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Analytic order-isomorphisms of countable dense subsets of the unit circle

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Abstract. For functions in $C^k(\mathbb{R})$ which commute with a translation, we prove a theorem on approximation by entire functions which commute with the same translation, with a requirement that the values of the entire function and its derivatives on a specified countable set belong to specified dense sets. Using this theorem, we show that if A and B are countable dense subsets of the unit circle $T \subseteq \mathbb{C}$ with $1 \notin A$, $1 \notin B$, then there is an analytic function $h: \mathbb{C} \setminus \{0\} \to \mathbb{C}$ that restricts to an order isomorphism of the arc $T \setminus \{1\}$ onto itself and satisfies $h(A) = B$ and $h'(z) \neq 0$ when $z \in T$. This answers a question of P. M. Gauthier.

1 Introduction

The Barth–Schneider theorem [1] states that whenever A, B are countable dense subsets of $\mathbb{R}$, there is an entire function f which restricts to an order-isomorphism of A onto B. This generalizes Cantor's theorem which gives an order-isomorphism of A onto B which then extends uniquely to an order-isomorphism of $\mathbb{R}$ onto itself. P. M. Gauthier noted (private communication) that if A and B are countable dense subsets of the unit circle $T \subseteq \mathbb{C}$, then there is a diffeomorphism $T \to T$ mapping A onto B and asked whether this diffeomorphism can be taken to be analytic.

It is a standard exercise (cf. [4, Exercise 4, p. 264 and Exercise 25, p. 295]) that the entire functions that map the unit circle T into itself are the functions $f(z) = az^n$, $|a| = 1$, $n = 0, 1, 2, \ldots$. These map T bijectively onto itself only when they are rotations, i.e., when $n = 1$. Thus they give a bijection satisfying $f(A) = B$ only when B is the image of A under a rotation.

We will prove a version (Theorem 3.6) of the Barth–Schneider theorem for sets A, B which are invariant under a translation $\sigma(x) = x + t$ and use it to show (Theorem 4.1) that if A and B are countable dense subsets of $T \setminus \{1\}$, where $T \subseteq \mathbb{C}$ is the unit circle $|z| = 1$, then there is an analytic function $h: \mathbb{C} \setminus \{0\} \to \mathbb{C}$ that restricts to an order isomorphism of the arc $T \setminus \{1\}$ onto itself $T \setminus \{1\}$ and satisfies $h(A) = B$ and $h'(z) \neq 0$ when $z \in T$. The proof uses a version (Theorem 3.3 and Corollary 3.5) of Theorem 3.2 of [2], adapted to produce functions which commute with a translation.

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In this paper, we fix a positive real number t, and the translation σ given by $\sigma(z) = z + t$. Note that $f\sigma = f$ says that f is periodic with period t, where $f\sigma$ denotes the composition $f \circ \sigma$.

2 Preliminary results

In this section, we prove two technical facts needed in the next section. The first is an analog of a result of Walsh (cf. [3, Corollaries 1.2 and 1.3]). This is likely known, but we have not found it in the literature except for $k = 0$, so we give a proof. The second is adapting Proposition 2.1 of [2] to our present context.

Theorem 2.1 *Let k be a nonnegative integer, $\sigma(z) = z + t$. If $f\colon \mathbb{R} \to \mathbb{R}$ is a C^k function which satisfies $f\sigma = f$, and $F \subseteq \mathbb{R}$ is finite, then there is an entire function g such that*

(a) $g(\mathbb{R}) \subseteq \mathbb{R}$ *and* $g\sigma = g$
(b) $|D^i g(x) - D^i f(x)| < \varepsilon$, $x \in \mathbb{R}$, $i = 0, \ldots, k$
(c) $D^i g(x) = D^i f(x)$ *for* $x \in F$, $i = 0, \ldots, k$

Proof If we can arrange (a) and (b), then it follows that we can also arrange (c) by Theorem 0 of [3] applied to the space $X_k = \{f \in C^k(\mathbb{R}) : f\sigma = f\}$ with the norm $\|f\| = \sum_{i=0}^{k} \|D^i f\|_\infty$. (Cf. [3, Example (1), p. 1183, and Corollary 1.3]) So we need only arrange (a) and (b).

It is standard that in X_0 the functions $g(2\pi z/t)$, where g is a trigonometric polynomial with real coefficients, are dense (e.g., see [3, Example (2), p. 1183] or [5, Remark 9.4.24 (2)]).

For the general case, we proceed by induction on k. When $k > 0$, fix $f \in X_k$ and apply the induction hypothesis to Df to get an entire function g_1 satisfying $g_1(\mathbb{R}) \subseteq \mathbb{R}$, $g_1\sigma = g_1$ and $|D^{i-1}g_1(x) - D^i f(x)| < \delta = \min(\varepsilon/2, \varepsilon/(2t))$ for $x \in \mathbb{R}$ and $i = 1, \ldots, k$. Define an entire function g by

$$g(z) = f(0) + \int_0^z g_1(s)\,ds - cz, \quad z \in \mathbb{C},$$

where $c = (1/t)\int_0^t g_1(s)\,ds$ is the average value of g_1 over $[0, t]$. Since $\int_0^t Df = 0$ by periodicity, we have

$$|c| = \left|\frac{1}{t}\int_0^t (g_1(s) - Df(s))\,ds\right| \le \delta.$$

To verify that $g\sigma = g$, it suffices to check that $g(x + t) = g(x)$ when x is real, and that follows by the following calculation.

$$g(x + t) = f(0) + \int_0^{x+t} g_1(s)\,ds - c(x + t) = g(x) + \int_x^{x+t} g_1(s)\,ds - ct = g(x).$$

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For $x \in [0,t]$, we also have

$$|g(x) - f(x)| = \left| \int_0^x (g_1(s) - Df(s))\, ds - cx \right| \\ \leq \int_0^x |g_1(s) - Df(s)|\, ds + |c|x \\ \leq t\delta + t\delta = 2t\delta < \varepsilon,$$

and so, by periodicity, $|g(x) - f(x)| < \varepsilon$ for all $x \in \mathbb{R}$. This takes care of the bound on $|D^i g(x) - D^i f(x)|$, $x \in \mathbb{R}$, when $i = 0$. When $i = 1$ we have

$$|Dg(x) - Df(x)| = |g_1(x) - c - Df(x)| \leq |g_1(x) - Df(x)| + |c| \leq 2\delta < \varepsilon,$$

and when $2 \leq i \leq k$, $|D^i g(x) - D^i f(x)| = |D^{i-1} g_1(x) - D^i f(x)| < \delta < \varepsilon$. ■

Lemma 2.2 *Let $\varepsilon > 0$. Let k be a nonnegative integer and let $F \subseteq [0,t]$ be a finite set such that $\{0,t\} \subseteq F$ or $\{0,t\} \cap F = \emptyset$. For each $p \in F$ and $i = 0, \dots k$, let $k_{p,i} \in \{-1,0,1\}$ with $k_{0,i} = k_{t,i}$ if $0 \in F$. Then there is an entire function $f\colon \mathbb{C} \to \mathbb{C}$ such that $f\sigma = f$, $f(\mathbb{R}) \subseteq \mathbb{R}$ and for $i = 0, \dots, k$ and $x \in \mathbb{R}$, we have $|(D^i f)(x)| < \varepsilon$ and for $p \in F$, $(D^i f)(p)$ is < 0, $= 0$, > 0 when $k_{p,i} = -1, 0, 1$, respectively.*

