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Relaxation of a non-ideal incompressible plasma with mass flow

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Abstract. The time evolution of an incompressible non-ideal magnetohydrodynamic (MHD), current-carrying plasma with mass flow is investigated. An approach for the reduction of the nonlinear vector MHD equations to a set of scalar partial differential equations is supposed. Analytical time-dependent solutions of this system are presented. They describe kinetic plasma equilibria both with well-defined nested-in magnetic and velocity surfaces and in the form of vortices. The obtained solutions may be called 'diffusion-like', since their temporal structure is very similar to the solutions of the diffusion problem. It is shown that the magnetic field and the velocity have different dumping rates. In the asymptotic limit $t \to \infty$, the plasma slowly relaxes towards the hydrostatic equilibrium of gravitating systems.

1. Introduction

A convenient approximation to the full magnetohydrodynamic (MHD) description is so-called 'reduced' MHD (Strauss 1976), which has been employed both analytically and numerically for the description of the equilibrium and the dynamical evolution of plasmas (Tsinganos 1981; Montgomery 1992). In the present paper, elementary arguments are presented, according to which a well-defined plasma system with appropriate boundary conditions relaxes in the long-time limit owing to dissipation towards the state with zero magnetic field and zero flow. This quiescent state can be identified with the state of hydrostatic equilibrium. The investigation of the time evolution is performed for a homogeneous, infinitely long cylindrical plasma column, based on the assumption of constant density, viscosity and scalar conductivity. It is known that such an idealization gives an elementary self-consistent model for a gravitating system in space that assists in understanding the basic features of its behaviour.

The large-scale vortical movement of plasmas is also investigated. It has been pointed out in Petviashvili et al. (1986) and Andrushchenko et al. (1993a) that for a fixed plasma boundary there exist localized vortex solutions. In the present investigation, it is shown that the 'reduced' MHD equations also allow the description of global vortices, where the size of each vortex in the cross-section of the plasma column is of the order of the plasma radius itself. Such vortices are not sensitive to small-scale perturbations of the environment, and for this reason they do not depend on the specific dispersion properties of the plasma and can be described within the framework of the single-fluid model.

2. Reduction of the MHD equations

The starting point of the investigations is the full set of non-ideal, nonlinear vector MHD equations for an incompressible plasma $\rho = \text{const}$ with constant kinematic and magnetic viscosities ν and μ in the form

$$\frac{\partial \mathbf{v}}{\partial t} + (\nabla \times \mathbf{v}) \times \mathbf{v} = -\nabla\left(\frac{p}{\rho} + \frac{v^2}{2} + U\right) + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi\rho} + \nu \Delta \mathbf{v}, \tag{1}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \mu \Delta \mathbf{B}, \tag{2}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{4}$$

$$\Delta U = -4\pi G \rho, \tag{5}$$

where ρ denotes the constant mass density, $\mathbf{v}$ the hydrodynamic velocity, $\mathbf{B}$ the magnetic field, p the plasma pressure, U the gravitational potential and G the gravitational constant. In this case, the pressure may first be eliminated from (1) by the differential operator $\nabla \times$ and can then afterwards be determined by one of the components of (1), if the solution for $\mathbf{v}$ and $\mathbf{B}$ is known. It may be noted that U enters only into the expression for the 'Bernoulli' pressure in the form $P^* = p + \frac{1}{2}\rho v^2 + \rho U$, so that the special form of U essentially only determines the pressure profile, provided that the solution for $\mathbf{v}$ is known.

Following (Cheremnykh et al. 1994), only translationally symmetric solutions $(\partial/\partial z = 0)$ of the set of equations (1)–(5) are considered in cylindrical coordinates (r, θ, z), assuming that there are no longitudinal components (along the z axis) of the velocity and the magnetic field. For the description of the resulting only two-dimensional motion of the plasma, the so-called Stokes potentials and ψ are introduced:

$$\mathbf{v} = \nabla \quad \times \mathbf{e}_z, \tag{6a}$$

$$\mathbf{B} = \nabla \psi \times \mathbf{e}_z, \tag{6b}$$

where $\quad = \quad (t, r, \theta)$ and $\psi = \psi(t, r, \theta)$ are scalar functions of the time and the polar coordinates in a plane $z = \text{const}$, and $\mathbf{e}_z$ is the unit vector in the z direction.

