call and one put, both with one year maturity and at the money with respect to the maturity's forward). We assume the insurer to hold the position in the straddle for one hedging period, then sell the options at then-current prices, and set up a new position in a then 1-year ATMF straddle. For each hedging period, the new straddle is denoted by $X(\cdot)$. We assume that the portion of the hedge portfolio that was not invested in either $S(\cdot)$ or $X(\cdot)$ is invested in (or borrowed from) the money market. Thus, the hedge portfolio at time t has the form

$$\Pi^{Hedge}(t) = \Delta_S(t)S(t) + \Delta_B(t)B(t) + \Delta_X(t)X(t), \quad (15)$$

where

$$\Delta_B(t) := \frac{\Pi^{Hedge}(t) - \Delta_S(t)S(t) - \Delta_X(t)X(t)}{B(t)}. \quad (16)$$

5.2. Dynamic Hedging Strategies

For both considered hedging models, Black-Scholes and Heston, we analyze three different types of (dynamic) hedging strategies.

No Hedge (NH)

The first strategy simply invests all guarantee fees in the money-market account. The strategy is obviously identical for both models.

Delta Hedge (D)

The second type of hedging strategy uses a position in the underlying in order to immunize the portfolio against small changes in the underlying's level. Within the Black-Scholes framework, assuming continuous trading and no transaction costs, such a position is sufficient to perform a perfect hedge. In reality however, time-discrete trading and transaction costs cause imperfections.

Using the Black-Scholes model as hedging model, in order to immunize the portfolio against small changes in the underlying's price (i.e. to attain delta-neutrality), Δ_S is chosen as the delta of $\Psi(\cdot)$, i.e. the partial derivative of $\Psi(\cdot)$ with respect to the underlying's spot price $S(\cdot)$.

While delta hedging under the Black-Scholes model (given the usual assumptions), constitutes a theoretically perfect hedge, it does not under the Heston model. This leads to (locally) risk minimizing strategies that aim at

minimizing the variance of the portfolio's instantaneous changes. Under the Heston model⁷, the problem

$$\text{var}(d\Pi(t)) \rightarrow \min, \quad \Delta_S(t) \in \mathbb{R}, \Delta_X(t) \equiv 0 \quad (17)$$

has the solution (see e.g. Ewald et al. (2009))

$$\Delta_S(t) = \frac{\partial \Psi^{\text{Heston}}(t, S(t), V(t))}{\partial S(t)} + \frac{\rho \sigma_V}{S(t)} \frac{\partial \Psi^{\text{Heston}}(t, S(t), V(t))}{\partial V(t)}. \quad (18)$$

To keep notation simple, this (locally) risk minimizing strategy using the Heston model is also referred to as delta hedge.

Delta and Vega (DV)

The third type of hedging strategies incorporates the use of the straddle option $X(\cdot)$, exploiting its sensitivity to changes in volatility for the sake of neutralizing the portfolio's exposure to changes in volatility.

Under the Black-Scholes model, volatility is assumed to be constant; therefore using it to hedge against a changing volatility appears rather counterintuitive. Nevertheless, following Taleb (1997), we analyze some kind of *ad-hoc* vega hedge in our simulations, which aims at compensating the deficiencies of the Black-Scholes model: For the vega hedge, we do not compute the Black-Scholes vega of the guarantee's value $\Psi(\cdot)$ and compare it to the corresponding Black-Scholes vega of the option's value $X(\cdot)$, but, instead, we use the so-called *modified vega* of $\Psi(\cdot)$ for comparison. Since all maturities cannot be expected to react the same way to changes in today's volatility, the modified vega applies a different weighting to the respective vega of each maturity. We use the inverse of square root of time as simple weighting method and use the maturity of the hedging instrument $X(\cdot)$, i.e. one year, as benchmark maturity. The modified vega of $\Psi(\cdot)$ at a policy calculation date τ then has the form

$$\text{ModVega}(\tau) = \sum_{t=\tau+1}^T v_t \frac{1}{\sqrt{t-\tau}}, \quad (19)$$

where v_t denotes the respective Black-Scholes vega of the expected discounted cash flow at time t of the pool of policies. The ratio between the modified vega and the vega of the straddle option determines the quantity of straddle options in the hedge position of the portfolio.

