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# GROWTH AND FIRM DYNAMICS WITH HORIZONTAL AND VERTICAL R&D

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A negative or nonsignificant empirical correlation between aggregate R&D intensity and the economic growth rate is a well-known fact in the empirical growth literature, but scarcely addressed in the theoretical growth literature. This paper develops an endogenous-growth model that explores the interrelation between horizontal and vertical R&D under a lab-equipment specification that is consistent with that stylized fact. A key feature is that the growth rate is fully endogenous both on the intensive and on the extensive margin. Strong composition effects between horizontal and vertical R&D, along both transition and the balanced-growth path, then emerge as the main mechanism producing those results. This setting also allows us to obtain a relationship between economic growth and firm dynamics that is consistent with the empirical facts.

**Keywords:** Endogenous Growth, Vertical and Horizontal R&D, Firm Dynamics, Piecewise Smooth Dynamics

## 1. INTRODUCTION

In the empirical growth literature, it is a well-known fact that R&D intensity, measured as the fraction of output devoted to R&D, and the economic growth rate tend to exhibit a nonsignificant or negative correlation, both in the

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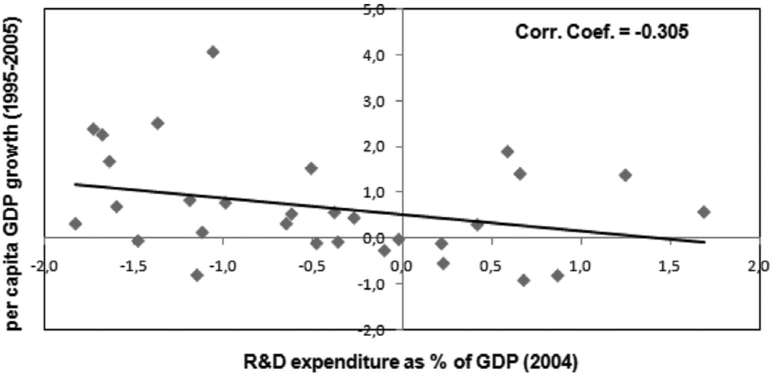


**FIGURE 1.** R&D expenditure over GDP versus per capita GDP growth rate—cross-sectional evidence for the OECD countries. *Source*: *OECD in Figures 2006–2007*, available online at www.oecd.org/infigures.

cross-sectional and in the time-series perspective [e.g., Backus et al. (1992); Bassanini et al. (2000); Pintea and Thompson (2007)]. For example, in a cross section of OECD countries, the simple correlation between business R&D expenditure over GDP and total factor productivity growth was −0.03 in the 1980s and 0.03 in the 1990s [Bassanini et al. (2000)], and an even stronger negative correlation has been observed between R&D expenditure over GDP and the per capita GDP growth, for more recent years (Figure 1). As regards time series data, business R&D spending has been growing faster than GDP in the United States and other developed countries over the past decades. In the United States, for instance, business R&D expenditures have grown from 0.63% of GDP in 1953 to 2.01% of GDP in 2008. At the same time, the per capita GDP growth rate has been declining slightly [Pintea and Thompson (2007)]. Figure 2 depicts similar data for Japan and Denmark, where a growing R&D intensity has been observed together with a decreasing or stagnant economic growth rate.

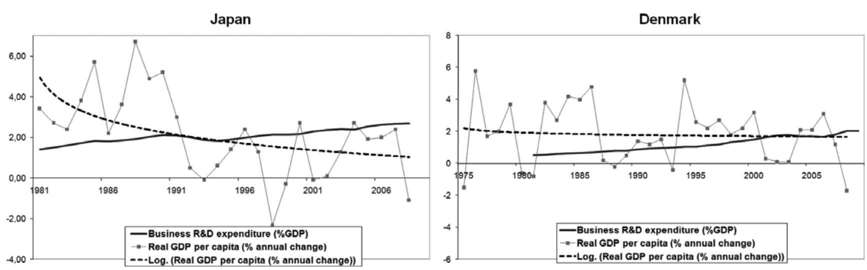


**FIGURE 2.** R&D expenditure over GDP and per capita GDP growth rate—time-series evidence for Japan and Denmark. *Source*: Eurostat online database, available at http://epp.eurostat.ec.europa.eu.

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Despite these well-known facts, few explanations have been put forward for the growth rate–R&D intensity relationship.[1]

A notable exception is the paper by Pintea and Thompson (2007). However, this paper addresses the time-series perspective, but is silent as regards the cross-sectional relationship. It proposes an endogenous-growth model of vertical R&D, skills accumulation, and learning, whose transitional dynamics can be confronted with the data. New product generations, which raise the quality of a product line, arrive according to an exogenous process. While manufacturing, firms learn how to further enhance the relative quality of their products by employing skilled labor. R&D intensity is measured as the fraction of skilled labor devoted to (quality-improving) R&D. A one-time increase in complexity because of the arrival of a new technological paradigm slowly diffuses (also according to an exogenous process) through the economy as firms gradually adopt new product generations embodying the new paradigm. As a result, despite the rise in R&D intensity, because of the increase in the number of adopters, the economic growth rate declines, because diffusion involves aggregate diminishing returns.

In our paper, an alternative—entirely endogenous—analytical mechanism is considered, within an endogenous-growth model with simultaneous horizontal (expanding number of varieties) and vertical R&D (quality ladders). The transitional dynamics and the long-run (balanced-growth path, BGP) behavior of our model are consistent with both the time-series and the cross-sectional evidence. We measure the R&D intensity by the sum of vertical and horizontal R&D as a fraction of output. The negative relationship between the economic growth rate and the R&D intensity then stems from the (endogenous) interrelation between vertical and horizontal R&D, both operating along time and across economies.

Consider an economy out of the BGP characterized by, for instance, a shallow-market regime, displaying a small number of varieties relative to the technological-knowledge stock (measured by the economywide quality index). Such an economy starts with a high growth rate for the number of varieties, a small vertical-innovation rate, and a high economic growth rate relative to the BGP. Along the transition path, the resources allocated to horizontal R&D are gradually retargeted to vertical R&D. However, the positive impact of the latter on the economic growth rate is more than compensated for by the downward movement in the former. This negative net result reflects the asymmetry between the influence of the specialization effect related to the number of varieties, which has a one-to-one impact on economic growth, and the efficiency effect of higher quality levels, which is dampened by a creative-destruction effect. On the other hand, a falling consumption rate implies an increasing investment rate and hence an increase in R&D intensity. Thus, along the transition path, the economic growth rate falls, whereas R&D intensity increases. Alternatively, if the economy starts from a deep-market regime, a negative relationship between R&D intensity and the economic growth rate will also arise, but now with the former falling and the latter increasing along the adjustment path.

In turn, the long-run relationship between the economic growth rate and R&D intensity is assessed by performing a comparative-statics analysis where the BGP is shifted by technological parameter changes. Thus, we can compare the results of the model with the data from the perspective of the cross-sectional correlation between those variables, under the assumption that there is heterogeneity across countries in any or all of those parameters. A mixed result is obtained, as the sign of the comparative-statics results depends upon the source of the change in the general equilibrium, thereby lending theoretical support to the previously highlighted lack of a clear-cut empirical relationship between those variables. On one hand, the same mechanism that generates the negative correlation between the economic growth rate and R&D intensity in the transitional dynamics also applies to the comparative-statics results. However, the response of the effective rate of return rewarding R&D activity to variations in some parameters is a new effect arising in this context, and may induce a reversal of that correlation.

The joint consideration of horizontal and vertical R&D allows us also to address the relationship between economic growth and firm dynamics, measured by size and/or the number of firms. The literature identifies an unclear empirical relationship between the economic growth rate and firm size [see Pagano and Schivardi (2003)], and our model allows a discussion of the origins of that fact.

Summing up, we show that considering innovation as the ultimate source of economic growth [see, e.g., Samaniego (2007)] is compatible with the empirical evidence concerning the link between R&D intensity and both firm size and the economic growth rate.

Concerning the formal setup, we develop a dynamic general equilibrium model where we consider two R&D sectors, one targeting vertical innovation, by which entrants increase the quality of an existing variety and hence substitute for the incumbent—i.e., there is a creative-destruction effect [e.g., Aghion and Howitt (1992)]—and the other targeting horizontal innovation, by which entrants create a new variety [e.g., Romer (1990)].[2]

We use a lab-equipment R&D specification, whereas the literature on simultaneous vertical and horizontal R&D typically assumes that R&D is knowledge-driven.[3] In the latter, the allocation of resources between vertical and horizontal R&D implies a division of labor between the two types of R&D. Because the total labor level is determined exogenously, there is a shortcoming: in the end, the rate of extensive growth is exogenous—i.e., the BGP flow of new goods occurs at the same rate as (or is proportional to) population growth. The alternative assumption that R&D is of the lab-equipment type implies that the allocation of resources between vertical and horizontal innovation is related to the splitting of R&D expenditures, which are fully endogenous. Therefore, we endogenize the rate of extensive growth, which will be a crucial feature in generating the inter-R&D composition effects and hence the negative correlation between R&D intensity and the economic growth rate that characterizes our results. In the existing models featuring a knowledge-driven specification [e.g., Dinopoulos and Thompson (1998); Peretto (1998); Howitt (1999); Peretto and Smulders (2002); Peretto and

Connolly (2007)], inter-R&D composition effects are not able to produce that kind of negative correlation, because the growth rate tends to change in response to vertical innovation only.[4]

The remainder of the paper is organized as follows. In Section 2, we present the model, giving a detailed account of the production, price, and R&D decisions, and derive the dynamic general equilibrium. In Section 3, we characterize the interior BGP and the transitional dynamics, and then discuss their consistency with the empirical literature. In Section 4, we briefly discuss Pareto optimality. Section 5 concludes.

## 2. THE MODEL

We consider a closed economy where a single competitively produced final good can be used in consumption, production of intermediate goods, and R&D. The final good is produced by a (large) number of firms, each using labor and a continuum of intermediate goods indexed by $\omega \in [0, N]$. The economy is populated by $L$ identical dynastic families, each endowed with one unit of labor that is inelastically supplied to final-good firms. Thus, the total labor level is $L$, which, by assumption, is constant over time. In turn, families make consumption decisions and invest in firms' equity.