With the help of the representations (6a, b) the vector MHD equations (1)–(4) can, by some tedious manipulations, be reduced to the following set of scalar partial differential equations (Cheremnykh et al. 1994):

$$\frac{\partial}{\partial t}\omega + \{\omega, \quad\} = \frac{1}{4\pi\rho}\{j, \psi\} + \nu \Delta_\perp \omega, \tag{7}$$

$$\frac{\partial \psi}{\partial t} = \{\quad, \psi\} - \mu j, \tag{8}$$

$$\frac{\partial}{\partial r}\left(\frac{p}{\rho} + \frac{v^2}{2} + U\right) + \frac{1}{4\pi\rho}\frac{\partial \psi}{\partial r}\Delta_\perp \psi - \frac{\partial}{\partial r}\Delta_\perp \quad = -\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial}{\partial t} - \nu \Delta_\perp \quad\right), \tag{9}$$

$$\left\{\frac{p}{\rho} + \frac{v^2}{2} + U, \psi\right\} = \left(\frac{\partial \psi}{\partial t} - \mu \Delta_\perp \psi\right)\Delta_\perp \quad - \left(\nabla_\perp\left(\frac{\partial}{\partial t} - \nu \Delta_\perp \quad\right) \cdot \nabla_\perp \psi\right), \tag{10}$$

where $j = -\Delta_\perp \psi$ is the current density and $\omega = -\Delta_\perp \quad$ is the flow vorticity. Furthermore, the following definitions have been employed:

$$\boldsymbol{\nabla}_\perp = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta},$$

$$\Delta_\perp = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2},$$

$$\{\quad, \psi\} = \frac{1}{r}\left(\frac{\partial}{\partial r}\frac{\partial \psi}{\partial \theta} - \frac{\partial}{\partial \theta}\frac{\partial \psi}{\partial r}\right),$$

together with the boundary conditions $\quad = \psi = \text{const}$ at $r = a$. Note that these conditions are equivalent with the requirement of vanishing normal components of the velocity and magnetic field, i.e. $v_r = 0$, $B_r = 0$ at $r = a$.

3. Analytical solutions for several special cases

Let us now consider some special cases of the general non-ideal single-fluid equations (1)–(5), which result in a corresponding simplified set of equations (7)–(10), allowing a further analytical treatment.

3.1. $\partial/\partial t = \mu = \nu = \mathbf{v} = 0$

This case corresponds to the static ideal-MHD equilibria, and the system (7)–(10) degenerates to the Grad–Shafranov equation

$$\Delta_\perp \psi = -\frac{1}{4\pi}\frac{d}{d\psi}(p + \rho U) \tag{11}$$

and the Strauss equation

$$\{\psi, j\} = 0. \tag{12}$$

3.2. $\partial/\partial t = \mu = \nu = 0$, $\mathbf{v} \neq 0$

Here we have the case of the ideal-MHD equilibria with flow, and (7)–(10) reduce to the following system of nonlinear equations:

$$\{\quad, \omega\} = \frac{1}{4\pi\rho}\{\psi, j\}, \tag{13a}$$

$$\{\quad, \psi\} = 0, \tag{13b}$$

$$\frac{d}{d\psi}\left(\frac{p}{\rho} + \frac{v^2}{2} + U\right) + \frac{1}{4\pi\rho}\Delta_\perp \psi - \frac{d}{d\psi}\Delta_\perp \quad = 0. \tag{13c}$$

If the two potentials $\quad$ and ψ are related by the linear dependence

$$\quad = \alpha(4\pi\rho)^{-1/2}\psi, \quad \alpha = \text{const}, \tag{14}$$

then the system (7)–(10) possesses a solution of the form (Andrushchenko et al. 1993b):

$$\Delta_\perp \psi = F(\psi), \tag{15a}$$

$$\frac{p}{\rho} + \frac{v^2}{2} + U - \frac{\alpha^2 - 1}{4\pi\rho}\int F(\psi)\, d\psi = \text{const}, \tag{15b}$$

where $F(\psi)$ is an arbitrary function of ψ.

