

ON THE INVERSE MULTIFRACTAL FORMALISM

L. OLSEN

Department of Mathematics, University of St. Andrews, St. Andrews, Fife KY16 9SS, Scotland
e-mail: lo@st-and.ac.uk

(Received 8 April 2008; 7 May 2009; accepted 25 August 2009)

Abstract. Two of the main objects of study in multifractal analysis of measures are the coarse multifractal spectra and the Rényi dimensions. In the 1980s it was conjectured in the physics literature that for ‘good’ measures the following result, relating the coarse multifractal spectra to the Legendre transform of the Rényi dimensions, holds, namely

‘the coarse multifractal spectra = the Legendre transforms of the Rényi dimensions’.

This result is known as the multifractal formalism and has now been verified for many classes of measures exhibiting some degree of self-similarity. However, it is also well known that there is an abundance of measures not satisfying the multifractal formalism and that, in general, the Legendre transforms of the Rényi dimensions provide only upper bounds for the coarse multifractal spectra. The purpose of this paper is to prove that even though the multifractal formalism fails in general, it is nevertheless true that all measures (satisfying a mild regularity condition) satisfy the inverse of the multifractal formalism, namely

‘the Rényi dimensions = the Legendre transforms of the coarse multifractal spectra’.

2000 *Mathematics Subject Classification.* 28A80.

1. The inverse multifractal formalism. For a probability measure μ on \mathbb{R}^d with compact support K , the Rényi dimensions of μ are defined as follows. For $r > 0$ and a real number q write

$$M(r; q) = \sup_{\substack{(B(x_i, r))_i \text{ is a centred} \\ \text{packing of } K}} \sum_i \mu(B(x_i, r))^q. \quad (1.1)$$

(Recall that a finite or countable family of balls $(B(x_i, r))_i$ is called a centred packing of K if $x_i \in K$ for all i and $B(x_i, r) \cap B(x_j, r) = \emptyset$ for all $i \neq j$.) The lower and upper Rényi dimensions of order q are now defined by

$$\begin{aligned} \underline{\tau}(q) &= \liminf_{r \searrow 0} \frac{\log M(r; q)}{-\log r}, \\ \bar{\tau}(q) &= \limsup_{r \searrow 0} \frac{\log M(r; q)}{-\log r}. \end{aligned} \quad (1.2)$$

These dimensions were essentially introduced by Rényi [10, 11] in the 1960s as a tool for analysing various problems in information theory. In fact, for a probability vector $\mathbf{p} = (p_1, \dots, p_n)$ and $q \in \mathbb{R}$, Rényi defined the q -entropy $H_{\mathbf{p}}(q)$ of \mathbf{p} by $H_{\mathbf{p}}(q) = \frac{1}{1-q} \log \sum_i p_i^q$ for $q \neq 1$, and $H_{\mathbf{p}}(1) = -\sum_i p_i \log p_i$.

The main significance of the Rényi dimensions is their relationship with the coarse multifractal spectra. For a probability measure μ on \mathbb{R}^d with support equal to K , we define the coarse multifractal spectra as follows. Let $\alpha \in \mathbb{R}$. For $r, \varepsilon > 0$, let

$$N(\varepsilon, r; \alpha) = \sup \left\{ n \in \mathbb{N} \mid \begin{array}{l} (B(x_i, r))_{i=1}^n \text{ is a centred packing of } K \text{ such that} \\ \alpha - \varepsilon \leq \frac{\log \mu(B(x_i, r))}{\log r} \leq \alpha + \varepsilon \quad \text{for all } i = 1, \dots, n \end{array} \right\}. \tag{1.3}$$

In analogy with the definition of the box dimension, we now define the lower and upper coarse multifractal spectra of the measure μ by

$$\begin{aligned} \underline{f}(\alpha) &= \liminf_{\varepsilon \searrow 0} \liminf_{r \searrow 0} \frac{\log N(\varepsilon, r; \alpha)}{-\log r}, \\ \bar{f}(\alpha) &= \liminf_{\varepsilon \searrow 0} \limsup_{r \searrow 0} \frac{\log N(\varepsilon, r; \alpha)}{-\log r} \end{aligned} \tag{1.4}$$