A potential entrant can devote resources either to horizontal or to vertical R&D. Horizontal R&D increases the number of intermediate-good industries $N$, whereas vertical R&D increases the quality of the good of an existing industry, indexed by $j(\omega)$. The number of intermediate-good varieties can be taken as a measure of a specialization effect on final-good productivity. The quality level $j(\omega)$ of each variety translates into productivity through an efficiency effect from using the good produced by industry $\omega$, $\lambda^{j(\omega)}$, where $\lambda > 1$ is a parameter measuring the size of every quality upgrade. By improving on the current best-quality index $j$, a successful R&D firm will introduce the leading-edge quality $j(\omega) + 1$ and hence render the existing input inefficient. Thus, the successful innovator will become a monopolist in $\omega$. However, this monopoly, and the monopolist earnings that come with it, are temporary, because a new successful innovator will eventually replace the incumbent.

### 2.1. Production and Price Decisions

The final-good firm has a constant–returns to scale technology using labor and a continuum of intermediate goods with measure $N(t)$. The final-good output at time $t$ is

$$Y(t) = A_0 \cdot L^{1-\alpha} \cdot \int_0^{N(t)} \left[\lambda^{j(\omega,t)} \cdot X(\omega, t)\right]^\alpha d\omega, \quad 0 < \alpha < 1, \quad \lambda > 1, \quad \textbf{(1)}$$

where $A_0 > 0$ is the total factor productivity, $L$ is the labor input, and $1 - \alpha$ is the labor share in production, and $\lambda^{j(\omega,t)} \cdot X(\omega, t)$ is the input of intermediate

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good $\omega$ measured in efficiency units at time $t$.[5] That is, we integrate the final-producer technologies that are considered in variety-expansion [Barro and Sala-i-Martin (2004, Ch. 6)] and quality-ladders [Barro and Sala-i-Martin (2004, Ch. 7)] models.

Final-good producers are price-takers in all the markets they participate in. They take wages, $w(t)$, and input prices, $p(\omega, t)$, as given and sell their output at a price equal to unity. From the profit-maximization conditions, we determine the demand for the intermediate good $\omega$:

$$X(\omega, t) = L \cdot \left[\frac{A_0 \cdot \alpha \cdot \lambda^{j(\omega,t)\alpha}}{p(\omega, t)}\right]^{\frac{1}{1-\alpha}}, \ \omega \in [0, N(t)]. \quad \mathbf{(2)}$$

The intermediate-good sector consists of a continuum of $N(t)$ industries. There is monopolistic competition if we consider the whole sector: the monopolist in industry $\omega$ fixes the price $p(\omega, t)$ but faces the isoelastic demand curve (2). We assume that the intermediate good is nondurable and entails a unit marginal cost of production, in terms of the final good, whose price is taken as given. Profit in industry $\omega$ is thus $\pi(\omega, t) = [p(\omega, t)-1]\cdot X(\omega, t)$, and the profit-maximizing price is a constant markup over marginal cost, $p(\omega, t) \equiv p = 1/\alpha > 1$.[6] We denote the quality of the intermediate good $\omega$ by the index $q(\omega, t) = q(j) \equiv \lambda^{j(\omega,t)\alpha/(1-\alpha)}$. Then the production

$$X(\omega, t) = L \cdot (A_0 \cdot \alpha^2)^{\frac{1}{1-\alpha}} q(\omega, t), \quad \mathbf{(3)}$$

and the profit

$$\pi(\omega, t) = \pi_0 \cdot L \cdot q(\omega, t), \quad \mathbf{(4)}$$

accrued by the monopolist in $\omega$, are both linear functions of $q(\omega, t)$. The constant $\pi_0 \equiv (1-\alpha)\alpha^{2/(1-\alpha)} A_0^{1/(1-\alpha)}/\alpha$ can be seen as "basic" profit, which accrues when $j = 0$ (i.e., $q = 1$).

The aggregate quality index

$$Q(t) = \int_0^{N(t)} q(\omega, t) d\omega \quad \mathbf{(5)}$$

measures the technological-knowledge level of the economy. This implies that aggregate output,

$$Y(t) = \left(A_0^{\frac{1}{\alpha}} \alpha^2\right)^{\frac{\alpha}{1-\alpha}} \cdot L \cdot Q(t) = A_Y \cdot L \cdot Q(t), \quad \mathbf{(6)}$$

where $A_Y \equiv (A_0^{\frac{1}{\alpha}} \cdot \alpha^2)^{\frac{\alpha}{1-\alpha}}$, total resources devoted to intermediate-goods production,

$$X(t) = \int_0^{N(t)} X(\omega, t) d\omega = (A_0 \alpha^2)^{\frac{1}{1-\alpha}} \cdot L \cdot Q(t) = A_X \cdot L \cdot Q(t), \quad \mathbf{(7)}$$

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where $A_X \equiv (A_0 \cdot \alpha^2)^{\frac{1}{1-\alpha}}$, and total profits,

$$\Pi(t) = \int_0^{N(t)} \pi(\omega, t) d\omega = \pi_0 \cdot L \cdot Q(t), \qquad \textbf{(8)}$$

are all linear functions of $Q(t)$.

## 2.2. R&D

We consider two R&D sectors, one targeting vertical innovation—which can be seen as pertaining to process innovation or incremental product innovation—and the other targeting horizontal innovation—pertaining to radical product innovation. The technological-knowledge stock, $Q(t)$ in (5), has two components: a quantity component, $N$, and a quality component, $q$. The quantity component, which is targeted by horizontal R&D, behaves in a way similar to physical capital: it can be accumulated, in the sense that technological knowledge may be disseminated, and it can be reversed, in the sense that some sectors may be destroyed. The second component, $q$, is irreversible, in the sense that after the invention is done, it cannot be uninvented. Thus, there is asymmetry between horizontal and vertical R&D, between dissemination of technology and creation of new technologies, or between technology embodied in new products and in improved processes/products.

We assume that the pools of innovators performing the two types of R&D are different. We also make the simplifying assumptions that both vertical and horizontal R&D are performed by (potential) entrants, and that successful R&D leads to the setup of a new firm in either an existing or a new industry [e.g., Howitt (1999); Segerstrom (2000); Cozzi and Spinesi (2006); Strulik (2007)]. There is perfect competition among entrants and free entry in the R&D business.

*Vertical R&D.* As is common in the literature [e.g., Aghion and Howitt (1992); Barro and Sala-i-Martin (2004, Ch. 7)], every new design is granted a patent and thus a successful innovator retains exclusive rights over the use of his/her good. By improving on the current top quality level $j(\omega, t)$, a successful R&D firm earns monopoly profits from selling the leading-edge input of quality $j(\omega, t) + 1$ to final-good firms. A successful innovation will instantaneously increase the quality index in $\omega$ from $q(j)$ to $q(j+1) = \lambda^{\alpha/(1-\alpha)} q(j)$. At equilibrium, lower qualities within sector $\omega$ are priced out of business.

Let $I_i(j)$ denote the Poisson arrival rate of vertical innovation introduced by potential entrant $i$ in industry $\omega$ when the highest quality is $j$. The rate $I_i(j)$ is independently distributed across firms, across industries, and over time and depends on the flow of resources $R_{vi}(j)$ committed by entrants at time $t$. Moreover, $I_i(j)$ features constant returns in R&D expenditures, $I_i(j) = R_{vi}(j) \cdot \Phi(j)$, where $\Phi(j)$ is the R&D productivity factor, which is assumed to be homogeneous across

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$i$ in $\omega$. We assume

$$\Phi(j) = \frac{1}{\zeta \cdot L \cdot q(j+1)}, \tag{9}$$

where $\zeta > 0$ is a constant fixed (flow) cost. Equation (9) incorporates a complexity effect [e.g., Barro and Sala-i-Martin (2004, Ch. 7); Etro (2008)], implying the existence of vertical-R&D dynamic decreasing returns to scale (i.e., decreasing returns to cumulated R&D). That is, the higher the level of quality, $q$, the costlier it is to improve it further.[7] Moreover, (9) also implies that an increase in market scale, $L$, dilutes the effect of R&D outlays on innovation probability. Overall, this may happen because of coordination, organizational, and transportation costs [e.g. Dinopoulos and Thompson (1999)] and rental protection actions by incumbents [e.g., Sener (2008)] which are (positively) related to market size. These assumptions allow us to avoid the usual scale effect arising from the aggregate labor level. Aggregating across $i$ in $\omega$, we get $R_v(j) = \sum_i R_{vi}(j)$ and $I(j) = \sum_i I_i(j)$, and thus the vertical-innovation rate becomes

$$I(j) = R_v(j) \cdot \frac{1}{\zeta \cdot L \cdot q(j+1)}, \tag{10}$$

where $I(j) = I(\omega, t)$ is time-varying.

As the terminal date of an existing monopoly arrives as a Poisson process with frequency $I(j)$ per (infinitesimal) increment of time, the present value of a monopolist's profits is a random variable. Let $V(j)$ denote the expected value of an incumbent firm with current quality level $j(\omega, t)$ [see, e.g., Diewert and Huang (2011)],[8]

$$V(j) = \pi(j) \int_t^{\infty} e^{-\int_t^s [r(v)+I(j)]dv} ds, \tag{11}$$

where $r$ is the equilibrium market real interest rate and $\pi(j)$, given by (4), is constant between innovations. Free entry prevails in vertical R&D such that the condition $I(j) \cdot V(j+1) = R_v(j)$ holds. Thus,

$$V(j+1) = \frac{1}{\Phi(j)} = \zeta \cdot L \cdot q(j+1), \tag{12}$$

where $V(j+1)$ is analogous to (11).

The arbitrage condition facing a vertical innovator,

$$r(t) + I(t) = r_0 \equiv \frac{\pi_0}{\zeta}, \tag{13}$$

can be obtained by time-differentiating $V(j+1)$ and considering equations (12) and (4).[9] This equation has several implications. First, the effective rate of return for introducing an innovation , $r + I$, should be equal to the "basic" rate of return associated with the existing technology, or equivalently the cost of entry should be equal to the present value of the "basic" profit using the effective interest rate as a discount factor. Second, the rates of entry are symmetric across industries,

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$I(\omega, t) = I(t)$. Last, the effective discount rate is constant over time. Thus, the vertical-innovation rate is perfectly negatively correlated with the rate of return $I(t) = r_0 - r(t)$, and takes only place if $r_0 > r(t)$.

After solving (10) for $R_v(\omega, t) = R_v(j)$ and aggregating across industries $\omega$, we determine the aggregate expenditure devoted to vertical R&D, $R_v(t) = \int_0^{N(t)} R_v(\omega, t) d\omega = \int_0^{N(t)} \zeta \cdot L \cdot q(j(\omega, t) + 1) \cdot I(\omega, t) d\omega$. As the innovation rate is industry-independent, the cost function for vertical R&D is

$$R_v(t) = \max\{\zeta \cdot \lambda^{\frac{\alpha}{1-\alpha}} \cdot (r_0 - r(t)) \cdot L \cdot Q(t), 0\}. \quad \textbf{(14)}$$

Vertical innovation is only performed if the rate of return from becoming a monopolist is higher than the macroeconomic rate of return. There is irreversibility as far as vertical innovation is concerned because, after an innovation has been introduced in a particular sector, it cannot be uninvented.