Proof For $r > 0$ and $p \in \mathbb{R}$, write $I_r(p) = (p - r, p + r)$. For each $p \in F$, choose $0 < r_p < 1$, so that the intervals $I_{r_p}(p)$ are contained in $(0,t)$ when $0 < p < t$, $I_{r_t}(t) = I_{r_0}(0) + t$ if $0 \in F$, and the intervals $I_{r_p}(p)$ for $p \in F$ and have closures which are disjoint from each other. For each $p \in F$, choose a C^∞ bump function φ_p with support equal to the closure of $I_{r_p}(p)$, so that $0 \leq \varphi_p \leq 1$, $\varphi_p(p) = 1$, φ_p has a flat point at p (i.e., $(D^i \varphi_p)(p) = 0$ for all nonzero i), taking $\varphi_t(x) = \varphi_0(x - t)$ if $0 \in F$. Let θ_p be a C^∞ function whose derivatives at p follow the requisite pattern, with $\theta_t(x) = \theta_0(x - t)$ if $0 \in F$, for example, take $\theta_p(x) = \sum_{i=0}^k k_{p,i}(x - p)^i$. For suitably chosen positive constants λ_p, we set $g(x) = \sum_{p \in F} \lambda_p \theta_p(x) \varphi_p(x)$, $0 \leq x \leq t$, and extend g by periodicity to all of $\mathbb{R}$ so that $g\sigma = g$. This function g is C^∞ and its derivatives have the requisite pattern at the points $p \in F$ since for $i = 0, \dots, k$ we have $(D^i g)(p) = \lambda_p (D^i(\varphi_p \theta_p))(p) = \lambda_p D^i \theta_p(p)$. Choose the constants λ_p, $p \in F$, small enough so that for each $i = 0, \dots, k$,

$$\lambda_p \| D^i(\varphi_p \theta_p) \|_\infty < \varepsilon/2.$$

Then g is a C^∞ function which satisfies the conclusion in the place of f and with ε replaced by $\varepsilon/2$. Get the desired entire function f by applying Theorem 2.1 to g with $\varepsilon/2$ in the place of ε. ■

3 Barth–Schneider for periodic functions

Definition 3.1 A *fiber-preserving local homeomorphism* of $\mathbb{R}^2 \cong \mathbb{R} \times \mathbb{R}$ is a homeomorphism $h\colon G_h^1 \to G_h^2$ between two open sets $G_h^1, G_h^2 \subseteq \mathbb{R}^2$ such that h has the form $h(x,y) = (x, h^*(x,y))$ for some continuous map $h^*\colon G_h^1 \to \mathbb{R}$. We write k_h for the inverse of h.

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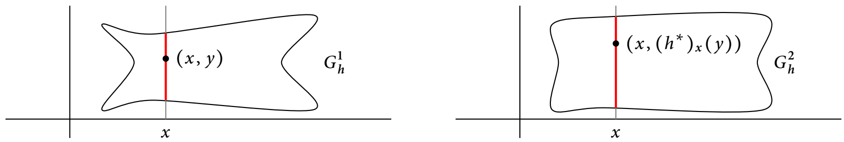
We can identify a fiber-preserving homeomorphism h with the family $\{(h^*)_x\}$ of vertical sections of h^*, given by $(h^*)_x(y) = h^*(x, y)$. These are homeomorphisms $(h^*)_x : (G^1_h)_x \to (G^2_h)_x$, where $(G^1_h)_x$, $(G^2_h)_x$ are the vertical sections at x of G^1_h, G^2_h.



It is separate continuity rather than joint continuity that we require for these fiber-preserving maps. In our context, however, this is not a weaker property as the following proposition shows.

Proposition 3.1 *Let X be any topological space, $G \subseteq X \times \mathbb{R}$ an open set, and let $h: G \to X \times \mathbb{R}$. Assume that for each $x \in X$, $h(G_x) \subseteq \{x\} \times \mathbb{R}$, h is one-to-one and h is separately continuous, i.e., the functions $x \mapsto h(x, y)$, $y \mapsto h(x, y)$ are continuous on their domains. Then h is continuous.*

Taking $h(x, y) = (x, h^*(x, y))$ on $\mathbb{R} \times \mathbb{R}$ with $h^*(x, y) = xy/(x^2 + y^2)$ when $(x, y) \neq (0, 0)$, $h^*(0, 0) = 0$, shows that the assumption that h is one-to-one cannot be omitted.

Proof Write $h(x, y) = (x, h^*(x, y))$. The continuity of h is equivalent to that of h^*. Also note that because h is one-to-one, so is $y \mapsto h^*(x, y)$ for each fixed x. Let $a \in X$, $b, c \in \mathbb{R}$ with $(a, b) \in G$ and $h^*(a, b) = c$. Fix an open neighborhood U of a and an open interval V containing b such that $U \times V \subseteq G$. Choose an open interval W with $c \in W$. Since $y \mapsto h^*(a, y)$ is continuous, there is a $\delta > 0$ such that $|y - b| \leq \delta$ implies $y \in V$ and $h^*(a, y) \in W$. The two functions $x \mapsto h^*(x, b \pm \delta)$ are continuous at a and hence there is an open neighborhood $U' \subseteq U$ of a such that $x \in U'$ implies $h^*(x, b \pm \delta) \in W$. For $x \in U'$, $y \mapsto h^*(x, y)$ is a continuous one-to-one function on the interval $[b - \delta, b + \delta]$, so that the image of this interval is an interval with endpoints $h^*(x, b \pm \delta)$, and hence is contained in W. Thus, $h^*(U' \times (b - \delta, b + \delta)) \subseteq W$ and therefore h^* is continuous at (a, b). ■

Remark 3.2 The inverse k_h of a fiber-preserving local homeomorphism h is also a fiber-preserving local homeomorphism and is related to h by the fact that $(x, y) \in G^1_h$ and $h^*(x, y) = z$ if and only if $(x, z) \in G^2_h$ and $k^*_h(x, z) = y$. As pointed out in the proof of Proposition 3.1, the vertical sections of h^* are one-to-one since h is one-to-one. The main examples for our purposes are $h = \mathrm{id}_{\mathbb{R}^2}$, the identity map on $\mathbb{R}^2$ (for which $(h^*)_x(y) = y$ is the identity map for each $x \in \mathbb{R}$), and for a given a continuous map $g: \mathbb{R} \to \mathbb{R}$, $h^*(x, y) = g(x) + y$. Both of these have $G^1_h = G^2_h = \mathbb{R}^2$.

We shall prove our main results using the following version of Theorem 3.2 of [2]. The proof is similar to that of the original version, but we repeat the argument with the necessary changes in order to be clear.