(if $N(\varepsilon, r; \alpha) = 0$, then we put $\frac{\log N(\varepsilon, r; \alpha)}{-\log r} = -\infty$). In particular, it follows immediately from the definitions that

$$\underline{f}(\alpha) = \bar{f}(\alpha) = -\infty$$

for all $\alpha < 0$. We also note that

$$\underline{f}(\alpha) \leq \bar{f}(\alpha) \leq \alpha < \infty$$

for all $\alpha \geq 0$; indeed, since $\bar{\tau}(1) \leq 0$ (because $M(r; 1) \leq 1$ for all $r > 0$) and $\bar{f}(\alpha) \leq \bar{\tau}^*(\alpha)$ (see [3] or Theorem 2.1 below), we conclude that $\underline{f}(\alpha) \leq \bar{f}(\alpha) \leq \bar{\tau}^*(\alpha) = \inf_q (q\alpha + \bar{\tau}(q)) \leq 1 \cdot \alpha + \bar{\tau}(1) \leq 1$.

In the 1980s it was conjectured in the physics literature [4, 5] that for ‘good’ measures, the following result, relating the coarse multifractal spectra to the Legendre transform of the Rényi dimensions, holds. This result is known as the ‘multifractal formalism’ and is stated below. In order to state the ‘multifractal formalism’ we need the notion of the Legendre transform of a function. For a function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ we define the Legendre transform $\varphi^* : \mathbb{R} \rightarrow [-\infty, \infty]$ by

$$\varphi^*(x) = \inf_y (xy + \varphi(y)). \tag{1.5}$$

We can now state the multifractal formalism.

DEFINITION (The multifractal formalism). *A probability measure μ on \mathbb{R}^d is said to satisfy the multifractal formalism if*

‘the coarse multifractal spectra = the Legendre transforms of the Rényi dimensions’,

i.e. if

$$\begin{aligned} \underline{f}(\alpha) &= \underline{\tau}^*(\alpha), \\ \overline{f}(\alpha) &= \overline{\tau}^*(\alpha), \end{aligned}$$

for all $\alpha \in \mathbb{R}$.

During the 1990s there has been an enormous interest in verifying the multifractal formalism and computing the multifractal spectra of measures in the mathematical literature, and within the last 10–15 years the multifractal spectra of various classes of measures in \mathbb{R}^d exhibiting some degree of self-similarity have been computed rigorously (cf. [3, 7] and the references therein). However, it is also known that there is an abundance of measures not satisfying the multifractal formalism (indeed, since the function $\overline{\tau}^*$ is always concave it follows that if μ is a measure for which \overline{f} is not concave, then μ does not satisfy the multifractal formalism) and that, in general, the Legendre transforms of the Rényi dimensions provide only upper bounds for the coarse multifractal spectra. This is the content of the next theorem.

THEOREM A [3, 6]. *Let μ be a probability measure on \mathbb{R}^d with compact support. We have*

$$\begin{aligned} \underline{f}(\alpha) &\leq \underline{\tau}^*(\alpha), \\ \overline{f}(\alpha) &\leq \overline{\tau}^*(\alpha), \end{aligned}$$

for all $\alpha \in \mathbb{R}$.

In physics literature one is often interested in computing the coarse multifractal spectra $\underline{f}(\alpha)$ and $\overline{f}(\alpha)$ [1, 2, 4, 5]. Unfortunately, the coarse multifractal spectra cannot easily be computed numerically. However, the Rényi dimensions $\underline{\tau}(q)$ and $\overline{\tau}(q)$ can relatively easily be computed numerically, and physicists therefore often compute $\underline{f}(\alpha)$ from $\underline{\tau}(q)$ (and $\overline{f}(\alpha)$ from $\overline{\tau}(q)$) using the multifractal formalism [1, 2, 4, 5]. This is clearly unfortunate since the multifractal formalism is false in general. The purpose of this paper is to prove that even though the multifractal formalism fails in general, it is nevertheless true that the Legendre transforms of $\underline{f}(\alpha)$ and $\overline{f}(\alpha)$ can be computed from $\underline{\tau}(q)$ and $\overline{\tau}(q)$, respectively. Indeed, we prove that all measures (satisfying a mild regularity condition) satisfy the inverse of the multifractal formalism namely the Rényi dimensions coincide with the Legendre transforms of the coarse multifractal spectra.

THEOREM 1.1 (The inverse multifractal formalism). *Let μ be a probability measure on \mathbb{R}^d with compact support. Then*

‘the Rényi dimensions = the Legendre transforms of the coarse multifractal spectra.’