*Horizontal R&D.* Variety expansion arises from R&D aimed at creating a new intermediate good. Again, innovation is performed by a potential entrant, which means that, because there is free entry, the new good is produced by new firms. Under perfect competition among R&D firms and constant returns to scale at the firm level, instantaneous entry is obtained as $\dot{N}_e(t) = R_{ne}(t)/\eta(t)$, where $\dot{N}_e(t)$ is the contribution to the instantaneous flow of new varieties by R&D firm $e$ at a cost of $\eta(t)$ units of the final good and $R_{ne}(t)$ is the flow of resources devoted to horizontal R&D by innovator $e$ at time $t$. The cost $\eta$ is assumed to be symmetric. Thus, $R_n = \sum_e R_{ne}$ and $\dot{N}(t) = \sum_e \dot{N}_e(t)$, implying that

$$R_n(t) = \eta(t) \cdot \dot{N}(t). \quad \textbf{(15)}$$

We assume that the cost of setting up a new variety (cost of horizontal entry) is increasing in both the number of existing varieties, $N$, and the number of new entrants, $\dot{N}$,

$$\eta(t) = \phi \cdot N(t)^{\sigma} \cdot \dot{N}(t)^{\gamma}, \quad \textbf{(16)}$$

where $\phi > 0$ is a constant fixed (flow) cost, and $\sigma > 0$ and $\gamma > 0$. Equation (16) introduces two types of decreasing returns associated with horizontal innovation: dynamic and static. Dynamic decreasing returns to scale are modeled by the dependence of $\eta$ on $N$ and result from complexity [e.g., Evans et al. (1998); Barro and Sala-i-Martin (2004, Ch. 6)], in the sense that the larger the number of existing varieties, the costlier it is to introduce new varieties. Again, this may happen because of coordination, organizational, and transportation costs related to market size [e.g., Dinopoulos and Thompson (1999)]. Static decreasing returns to scale are modeled by the dependence of $\eta$ on $\dot{N}$ and mean that one potential entrant exerts an externality on other entrants (e.g., because of congestion effects). This externality is compatible with the previous assumption of constant returns to scale at the firm level [e.g., Arnold (1998); Jones and Williams (2000)].[10] The dependence of the entry cost on the number of entrants introduces dynamic

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second-order effects from entry, implying that new varieties are brought to the market gradually, instead of through a lumpy adjustment. This is in line with the stylized facts on entry [e.g., Geroski (1995)]: entry occurs mostly on a small scale because adjustment costs penalize large-scale entry.

Every horizontal innovation results in a new intermediate good whose quality level is drawn randomly from the distribution of existing varieties [e.g., Dinopoulos and Thompson (1998); Howitt (1999)]. Thus, the expected quality level of the horizontal innovator is

$$\bar{q}(t) = \int_0^{N(t)} \frac{q(\omega, t)}{N(t)} d\omega = \frac{Q(t)}{N(t)}. \quad \textbf{(17)}$$

As his/her monopoly power will be also terminated by the arrival of a successful vertical innovator in the future, the benefits from entry are given by

$$V(\bar{q}) = \bar{\pi}(t) \int_t^{\infty} e^{-\int_t^s [r(v)+I(\bar{q})]dv} ds, \quad \textbf{(18)}$$

where $\bar{\pi} = \pi_0 L \bar{q}$. The free-entry condition, $\dot{N} \cdot V(\bar{q}) = R_n$, by (15), simplifies to

$$V(\bar{q}) = \eta(t). \quad \textbf{(19)}$$

Substituting (18) into (19) and time-differentiating the resulting expression yields the arbitrage equation facing a horizontal innovator:

$$r(t) + I(t) = \frac{\bar{\pi}(t)}{\eta(t)}. \quad \textbf{(20)}$$

Thus, the cost function for horizontal R&D is

$$R_n(t) = \frac{\pi_0 \cdot L \cdot Q(t)}{r(t) + I(t)} \cdot \frac{\dot{N}(t)}{N(t)}. \quad \textbf{(21)}$$

*No-Arbitrage Condition.* No arbitrage in the market for innovations requires that the two types of entry, into vertical and horizontal R&D, yield equal rates of return; otherwise, a corner solution obtains. By considering (13) and (20), and equating the effective rates of return $r + I$ for both types of entry, there is no arbitrage in the market for R&D if and only if

$$\bar{q}(t) = \frac{Q(t)}{N(t)} = \frac{\eta(t)}{\zeta \cdot L}. \quad \textbf{(22)}$$

This condition is one of the key ingredients of the model. It equates the average cost of horizontal R&D, $\eta$, to the average cost of vertical R&D, $\bar{q}\zeta L$.

From (22), we obtain the dynamic equation for the aggregate number of varieties,

$$\dot{N}(t) = x(Q(t), N(t)) \cdot N(t), \quad \textbf{(23)}$$

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where

$$x(Q,N)=\left(\frac{\zeta\cdot L}{\phi}\right)^{\frac{1}{\gamma}}\cdot Q^{\frac{1}{\gamma}}\cdot N^{-\left(\frac{\sigma+\gamma+1}{\gamma}\right)} \tag{24}$$

expresses the mechanism of entry adopted in our model, incorporating a channel between vertical innovation and firm dynamics. On the other hand, time-differentiating (5) and using (23) yields a dynamic equation for the technological-knowledge stock,

$$\dot{Q}(t)=\{\Xi\cdot I(t)+x[Q(t),N(t)]\}\cdot Q(t), \tag{25}$$

where $\Xi\equiv[q(j+1)-q(j)]/q(j)=\lambda^{\frac{\alpha}{1-\alpha}}-1$ is the quality shift that is generated by successful vertical R&D. We call $\Xi$ the Schumpeterian push because it is equal to the difference between the quality associated with the entrant and the quality associated with the firm that is destroyed upon entry (creative-destruction effect). The vertical-innovation rate, $I$, is endogenous and will be determined later as an economywide function. Equation (25) expresses a second dynamic interaction between the two types of entry, in this case between the number of varieties and the quality index of the economy.

Thus, the instantaneous growth rate of average quality $\overline{q}$ is a linear function of the vertical-innovation rate,

$$\frac{\dot{\overline{q}}}{\overline{q}}=\frac{\dot{Q}}{Q}-\frac{\dot{N}}{N}=\Xi\cdot I(t). \tag{26}$$

Observe that both the vertical-innovation rate and the quality shift are industry-independent. Given the irreversibility of vertical innovation, $I\geq 0$. If $I=0$, there is no vertical R&D (no Schumpeterian push) and the technological knowledge varies only by successful horizontal R&D. If the vertical-innovation rate is positive, $I>0$, meaning that vertical R&D also becomes profitable, then there will be a positive Schumpeterian push and the technological-knowledge stock, $Q$, increases by both increases in quality and number of varieties.

Because $x>0$ from (24), we have $\dot{N}>0$ along the equilibrium with simultaneous vertical and horizontal R&D, although no irreversibility constraint has been imposed. If we rewrite $x$ as

$$x(\overline{q},N)=\left(\frac{\zeta\cdot L}{\phi}\right)^{\frac{1}{\gamma}}\cdot\overline{q}(t)^{\frac{1}{\gamma}}\cdot N(t)^{-\left(\frac{\sigma+\gamma}{\gamma}\right)},$$

then we infer that, if the vertical-innovation rate is equal to zero, the growth rate of the technological-knowledge stock is equal to the horizontal-entry rate, $\dot{N}/N$ [see (25)], and they both depend positively on the average quality level, $\bar{q}$, and negatively on the number of varieties, $N$. The first effect represents *complementarity going from vertical innovation to the horizontal-entry rate*, and the second results from the complexity and the congestion effects in horizontal entry [see (16)]. If the rate of vertical innovation is positive, there is an increase in average quality and this effect accelerates horizontal entry. Therefore, there is also a dynamic complementarity effect of quality on the number of varieties.

## 2.3. Households

The economy has $L$ identical dynastic families who consume and collect income (dividends) from investments in financial assets (equity) and from labor. Each family is endowed with one unit of labor, which is inelastically supplied. Thus, total labor supply, $L$, is exogenous and constant. We assume that consumers have perfect foresight concerning technological change over time and every household chooses the path of consumption $[C(t), t \geq 0]$ to maximize discounted lifetime utility,

$$U = \int_0^{\infty} \left( \frac{C(t)^{1-\theta} - 1}{1-\theta} \right) e^{-\rho t} dt, \tag{27}$$

where $\rho > 0$ is the subjective discount rate and $\theta > 0$ is the inverse of the intertemporal elasticity of substitution, subject to the flow budget constraint

$$\dot{a}(t) = r(t) \cdot a(t) + w(t) - C(t), \tag{28}$$

where $a$ denotes the household's real financial assets holdings. The initial level of wealth $a(0)$ is given and the non–Ponzi games condition $\lim_{t\to\infty} e^{-\int_0^t r(s)ds} a(t) \geq 0$ is also imposed. The Euler equation for consumption and the transversality condition are standard:

$$\frac{\dot{C}(t)}{C(t)} = \frac{1}{\theta} \cdot (r(t) - \rho), \tag{29}$$

$$\lim_{t\to\infty} e^{-\rho t} \cdot C(t)^{-\theta} \cdot a(t) = 0. \tag{30}$$

## 2.4. Equilibrium Vertical-Innovation Rate and R&D Intensity

We determine the equilibrium vertical-innovation rate from the equilibrium condition in the market for the final good, or, equivalently, from the macroeconomic aggregate constraint. The aggregate financial wealth held by all households is $L \cdot a(t) = \int_0^{N(t)} V(\omega, t) d\omega$, which, from the arbitrage condition between vertical and horizontal entry, yields $L \cdot a(t) = \eta(t) \cdot N(t)$. Taking time derivatives and comparing with (28), we get an expression for the aggregate flow budget constraint that is equivalent to the product market equilibrium condition [see Gil et al. (2010)],

$$Y(t) = L \cdot C(t) + X(t) + R_v(t) + R_n(t). \tag{31}$$

If we substitute the expressions for the aggregate outputs (6) and (7) and for total R&D expenditures (14) and (15), we have

$$A \cdot L \cdot Q(t) = L \cdot C(t) + \eta(t) \cdot \dot{N}(t) + \zeta \cdot L \cdot \lambda^{\frac{\alpha}{1-\alpha}} \cdot I(t) \cdot Q(t), \tag{32}$$