⁷ Note that a (time-continuously) delta-hedged portfolio under the Black-Scholes model is already risk-free. Therefore for the Black-Scholes model, the delta-hedging strategy coincides with the locally risk minimizing strategy.

TABLE 7

HEDGE RATIOS FOR ALL THREE STRATEGIES IF THE BLACK-SCHOLES MODEL IS USED AS HEDGING MODEL.

	Δ_S	Δ_X
(NH)	0	0
(D-BS)	$\frac{\partial \Psi^{BS}(t, S(t), \sigma^{BS})}{\partial S(t)}$	0
(DV-BS)	$\frac{\partial \Psi^{BS}(t, S(t), \sigma^{BS})}{\partial S(t)} - \Delta_X \frac{\partial X^{BS}(t, S(t), \sigma^{BS})}{\partial S(t)}$	$ModVega(t) / \frac{\partial X^{BS}(t, S(t), \sigma_X^{BS})}{\partial V(t)}$

TABLE 8

HEDGE RATIOS FOR ALL THREE STRATEGIES IF THE HESTON MODEL IS USED AS HEDGING MODEL.

	Δ_S	Δ_X
(NH)	0	0
(D-H)	$\frac{\partial \Psi^{Heston}(t, S(t), V(t))}{\partial S(t)} + \frac{\rho \sigma_v}{S(t)} \frac{\partial \Psi^{Heston}(t, S(t), V(t))}{\partial V(t)}$	0
(DV-H)	$\frac{\partial \Psi^{Heston}(t, S(t), V(t))}{\partial S(t)} - \Delta_X \frac{\partial X^{Heston}(t, S(t), V(t))}{\partial S(t)} + \frac{\rho \sigma_v}{S(t)} \left(\frac{\partial \Psi^{Heston}(t, S(t), V(t))}{\partial V(t)} - \Delta_X \frac{\partial X^{Heston}(t, S(t), V(t))}{\partial V(t)} \right)$	$\frac{\frac{\partial \Psi^{Heston}(t, S(t), V(t))}{\partial V(t)}}{\frac{\partial X^{Heston}(t, S(t), V(t))}{\partial V(t)}}$

Under the Heston model, we compare the two derivatives of $\Psi(\cdot)$ and $X(\cdot)$ with respect to the current local variance $V(\cdot)$ and then analogously determine the straddle option position of the hedge portfolio.

Of course, under both hedging models, the position in the underlying must be adjusted for the delta of the straddle position $\Delta_X X(\cdot)$.

The hedge ratios for all three strategies used in our simulations are summarized in Table 7 for the Black-Scholes model, and in Table 8 for the Heston model.

Additionally, for all dynamic hedging strategies (Delta and Delta-Vega), we assume that the hedger buys 1-year European put options at each policy anniversary, such that the possible guarantee payments for the next policy anniversary are fully hedged by the put options (assuming surrender and mortality rates are deterministic and known). This strategy aims at avoiding having to hedge an option with short time to maturity and hence having to deal with a potentially rapidly alternating delta (high gamma) if the option is near the strike. This is possible for all four ratchet mechanisms, since the guaranteed withdrawal amount is known one year in advance.

For all considered hedging strategies we assume that the hedge portfolio is rebalanced on a monthly basis.

5.3. Simulation Results

We use the following three ratios to compare the different hedging strategies, all of which will be normalized as a percentage of the sum of the premiums paid to the insurer at $t = 0$:

- $E_P[e^{-rT}\Pi_T]$, the expectation of the discounted final value of the insurer's profit under the real-world measure P . This is a measure for the insurer's expected profit and constitutes the "performance" ratio in our context. A value of 1 means that, in expectation, for a single premium of 100 paid by the client, the insurance company's discounted profit is 1.
- $CTE_{1-\alpha}(\chi) = E_P[-\chi | -\chi \geq VaR_\alpha(\chi)]$, the conditional tail expectation of the random variable χ , where χ is defined as the minimum of the discounted values of the insurer's profit/loss at all policy calculation dates, i.e. $\chi = \min\{e^{-rt}\Pi_t | t = 0, \dots, T\}$, and VaR denotes the Value at Risk. This is a measure for the corresponding insurer's risk to a certain hedging strategy: it can be interpreted as the additional amount of money that would be necessary at outset such that the insurer's portfolio would never become negative over the life of the contract, even if the market develops according to the average of the α (e.g. 10%) worst scenarios in the stochastic model. Thus a value of 1 means that, in expectation over the α worst scenarios, for a single premium of 100 paid by the client, the insurance company would need to hold 1 additional unit of capital upfront.
- $CTE_{1-\alpha}(e^{-rT}\Pi_T) = E_P[e^{-rT}\Pi_T | -e^{-rT}\Pi_T \geq VaR_\alpha(e^{-rT}\Pi_T)]$, the conditional tail expectation of the discounted profit/loss' final value. This is also a risk measure which, however, focuses on the value of the profit/loss at time T , i.e. after all liabilities have been met, and does not care about negative portfolio values over time. Thus a value of 1 in the above table means that, in expectation over the α worst scenarios, for a premium of 100 paid by the client, the insurance company's expected discounted loss is 1. By definition, of course, $CTE_{1-\alpha}(\chi) \geq CTE_{1-\alpha}(e^{-rT}\Pi_T)$.

In the numerical analyses below, we set $\alpha = 10\%$ for both risk measures and assume a pool of identical policies with parameters as given in Section 4 and assuming that the policyholders do not surrender. We assume that mortality within the population of insured happens exactly according to the best-estimate probabilities given in the DAV 2004R table. Our analysis focuses on model risk rather than parameter risk. Therefore, we use the same parameters for the capital market models for both, the hedging and the data-generating model.

We start our analyses using Black-Scholes as data-generating model with a risk-free rate of interest $r = 4\%$, an underlying's drift $\mu = 7\%$, and constant equity volatilities of $\sigma_{BS} = 22\%$ and $\sigma_{BS} = 25\%$, respectively. Table 9 gives the

TABLE 9

RESULTS FOR THE TWO HEDGING STRATEGIES (NH) AND (D-BS) USING THE BLACK-SCHOLES MODEL AS DATA-GENERATING MODEL WITH $r = 4\%$, $\mu = 7\%$, AND VOLATILITY PARAMETERS $\sigma_{BS} = 22\%$ AND $\sigma_{BS} = 25\%$. RESULTS ARE EXPRESSED AS A PERCENTAGE OF THE SINGLE PREMIUM PAID BY THE CLIENT.

		Data-Generating model							
		Black-Scholes ($\sigma_{BS} = 22\%$)				Black-Scholes ($\sigma_{BS} = 25\%$)			
		Product				Product			
		I	II	III	IV	I	II	III	IV
No hedge (NH)	$E_P[e^{-rT}\Pi_T]$	10.4	7.9	6.8	4.1	10.2	7.5	6.6	4.1
	$CTE_{1-\alpha}(\chi)$	25.9	20.3	17.8	15.6	27.3	21.2	19.1	16.4
	$CTE_{1-\alpha}(e^{-rT}\Pi_T)$	23.2	17.8	15.3	13.0	25.1	18.9	16.8	14.1
Delta hedge Black-Scholes (D-BS)	$E_P[e^{-rT}\Pi_T]$	0.4	0.3	0.3	0.2	0.6	0.4	0.4	0.3
	$CTE_{1-\alpha}(\chi)$	1.5	2.7	2.4	1.7	1.6	2.8	2.6	1.8
	$CTE_{1-\alpha}(e^{-rT}\Pi_T)$	1.3	2.3	2.1	1.5	1.4	2.4	2.3	1.6

results for different hedging strategies and both volatility parameters as a percentage of the single premium paid by the client.