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Theorem 3.3 *Let $E_i \subseteq [0,t]$, $i = 0,\dots,k$, be countable sets with $\{0,t\} \subseteq E_i$ or $E_i \cap \{0,t\} = \emptyset$. Let $F \subseteq [0,t]$ be a finite set disjoint from $\bigcup_{i=0}^{k} E_i$ with $\{0,t\} \subseteq F$ or $F \cap \{0,t\} = \emptyset$. For each $p \in E_i$, let $A_{p,i} \subseteq \mathbb{R}$ be a countable dense set, with $A_{0,i} = A_{t,i}$ if $\{0,t\} \subseteq A$. Let $\varepsilon > 0$. Let $\mathscr{H}$ be a countable family of fiber-preserving local homeomorphisms of $\mathbb{R}^2$. There exists an entire function $f: \mathbb{C} \to \mathbb{C}$ such that $f(\mathbb{R}) \subseteq \mathbb{R}$, $f\sigma = f$ and for all $x \in \mathbb{R}$ and $i = 0,\dots,k$,*

(a) *$|(D^i f)(x)| < \varepsilon$, and if $x \in F$ then $D^i f(x) = 0$.*
(b) *for each $p \in E_i$, $(D^i f)(p) \in A_{p,i}$.*
(c) *for every $q \in \mathbb{R}$, $h \in \mathscr{H}$ and every open interval $U \subseteq \mathbb{R} \setminus F$, if*

$$(x, (D^i f)(x)) \in G_h^1 \quad \text{and} \quad q = h^*(x, (D^i f)(x))$$

for some $x \in U \cap \operatorname{cl} Y_{h,q,i}$, where $Y_{h,q,i} = \{p \in E_i : \text{for some } q' \in A_{p,i}, (p,q') \in G_h^1$ and $q = h^*(p,q')\}$, then $q = h^*(p, (D^i f)(p))$ for some $p \in U \cap E_i$.

Remark 3.4 (i) For the hypothesis in (c) to be nonvacuous, there must be a point $x \in U \cap \operatorname{cl} E_i \subseteq U \cap [0,t]$, so only intervals which intersect $[0,t]$ are relevant.

(ii) It is sometimes useful to reword clause (c) in an equivalent form. The criterion for $p \in Y_{h,q,i}$ is that $p \in E_i$ and $q \in (h^*)_p(A_{p,i})$. Clause (c) says equivalently that for $x \in [0,t] \setminus F$, if $q = (h^*)_x((D^i f)(x))$ and arbitrarily close to x there are points $p \in E_i$ such that $q \in (h^*)_p(A_{p,i})$, then arbitrarily close to x there are points $p \in E_i$ such that $q = (h^*)_p((D^i f)(p))$.

(iii) If we require $\mathrm{id}_{\mathbb{R}^2} \in \mathscr{H}$ then this ensures that either f is constant or for $x \in [0,t] \setminus F$, $i = 0,\dots,k$, if $q = (D^i f)(x)$ and arbitrarily close to x there are points $p \in E_i$ such that $q \in A_{p,i}$, then $x \in E_i$. To see why, apply the reworded clause (c) in (ii) with h being the identity map to get that if $q = (D^i f)(x)$ and arbitrarily close to x there are points $p \in E_i$ such that $q \in A_{p,i}$, then arbitrarily close to x there are points $p \in E_i$ such that $q = (D^i f)(p)$. If x itself does not belong to E_i, then since $D^i f$ is entire, $D^i f$ is constant with value q and hence f is a polynomial. Since f is periodic, it then follows that f is constant.

Proof We may assume that $\bigcup_{i=0}^{k} E_i \neq \emptyset$ (otherwise take $f = 0$) and that $\varepsilon \leq 1$. Let $\mathscr{B} = \{U_r : r = 1,2,\dots\}$ be a one-to-one enumeration of a base of bounded open intervals for $\mathbb{R} \setminus F$. For each r, write $U_r = \bigcup_{n=1}^{\infty} U_{r,n}$, the union of an increasing sequence of concentric intervals so that $\operatorname{cl} U_{r,n} \subseteq U_{r,n+1}$. Let $\{(i_n, s_n) : n = 1,2,\dots\}$ list all pairs (i,s) consisting of an $i = 0,\dots,k$ and a point $s \in E_i$. Let $Q = \{h^*(p,q') : h \in \mathscr{H}, p \in E_i, q' \in A_{p,i}, (p,q') \in G_h^1, i = 0,\dots,k\}$. List the quadruples (h,j,q,V) consisting of an $h \in \mathscr{H}$, a $j = 0,\dots,k$ and elements $q \in Q$, $V \in \mathscr{B}$ as $\{(h_m, j_m, q_m, V_m) : m = 1,2,\dots\}$ with each quadruple listed infinitely many times. As with the U_r above, we write $V_m = \bigcup_{n=1}^{\infty} V_{m,n}$ where if $V_m = U_r$ then $V_{m,n} = U_{r,n}$. We will write G_m^1, G_m^2, k_m for $G_{h_m}^1$, $G_{h_m}^2$, k_{h_m}, respectively.

We will build the required function as $f = \sum_{n=1}^{\infty} \lambda_n u_n$, where for each $n \in \mathbb{N}$, $\lambda_n \in \mathbb{R}$ satisfies $|\lambda_n| \leq 1$ and $u_n: \mathbb{C} \to \mathbb{C}$ is an entire function such that $u_n(\mathbb{R}) \subseteq \mathbb{R}$. We recursively define the following.

(i) λ_n and u_n.

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(ii) An increasing sequence of finite sets $\emptyset = K_0 \subseteq K_1 \subseteq \dots$ of pairs (w, p) consisting of a $w = 0, \dots, k$ and a point $p \in E_w$.
(iii) A decreasing sequence of positive numbers $1 = \delta_0 > \delta_1 > \dots$.

We will arrange that the following properties hold for $n = 1, 2, \dots$. In this list, f_n denotes the sum $\sum_{k=1}^{n} \lambda_k u_k$, and $f_0 = 0$.