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where $A \equiv A_Y - A_X = (\alpha^{\alpha-1} - 1) \cdot (\alpha \cdot A_0)^{\frac{1}{\alpha}} > 0$. Solving for $I$, and using (22) and (23), we get the endogenous vertical-innovation rate at equilibrium,

$$I(Q, N, C) = \frac{1}{\zeta \cdot \lambda^{\frac{\alpha}{1-\alpha}}} \cdot \left[ A - \frac{C}{Q} - \zeta \cdot x(Q, N) \right], \qquad \textbf{(33)}$$

which is decreasing in consumption, increasing in the number of varieties, and related in an ambiguous way to the aggregate quality level.[11] As the function $I(Q, N, C)$ may be negative, the relevant vertical-innovation rate at the macroeconomic level is

$$I^{+}(Q, N, C) = \max\{I(Q, N, C), 0\}. \qquad \textbf{(34)}$$

Thus, there is also a *complementary effect of horizontal innovation on vertical innovation*: if the number of varieties is too *low*, vertical R&D shuts down. This contrasts with the result under the knowledge-driven framework [e.g., Peretto and Connolly (2007)] that vertical R&D falls to zero when the number of varieties becomes too *high*, which basically reflects the assumption that horizontal R&D competes away scarce resources from vertical R&D.[12]

### 2.5. The Dynamic General Equilibrium

The dynamic general equilibrium is defined by the allocation $[X(\omega, t), \omega \in [0, N(t)], t \geq 0]$, the prices $[p(\omega, t), \omega \in [0, N(t)], t \geq 0]$, and the aggregate paths $[C(t), N(t), Q(t), I(t), r(t),\ t \geq 0]$, such that (i) consumers, final-good firms, and intermediate-good firms solve their problems; (ii) there is free entry and absence of arbitrage opportunities both within and between the markets for horizontal and vertical innovations; and (iii) markets clear. The equilibrium paths solve the piecewise-smooth system[13]

$$\dot{C} = \begin{cases} \frac{1}{\theta} \cdot (r_0 - \rho) \cdot C & \text{if } I(Q, N, C) \leq 0 \\ \frac{1}{\theta} \cdot [r(Q, N, C) - \rho] \cdot C & \text{if } I(Q, N, C) > 0 \end{cases}, \qquad \textbf{(35a)}$$

$$\dot{Q} = \begin{cases} x(Q, N) \cdot Q & \text{if } I(Q, N, C) \leq 0 \\ [\Xi \cdot I(Q, N, C) + x(Q, N)] \cdot Q & \text{if } I(Q, N, C) > 0 \end{cases}, \qquad \textbf{(35b)}$$

$$\dot{N} = x(Q, N) \cdot N, \qquad \textbf{(35c)}$$

together with the initial $Q(0)$ and $N(0)$ and the transversality conditions,

$$\lim_{t \to \infty} e^{-\rho t} C(t)^{-\theta} \zeta \cdot L \cdot Q(t) = 0. \qquad \textbf{(36)}$$

When necessary, we will consider a convenient equivalent representation of system (35a)–(35c) in the detrended variables, $x$ as in (24), and $z \equiv C/Q$:

$$\dot{x} = \begin{cases} -\left(\frac{\sigma}{\gamma}+1\right)\cdot x^2 & \text{if } I(x,z) \le 0 \\ \left[I(x,z)\cdot \Xi \cdot \frac{1}{\gamma} - \left(\frac{\sigma}{\gamma}+1\right)\cdot x\right]\cdot x & \text{if } I(x,z) > 0 \end{cases}, \quad \textbf{(37a)}$$

$$\dot{z} = \begin{cases} (\mu - x)\, z & \text{if } I(x,z) \le 0 \\ \left[\mu - \left(\frac{1}{\theta}+\Xi\right)\cdot I(x,z) - x\right]\cdot z & \text{if } I(x,z) > 0 \end{cases}, \quad \textbf{(37b)}$$

where the innovation rate becomes a linear function of $(z, x)$,

$$I(x,z) = \frac{1}{\zeta \cdot \lambda^{\alpha/(1-\alpha)}}\,(A - z - \zeta \cdot x). \quad \textbf{(38)}$$

If we define the R&D intensity as $R \equiv (R_n + R_v)/(Y - X)$ and use (14) and (21), the equilibrium R&D intensity function is

$$R = \begin{cases} \zeta \cdot x/A, & \text{if } I(x,z) \le 0, \\ 1 - z/A, & \text{if } I(x,z) > 0, \end{cases} \quad \textbf{(39)}$$

meaning that, when there is no vertical innovation, the R&D intensity is driven solely by the change in the number of varieties; thus, $R = \max\{\zeta \cdot x/A, 1 - z/A\}$.

The economy can operate in one of two regimes: in a shallow-market regime, in which the number of differentiated goods is small relative to the technological-knowledge stock (and the nonconsumed part of the final good is low) and there is only horizontal R&D, or in a deep-market regime, in which the relative number of differentiated goods is large and there is also vertical R&D. As both regimes may operate with the same level of technological-knowledge stock, the distinction between the regimes is not related to the existence of a poverty trap. If we measure the average size of the firms by $\overline{q} = Q/N$, firms will be relatively large in the first regime and relatively small in the second. Nevertheless, we prove in the next section that the first regime, in which large firms proliferate, is only transient.

## 3. EQUILIBRIUM DYNAMICS

Next, we deal with the existence, uniqueness, and characterization of a BGP and compare the behavior of our model and the stylized facts presented in the Introduction from both the long-run and the transitional-dynamics perspectives.

### 3.1. The Balanced-Growth Path

The BGP, denoted by $*$, is the path $[(C^*(t), Q^*(t), N^*(t), t \ge 0]$ such that the growth rates $g_C^*$, $g_Q^*$, and $g_N^*$ are constants. As the functions in system

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(35a)–(35c) are homogeneous, a BGP exists only if (i) the asymptotic growth rates of consumption, $g_C$, and of the quality index, $g_Q$, are equal to the economic growth rate (GDP growth rate), $g_C = g_Q = g$; (ii) the asymptotic growth rate of the number of varieties is monotonically related to $g$, $g_N = g/(\sigma + \gamma + 1)$; and (iii) the vertical-innovation rate, $I$, is asymptotically trendless.

PROPOSITION 1 (Balanced-Growth Path). *Assume that* $\mu \equiv (r_0 - \rho)/\theta > 0$, *that* $\theta \geq 1$, *and that* $0 < \frac{\mu(\sigma+\gamma)}{\Xi(\sigma+\gamma+1)+(\sigma+\gamma)/\theta} < \overline{I} \equiv \frac{A(\sigma+\gamma)/\zeta}{\Xi(\sigma+\gamma+1)+\sigma+\gamma}$. *Then a BGP exists and is unique. Along the BGP, the vertical-innovation rate is positive but bounded,*

$$I^* = \frac{\mu\,(\sigma+\gamma)}{\Xi\,(\sigma+\gamma+1) + \frac{1}{\theta}\,(\sigma+\gamma)} \in (0, \overline{I}); \tag{40}$$

*the endogenous economic growth rate is also positive (and bounded),*

$$g^* = \left(\frac{\sigma+\gamma+1}{\sigma+\gamma}\right) \cdot \Xi \cdot I^*; \tag{41}$$

*and*

$$g_N^* = \frac{\Xi \cdot I^*}{\sigma+\gamma}. \tag{42}$$

*The level variables are* $C^*$, $Q^*$, *and* $N^*$, *where* $Q^*$ *is undetermined and*

$$C^* = z^* Q^*, \tag{43}$$

$$N^* = \left(\frac{\zeta \cdot L}{\phi}\right)^{\frac{1}{\sigma+\gamma+1}} (x^*)^{\frac{-\gamma}{\sigma+\gamma+1}} (Q^*)^{\frac{1}{\sigma+\gamma+1}}, \tag{44}$$

*with*

$$z^* = \zeta \left(1 + \frac{\sigma+\gamma+1}{\sigma+\gamma}\Xi\right) \cdot \left(\overline{I} - I^*\right) > 0 \tag{45}$$

*and*

$$x^* = g_N^*. \tag{46}$$

See the Proof in Appendix A. Under the conditions of this proposition, $g_C = g_C^*$, $g_Q = g_Q^*$, and $g_N = g_N^*$, and $x$ and $z$ are asymptotically trendless.

We need to emphasize the result that a BGP will only exist if there is a positive vertical-innovation rate, $I > 0$, and there is no BGP in which $I = 0$. This means that we can only have $I = 0$ as a transient state of the economy (see the next section). In this case, the growth rate $g$ is positive and is equivalent to

$$g^* = g_N^* + \Xi \cdot I^*. \tag{47}$$

Therefore, under a sufficiently productive technology, our model predicts a BGP with constant positive growth rates, $g$ and $g_N$, where the former exceeds the latter by an amount corresponding to the growth of intermediate-good quality, and is driven by the expected productivity push resulting from vertical innovation.

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This is consistent with the idea that industrial growth proceeds along both an intensive and an extensive margin. However, from equation (41), we observe that the ultimate source of growth is the expected Schumpeterian push, i.e., the (exogenous) increase in productivity generated by vertical innovation, $\Xi$, times its probability of occurrence, $I$ (which is itself a function of the Schumpeterian push, $\Xi$).

From equation (42), we observe that, differently from the knowledge-driven literature, a positive $g_N^*$ does not depend on the existence of a positive (exogenous) growth rate of the population. The negative externality effect in (16) can only generate a constant $N$ along the BGP [provided population growth is zero; see Barro and Sala-i-Martin (2004, Ch. 6)];[14] however, the variety expansion is sustained by technological-knowledge accumulation (independent of population growth), because the expected growth of intermediate-good quality due to vertical R&D makes it attractive, in terms of intertemporal profits, for potential entrants always to put up a (horizontal) entry cost, even if it is increasing with $N$.

### 3.2. Long-Run R&D Intensity, Firm Size, and the Economic Growth Rate

We now focus on the long-run growth effects of changes in the technological parameters. The effect of changes in the Schumpeterian push, $\Xi$, the fixed cost of vertical and horizontal R&D, $\zeta$ and $\phi$, respectively, and the elasticities of the horizontal entry cost, $\sigma$ and $\gamma$, on the vertical-innovation rate, $I^*$, the growth rate of varieties, $g_N^*$, and the economic growth rate, $g^*$, are gathered in the next proposition:

PROPOSITION 2 (Comparative Statics Results for $I^*$, $g_N^*$, and $g^*$). *A permanent and unanticipated increase in*

(a) *The Schumpeterian push (the elasticities of the horizontal entry cost) will decrease (increase) the vertical-innovation rate, but will increase (decrease) both the economic growth rate and the growth rate of varieties.*
(b) *The fixed cost of vertical R&D will decrease the vertical-innovation rate and both the economic growth rate and the growth rate of varieties. The fixed cost of horizontal R&D and the level of the labor force, $L$, have no growth effects.*

A positive shock in the Schumpeterian push, $\Xi$, decreases the long-run vertical-innovation rate, $I^*(\Xi)$, because fewer resources are necessary to generate a given upward push in the quality level. This decreases the probability of arrival of a vertical innovation.