The solution (15a, b) describes arbitrary stationary equilibrium states (both with magnetic islands and with only one extremum of the magnetic potential) at which the plasma flows along the magnetic surfaces $\psi = \text{const}$, i.e. the flow surfaces coincide with the magnetic ones (Cheremnykh et al. 1994; Andrushchenko et al. 1993b).

If one chooses for $F(\Psi)$ a linear dependence

$$F(\psi) = -\beta^2\psi \tag{16}$$

then the solution for ψ is obtained in the form

$$\psi = \text{const} \times \left[J_0\left(\frac{\kappa_m r}{a}\right) + bJ_m\left(\frac{\kappa_m r}{a}\right)\cos m\theta \right], \tag{17}$$

where J_m is the mth-order Bessel function, κ_m is its first root, $\beta = \kappa_m/a$, and b is a constant. The other equilibrium quantities may then be obtained from (5), (13) and (14).

In the case $\alpha = \pm 1$, one recovers from (14) and (15) Chandrasekhar's solution

$$\mathbf{v} = \pm\frac{\mathbf{B}}{(4\pi\rho)^{1/2}}, \tag{18}$$

$$\frac{p}{\rho} + \frac{v^2}{2} + U = \text{const}, \tag{19}$$

which is independent of the special form of the potential ψ. In the case $\alpha = 0$, the solution (15) reduces to the magnetostatic equilibria of Sec. 3.1.

3.3. $\mu = \nu = 0$, $\partial/\partial t = -\Omega\partial/\partial\theta$

This case, which is also subject to the ideal-MHD equations, corresponds to the situation where both the plasma and the magnetic field rotate at constant angular velocity Ω, superimposed on a stationary equilibrium state. The solution of (7)–(10) is sought in the form of a nonlinear wave rotating with a constant angular velocity Ω, suggesting the ansatz

$$\quad = \quad (r, \theta - \Omega t), \quad \psi = \psi(r, \theta - \Omega t).$$

Equations (7) and (8) then become

$$\{\Phi, \Delta_\perp \Phi\} = \{A, \Delta_\perp A\}, \tag{20}$$

$$\{\Phi, A\} = 0, \tag{21}$$

where A and Φ are defined by

$$A = \frac{\psi}{(4\pi\rho)^{1/2}}, \tag{22a}$$

$$\Phi = \quad + \tfrac{1}{2}\Omega r^2. \tag{22b}$$

It turns out that (20) and (21) admit solutions in the form of localized vortices. In order to obtain a solution for (20) and (21), one chooses for $\Delta_\perp \Phi$ and A the following ansatz (Andrushchenko et al. 1993a):

$$\Delta_\perp \Phi = -b^2\Phi + f(r), \qquad A = c\Phi, \tag{23}$$

where b and c are constants and f is an arbitrary function of r.

If one takes into account the boundary conditions then the solution of (20) and (21) can be represented in the form

$$\psi = (4\pi\rho)^{1/2}\left[J_0\left(\frac{\kappa_m r}{a}\right)+bJ_m\left(\frac{\kappa_m r}{a}\right)\cos m(\theta-\Omega t)\right], \tag{24a}$$

$$= \alpha\left[J_0\left(\frac{\kappa_m r}{a}\right)+bJ_m\left(\frac{\kappa_m r}{a}\right)\cos m(\theta-\Omega t)\right]+\tfrac{1}{2}\Omega(r^2-a^2), \tag{24b}$$

where κ_m is a zero of the mth-order Bessel function $J_m(x)$.

3.4. Diffusion-like solutions of the non-ideal MHD equations

Besides the solutions of the preceding sections, which have for the special case $U = 0$ already been partly discussed in previous publications and which all belong to ideal-MHD with $\mu = \nu = 0$, there also exist diffusion-like solutions of the non-ideal MHD equations ($\mu \neq 0$ and $\nu \neq 0$) for the special case where the velocity and the magnetic field potentials are restricted by the requirements

$$\frac{\partial \quad}{\partial t}-\nu\Delta_\perp \quad = 0, \tag{25a}$$

$$\frac{\partial \psi}{\partial t}-\mu\Delta_\perp \psi = 0. \tag{25b}$$

In this case, the system (7)–(10) can be transformed to the following set of equations:

$$\{\quad,\psi\} = 0, \tag{26}$$

$$\left\{\frac{p}{\rho}+\frac{v^2}{2}+U,\psi\right\} = 0, \tag{27}$$

$$\frac{\partial\psi}{\partial r}\left[\frac{\partial\psi}{\partial t}\frac{1}{4\pi\rho\mu}+\frac{d}{d\psi}\left(\frac{p}{\rho}+\frac{v^2}{2}+U\right)\right] = \frac{\partial \quad}{\partial r}\frac{1}{\nu}\frac{\partial \quad}{\partial t}, \tag{28}$$

$$\{\quad,\Delta_\perp \quad\} = \frac{1}{4\pi\rho}\{\psi,\Delta_\perp\psi\}. \tag{29}$$

It follows from (26) that $\quad = F(t,\psi(t,r,\theta))$, whereas (27) and (28) may be used to determine the pressure. In the following, the condition (29) is discussed in more detail.

3.4.1. $\{\psi,\Delta_\perp\psi\} \neq 0$. For this case, one can derive from (28) the relation

$$F''_{\psi\psi}F'_\psi\{\psi,(\boldsymbol{\nabla}_\perp\psi)^2\}+\left[(F'_\psi)^2-\frac{1}{4\pi\rho}\right]\{\psi,\Delta_\perp\psi\} = 0. \tag{30}$$

If $F''_{\psi\psi} \neq 0$ then one arrives at the Poisson-bracket relation

$$\left\{\psi,\frac{\{\psi,(\boldsymbol{\nabla}_\perp\psi)^2\}}{\{\psi,\Delta_\perp\psi\}}\right\} = 0, \tag{31}$$

which is equivalent to

$$\det\begin{Vmatrix}\{\psi,\{\psi,(\boldsymbol{\nabla}_\perp\psi)^2\}\} & \{\psi,\{\psi,\Delta_\perp\psi\}\} \\ \{\psi,(\boldsymbol{\nabla}_\perp\psi)^2\} & \{\psi,\Delta_\perp\psi\}\end{Vmatrix} = 0.$$

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If $F''_{\psi\psi} = 0$ then again Chandrasekhar's solution is obtained in the form

$$= \frac{\psi}{(4\pi\rho)^{1/2}}. \tag{32}$$

3.4.2. $\{\psi, \Delta_\perp \psi\} = 0$. For this case, one obtains from (25), (28) and (29) the two Poisson-bracket relations

$$\left\{\psi, \frac{\partial \psi}{\partial t}\right\} = 0, \tag{33a}$$

$$\left\{\ , \frac{\partial}{\partial t}\right\} = 0, \tag{33b}$$

implying that the magnetic field lines are temporal constants. With the help of the definition $\xi(r, \theta) = \psi(0, r, \theta)$, one then arrives at the conclusion

$$\psi(t, r, \theta) = \Psi(t, \xi(r, \theta)),$$

$$(t, r, \theta) = F(t, \xi(r, \theta)),$$

so that the condition 3.4.2 can be presented in the form

$$\Psi''_{\xi\xi}\{\xi, (\boldsymbol{\nabla}_\perp \xi)^2\} = 0. \tag{34}$$

From this equation, it follows that either $\{\xi, (\boldsymbol{\nabla}_\perp \xi)^2\} = 0$, implying that the magnetic (flow) lines are equidistant, or $\Psi''_{\xi\xi} = 0$, implying that $\Psi(t, \xi) = f(t)\,\xi$, so that (25) can be solved by separating Φ and Ψ into space- and time-dependent parts. In the latter case, the obtained solutions are given by

$$\psi_m = \text{const} \times J_m\left(\frac{\kappa_m r}{a}\right) \cos m\theta \exp\left(-\frac{\kappa_m^2 \mu t}{a^2}\right), \tag{35a}$$

$$_m = \text{const} \times J_m\left(\frac{\kappa_m r}{a}\right) \cos m\theta \exp\left(-\frac{\kappa_m^2 \nu t}{a^2}\right), \tag{35b}$$

with $m = 1, 2, \ldots$, and

$$\psi_0 = \text{const} \times J_0(\beta r) \exp(-\beta^2 \mu t), \tag{36a}$$

$$_0 = \text{const} \times J_0(\beta r) \exp(-\beta^2 \nu t), \tag{36b}$$

where κ_m is the lowest zero of the mth-order Bessel function J_m and β is an arbitrary constant.