(1) $u_n \sigma = u_n$ and $|(D^i u_n)(x)| < 2^{-n-1}\delta_{n-1}\varepsilon$ for $x \in \mathbb{R}$, $i = 0, \dots, k$.
(2) $(D^w u_n)(p) = 0$ for $(w, p) \in K_{n-1}$.
(3) $(D^i u_n)(x) = 0$ whenever $x \in F$, $i = 0, \dots, k$.
(4) $|u_n(z)| < 2^{-n}$, for $|z| \le n$.
(5) For each $(w, p) \in K_n$, $(D^w f_n)(p) \in A_{p,w}$.
(6) If n is odd then $K_n = K_{n-1} \cup \{(i_\ell, s_\ell)\}$ for the least ℓ such that $(i_\ell, s_\ell) \notin K_{n-1}$. We have $(D^{i_\ell} u_n)(s_\ell) \ne 0$.
(7) If $n = 2m$ is even, we have the following. Suppose that

$$(x, (D^{j_m} f_{n-1})(x)) \in G_m^1 \text{ and } (x, q_m) \in G_m^2 \text{ whenever } x \in V_m.$$

(a) If $q_m = h_m^*(x, (D^{j_m} f_{n-1})(x))$ for some $x \in V_m \cap \operatorname{cl} Y_{h_m,q_m,j_m}$ then for some $p \in V_m \cap Y_{h_m,q_m,j_m}$, $h_m^*(p, (D^{j_m} f_n)(p)) = q_m$ and $K_n = K_{n-1} \cup \{(j_m, p)\}$.
(b) If $q_m \ne h_m^*(x, (D^{j_m} f_{n-1})(x))$ for all $x \in V_m \cap \operatorname{cl} Y_{h_m,q_m,j_m}$ then $u_n = 0$, $K_n = K_{n-1}$ and $\delta_n < \inf\{|k_m^*(x, q_m) - (D^{j_m} f_{n-1})(x)| : x \in V_{m,n} \cap \operatorname{cl} Y_{h_m,q_m,j_m}\}$.

We now explain how to carry out the construction at odd stages and at even stages.

First suppose n is odd. This stage includes the initial step $n = 1$. Set $K_n = K_{n-1} \cup \{(i_\ell, s_\ell)\}$ for the least ℓ such that $(i_\ell, s_\ell) \notin K_{n-1}$. Apply Lemma 2.2 to get an entire function $u_n : \mathbb{C} \to \mathbb{C}$ satisfying (1), (2), (3), and (6). Arrange (4) by replacing u_n by a smaller positive multiple of itself if necessary. Since $(D^{i_\ell} u_n)(s_\ell) \ne 0$ by (6), and A_{s_ℓ,i_ℓ} is dense in $\mathbb{R}$, we may choose λ_n so that $0 < \lambda_n < 1$ and $(D^{i_\ell} f_n)(s_\ell) = (D^{i_\ell} f_{n-1})(s_\ell) + \lambda_n (D^{i_\ell} u_n)(s_\ell) \in A_{s_\ell,i_\ell}$. This gives (5) because if $(w, p) \in K_{n-1}$ then $(D^w u_n)(p) = 0$ by (2), so $(D^w f_n)(p) = (D^w f_{n-1})(p)$ which by (5) for $n-1$ belongs to $A_{p,w}$. For δ_n, choose any positive number satisfying $\delta_n < \delta_{n-1}$.

Now suppose that $n = 2m$ is even. If the assumption of (7) fails, take $u_n = 0$, $\lambda_n = 0$, $K_n = K_{n-1}$, and let δ_n be any number such that $0 < \delta_n < \delta_{n-1}$. Clearly (1)–(7) hold. Now suppose that the assumption is satisfied, i.e., $(x, (D^{j_m} f_{n-1})(x)) \in G_m^1$ and $(x, q_m) \in G_m^2$ whenever $x \in V_m$.

Case 1. $q_m \ne h_m^*(x, (D^{j_m} f_{n-1})(x))$ for all $x \in V_m \cap \operatorname{cl} Y_{h_m,q_m,j_m}$.

Let $u_n = 0$, $\lambda_n = 0$, $K_n = K_{n-1}$. Clearly (1)– (5) hold. For (7), choose any positive number $\delta_n < \delta_{n-1}$ so that $\delta_n < \inf\{|k_m^*(x, q_m) - (D^{j_m} f_{n-1})(x)| : x \in V_{m,n} \cap \operatorname{cl} Y_{h_m,q_m,j_m}\}$ where the right-hand side is positive because $\operatorname{cl}(V_{m,n}) \cap \operatorname{cl}(Y_{h_m,q_m,j_m})$ is compact and contained in $V_m \cap \operatorname{cl} Y_{h_m,q_m,j_m}$.

Case 2. $h_m^*(p, (D^{j_m} f_{n-1})(p)) = q_m$ for some $p \in V_m \cap Y_{h_m,q_m,j_m}$.

By definition, $p \in Y_{h_m,q_m,j_m}$ means that $p \in E_{j_m}$ and there is a (unique) $q' \in A_{p,j_m}$ such that $(p, q') \in G_m^1$ and $q_m = h_m^*(p, q')$. Since the vertical sections of h_m^* are one-to-one, we must have $(D^{j_m} f_{n-1})(p) = q'$ and hence $(D^{j_m} f_{n-1})(p) \in A_{p,j_m}$. Let $u_n = 0$, $\lambda_n = 0$, $K_n = K_{n-1} \cup \{(j_m, p)\}$. Clearly (1)–(5) and (7) hold. For δ_n, choose any positive number satisfying $\delta_n < \delta_{n-1}$.

Case 3. $h_m^*(x, (D^{j_m} f_{n-1})(x)) \neq q_m$ for all $x \in V_m \cap Y_{h_m,q_m,j_m}$, but

$$h_m^*(p, (D^{j_m} f_{n-1})(p)) = q_m$$

for some $p \in V_m \cap \operatorname{cl} Y_{h_m,q_m,j_m}$.

The assumptions give $h_m^*(p, (D^{j_m} f_{n-1})(p)) = q_m$ and $p \notin V_m \cap Y_{h_m,q_m,j_m}$, and since $p \in V_m$ this gives $p \notin Y_{h_m,q_m,j_m}$. It follows from (5) that $(j_m, p) \notin K_{n-1}$ because if $(j_m, p) \in K_{n-1}$ then $p \in E_{j_m}$ and by (5) we have

$$(D^{j_m} f_{n-1})(p) \in A_{p,j_m},$$

which together with $h_m^*(p, (D^{j_m} f_{n-1})(p)) = q_m$ gives $p \in Y_{h_m,q_m,j_m}$, contradiction. Apply Lemma 2.2 to get an entire function $u_n : \mathbb{C} \to \mathbb{C}$ satisfying (1), (2), (3) and $(D^{j_m} u_n)(p) > 0$. Arrange (4) by replacing u_n by a smaller positive multiple of itself if necessary. The function $x \mapsto (k_m^*(x, q_m) - (D^{j_m} f_{n-1})(x))/(D^{j_m} u_n)(x)$ is continuous at p with value 0 there, and $p \in V_m \cap \operatorname{cl} Y_{h_m,q_m,j_m}$, so we can pick an element p' of $V_m \cap Y_{h_m,q_m,j_m}$ so close to p that the number

$$\lambda_n = \frac{k_m^*(p', q_m) - (D^{j_m} f_{n-1})(p')}{(D^{j_m} u_n)(p')}$$

satisfies $|\lambda_n| < 1$. With this value of λ_n, we have $k_m^*(p', q_m) = (D^{j_m} f_n)(p')$ and therefore

$$h_m^*(p', (D^{j_m} f_n)(p')) = q_m,$$

and hence (7) holds if we take $K_n = K_{n-1} \cup \{(j_m, p')\}$. Note that (5) is satisfied. For δ_n, choose any positive number satisfying $\delta_n < \delta_{n-1}$.