However, there are two effects of an increase in the Schumpeterian push over the two growth rates: a positive, direct effect that increases the growth rates because inputs become more productive, and a negative, indirect effect associated with the decrease in the vertical-innovation rate. Because the direct effect dominates, both the aggregate and the extensive growth rate increase because $\Xi \cdot I^*(\Xi)$ increases. This means that the expected value of a Schumpeterian push is positive.

The lack of relationship between the growth rates and the fixed cost of horizontal R&D, $\phi$, is noteworthy. Intuitively, it results from the dominant effect exerted by the vertical-innovation mechanism (the intensive margin) over the horizontal-entry dynamics (the extensive margin). Given the postulated horizontal entry technology, an equilibrium BGP with positive net entry occurs ultimately because entrants expect incumbency value to grow, propelled by quality-enhancing R&D.

Now, in order to associate our results with the empirical facts that were mentioned in the Introduction, we analyze the theoretical counterparts to the R&D intensity,

$$R^* = 1 - \frac{z^*}{A} = \frac{\zeta}{A}\left[\frac{1}{\sigma+\gamma+1}\cdot g^* + (\Xi+1)\cdot I^*\right], \qquad \textbf{(48)}$$

and to the long-run average firm size $\overline{q}^* = Q^*/N^*$ ,

$$\overline{q}^* = \left(\frac{\phi}{\zeta\cdot L}\right)^{\frac{1}{\sigma+\gamma+1}}\cdot\left(\frac{\Xi\cdot I^*}{\sigma+\gamma}\right)^{\frac{\gamma}{\sigma+\gamma+1}}(Q^*)^{\frac{\sigma+\gamma}{\sigma+\gamma+1}}. \qquad \textbf{(49)}$$

Firm size is thus measured as technological-knowledge stock per firm (or firm size relative to market size is measured as $1/N^*$), which relates closely to production (sales) per firm or financial assets per firm.[15] Because $\dot{\overline{q}}/\overline{q} = \Xi\cdot I > 0$, firm size expands at the growth rate of intermediate-good quality along the BGP. These results are broadly supported by historical empirical evidence. The increase in sales per firm over time is referred to, e.g., by Jovanovic (1993) for the United States. The increase in the number of firms and establishments over the long run is reported, e.g., by Maddison (1994).[16]

The next proposition summarizes the long-run effects of changes in the technological parameters on R&D intensity, $R^*$, and on firm size, $\overline{q}^*$.

PROPOSITION 3 (Comparative Statics Results for $R^*$ and $\overline{q}^*$).

(a) *Let $\theta \geq 1$. Then a permanent and nonanticipated increase in the Schumpeterian push, $\Xi$ (the elasticities of the horizontal entry, $\sigma$ and $\gamma$), will decrease (increase) the R&D intensity. An increase in the fixed cost of vertical R&D, $\zeta$, will decrease the R&D intensity. Both the fixed cost of horizontal R&D, $\phi$, and the level of the labor force, $L$, have no effect on R&D intensity.*

(b) *For a given $Q^*$, a permanent and nonanticipated increase in $\sigma$, $\Xi$, and $\phi$ (in $\gamma$, $\zeta$, and $L$) will increase (decrease) the firm size.*

Propositions 2 and 3 offer a mixed picture with respect to the correlation between the economic growth rate, $g^*$, and both the R&D intensity, $R^*$, and the firm size, $\bar{q}^*$, because comparative-statics results depend upon the source of the shift in the general equilibrium.

COROLLARY 1 (Long-Run Relationships between $g^*$, $R^*$, and $\bar{q}^*$).

(a) *Changes in the Schumpeterian push, $\Xi$, or in the elasticities of the horizontal entry cost, $\sigma$ and $\gamma$ (the fixed cost of vertical R&D, $\zeta$), yield a negative (positive) relationship between $g^*$ and $R^*$.*

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(b) *Changes in $\sigma$ yield a negative relationship between $g^*$ and $\bar{q}^*$, but changes in $\Xi$, $\gamma$, and $\zeta$ ($\phi$ and $L$) give rise to a positive (null) correlation.*

An increase in $\zeta$ induces a rise in the number of firms, for a given $\tilde{q}^*$, such that the initial increase in $\zeta$ is matched by a decrease in average quality [see (22)] and hence firm size. The increase in $\zeta$ also reduces the effective rate of return $I^* + r^*$ [see (13)], which leads to fewer resources being allocated to investment (vertical and horizontal R&D) in favor of present consumption [and hence granting smaller consumption growth—see the impact of $\zeta$ on (29)]. This then implies a reduction in both the growth rate of the number of firms, $g_N^*$, and the Poisson rate, $I^*$. Thus, both R&D intensity and the economic growth rate are decreased.

In turn, an increase in, say, the horizontal entry cost elasticity $\sigma$ implies a decrease in the number of firms, although having no impact on the effective rate of return. However, because a change in $\sigma$ alters the balance between the growth rate of the number of firms and the growth rate of quality [recall that $g_Q/g_N = (\sigma + \gamma + 1)$ on the BGP], there will be a shift of resources from horizontal R&D, $R_n^*$, to vertical R&D, $R_v^*$, and hence a reduction in the growth rate of the number of firms and an increase in the Poisson rate. However, the effect on the economic growth rate, $g^*$, of the increment of the intensive margin is dominated by the decrease in the extensive one. In the end, the observed negative correlation between $g^*$ and $R^*$ reflects the negative correlation between $g^*$ and $R_v^*$, which outweighs the positive relationship between $g^*$ and $R_n^*$.

Also in Dinopoulos and Thompson (1998), Peretto (1998), and others, a change in horizontal entry costs induces a change of the vertical-innovation rate in the same direction. However, this implies a positive correlation with the economic growth rate, because the latter changes solely because of the intensive margin (the extensive margin is linked to the exogenous population growth rate).

We can compare the results of the model with the data from the perspective of the cross-section correlation between those variables, in the following sense: if there is heterogeneity across countries in any or all the (technological) parameters, we expect that the data exhibit an unclear relationship between the economic growth rate and both the R&D intensity and the firm size.

Empirical studies in general find a strong positive relationship between R&D intensity and the growth rate at the sectoral and firm level, but a clear link is usually difficult to establish at the aggregate (cross-country) level [e.g., Backus et al. (1992); Bassanini et al. (2000)]. On the other hand, although recent empirical work has found a positive relationship between firm size and the growth rate at the aggregate level, the majority of the empirical literature still gives little support for this view [see Pagano and Schivardi (2003)].

The empirical literature often places the emphasis on the complexity of the link between these variables [Audretsch and Keilbacha (2008)] and the several conceptual and measurement problems that still afflict empirical analysis in this field [Bassanini et al. (2000)] to explain the lack of robust results. According to our model, the underlying positive relationship between R&D and growth may

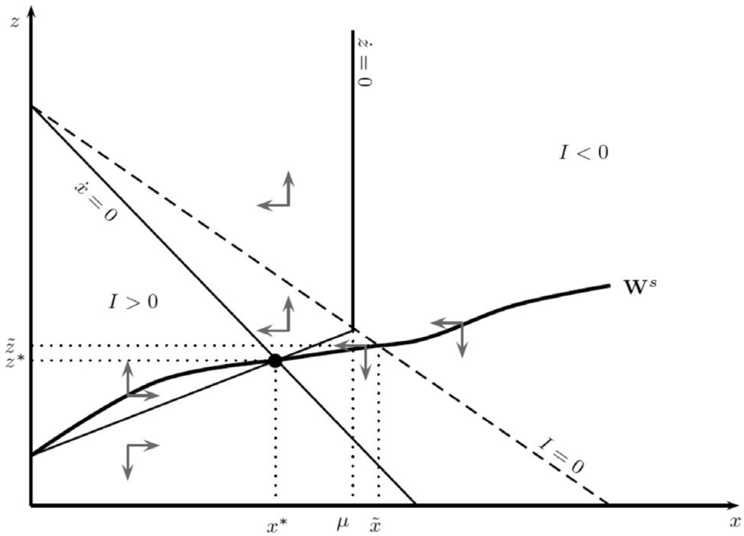


**FIGURE 3.** Phase diagram in the detrended variables $(x, z)$ for the piecewise smooth system (37a)–(37b). Curves $\dot{x} = 0$ and $\dot{z} = 0$ are the isoclines and curve $\mathbf{W}^s$ is the stable manifold, which is the only equilibrium trajectory.

be counteracted by the the shift of resources between vertical and horizontal R&D. Therefore, the existence of an endogenous inter-R&D composition effect causes those ambiguous relationships and possibly the lack of clear-cut empirical findings.

### 3.3. Aggregate Transitional Dynamics

Because the BGP level is indeterminate, as is usual in endogenous-growth models, we set $Q^* = Q(0)$, which is given. The economy will follow along the BGP if $N(0) = \tilde{N}(0) = (\zeta \cdot L/\phi)^{1/(\sigma+\gamma+1)}(x^*)^{-\gamma/(\sigma+\gamma+1)}[Q(0)]^{1/(\sigma+\gamma+1)}$ and $C(0) = z^* Q(0)$. If, given $Q(0)$, the initial number of varieties is different from $\tilde{N}(0)$, then a transitional dynamics path will unfold. The qualitative dynamics is easier to characterize if we study it using the detrended system (37a)–(37b).

PROPOSITION 4 [Transitional Dynamics (See Figure 3)]. *Under the previous assumptions, the BGP is determinate and is saddlepoint stable. There is a piecewise smooth continuous stable manifold that is positively sloped, implying there is a positive correlation between x and z along the transition to the BGP. In addition, there is a magnitude* $\tilde{x} \in (\mu, \frac{\Xi(\sigma+\gamma+1)+\sigma+\gamma}{\Xi(\sigma+\gamma+1)+(\sigma+\gamma)/\theta}\mu)$ *such that*

(a) *If* $x(0) > \tilde{x}$, *then there is a point* $z(0)$ *such that the pair* $(x(t), z(t))$ *diminishes through time until the economy crosses the point* $(\tilde{x}, \tilde{z})$ *at time* $t_0 > 0$. *When* $t \in [0, t_0]$, *we have* $I(t) = 0$. *From* $t_0$ *onward,* $I(t)$ *increases and the pair* $(x(t), z(t))$ *will still diminish in time until it converges asymptotically to the BGP levels* $(x^*, z^*)$.