More precisely, the following holds:

(1) *If $0 < q$ and $\underline{f} \neq -\infty$, then*

$$-\underline{\tau}(q) = (-\underline{f})^*(q).$$

(2) *If $0 < q$, then*

$$-\overline{\tau}(q) = (-\overline{f})^*(q).$$

In fact, Theorem 1.1 follows from more general version of the inverse multifractal formalism. This is explained in the next section. We also note that the requirement $\underline{f} \neq -\infty$ cannot be omitted. Indeed, in Example 2 below we construct a probability measure on $[0, 1]$ such that $\underline{f} = -\infty$ and the inverse multifractal formalism for $\underline{\tau}$ and \underline{f} fails, in fact, $-\underline{\tau}(q) < (-\underline{f})^*(q)$ for all $q \in \mathbb{R}$.

2. The mixed inverse multifractal formalism. Recently mixed (or simultaneous) multifractal spectra have generated an enormous interest in the mathematical literature (cf. [8, 9] and the references therein). Previously, only the scaling behaviour of a single measure μ has been investigated (see [3] and the references therein). However, mixed multifractal analysis investigates the *simultaneous* scaling behaviour of finitely many measures μ_1, \dots, μ_k . Mixed multifractal analysis thus combines local characteristics which depend simultaneously on various different aspects of the underlying dynamical system, and provides the basis for a significantly better understanding of the underlying dynamics. We will now give the definitions of the mixed Rényi dimensions and the mixed coarse multifractal spectra extending (1.2) and (1.4), and state a mixed version of the inverse multifractal formalism.

2.1. Mixed Rényi dimensions. Let μ_1, \dots, μ_k be probability measures on \mathbb{R}^d with common support equal to K . For $\mathbf{q} = (q_1, \dots, q_k) \in \mathbb{R}^k$, we define the mixed moment scaling function of the measures μ_1, \dots, μ_k by

$$M(r; \mathbf{q}) = \sup_{\substack{(B(x_i, r))_i \\ \text{is a centred} \\ \text{packing of } K}} \sum_i \mu_1(B(x_i, r))^{q_1} \cdots \mu_k(B(x_i, r))^{q_k}. \tag{2.1}$$

The lower and upper mixed Rényi dimensions, denoted by $\underline{\tau}(\mathbf{q})$ and $\bar{\tau}(\mathbf{q})$, of μ_1, \dots, μ_k are defined by

$$\begin{aligned} \underline{\tau}(\mathbf{q}) &= \liminf_{r \searrow 0} \frac{\log M(r; \mathbf{q})}{-\log r}, \\ \bar{\tau}(\mathbf{q}) &= \limsup_{r \searrow 0} \frac{\log M(r; \mathbf{q})}{-\log r}. \end{aligned} \tag{2.2}$$

The reader will observe that for $k = 1$, these definitions reduce to (1.1) and (1.2).

2.2. Mixed coarse multifractal spectra. Let μ_1, \dots, μ_k be probability measures on \mathbb{R}^d with common support equal to K . In order to define the mixed coarse multifractal spectra of the measures μ_1, \dots, μ_k , we fix $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k$. For $r, \varepsilon > 0$, let

$$N(\varepsilon, r; \boldsymbol{\alpha}) = \sup \left\{ n \in \mathbb{N} \mid \begin{array}{l} (B(x_i, r))_{i=1}^n \text{ is a centred packing of } K \text{ such that} \\ \alpha_j - \varepsilon \leq \frac{\log \mu_j(B(x_i, r))}{\log r} \leq \alpha_j + \varepsilon \text{ for all } i = 1, \dots, n \text{ and all } j = 1, \dots, k \end{array} \right\}. \tag{2.3}$$

In analogy with the definition of the box dimension, we now define the lower and upper mixed coarse spectra of the measures μ_1, \dots, μ_k by

$$\begin{aligned} \underline{f}(\boldsymbol{\alpha}) &= \liminf_{\varepsilon \searrow 0} \liminf_{r \searrow 0} \frac{\log N(\varepsilon, r; \boldsymbol{\alpha})}{-\log r}, \\ \bar{f}(\boldsymbol{\alpha}) &= \liminf_{\varepsilon \searrow 0} \limsup_{r \searrow 0} \frac{\log N(\varepsilon, r; \boldsymbol{\alpha})}{-\log r}. \end{aligned} \tag{2.4}$$

Again, the reader will observe that for $k = 1$, these definitions reduce to (1.3) and (1.4).