If no hedging is in place, the insurance company has a long position in the underlying and thus faces a rather high expected return combined with high risk. No hedging effectively means that the insurance company, on average over the worst ten percent of the scenarios, would need additional capital between 15 and 28 percent of the premium volume paid by the clients in order to avoid a negative hedge portfolio over time. The $CTE_{1-\alpha}(e^{-rT}\Pi_t)$ are around 23-25 for product I (No Ratchet) and around 13-14 for product IV (Performance Bonus) under both volatility parameter assumptions. The corresponding values for the products with ratchet lie in between.

If the insurance company sets up a delta-hedging strategy based on the Black-Scholes model, risk is significantly reduced for all products and both volatility parameter assumptions. Whereas without hedging, the No Ratchet product appeared to be the riskiest, with delta hedging the products with a ratchet (Lookback Ratchet and Remaining WBB Ratchet) now are the riskiest. The reason for this is that delta is rather “volatile” for the products with ratchet, cf. Figure 3 in Section 4. Since fast changes in the delta lead to potential losses due to hedging errors, this increases the riskiness of the ratchet type products. This basically shows the effect of a high gamma (second order derivative of the option value with respect to the underlying’s spot price). The higher the gamma, the higher are discretization errors and thus the risk of a delta-only hedge.

Comparing the left and right hand part of Table 9, we find that higher volatility values lead to larger hedging errors, which in turn results in a higher risk for the insurer. However, the hedging results show only very little sensitivity with respect to the volatility parameter, i.e. only a slight increase in risk and

return of the insurer. Overall, delta hedging seems to be effective in this setting, with risk being reduced to less than six percent when compared to the No hedge strategy in case of product I and to around 11 to 14 percent for the other three designs. This goes hand in hand with a reduction of the expected profit of the insurer.

We now describe how the results of the Black-Scholes Delta hedge change and how other hedging strategies perform if the Heston model with stochastic equity volatility is used as data-generating model instead of the Black-Scholes model. This gives an indication on the model risk that arises from neglecting stochastic equity volatility in the risk management process while, in the real world, volatility is stochastic.

The following Table 10 gives the results for all considered hedging strategies using the Heston model as data-generating model with $r = 4\%$, $\mu = 7\%$ and volatility parameters as stated in Table 4 as the base case. As “stressed” volatility parameters we use the values for mean reversion and long-term volatility that

TABLE 10

RESULTS FOR ALL CONSIDERED HEDGING STRATEGIES USING THE HESTON MODEL AS DATA-GENERATING MODEL, WITH $r = 4\%$, $\mu = 7\%$, VOLATILITY PARAMETERS AS STATED IN TABLE 4 AND WITH STRESSED VOLATILITY PARAMETERS THAT CORRESPOND TO THE Q -PARAMETERS WITH $\lambda = -2$ AS GIVEN IN TABLE 5. RESULTS ARE EXPRESSED AS A PERCENTAGE OF THE SINGLE PREMIUM PAID BY THE CLIENT.