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(b) *If* $x^* < x(0) < \tilde{x}$*, we have* $0 < I(0) < I^*$ *and* $I(t)$ *increases, whereas the pair* $(x(t), z(t))$ *diminishes and converges asymptotically to the BGP levels.*
(c) *If* $0 < x(0) < x^*$*, then* $I(0) > I^* > 0$ *and* $I(t)$ *decreases, whereas the pair* $(x(t), z(t))$ *increases and converges asymptotically to the BGP levels.*

See the Proof in Appendix B.

We focus in particular on the deviations of $x$ above its steady-state value, $x(0) > x^*$, where

$$x(0) = \left(\frac{\zeta \cdot L}{\phi}\right)^{\frac{1}{\gamma}} \cdot \overline{q}(0)^{\frac{1}{\gamma}} \cdot N(0)^{\frac{-(\sigma+\gamma)}{\gamma}}. \qquad \textbf{(50)}$$

This would be the case for an economy with a low $N(0)$ relative to $Q(0)$ and, thus, having a large firm size and/or displaying a shallow market for differentiated goods. Proposition 4 makes clear the *complementarity* between horizontal and vertical innovation by showing that, when the number of varieties is too *low*, no vertical R&D is performed. This result is supported by, e.g., Ciccone and Matsuyama (1996), who analyze the existence of a no-growth trap if there are initially a small number of varieties of innovative goods. This complementarity result can also be seen as the dynamical counterpart of the prediction that vertical R&D allows sustained horizontal R&D along the BGP, which was mentioned in Subsection 3.1.

Alternatively, consider an economy that is initially endowed with $N(0)$, such that $x^* < x(0) < \tilde{x}$ and $\theta > 1$. An economy with few varieties relative to the technological-knowledge stock (i.e., with a large firm size) starts with a small vertical-innovation rate, $I$, high growth rates, $g_N$ and $g$, and a high interest rate, $r$, relative to the BGP. A high interest rate implies a low vertical-innovation rate, $I$ [see (13)], and a low vertical R&D expenditure, $R_v$ [see (10)], freeing resources to be allocated to horizontal R&D, $R_n$. The higher $g$ is justified solely by the higher rate of variety expansion, $g_N$.

The transitional dynamics of $g$ and $R$ is then characterized as follows:

COROLLARY 2 (Co-movement of $g$ and $R$ in the Neighborhood of the Steady State). *Let* $\theta > 1$*; then the economic growth rate is negatively correlated with R&D intensity along the transition path converging to the BGP.*

See the Proof in Appendix C. The economy experiences a decreasing $z$ and $x$ ($= g_N$), which implies that more resources become available to vertical innovation, boosting $I$ [see (38)] and reducing $r$ [see (13)]. Hence, part of the resources allocated to $R_n$ are gradually re-targeted to $R_v$. However, the positive impact of the intensive margin on $g$ is more than compensated for by the downward movement on the extensive margin. This negative net effect reflects the asymmetric impact of the extensive and the intensive margin on economic growth: (i) the former works its way through a specialization effect related to the number of varieties, which has a one-to-one impact on economic growth; (ii) the latter is based on an efficiency

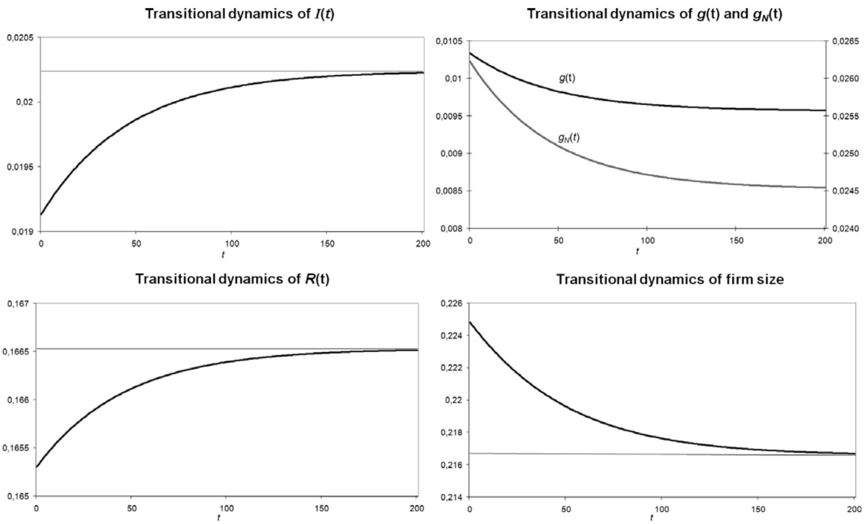


**FIGURE 4.** Transitional dynamics of the growth rates, $g$ and $g_N$, the vertical-innovation rate, $I$, the R&D intensity, $R$, and the firm size, $\tilde{q}/n$, when $x^* < x(0) < \tilde{x}$.

effect of higher quality levels, which is dampened by the creative-destruction effect [see (25)]. On the other hand, the fall in the consumption rate, $z$, and the corresponding increase in the savings rate finance an expansion in $R$ [see (39)]. Thus, along the transition path, $g$ falls, whereas $R$ increases (see Figure 4).[17]

Dinopoulos and Thompson (1998) report a similar rebalancing effect between vertical and horizontal R&D; however, their vertical-innovation rate, as well as total R&D intensity, falls parallel to aggregate growth and the interest rate along the transition path [as in Peretto (1998)]. In contrast, the medium-run negative relationship between aggregate growth and the vertical-innovation rate is also apparent in Aghion and Howitt (1998, Ch. 3), but only for a specific set of parameter values.[18]

We can study the comparative dynamics consequences from an exogenous increase in technological complexity, brought about as a drop in the Schumpeterian push, $\Xi$, as in Pintea and Thompson (2007), or from an increase in the elasticities of the horizontal entry technology, $\sigma$ and $\gamma$. In any of those cases, the stable manifold in Figure 3 will shift southwest. The short-term response will be a discrete fall in the consumption rate, $z$, and hence an increase in R&D intensity, $R$. The medium-term adjustment will then consist of a gradual further increase in R&D intensity toward the new BGP, whereas the growth rate will follow a downward path.[19]

The results described offer a theoretical explanation for the negative relationship between total R&D intensity and the per capita GDP growth rate found in the United States and other developed countries over the past decades [e.g., Pintea and Thompson (2007)]. Our own calculations, based on data for the 1975–2010 period,[20] confirm the view that a growing R&D intensity has been paralleled by a decreasing or stagnating economic growth rate in an array of countries besides the United States, such as Japan, Denmark, France, and Italy.

Finally, our model also allows one to analyze the correlation between firm size and the economic growth rate along the transition path. To that end, it is convenient to compute the number of firms, $N$, and the technological-knowledge stock, $Q$, in stationarized terms; i.e., we define $n$ and $\tilde{q}$ such that $n(t) = N(t) \cdot e^{-g_N^* \cdot t}$ and $\tilde{q}(t) = Q(t) \cdot e^{-g^* \cdot t}$. When $x^* < x(0) < \tilde{x}$, both $n$ and $\tilde{q}$ grow along the transition path, but the former grows more than the latter, implying a falling firm size.[21] Thus, firm size and the economic growth rate are positively related along the transition path (see Figure 4).

In Aghion and Howitt (1998, Ch. 12), firm size, measured as the physical-capital stock per firm in efficiency units, is commanded by the physical-capital stock along the transition path. As long as the model exhibits the convergence property, it produces a positive relationship between firm size and aggregate growth. The models of Arnold (1998) and Peretto (1998) also generate a positive relationship along the transition path between aggregate growth and firm size, measured as employment per firm in the former and human-capital stock per firm in the latter.

Empirical evidence of a positive correlation between aggregate growth and firm size in the medium run is provided by Jovanovic (1993) and Laincz and Peretto (2006), whereas Campbell (1998) reports evidence for a positive correlation between aggregate growth and the rate of entry (corresponding, in our model, to $g_N$). It is noteworthy that the less than one-to-one relationship predicted by our model is clearly consistent with the empirical findings in Campbell (1998).

## 4. OPTIMALITY

Our model inherits the well-known features found in the standard R&D endogenous-growth models [e.g., Barro and Sala-i-Martin (2004)] as regards Pareto optimality of the equilibrium. It can be shown that both static and dynamic social inefficiencies arise from the monopolistic structure in the intermediate-good sector. The dynamic effect is related to the finite duration of the monopoly (creative-destruction effect), implying that the resources allocated to R&D are too high. The static effect is related to the monopoly power, implying that the production of intermediate goods is too low from a social perspective; this then implies that the resources allocated to R&D are too low. As a result, the optimal economic growth rate may be either lower or higher than the (decentralized equilibrium) growth rate in our model. To achieve the first best, a subsidy to intermediate-good production must be combined with a mechanism that implements a transfer from the entrant to the exiting incumbent.

However, in our model, there are also spillover effects in the horizontal entry mechanism in addition to the standard effects referred to previously. It can be shown that these effects are only relevant for the gap between the decentralized equilibrium and the social optimum, because there is an interaction with the way resources are endogenously allocated between vertical and horizontal R&D. This allocation involves either the interplay of the free-entry conditions (and the associated creative-destruction effect), in the decentralized equilibrium, or the elimination of (vertical) R&D competition, in the centralized equilibrium. These spillover effects, per se, imply that the optimal growth rate is higher than the decentralized equilibrium growth rate.

## 5. CONCLUSION

This paper develops a nonscale tournament endogenous-growth model, with simultaneous expanding variety and quality ladders under a lab-equipment specification. The choice between vertical and horizontal innovation is related to the splitting of R&D expenditures, which are fully endogenous. Therefore, we endogenize the rate of extensive growth. There is complementarity between vertical and horizontal R&D acting in both senses: vertical R&D allows sustained horizontal R&D, whereas positive vertical R&D occurs when the number of varieties is *above* a certain threshold.

Our framework gives rise to strong endogenous inter-R&D composition effects and makes economic growth and firm dynamics closely related: vertical R&D is the ultimate growth engine, whereas horizontal R&D builds an explicit link between aggregate and firm-dynamics variables.