2.3. The mixed inverse multifractal formalism. To state our main results we need the notion of the Legendre transform of a function defined on \mathbb{R}^k . For a function $\varphi : \mathbb{R}^k \rightarrow \mathbb{R}$ we define the Legendre transform $\varphi^* : \mathbb{R}^k \rightarrow [-\infty, \infty]$ by

$$\varphi^*(\mathbf{x}) = \inf_{\mathbf{y}} (\langle \mathbf{x} | \mathbf{y} \rangle + \varphi(\mathbf{y})), \tag{2.5}$$

where $\langle \cdot | \cdot \rangle$ denotes the usual inner product in \mathbb{R}^k . Observe that this definition coincides with definition (1.5) for $k = 1$.

Our first result says that the statement in Theorem A holds for arbitrary positive integers k , i.e. for all positive integers k , the Legendre transforms of the mixed Rényi dimensions provide upper bounds for the mixed coarse multifractal spectra.

THEOREM 2.1. *Let μ_1, \dots, μ_k be probability measures on \mathbb{R}^d with common compact support. We have*

$$\begin{aligned} \underline{f}(\boldsymbol{\alpha}) &\leq \underline{\tau}^*(\boldsymbol{\alpha}), \\ \bar{f}(\boldsymbol{\alpha}) &\leq \bar{\tau}^*(\boldsymbol{\alpha}), \end{aligned}$$

for all $\boldsymbol{\alpha} \in \mathbb{R}^k$.

The proof of Theorem 2.1 is given in Section 3. Our second main result (Theorem 2.2) says that even though the mixed coarse multifractal spectra, in general, do not coincide with the Legendre transforms of the mixed Rényi dimensions, all measures satisfy the inverse of the multifractal formalism, namely, the mixed Rényi dimensions coincide with the Legendre transforms of the mixed coarse multifractal spectra. In Theorem 2.2 we use the following notation. If $\mathbf{x} = (x_1, \dots, x_k)$, $\mathbf{y} = (y_1, \dots, y_k) \in \mathbb{R}^k$, then we write $\mathbf{x} \leq \mathbf{y}$ if $x_j \leq y_j$ for all j , and we write $\mathbf{x} < \mathbf{y}$ if $x_j < y_j$ for all j . Also, we write $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^k$.

THEOREM 2.2 (The mixed inverse multifractal formalism). *Let μ_1, \dots, μ_k be probability measures on \mathbb{R}^d with common compact support. Write*

$$A = \max_{j=1, \dots, k} \sup_{\substack{\mathbf{x} \in K \\ r > 0}} \frac{\log \mu_j(B(\mathbf{x}, r))}{\log r},$$

where K denotes the common support of the measures μ_1, \dots, μ_k . Let $\mathbf{q} \in \mathbb{R}^k$.

(1) *If (i) $\mathbf{0} < \mathbf{q}$ and $\underline{f} \neq -\infty$, or (ii) $A < \infty$, then*

$$-\underline{\tau}(\mathbf{q}) = (-\underline{f})^*(\mathbf{q}).$$

(2) If (i) $0 < \mathbf{q}$ and $\bar{f} \neq -\infty$, or (ii) $A < \infty$, then

$$-\bar{\tau}(\mathbf{q}) = (-\bar{f})^*(\mathbf{q}).$$

The proof of Theorem 2.2 is given in Section 4. If $k = 1$, then $\bar{f} \neq -\infty$ (see Proposition 2.3). Hence, for $k = 1$, the requirement $\bar{f} \neq -\infty$ can be omitted. Proposition 2.3 also shows that Theorem 2.1 follows from Theorem 2.2. Unfortunately, we have not been able to show that $\bar{f} \neq -\infty$ for $k \geq 2$. We also note that it follows from Example 2 below that, even for $k = 1$, the lower coarse multifractal spectrum \underline{f} may equal $-\infty$.

PROPOSITION 2.3. *If $k = 1$, then $\bar{f} \neq -\infty$.*

Proof. Let μ be a probability measure on \mathbb{R}^d . We must now prove that the upper coarse multifractal spectrum \bar{f} of μ is not constantly equal to $-\infty$.