		Data-Generating model							
		Heston (base case)				Heston (stressed parameters)			
		Product				Product			
		I	II	III	IV	I	II	III	IV
No hedge (NH)	$E_P[e^{-rT}\Pi_T]$	10.3	7.5	6.5	3.8	10.3	7.2	6.5	3.7
	$CTE_{1-\alpha}(\chi)$	28.0	23.4	20.8	17.5	29.4	24.4	22.2	18.5
	$CTE_{1-\alpha}(e^{-rT}\Pi_T)$	25.4	20.9	18.4	15.1	27.2	22.2	20.1	16.4
Delta hedge Black-Scholes (D-BS)	$E_P[e^{-rT}\Pi_T]$	0.4	0.2	0.1	0.1	0.6	0.3	0.3	0.2
	$CTE_{1-\alpha}(\chi)$	2.7	4.8	4.5	3.3	2.9	5.3	5.0	3.7
	$CTE_{1-\alpha}(e^{-rT}\Pi_T)$	2.4	4.3	4.1	3.0	2.6	4.8	4.6	3.4
Delta hedge Heston (D-H)	$E_P[e^{-rT}\Pi_T]$	0.2	-0.2	-0.2	0.0	0.4	0.0	0.0	0.1
	$CTE_{1-\alpha}(\chi)$	2.6	4.8	4.5	3.3	2.9	5.3	5.0	3.7
	$CTE_{1-\alpha}(e^{-rT}\Pi_T)$	2.4	4.2	4.0	3.0	2.6	4.7	4.5	3.4
Delta-Vega hedge Black-Scholes (DV-BS)	$E_P[e^{-rT}\Pi_T]$	0.7	0.8	0.9	0.5	1.0	1.1	1.1	0.6
	$CTE_{1-\alpha}(\chi)$	1.6	2.1	2.5	1.8	1.6	2.4	2.9	1.9
	$CTE_{1-\alpha}(e^{-rT}\Pi_T)$	1.3	1.7	2.0	1.4	1.3	1.9	2.2	1.6
Delta-Vega hedge Heston (DV-H)	$E_P[e^{-rT}\Pi_T]$	0.5	0.4	0.4	0.3	0.7	0.6	0.6	0.4
	$CTE_{1-\alpha}(\chi)$	1.2	1.3	1.3	1.2	1.2	1.3	1.3	1.2
	$CTE_{1-\alpha}(e^{-rT}\Pi_T)$	1.0	1.1	1.1	1.0	1.0	1.1	1.1	1.1

correspond to the Q -parameters for $\lambda = -2$ (cf. Table 5). As in Table 9, results are stated as a percentage of the single premium paid by the client.

The risk for the No hedge strategy increases by around 10 to 20 percent in comparison to the results where the Black-Scholes model is used as a data-generating model.

For the Black-Scholes Delta hedge, by introducing stochastic volatility to the capital market, the insurance company's expected profit hardly changes. However, the insurer's risk is increased by roughly 80 to 100 percent throughout all product types and for both volatility parameter assumptions. In contrast, if the calculation of the hedge position of the Delta hedge is performed within the Heston model (D-H), risk (and the expected profit) is only reduced by a small amount. Thus, we can conclude that the data-generating model has a huge impact on the insurer's risk (since risk hugely differs if the world behaves according to the Heston model instead of the Black-Scholes model), whereas for a given data-generating model, the choice of the hedging model⁸ is of lesser importance.

We now analyze the two strategies in which volatility risk is also hedged. The DV-BS hedge reduces risk significantly compared to the two delta-only hedges, even though the hedge is set up under a model with deterministic volatility. Risk is reduced by around 50% and some of the results are even better than a D-BS hedge under the Black-Scholes data-generating model, which is not surprising, as the hedge instrument used for vega hedging – a straddle option – also introduces a partial hedge of the gamma of the insurer's liability.

If the vega hedge is set up within the Heston model, results improve even further, reducing risk by around 55 to 75 percent compared to the delta-only hedge. Especially the two designs with ratchets (II and III) seem to benefit, as their risk now is only slightly higher than that of the products without ratchet mechanism. Also, with this hedging strategy, sensitivity with regard to the volatility parameters seems to be lower than with the other strategies. Market risk within our model is now below 2% of the initial single premium paid by the client.

We would like to close this section with some comments on vega hedging: First, we would like to stress that – since on the one side there are different types of volatility (e.g. actual vs. implied), which can change with respect to their level, skew, slope, convexity, etc., and on the other side there is a great variety of hedging instruments in the market that exhibit some kind of sensitivity to changes in volatility – a unique vega hedging strategy does not exist. Second, we would like to point out the shortcomings of a somewhat intuitive and straightforward (but unfortunately ill-advised) way of setting up a vega hedge portfolio within the Black-Scholes model: One could simply calculate the first order derivative of the option value with respect to the volatility parameter and use this number to set up a vega hedge portfolio. However, in our model framework (and potentially in a real-world scenario, too) this would result in a rather bad hedge performance due to the following reasons: A change in

⁸ Note that we here refer to the hedging model (i.e. Black-Scholes or Heston) and not the hedging strategy (e.g. Delta, Delta-Vega).