This completes the construction. Property (4) ensures that the formula $f = \sum_{n=1}^{\infty} \lambda_n u_n$ defines an entire function and that

$$(D^i f)(z) = \sum_{n=1}^{\infty} \lambda_n (D^i u_n)(z)$$

for all $i = 0, \dots, k$ and $z \in \mathbb{C}$. Clearly $f(\mathbb{R}) \subseteq \mathbb{R}$ and by (1), we have $f\sigma = \sum_{n=1}^{\infty} \lambda_n u_n \sigma = f$. We now verify (a)–(c).

(a) When $i = 0, \dots, k$, $x \in \mathbb{R}$, we have using (1) that $|(D^i f)(x)| \leq \sum_{i=1}^{\infty} |(D^i u_i)(x)| < \sum_{i=1}^{\infty} 2^{-i-1}\varepsilon = \varepsilon$ and, using (3), when $x \in F$ we have $(D^i f)(x) = 0$.

(b) When $p \in E_i$, $i = 0, \dots, k$, (6) ensures that $(i, p) \in K_n$ if n is sufficiently large. Then (5) gives $(D^i f_n)(p) \in A_{p,i}$. From (2), we get that $(D^i u_j)(p) = 0$, $j \geq n+1$, and hence $(D^i f)(p) = (D^i f_n)(p) \in A_{p,i}$.

(c) Suppose $i = 0, \dots, k$, $q \in \mathbb{R}$, $U \subseteq \mathbb{R} \setminus F$ is an open interval, and we have $(p, (D^i f)(p)) \in G_h^1$, and $q = h^*(p, (D^i f)(p))$ for some $p \in U \cap \operatorname{cl} Y_{h,q,i}$. We wish to show that $q = h^*(\bar{p}, (D^i f)(y))$ for some $y \in U \cap E_i$. We may assume that $U \in \mathscr{B}$ and for some open intervals $W_1, W_2 \subseteq \mathbb{R}$, for all $x \in U$,

- $(D^i f)(x) \in W_1 \subseteq \operatorname{cl} W_1 \subseteq W_2$, and $U \times W_2 \subseteq G_h^1$ and
- $(x, q) \in G_h^2$.

Since $D^i f_n \to D^i f$ uniformly on compact sets, for large enough n we have for $x \in U$ that $(D^i f_n)(x) \in W_2$ and hence $(x, (D^i f_n)(x)) \in G_h^1$. By assumption, $p \in \operatorname{cl} Y_{h,q,i}$, so $Y_{h,q,i} \neq \emptyset$ and hence $q \in Q$. Fix r such that $U = U_r$. Choose an even $n = 2m$ with $(h_m, j_m, q_m, V_m) = (h, i, q, U_r)$, n large enough so that $p \in U_{r,n} = V_{m,n}$ and $(x, (D^i f_{n-1})(x)) \in G_h^1$ for all $x \in U$.

Claim. $q = h^*(x, (D^i f_{n-1})(x))$ for some $x \in U \cap \operatorname{cl} Y_{h,q,i}$.

If not, then by (7), $\delta_n < \inf\{|k_m^*(x, q_m) - (D^{j_m} f_{n-1})(x)| : x \in V_{m,n} \cap \operatorname{cl} Y_{h_m,q_m,j_m}\}$ and $u_n = 0$, so that $f_n = f_{n-1}$. We have that for $j > n$, $|(D^i u_j)(p)| < 2^{-j-1}\delta_n$. This gives

$$|(D^i f)(p) - (D^i f_{n-1})(p)| \leq \sum_{j=n}^{\infty} 2^{-j-1}\delta_n < \delta_n < |k_m^*(p, q) - (D^i f_{n-1})(p)|.$$

This contradicts $h^*(p, (D^i f)(p)) = q$ (which is equivalent to $k_h^*(p, q) = (D^i f)(p)$). This completes the proof of the claim.

The claim says that $q_m = h_m^*(x, (D^{j_m} f_{n-1})(x))$ for some $x \in V_m \cap \operatorname{cl} Y_{h_m,q_m,j_m}$, so by (7), for some $y \in V_m \cap Y_{h_m,q_m,j_m}$, $h_m^*(y, (D^{j_m} f_n)(y)) = q_m$ and $K_n = K_{n-1} \cup \{(j_m, y)\}$. But then by (2), $(D^{j_m} u_k)(y) = 0$ if $k > n$, so $h_m^*(y, (D^{j_m} f)(y)) = h_m^*(y, (D^{j_m} f_n)(y)) = q_m$. ■

The next corollary incorporates into the theorem the ability to approximate a given function g such that either $g\sigma = g$ or $g\sigma = \sigma g$. These two types of functions are related by the fact that for a function g, if we write $g = g_1 + \mathrm{id}$, i.e., $g(x) = g_1(x) + x$, then $\sigma g = g\sigma$ if and only if $g_1\sigma = g_1$. Also note that if either $g\sigma = g$ or $g\sigma = \sigma g$ then $(D^i g)\sigma = D^i g$ for $i > 0$ when the derivatives exist.

Corollary 3.5 *(A) Let $g \in C^k(\mathbb{R})$ satisfy $g\sigma = g$. Let $E_i \subseteq [0, t]$, $i = 0, \ldots, k$, be countable sets with $\{0, t\} \subseteq E_i$ or $E_i \cap \{0, t\} = \emptyset$. Let $F \subseteq [0, t]$ be a finite set disjoint from $\bigcup_{i=0}^k E_i$ with $\{0, t\} \subseteq F$ or $F \cap \{0, t\} = \emptyset$. For each $p \in E_i$, let $A_{p,i} \subseteq \mathbb{R}$ be a countable dense set, with $A_{0,i} = A_{t,i}$ if $\{0, t\} \subseteq E_i$. Let $\varepsilon > 0$. Let $\mathscr{H}$ be a countable family of fiber-preserving local homeomorphisms of $\mathbb{R}^2$. There exists an entire function $f : \mathbb{C} \to \mathbb{C}$ such that $f(\mathbb{R}) \subseteq \mathbb{R}$, $f\sigma = f$ and for all $x \in \mathbb{R}$ and $i = 0, \ldots, k$,*

(a) *$|(D^i f)(x) - (D^i g)(x)| < \varepsilon$, and if $x \in F$ then $D^i f(x) = D^i g(x)$.*
(b) *for each $p \in E_i$, $(D^i f)(p) \in A_{p,i}$.*
(c) *for every $q \in \mathbb{R}$, $h \in \mathscr{H}$ and every open interval $U \subseteq \mathbb{R} \setminus F$, if*

$$(x, (D^i f)(x)) \in G_h^1 \text{ and } q = h^*(x, (D^i f)(x))$$

for some $x \in U \cap \operatorname{cl} Y_{h,q,i}$, where $Y_{h,q,i} = \{p \in E_i : \text{for some } q' \in A_{p,i}, (p, q') \in G_h^1 \text{ and } q = h^*(p, q')\}$, then $q = h^*(p, (D^i f)(p))$ for some $p \in U \cap E_i$.