The model predicts, under a sufficiently productive technology, a BGP with constant positive growth rates, and in which the consumption growth rate equals the growth rate of the number of varieties plus the growth rate of intermediate-good quality, in line with the general view that industrial growth proceeds along both an intensive and an extensive margin. The growth of the number of varieties is sustained by technological-knowledge accumulation, as the expected growth of intermediate-good quality makes it attractive for potential entrants always to put up an entry cost, in spite of its upward trend.

We obtain specific results with respect to the impact of changes in the entry-cost parameters both on the economic growth rate and on the market structure along the BGP. We highlight (i) the lack of relationship between the economic growth and the fixed horizontal entry cost, but the positive relation between the latter and firm size, (ii) the contrasting effect of changes in the two elasticity parameters of the entry cost function on firm size, and (iii) the mixed result with respect to the BGP relation between the economic growth rate and both R&D intensity and firm size.

Along the transition path, the model produces results that differ from (or expand the results of) the early models of simultaneous vertical and horizontal R&D. In particular, we obtain as a general medium-term result that economic growth and firm size are positively correlated, whereas R&D intensity and both economic growth and firm size move in opposite directions. The former result adds to the theoretical predictions already found in the literature, of a positive correlation between economic growth and firm size measured either as employment per firm,

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human-capital stock per firm, or physical-capital stock per firm in efficiency units, and which have had wide empirical support. Importantly, the last result—together with (iii), earlier—offers a theoretical explanation for the nonsignificant or negative empirical correlation between aggregate R&D intensity and both the economic growth rate and firm size, a well-known fact in the growth literature.

Our framework is still quite stylized and encourages extensions in several directions. For example, although R&D is a key driver of economic growth, our theoretical framework can be extended to include human-capital accumulation. It can be also extended to explore the interrelation between horizontal and vertical innovation within a nontournament setting, in this way accommodating R&D by both incumbents and entrants.

## NOTES

1. Most R&D models predict a positive correlation between the economic growth rate and R&D intensity [see, e.g., Aghion and Howitt (1998)].
2. It is well known that investment in R&D is actually done by both incumbents and entrants. A large fraction of new innovators are occasional innovators that exit soon from the innovative scene, whereas only a small part of entrants survive; these firms then become persistent innovators [e.g., Malerba and Orsenigo (1999)]. Because, on one hand, our focus is not on the strategic interaction between incumbents and entrants over the R&D process [in contrast with, e.g., Segerstrom (2007) and Etro (2008)], and, on the other, we wish to highlight the role of the complementarity between horizontal and vertical R&D, we choose the simplifying assumption that only entrants perform (vertical) R&D, following the standard approach in the literature [e.g., Aghion and Howitt (1992); Howitt (1999); Segerstrom (2000); Strulik (2007); Francois and Lloyd-Ellis (2009)].
3. Using Rivera-Batiz and Romer (1991)'s terminology, the assumption that the homogeneous final good (which is the numeraire) is the R&D input means that one adopts the "lab-equipment" version of R&D, instead of the "knowledge-driven" specification, in which labor is ultimately the only input. In the latter case, R&D intensity is usually measured as labor devoted to R&D as a share of total labor force.
4. Alternatively, an endogenous extensive margin may be obtained under a knowledge-driven setup where (endogenous) human capital, instead of labor, is the input in both vertical and horizontal R&D. However, the presence of one more state variable (human capital)—in comparison with our lab-equipment setup—would render the analytical study of the model a lot more cumbersome, without new insights regarding the major focus of our paper.
5. In equilibrium, only the top quality for every industry $\omega$ is produced and used; thus, $X(j,\omega,t) = X(\omega,t)$.
6. We assume that innovations are drastic, i.e., $1/\alpha < \lambda$, such that existing monopolies do not need to limit price and can instead charge the unconstrained monopoly price.
7. The way $\Phi$ depends on $j$ implies that the increasing difficulty of creating new qualities exactly offsets the increased rewards from marketing higher qualities—see (9) and (4). This allows a constant vertical-innovation rate over $t$ and across $\omega$ along the BGP, i.e., a symmetric equilibrium.
8. We assume that entrants are risk-neutral and, thus, only care about the expected value.
9. Observe that, from (4) and (10), we have $\dot{\pi}(\omega,t)/\pi(\omega,t) = I(\omega,t)\cdot[\dot{j}(\omega,t)\cdot(\frac{\alpha}{1-\alpha})\cdot\ln\lambda]$ and $\dot{R}_v(\omega,t)/R_v(\omega,t) - \dot{I}(\omega,t)/I(\omega,t) = I(\omega,t)\cdot[\dot{j}(\omega,t)\cdot(\frac{\alpha}{1-\alpha})\cdot\ln\lambda]$. Thus, if we time-differentiate (12) by considering (11) and the equations preceding, we get $r(t) = \frac{\pi(j+1)\cdot I(j)}{R_v(j)} - I(j+1)$, which can then be rewritten as (13).
10. We depart from Howitt (1999) [see also Segerstrom (2000)] because he hypothesizes decreasing returns to scale to R&D at the firm level. Such an entry technology implies that (keeping our notation)

$V = \eta = \frac{dR_n}{d\dot{N}} > \frac{R_n}{\dot{N}}$. In contrast, given our assumption of constant returns to scale, we have $V = \eta = \frac{R_n}{\dot{N}} = \frac{dR_n}{d\dot{N}}$ [see (19)]. Note that the price of entry, $V$, equals the marginal cost of entry, $\eta$, in both cases considered earlier; nevertheless, the assumption of constant returns eschews positive profits from entering, because $V = \frac{R_n}{\dot{N}}$.

11. The partial derivative with $Q$ has the sign of $C(t)/Q(t) - \zeta/\gamma$, meaning that it may change over $t$.

12. In contrast, Peretto and Smulders (2002) obtain a result similar to ours under a knowledge-driven setup, but only for a specific set of parameter values, by assuming a feedback from the number of varieties to vertical R&D through a network spillover effect.

13. From (13), we get equivalently the innovation rate as a discount over the baseline rate of interest: $I^{+}(Q, N, C) = r_0 - r(Q, N, C)$.

14. In fact, the dependence of $\eta$ on $N$ is necessary to eschew the explosive growth that would occur if $\eta$ were constant over $t$, or depended solely on $\dot{N}$, thus implying that a BGP would not exist. This is not the case in Barro and Sala-i-Martin (2004)'s basic model of pure expanding variety. It can be shown that the specification $\eta \equiv \eta(Q)$, $\eta' > 0$, $\eta'' < 0$ produces a similar result in our model.

15. Observe that production (or sales) per firm is given by $X/N = A_X Q/N$ [see (7)], whereas financial assets per firm is given by $a/N = \eta = \zeta L Q/N$ [see (22)]. On the other hand, although empirical evidence usually relates directly to size, measured as employment per firm, a number of recent papers address the sensitivity of firm dynamics to different measures of size (employment, sales, capital and value added). The evidence is qualitatively similar to that obtained when employment is the measure of firm size [e.g., Bottazzi et al. (2007)].

16. See Gil (2010) for a detailed discussion of the empirical literature relating firm dynamics to long-run economic growth.

17. We use the following set of baseline parameter values to illustrate the transitional dynamics: $\gamma = 1.2$, $\sigma = 1.2$, $\phi = 1$, $\zeta = 0.9$, $\lambda = 2.5$, $\rho = 0.02$, $\theta = 1.5$, $\alpha = 0.4$, $A_0 = 1$, $L = 1$. Given that along the BGP, $g_Q - g_N = (\sigma + \gamma)g_N$, the choice of values for $\sigma$ and $\gamma$ is such that $(\sigma + \gamma) = 2.4$, which is the ratio between the growth rate of the average firm size and the growth rate of the number of firms we have found in the empirical data (the data, which are available from the authors upon request, concern 23 European countries in the period 1995–2005 and were taken from the Eurostat online database, available from http://epp.eurostat.ec.europa.eu). The values for $\lambda$, $\theta$, $\rho$, and $\alpha$ were set in line with previous work on growth and guided either by empirical findings or by theoretical specification, whereas the normalization of $A_0$ and $L$ to unity at every $t$ implies that all aggregate magnitudes can be interpreted as per capita magnitudes. The values of the remaining parameters were chosen to calibrate the BGP aggregate growth rate around 2.5%/year.

18. The mechanism in Aghion and Howitt (1998, Ch. 3) is different from ours. They develop a quality-ladders model with physical capital as an input to R&D. Innovation, and hence aggregate growth, is stimulated by a rise in capital intensity toward its BGP level, whereas diminishing marginal returns to physical capital imply per se a fall in the aggregate-growth rate. For some parameter values, economic growth and the innovation rate move in opposite directions along the transition path.

19. The negative relationship between the growth rate and R&D intensity obtained in Pintea and Thompson's model hinges on the negative (exogenous) impact of the increase in technological complexity on "passive learning" (the process by which skilled workers secure increases in product quality). In our paper, that relationship stems from the (endogenous) interrelation between vertical and horizontal R&D along the adjustment path, as already explained.

20. The data were taken from the Eurostat on-line database, available from http://epp.eurostat.ec.europa.eu.

21. To understand this result, first let $Q(t) = Q_0 e^{g_Q(t)t}$, $Q_0 > 0$, and $N(t) = N_0 e^{g_N(t)t}$, $N_0 > 0$. Together with $Q(t) = \tilde{q}(t) \cdot e^{g^* \cdot t}$ and $N(t) = n(t) \cdot e^{g_N^* \cdot t}$, we get $\tilde{q}(t) = Q_0 e^{(g_Q(t) - g^*)t}$ and $n(t) = N_0 e^{(g_N(t) - g_N^*)t}$. Because, if $x^* < x(0) < \tilde{x}$, $g_N(t) - g_N^* > 0$ and $I(t) - I^* < 0$ for $t \geq 0$, whereas, from (25), $g_Q(t) = \Xi \cdot I(t) + g_N(t)$, we have $g_N(t) - g_N^* > g_Q(t) - g^*$ along the transition path.

22. We draw on results obtained in a recent applied mathematical literature presented in Di Bernardo et al. (2008). This is a Lie derivative calculated along the switching boundary.