TABLE 11

RESULTS OF THE BLACK-SCHOLES DELTA-VEGA HEDGE USING THE UNMODIFIED VEGA.

	Data-Generating model							
	Heston (base case)				Heston (stressed parameters)			
	Product				Product			
	I	II	III	IV	I	II	III	IV
$E_p[e^{-rT}\Pi_T]$	1.4	2.4	2.5	1.5	1.8	2.8	2.9	1.6
$CTE_{1-\alpha}(\chi)$	7.4	15.6	17.5	9.6	8.3	19.4	21.1	10.7
$CTE_{1-\alpha}(e^{-rT}\Pi_T)$	6.0	12.7	14.2	8.0	6.8	15.6	17.0	9.0

current asset volatility under the Heston model would mean a change in short-term volatility and a much smaller change in long-term volatility. Since volatility (and therefore also the change in volatility) in the Black-Scholes model is assumed to be constant over time, the change in a long-term option's value due to changes in volatility would be significantly overestimated. The resulting hedge portfolio may lead to an increase in risk, foiling the very idea of hedging. To illustrate this effect, we calculated above risk measures for this "unmodified vega" hedge using the Heston model for data generation. The results are displayed in Table 11 and show that risk increases (in comparison to the values of the Black-Scholes Delta hedge shown in Table 10) due to the "over-hedging".

6. CONCLUSION

In the present paper, we have analyzed different types of guaranteed withdrawal benefits for life, the latest guarantee feature within variable annuities, both, from a client's perspective and from an insurer's perspective. We found that different ratchet and bonus features can lead to significantly different cash-flows to the insured. Both, the probability that guaranteed payments have to be paid and their amount vary significantly for the different products, even if they all come at the same fair guarantee fee.

The development over time of delta, rho and vega – i.e. the sensitivity of the value of the guarantees with respect to changes in the underlying's price, the interest rate level and the volatility, respectively – is also significantly different, depending on the selected product features. Thus both, the constitution of a hedging portfolio (following a certain hedging strategy) and the insurer's risk after hedging, differ significantly for the different products.

We found that the fair prices of the guarantees hardly change, when stochastic volatility is introduced. The insurer's risk however changes dramatically. We analyzed different hedging strategies (no hedging, delta only, delta and vega) to deal with that risk and analyzed the distribution of the insurer's cumulative

profit/loss and certain risk measures hereof. We found that the insurer's risk can be reduced significantly by implementing suitable hedging strategies.

We then quantified the model risk by using different capital market models for data generation and calculation of the hedge positions. This is an indication for the model risk an insurer faces by assuming a certain model whereas in the real world, capital markets display different properties. In this paper, we focused on the risk an insurer takes by assuming constant equity volatilities in the risk management model whilst, in the real world, volatilities are stochastic and showed that this risk can be substantial. While the benefits of using the "correct" model for hedging seem not to be very distinctive (at least for delta-only strategies), the differences between the considered data-generating models and the considered strategies (No hedge, delta only or delta and vega) are vast.

We were also able to show that a hedging strategy based on a modified version of vega can lead to a significant reduction of volatility risk even if a hedging model is used that only allows for deterministic equity volatility. On the other hand, a somewhat more intuitive and straightforward attempt to hedge against volatility risk, based on the unmodified vega, can lead to results inferior to the case with no vega hedging at all.

Our results – in particular with respect to model risk – should be of interest to both, insurers and regulators. The latter are in danger of systematically neglecting model risk if hedge efficiency is analyzed with the same model that the insurer uses as a hedging model.

Further research could aim at extending our findings to other products or other capital market models (e.g. with equity jumps, stochastic interest rates and/or other approaches to the stochasticity or uncertainty of actual and implied equity volatility). Also, a systematic analysis of parameter risk and robustness of the hedging strategies against policyholder behavior appears worthwhile.

Finally, it would be interesting to analyze how the insurer can reduce risk by product design, e.g. by offering funds as an underlying that are managed to meet some volatility target or by reserving the right to switch the insured's assets to less risky funds (e.g. bond or money market funds) if market volatilities increase. Such product features can already be observed in some insurance markets.

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