(B) The same statement is true with $g\sigma = g$, $f\sigma = f$ replaced by $g\sigma = \sigma g$, $f\sigma = \sigma f$, respectively, and the condition $A_{t,i} = A_{0,i}$ (when $\{0, t\} \subseteq E_i$) replaced by $A_{t,i} = A_{0,i}$ when $i \neq 0$, $A_{t,0} = A_{0,0} + t$.

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Proof (A) Using Theorem 2.1, we can approximate g by an entire function g_0 so that $g_0(\mathbb{R}) \subseteq \mathbb{R}$, $g_0\sigma = g_0$, and for $x \in \mathbb{R}$, $i = 0, \ldots, k$, $|D^i g_0(x) - D^i g(x)| < \varepsilon/2$ and if $x \in F$ then $D^i g_0(x) = D^i g(x)$. If we approximate g_0 as in Corollary 3.5, replacing ε by $\varepsilon/2$, we get the desired function f. Hence, we may assume that g is entire.

For $p \in E_i$, define $B_{p,i} = A_{p,i} - (D^i g)(p)$. Writing ξ_i for the fiber-preserving homeomorphism of $\mathbb{R}^2$ given by $\xi_i(x, y) = (x, y + (D^i g)(x))$, take $\mathscr{H}_i = \{h \circ \xi_i : h \in \mathscr{H}\}$ and apply Theorem 3.3 with $A_{p,i}$ replaced by $B_{p,i}$ and $\mathscr{H}$ replaced by $\overline{\mathscr{H}} = \bigcup_i \mathscr{H}_i$. (Here, $h \circ \xi_i : \xi_i^{-1}(G_h^1) \to G_h^2$.) If f_1 is the resulting entire function, the function $f = g + f_1$ is as desired. We have $f\sigma = g\sigma + f_1\sigma = g + f_1 = f$. Clauses (a) and (b) are immediate from the corresponding clauses of Theorem 3.3 and the definition of $B_{p,i}$.

For (c), fix an $i = 0, \ldots, k$, $q \in \mathbb{R}$, $h \in \mathscr{H}$, an open interval $U \subseteq \mathbb{R} \setminus F$, and assume that $(x, (D^i f)(x)) \in G_h^1$ and $q = h^*(x, (D^i f)(x))$ for some $x \in U \cap \operatorname{cl} Y_{h,q,i}$. Then $\xi_i(x, (D^i f_1)(x)) \in G_h^1$, so $(x, (D^i f_1)(x)) \in \xi_i^{-1}(G_h^1)$, and $q = (h^* \circ \xi_i)(x, (D^i f_1)(x))$.

Claim. $Y_{h,q,i} \subseteq Y_{h\circ\xi_i,q,i}$ where the set $Y_{h\circ\xi_i,q,i}$ is defined using the $B_{p,i}$ instead of the $A_{p,i}$.

Let $p \in Y_{h,q,i}$. Then $p \in E_i$ and for some $q' \in A_{p,i}$, $(p, q') \in G_h^1$ and $q = h^*(p, q')$. We then have $q' - (D^i g)(p) \in B_{p,i}$, $(p, q' - (D^i g)(p)) = \xi_i^{-1}(p, q') \in \xi_i^{-1}(G_h^1)$ and $q = (h^* \circ \xi_i)(p, q' - (D^i g)(p)) = (h \circ \xi_i)^*(p, q' - (D^i g)(p))$. Thus, $p \in Y_{h\circ\xi_i,q,i}$, which proves the claim.

By the claim, $x \in \operatorname{cl} Y_{h\circ\xi_i,q,i}$ and hence by (c) of Theorem 3.3, $q = (h \circ \xi_i)^*(p, (D^i f_1)(p)) = (h^* \circ \xi_i)(p, (D^i f_1)(p))$ for some $p \in U \cap E_i$. Then $h^*(p, (D^i f)(p)) = (h^* \circ \xi_i)(p, (D^i f_1)(p)) = q$.

(B) This part follows from (A) by an argument similar to that used for (A). Given $g \in C^k(\mathbb{R})$ satisfying $g\sigma = \sigma g$, write $g = g_1 + \mathrm{id}$. As pointed out above, we have $g_1\sigma = g_1$. For $p \in E_i$, define $B_{p,i} = A_{p,i} - (D^i \,\mathrm{id})(p)$. (So $B_{p,0} = A_{p,0} - p$, $B_{p,1} = A_{p,1} - 1$, $B_{p,i} = A_{p,i} (i > 1)$.) Writing ξ_i for the fiber-preserving homeomorphism of $\mathbb{R}^2$ given by $\xi_i(x, y) = (x, y + (D^i \,\mathrm{id})(x))$, take $\mathscr{H}_i = \{h \circ \xi_i : h \in \mathscr{H}\}$ and apply part (A) to g_1 with $A_{p,i}$ replaced by $B_{p,i}$ and $\mathscr{H}$ replaced by $\overline{\mathscr{H}} = \bigcup_i \mathscr{H}_i$. (Here, $h \circ \xi_i : \xi_i^{-1}(G_h^1) \to G_h^2$.) If f_1 is the resulting entire function, the function $f = \mathrm{id} + f_1$ is as desired. Since f_1 satisfies $f_1\sigma = f_1$, it follows that f satisfies $f\sigma = \sigma f$. Clauses (a) and (b) of part (B) are immediate from the corresponding clauses of part (A) and the definition of $B_{p,i}$. For (c), as in part (A), fix an $i = 0, \ldots, k$, $q \in \mathbb{R}$, $h \in \mathscr{H}$, an open interval $U \subseteq \mathbb{R} \setminus F$, and assume that $(x, (D^i f)(x)) \in G_h^1$ and $q = h^*(x, (D^i f)(x))$ for some $x \in U \cap \operatorname{cl} Y_{h,q,i}$. Then $\xi_i(x, (D^i f_1)(x)) \in G_h^1$, so $(x, (D^i f_1)(x)) \in \xi_i^{-1}(G_h^1)$, and $q = (h^* \circ \xi_i)(x, (D^i f_1)(x))$. As in the proof of (A), we get $Y_{h,q,i} \subseteq Y_{h\circ\xi_i,q,i}$ where the set $Y_{h\circ\xi_i,q,i}$ is defined using the $B_{p,i}$ instead of the $A_{p,i}$. (In the proof of the claim in part (A), replace the four g's by id.) Then finish exactly as in part (A), reading "by (c) of part (A)" instead of "by (c) of Theorem 3.3." ∎

Theorem 3.6 *Let (A_n^i, B_n^i), $i = 0, \ldots, k$, $n \in \mathbb{N}$, be pairs of countable dense subsets of $\mathbb{R}$ invariant under σ such that for each fixed i, the A_n^i are pairwise disjoint. Assume also that the B_n^0 are pairwise disjoint. Let $F \subseteq \mathbb{R}$ be a finite set disjoint from each A_n^i. Fix $\varepsilon > 0$.*