It may be worthwhile to note the 'indirect' presence of a nonlinearity: a superposition of two arbitrarily taken solutions (35) in general is no solution of the problem, since (28) and (29) will not be fulfilled. Nevertheless, there exist special superpositions such as

$$\psi_m^* = \text{const} \times \left[J_0\left(\frac{\kappa_m r}{a}\right) + b J_m\left(\frac{\kappa_m r}{a}\right) \cos m\theta\right] \exp\left(-\frac{\kappa_m^2 \mu t}{a^2}\right), \tag{37}$$

$$_m^* = \text{const} \times \left[J_0\left(\frac{\kappa_m r}{a}\right) + b J_m\left(\frac{\kappa_m r}{a}\right) \cos m\theta\right] \exp\left(-\frac{\kappa_m^2 \nu t}{a^2}\right), \tag{38}$$

with $m = 1, 2, \ldots$, which are solutions, owing to the fact that the constant β in (36) can be chosen appropriately. It can be seen that these solutions reduce in the limit $\mu \to 0$ and $\nu \to 0$ or $t = \text{const}$ to the stationary equilibrium states described by (14) and (17). Thus (37) and (38) describe the relaxation of a non-ideal plasma through a sequence of stationary equilibria, with the help of the resulting equation (28), finally leading to the well-known hydrostatic equilibrium of gravitating systems:

$$\nabla\left(\frac{p}{\rho}+U\right) = 0. \tag{39}$$

Since $\kappa_0 < \kappa_m$ for all m, one can conclude that on an intermediate timescale the plasma first relaxes towards the radially symmetrical state given by

$$\psi = (4\pi\rho)^{1/2} J_0\left(\frac{\kappa_0 r}{a}\right)\exp\left(-\kappa_0^2\frac{\mu t}{a^2}\right), \tag{40a}$$

$$= \alpha J_0\left(\frac{\kappa_0 r}{a}\right)\exp\left(-\kappa_0^2\frac{\nu t}{a^2}\right). \tag{40b}$$

It may be remarked that the relaxations of the velocity and magnetic field possess different dumping rates, so that on an even longer timescale the plasma arrives at the intermediate states $= 0$, $\psi \neq 0$, i.e. $\mathbf{v} = 0$, $\mathbf{B} \neq 0$ for $\mu \ll \nu$, and vice versa at $\neq 0$, $\psi = 0$, i.e. $\mathbf{v} \neq 0$, $\mathbf{B} = 0$ for $\mu \gg \nu$, before it relaxes in the long-time limit towards the purely hydrostatic equilibria of (39).

4. Conclusions

It has been shown that the system of reduced MHD equations permits the description of various stationary states: static equilibria, stationary flows, and a rotating plasma in the form of a nonlinear wave. However, the stability problem of the obtained dynamic equilibrium states has not yet been investigated. For this reason, it is impossible to make any predictions about the existence of all these states in nature or their experimental realization.

In the present work, attention has also been paid to the derivation of exact solutions of the reduced MHD equations for a non-ideal plasma. A new class of diffusion-like solutions corresponding to slowly varying states has been found. On short timescales satisfying the requirements

$$\frac{\kappa_m^2 \mu t}{a^2} \ll 1, \quad \frac{\kappa_m^2 \nu t}{a^2} \ll 1,$$

and owing to the small magnitudes of ν and μ, the obtained diffusion-like solutions permit a good description of equilibria with a stationary flow, and do not belong to the corresponding solutions of a nonlinear wave.

Finally, it has also been shown that in the long-timescale limit, the diffusion-like solutions relax towards purely hydrostatic equilibria of gravitating systems, with both zero flow and magnetic field, a result in agreement with that obtained by Montgomery (1992), who has also analysed the long-time behaviour of reduced MHD. If external sources and fields are absent then this relaxation to static equilibrium states seems to be a general feature of many nonlinear systems, as shown by direct three-dimensional numerical modelling (Hayashi et al. 1998).

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