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# APPENDIX A: PROOF OF PROPOSITION 1

Using the Caballé and Santos (1993) decomposition, we write $C(t) = c(t) \cdot e^{g_C^* \cdot t}$, $Q(t) = \tilde{q}(t) \cdot e^{g_Q^* \cdot t}$, and $N(t) = n(t) \cdot e^{g_N^* \cdot t}$. We can build a detrended system in $(c, \tilde{q}, n)$ only if $g_C = g_Q = g$ and $g_Q = g_N(\sigma + \gamma + 1)$, such that $g_C = g_C^*$, $g_Q = g_Q^*$, and $g_N = g_N^*$. If those conditions hold, then $x(\tilde{q}, n) = x(Q, N)$ and $I^+(C, Q, N) = I^+(c, \tilde{q}, n)$, and an equivalent piecewise-continuous system (PWS) for the detrended variables is obtained:

$$\dot{n} = [x(\tilde{q}, n) - g_N] \cdot n, \tag{A.1}$$

$$\dot{c} = [\mu - I^{\pm}(c, \tilde{q}, n) - \theta \cdot g] \cdot (c/\theta), \tag{A.2}$$

$$\dot{\tilde{q}} = [I^{\pm}(c, \tilde{q}, n) \cdot \Xi + x(\tilde{q}, n) - g_Q] \cdot \tilde{q}. \tag{A.3}$$

The BGP is defined in the main text, where $(c^*, \tilde{q}^*, n^*, g^*)$ is obtained for the equilibrium points of the PWS system (A.1)–(A.3). It can be determined instead from the equivalent system (37a)–(37b), where the change in variables $z = c/\tilde{q} = C/Q$ is introduced.

First, the condition for the existence of a BGP on the branch $I < 0$ is $A - z^* - \zeta x^* < 0$; then $I = I^- = 0$, and there may exist only one steady state $x^* = z^* = 0$, which is impossible for $A > 0$. Thus there is no BGP on this branch. Second, the condition for the existence of a BGP on the branch $I > 0$ is $A - z^* - \zeta x^* > 0$. Thus, an admissible interior steady state $(x^*, z^*) \in \mathbf{R}_{++}^2$ verifies $I(x, z) = (\sigma + \gamma) \cdot x / \Xi = \theta(\mu - x)/(1 + \theta \Xi)$. We obtain $I^*$ and $x^*$ as in equations (40) and (42), which are positive if $\mu > 0$.

Substituting back into the expression for $I$, we get (45), which is positive if $I^* < \overline{I}$. We can obtain the long-run levels for $(c, \tilde{q})$ by transforming back $c^* = \tilde{q}^* \cdot z^*$, where $\tilde{q}^*$ verifies $(x^*)^\gamma = (\zeta \cdot L/\phi) \cdot \tilde{q}^* \cdot (n^*)^{-(1+\sigma+\gamma)}$, where $n^*$ is left undetermined. Then there is a unique BGP, $C^*(t) = c^* e^{g^* t}$, $Q^*(t) = \tilde{q}^* e^{g^* t}$, $N^*(t) = n^* e^{g_N^* t}$, on which the long-run growth rates $g^*$ and $g_N^*$, as in (41) and (42), are positive if $\mu > 0$. Last, the transversality condition $\lim_{t\to\infty} \zeta L (c^*)^{-\theta} \tilde{q}^* e^{-[\rho+(\theta-1)g^*]t} = 0$ holds if $\theta \geq 1$. ■

# APPENDIX B: PROOF OF PROPOSITION 4

Consider the system (37a)–(37b) and its admissible steady state $(x^*, z^*)$, where $I^* = I^+(x^*, z^*) > 0$. The Jacobian evaluated at that steady state is

$$J(x^*, z^*) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{1}{1+\Xi} \begin{pmatrix} \dfrac{1+\theta\Xi}{\theta\zeta} z^* & \dfrac{(1-\theta)}{\theta} z^* \\ -\dfrac{\Xi}{\gamma\zeta} x^* & -\dfrac{\Xi(\sigma+\gamma+1)+\sigma+\gamma}{\gamma} x^* \end{pmatrix}.$$

As $\det(J(x^*, z^*)) = -[\Xi(\sigma+\gamma+1) + (\sigma+\gamma)/\theta] \cdot z^* x^* / \zeta\gamma(1+\Xi) < 0$, $J(x^*, z^*)$ has one negative, $\lambda_s < 0$, and one positive, $\lambda_u > 0$, eigenvalue. The stable manifold $W^s$ has dimension one. The slope of the stable eigenspace, which is tangent to the stable manifold in the neighborhood of $(x^*, z^*)$, is $\frac{dz}{dx}\big|_{W^s} = \frac{\lambda_s - a_{22}}{a_{21}}$. As $a_{21} < 0$, the slope has the same sign as $a_{22} - \lambda_s$. After some algebra, we get

$$a_{22} - \lambda_s = -\frac{1}{2(1+\Xi)} \cdot \left[ \frac{(\Xi(\sigma+\gamma+1)+\sigma+\gamma)\, x^*}{\gamma} + \frac{(1+\theta\Xi) z^*}{\theta\zeta} \right] + \Delta(J(x^*, z^*))^{1/2},$$

where the discriminant of the Jacobian is equivalent to

$$\Delta(J(x^*, z^*)) = \left\{ \frac{1}{2(1+\Xi)} \cdot \left[ \frac{(\Xi(\sigma+\gamma+1)+\sigma+\gamma)\, x^*}{\gamma} + \frac{(1+\theta\Xi) z^*}{\theta\zeta} \right] \right\}^2 + \frac{z^* x^* \Xi}{\zeta\gamma\theta(1+\Xi)^2} (\theta - 1).$$

Then $a_{22} - \lambda_s \geq 0$ if $\theta \geq 1$ and the local stable manifold is positively (zero) sloped in the neighborhood of $(x^*, z^*)$ if $\theta > 1$ ($\theta = 1$).

The phase diagram in Figure 3 allows a geometric characterization of the dynamics of transition. Observe that the switching curve, $I(x, z) = 0$, divides the state space into two zones: in the northeast area, where $I(x, z) < 0$, we set $I = 0$ and the dynamics is given by the first branch, and in the southwest area, where $I(x, z) > 0$, the dynamics is given by the second branch $I = I^+$. The isocline $\dot{x} = 0$ is negatively sloped and lies entirely in the second branch. The isocline $\dot{z} = 0$ passes through the two branches: it may have a positive intercept in the second branch if $A > \zeta \cdot \theta \cdot (1+\Xi) \cdot \mu/(1+\theta\Xi)$ has a positive slope and cuts the switching curve at point $x = \mu$, where it is continuous but piecewise smooth, and it is vertical in the first branch. As $x$ is a predetermined variable, this means that if $x(0) < \mu$ then the transition path lies entirely on the second branch, and approaches the BGP values $(x^*, z^*)$ as shown in the figure because the slope of the stable

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manifold is flatter than the isocline $\dot{z} = 0$. This can easily be proved if it is observed that $\frac{dz}{dx}|_{\dot{z}=0} - \frac{dz}{dx}|_{W^s} = -\frac{a_{12}}{a_{11}} + \frac{a_{22}-\lambda_s}{a_{21}} = \frac{\lambda_u + a_{11}}{a_{11}a_{21}}\lambda_s > 0$. If $x(0) > \tilde{x}$, the transition path is a concatenation of a transition path lying in the first branch with the transition path in the second branch. This can be proved if we observe that, first, the paths on the first branch to the right of the isocline $\dot{z} = 0$ cross the switching curve at a point $\tilde{x}$ that is larger than $\mu$ but is smaller than $\frac{\Xi(\sigma+\gamma+1)+\sigma+\gamma}{\Xi(\sigma+\gamma+1)+(\sigma+\gamma)/\theta}\mu$, which is the projection of the steady state $z^*$ onto the switching curve (this is an implication of the slope of the stable manifold); second, the paths starting on the first branch to the right of the isocline $\dot{z} = 0$ approach the switching curve and cross it. We can prove that there is no other form of collision by computing the projections of the vector fields on both sides of the switching curve,[22]

$$\frac{\partial I^+}{\partial z}\dot{z}^- + \frac{\partial I^+}{\partial x}\dot{x}^- = \frac{\partial I^+}{\partial z}\dot{z}^+ + \frac{\partial I^+}{\partial x}\dot{x}^+ = \frac{1}{\zeta(1+\Xi)}\left[\left(\frac{\sigma+\gamma}{\gamma}x^2 + \zeta(x-\mu)z\right)\right] > 0,$$

because the collision should take place for $x > \mu$. This means that the path coming from the first branch approaches the switching curve with the same direction as the one defined by the vector field for $I^+ < 0$, that is, both $x$ and $z$ decrease, which means that it will cross the switching boundary and continue with the same direction inside the second branch. We cannot determine the crossing point exactly, but we know that it should be in the intersection of the stable manifold in the second branch with the switching boundary. This means that the stable manifold is piecewise smooth and lies on the two branches as depicted in Figure 3. ■

# APPENDIX C: PROOF OF COROLLARY 2

The dynamic system in the detrended variables $x$ and $z$ has no closed form solution. Nevertheless, Proposition 4 presents the qualitative dynamics in the neighborhood of the BGP. If we take $x$ as a predetermined variable, we already proved in the proof of Proposition 4 that $\frac{dz(t)}{dx(t)}|_{W^s} \geq 0$ if $\theta \geq 1$. As along the transitional dynamics we have $I(t) = (A - z(t) - \zeta \cdot x(t))/(\zeta \cdot (1+\Xi))$, $g(t) = \Xi \cdot I(t) + x(t)$, and $R(t) = 1 - z(t)/A$, we can determine the co-movement of the variables by determining their asymptotic co-movement with the predetermined variable $x$ in the neighborhood of the BGP. We obtain $\frac{dI(t)}{dx(t)}|_{W^s} = -\frac{1}{1+\Xi}(1 + \frac{1}{\zeta}\frac{dz(t)}{dx(t)}|_{W^s}) \leq -\frac{1}{1+\Xi} < 0$, $\frac{dR(t)}{dx(t)}|_{W^s} = -\frac{1}{A}\frac{dz(t)}{dx(t)}|_{W^s} \leq 0$, and

$$\begin{aligned}\left.\frac{dg(t)}{dx(t)}\right|_{W^s} &= \frac{1}{1+\Xi}\left(1 - \frac{\Xi}{\zeta}\left.\frac{dz(t)}{dx(t)}\right|_{W^s}\right) = \frac{\gamma \cdot \lambda^s + (\sigma+\gamma+1)x^*}{x^*} \\ &= \frac{T_0}{2} - \left[\left(\frac{T_0}{2}\right)^2 - D_0\right]^{1/2} > 0,\end{aligned}$$

because $T_0 = \frac{1}{1+\Xi}[(\frac{1+\theta\Xi}{\theta\zeta}z^* + (\frac{\Xi(\sigma+\gamma+1)+\sigma+\gamma+2}{\gamma})x^*)] > 0$ and $D_0 = \frac{x^*}{(1+\Xi)\gamma} \cdot (\frac{z^*}{\zeta\theta} + \frac{(\sigma+\gamma+1)x^*}{\gamma}) > 0$. ■

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