Then for each $g \in C^k(\mathbb{R})$ with $k \geq 1$ such that $g\sigma = \sigma g$, $Dg > 0$, and $g(F) \cap B_n^0 = \emptyset$ for all $n \in \mathbb{N}$, there is an entire function f such that for $i = 0, \ldots, k$, $n \in \mathbb{N}$, and $x \in \mathbb{R}$,

(a) $f(\mathbb{R}) \subseteq \mathbb{R}$, $f\sigma = \sigma f$, $Df(x) > 0$

(b) $|(D^i f)(x) - (D^i g)(x)| < \varepsilon$, *and if* $x \in F$ *then* $D^i f(x) = D^i g(x)$

(c) $D^i f(A_n^i) \subseteq B_n^i$, $f(A_n^0) = B_n^0$

Proof For $i = 0, \ldots, k$, let $E_i = [0, t] \cap \bigcup_{n=1}^{\infty} A_n^i$. For pairs (p, i) with $p \in E_i$, for some unique n we have $p \in A_n^i$. Define $A_{p,i} = B_n^i$. If $0, t \in E_i$, then for some n we have $0, t \in A_n^i$. Then $A_{0,i} = A_{t,i} = B_n^i$ and therefore also $\sigma(A_{0,i}) = A_{t,i}$ by invariance of B_n^i. The orbit of each point has at most two points in $[0, t]$, so $F_1 = [0, t] \cap \bigcup_{\ell \in \mathbb{Z}} \sigma^\ell(F)$ is finite. We have $E_i \cap F_1 = \emptyset$ for each i, because $E_i \cap F_1 \subseteq (\bigcup_{n=1}^{\infty} A_n^i) \cap (\bigcup_{\ell=1}^{\infty} \sigma^\ell(F))$ and for all n, ℓ, $A_n^i \cap \sigma^\ell(F) = \sigma^\ell(A_n^i \cap F) = \emptyset$. Similarly, $g(F_1) \cap B_n^0 = \emptyset$ because $g(\sigma^\ell(F)) \cap B_n^0 = \sigma^\ell(g(F) \cap B_n^0) = \emptyset$ for all n, ℓ. Take $\varepsilon' = \min(\varepsilon, \inf\{Dg(x) : x \in \mathbb{R}\})$ which is positive by periodicity of Dg and since $Dg > 0$. Corollary 3.5 (B) with $\mathscr{H} = \{\mathrm{id}_{\mathbb{R}^2}\}$ gives f such that $f(\mathbb{R}) \subseteq \mathbb{R}$, $f\sigma = \sigma f$ and for all $x \in \mathbb{R}$ and $i = 0, \ldots, k$,

(i) $|(D^i f)(x) - (D^i g)(x)| < \varepsilon'$, and if $x \in F_1$ then $D^i f(x) = D^i g(x)$.

(ii) for each $p \in E_i$, $(D^i f)(p) \in A_{p,i}$.

(iii) for any $q \in \mathbb{R}$, if $q = f(x)$ for some $x \in \mathrm{cl}\{p \in E_0 : q \in A_{p,0}\}$, $x \notin F_1$, then $q = f(p)$ for some $p \in E_0$. (Since f is injective, necessarily $x = p$.)

From (i) we get (b). From (i) we also get for $x \in \mathbb{R}$, $|Df(x) - Dg(x)| < \varepsilon' \leq Dg(x)$, so $Df(x) > 0$ giving (a). For (c), fix $i = 0, \ldots, k$, $n \in \mathbb{N}$. We want to show that $D^i f(A_n^i) \subseteq B_n^i$, and when $i = 0$, $f(A_n^0) = B_n^0$. For $p \in A_n^i$ we have for some ℓ that $\sigma^\ell(p) \in [0, t]$. When $i = 0$, we get, using (ii), $f(p) = \sigma^{-\ell} f(\sigma^\ell(p)) \in \sigma^{-\ell} A_{\sigma^\ell(p),0} = B_n^0$. When $i > 0$, we get similarly $D^i f(p) = D^i f(\sigma^\ell(p)) \in A_{\sigma^\ell(p),i} = B_n^i$.

There remains to show that $B_n^0 \subseteq f(A_n^0)$. Let $q \in B_n^0$. Since $f(t) = f(0) + t$, for some ℓ, $\sigma^\ell(q) \in [f(0), f(t)]$ and therefore, for some $x \in [0, t]$, $f(x) = \sigma^\ell(q)$. We have $x \notin F_1$ because otherwise $g(x) = f(x) = \sigma^\ell(q) \in B_n^0$, contradicting the fact that $g(F_1) \cap B_n^0 = \emptyset$. Since A_n^0 is dense, arbitrarily close to x there are $p \in A_n^0 \cap [0, t]$ and these satisfy $\sigma^\ell(q) \in B_n^0 = A_{p,0}$, so by (iii), there is a $p \in E_0$ such that $\sigma^\ell(q) = f(p)$ and therefore $f(\sigma^{-\ell}(p)) = q$. This gives $q \in f(A_n^0)$ as long as $p \in A_n^0$. Let m be such that $p \in A_m^0$. Then $\sigma^\ell(q) = f(p) \in f(A_m^0) \subseteq B_m^0$. Since B_m^0 and B_n^0 are disjoint, it follows that $m = n$. ■

4 Order-isomorphisms of the arc $T \setminus \{1\}$

Applying Theorem 3.6 we can transfer the Barth–Schneider theorem for the real line to the following analog for the arc $T \setminus \{1\}$.

Theorem 4.1 *Let (A_n, B_n), $n \in \mathbb{N}$, be countable dense subsets of $T \setminus \{1\}$. Assume that $A_n \cap A_m = \emptyset$ and $B_n \cap B_m = \emptyset$ when $n \neq m$. Let $g\colon T \to T$ satisfy $g(e^{i\theta}) = e^{i\beta(\theta)}$, $0 \leq \theta \leq 2\pi$, where $\beta\colon [0, 2\pi] \to [0, 2\pi]$ is a C^1 bijection such that $D\beta > 0$ and $D\beta(0) = D\beta(2\pi)$. Let $F \subseteq T \setminus \{1\}$ be a finite set so that F is disjoint from each A_n and $g(F)$ is disjoint from each B_n. Then there is an analytic function $h\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ that restricts to an order-preserving bijection of the arc $T \setminus \{1\}$ onto itself and satisfies the following.*

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(a) $|h(z) - g(z)| < \varepsilon$ *for* $z \in T$ *and* $h(z) = g(z)$ *for* $z \in F$.
(b) $h(A_n) = B_n (n \in \mathbb{N})$.
(c) $Dh(z) \neq 0$ *when* $z \in T$.

Remark 4.2 (i) It will be apparent from the proof that we can require that $h(e^{i\theta}) = e^{i\alpha(\theta)}$ where $\alpha: [0, 2\pi] \to [0, 2\pi]$ is a bijection which is the restriction of an entire function, and $D\alpha(\theta) > 0$ and $|D\alpha(\theta) - D\beta(\theta)| < \varepsilon$ for $\theta \in [0, 2\pi]$.

(ii) We take the ordering on the arc $T \setminus \{1\}$ to be counterclockwise. We could replace "order-preserving" by "order-reversing." In this context, $D\beta > 0$ should be replaced by $D\beta < 0$. To get the desired conclusion, define $\bar{A}_n = \{\bar{z} : z \in A_n\}$, $\bar{F} = \{\bar{z} : z \in F\}$. Then define $\tilde{\beta}(\theta) = \beta(2\pi - \theta)$, $\tilde{g}(z) = g(1/z) = e^{i\tilde{\beta}(\theta)}$. Note that $\bar{F} \cap \bar{A}_n = \overline{F \cap A_n} = \emptyset$ and $\tilde{g}(\bar{F}) \cap B_n = g(F) \cap B_n = \emptyset$. From Theorem 4.1, get $\tilde{h}$ satisfying (a)–(c) with respect to $\tilde{g}$, $\bar{F}$, and the pairs $(\bar{A}_n, B_n)$. Then $h(z) = \tilde{h}(1/z)$ gives the desired order-reversing bijection of the arc $T \setminus \{1\}$.

(iii) The order-preserving bijection from the arc $T \setminus \{1\}$ to itself can be replaced by an order-preserving bijection $T \setminus \{p\} \to T \setminus \{q\}$ (where $p \in T \setminus A_n$, $q \in T \setminus B_n$) by considering the rotated sets $\bar{p}A_n$ and $\bar{q}B_n$.

Proof Define $F' = \{0\} \cup \{\theta \in (0, 2\pi) : e^{i\theta} \in F\}$, and define

$$A'_n = \{\theta \in \mathbb{R} : e^{i\theta} \in A_n\}, \quad B'_n = \{\theta \in \mathbb{R} : e^{i\theta} \in B_n\}.$$

The assumption on β ensures that it extends to a C^1 function on $\mathbb{R}$ satisfying $\sigma\beta = \beta\sigma$. The sets A'_n, B'_n are countable, dense, and invariant under translation by $t = 2\pi$. The A'_n are pairwise disjoint, as are the B'_n, and the conditions $F \cap A_n = \emptyset$, $g(F) \cap B_n = \emptyset$, $1 \notin A_n, B_n$ imply the conditions $F' \cap A'_n = \emptyset$, $\beta(F') \cap B'_n = \emptyset$. Choose $\delta > 0$ so that $D\beta(\theta) > \delta$, $\theta \in \mathbb{R}$, and $|\theta_1 - \theta_2| < \delta$ implies $|e^{i\theta_1} - e^{i\theta_2}| < \varepsilon$. Apply Theorem 3.6 to get an entire function α such that such that $\alpha(\mathbb{R}) \subseteq \mathbb{R}$, $\alpha\sigma = \sigma\alpha$, and

(i) $|\alpha(\theta) - \beta(\theta)| < \delta$ for $\theta \in \mathbb{R}$ and $\alpha(\theta) = \beta(\theta)$ for $\theta \in F'$.
(ii) $|D\alpha(\theta) - D\beta(\theta)| < \delta$ for $\theta \in \mathbb{R}$.
(iii) $\alpha(A'_n) = B'_n (n \in \mathbb{N})$.
(iv) $D\alpha > 0$.

(Note that (iv) follows from (ii) and the choice of δ.) From (i) we get $\alpha(0) = \beta(0) = 0$. For the function $\gamma(z) = \alpha(z) - z$ we have $\gamma(z + 2\pi) = \gamma(z)$, $\alpha(z) = z + \gamma(z)$. Consider the branches of log

$$\begin{aligned} \operatorname{Log} z &= \ln|z| + i \operatorname{Arg} z \quad (-\pi < \operatorname{Arg} z < \pi), \\ \log z &= \ln|z| + i \arg z \quad (0 < \arg z < 2\pi). \end{aligned}$$

We have $\operatorname{Log} z = \log z$ when $\operatorname{Im} z > 0$ and $\log z = \operatorname{Log} z + 2\pi i$ when $\operatorname{Im} z < 0$. By the periodicity of γ we have

$$\gamma(-i \log z) = \gamma(-i \operatorname{Log} z) \quad (\operatorname{Im} z \neq 0). \tag{4.1}$$

Define

$$h(z) = e^{i\alpha(-i \log z)}$$

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for $z \neq 0$ not a positive real number. We have

$$h(z) = e^{i\alpha(-i\log z)} = e^{i(-i\log z)}e^{i\gamma(-i\log z)} = ze^{i\gamma(-i\log z)}.$$

Similarly, if we set $H(z) = e^{i\alpha(-i\,\mathrm{Log}\,z)}$ when $z \neq 0$ is not a negative real number, we get $H(z) = ze^{i\gamma(-i\,\mathrm{Log}\,z)}$, and by (4.1), $h(z) = H(z)$ for $\mathrm{Im}\,z \neq 0$. Thus, h extends to an analytic function defined on $\mathbb{C} \setminus \{0\}$ by setting $h(z) = H(z)$ when z is a positive real number.

When $z = e^{i\theta}$, $0 < \theta < 2\pi$, we have $-i\log z = \theta$, so $h(z) = e^{i\alpha(\theta)}$. As θ runs over $(0, 2\pi)$ from 0 to 2π, $\alpha(\theta)$ does the same since $\alpha(0) = 0$ and $\alpha(2\pi) = \alpha(0) + 2\pi = 2\pi$, so $h(z) = h(e^{i\theta}) = e^{i\alpha(\theta)}$ runs over the arc $T \setminus \{1\}$ counterclockwise from 1 to 1. From (i) and the choice of δ we get

$$|h(z) - g(z)| = |h(e^{i\theta}) - g(e^{i\theta})| = |e^{i\alpha(\theta)} - e^{i\beta(\theta)}| < \varepsilon$$

and this implies the first part of (a). If $\theta \in F'$ then $h(e^{i\theta}) = e^{i\alpha(\theta)} = e^{i\beta(\theta)} = g(e^{i\theta})$ and this implies the second part of (a).

Also, for $0 < \theta < 2\pi$ and $n \in \mathbb{N}$,

$$e^{i\theta} \in A_n \Leftrightarrow \theta \in A'_n \Leftrightarrow \alpha(\theta) \in B'_n \Leftrightarrow e^{i\alpha(\theta)} \in B_n,$$

so that for $z \in T \setminus \{1\}$, we have $z \in A_n$ if and only if $h(z) \in B_n$. Hence h restricts to an order-isomorphism of $T \setminus \{1\}$ mapping A_n onto B_n.

There remains to verify that $Dh(z) \neq 0$ when $|z| = 1$. When $|z| = 1$, $z \neq 1$, we have

$$Dh(z) = \frac{d}{dz}e^{i\alpha(-i\log z)} = e^{i\alpha(-i\log z)} \cdot i \cdot D\alpha(-i\log z) \cdot \frac{-i}{z},$$

which is nonzero since $-i\log z$ is real when $|z| = 1$ and $D\alpha(\theta) \neq 0$ for $\theta \in \mathbb{R}$. Writing Log instead of log we have the analogous computation when $|z| = 1$, $z \neq -1$. ∎

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