Define the lower local dimension of μ at $x \in \mathbb{R}^d$ by

$$\underline{\dim}_{\text{loc}}(\mu; x) = \liminf_{r \searrow 0} \frac{\log \mu(B(x, r))}{\log r}.$$

We also define the lower and upper Hausdorff dimensions of μ by $\underline{\dim}_{\text{H}}(\mu) = \inf_{\mu(E) > 0} \dim_{\text{H}}(E)$ and $\overline{\dim}_{\text{H}}(\mu) = \inf_{\mu(\mathbb{R}^d \setminus E) = 0} \dim_{\text{H}}(E)$, where \dim_{H} denotes the Hausdorff dimension. It follows from [3, Propositions 10.2 and 10.3] that

$$\underline{\dim}_{\text{H}}(\mu) \leq \underline{\dim}_{\text{loc}}(\mu; x) \leq \overline{\dim}_{\text{H}}(\mu)$$

for μ -a.a. $x \in \mathbb{R}^d$. In particular, we conclude that $\underline{\dim}_{\text{loc}}(\mu; x) \leq \overline{\dim}_{\text{H}}(\mu) \leq d$ for μ -a.a. $x \in \mathbb{R}^d$. Hence, we can find $x_0 \in \mathbb{R}^d$ and $\alpha_0 \in [0, d]$ such that $\underline{\dim}_{\text{loc}}(\mu; x) = \alpha_0$. This clearly implies that for all $\varepsilon > 0$ there is a sequence $(r_n(\varepsilon))_n$ of positive real numbers with $r_n(\varepsilon) \rightarrow 0$ as $n \rightarrow \infty$ such that $\alpha_0 - \varepsilon \leq \frac{\log \mu(B(x_0, r_n(\varepsilon)))}{\log r_n(\varepsilon)} \leq \alpha_0 + \varepsilon$, whence

$$N(\varepsilon, r_n(\varepsilon); \alpha_0) \geq 1$$

for all $\varepsilon > 0$ and all n . We deduce from this that

$$\limsup_{r \searrow 0} \frac{\log N(\varepsilon, r; \alpha_0)}{-\log r} \geq \limsup_{n \rightarrow \infty} \frac{\log N(\varepsilon, r_n(\varepsilon); \alpha_0)}{-\log r_n(\varepsilon)} \geq 0$$

for all $\varepsilon > 0$. We conclude from this that $\bar{f}(\alpha_0) \geq 0$. □

We now consider various examples.

EXAMPLE 1. We will now illustrate Theorem 2.2 by presenting a simple example of a measure that does not satisfy the multifractal formalism but satisfies the inverse multifractal formalism. Fix $t \in (0, 1)$, and define the probability measure μ on $[0, 1]$ by $\mu(A) = \int_A \frac{1-t}{x^t} dx$ for $A \subseteq [0, 1]$. It is not difficult to show that

$$\underline{\tau}(q) = \bar{\tau}(q) = \max(-(1-t)q, 1-q) = \begin{cases} 1-q & \text{for } q \in (-\infty, \frac{1}{t}); \\ -(1-t)q & \text{for } q \in [\frac{1}{t}, \infty), \end{cases} \quad (2.6)$$

and that

$$f(\alpha) = \bar{f}(\alpha) = \begin{cases} -\infty & \text{for } \alpha \in \mathbb{R} \setminus \{1 - t, 1\}; \\ 0 & \text{for } \alpha = 1 - t; \\ 1 & \text{for } \alpha = 1. \end{cases} \tag{2.7}$$

For brevity we will write $\tau(q)$ for the common value of $\underline{\tau}(q)$ and $\bar{\tau}(q)$, and we will write $f(\alpha)$ for the common value of $\underline{f}(\alpha)$ and $\bar{f}(\alpha)$. An easy calculation using (2.6) shows that

$$\tau^*(\alpha) = \begin{cases} -\infty & \text{for } \alpha \in \mathbb{R} \setminus [1 - t, 1]; \\ \frac{1}{t}\alpha + 1 - \frac{1}{t} & \text{for } \alpha \in [1 - t, 1]. \end{cases}$$

In particular, we see that μ does not satisfy the multifractal formalism. Indeed, $\tau^*(\alpha) \neq f(\alpha)$ for all $\alpha \in (1 - t, 1)$. However, since it is clear that $\sup_{x \in [0, 1], r > 0} \frac{\log \mu(B(x, r))}{\log r} < \infty$, it follows from Theorem 2.2 that μ satisfies the inverse multifractal formalism. Indeed, this is also easily verified directly from (2.6) and (2.7) since

$$\begin{aligned} (-f)^*(q) &= \inf_{\alpha \geq 0} (\alpha q - f(\alpha)) \\ &= \inf_{\alpha \in [1-t, 1]} (\alpha q - f(\alpha)) \\ &= \min((1 - t)q, -1 + q) \\ &= -\max(- (1 - t)q, 1 - q) \\ &= -\tau(q) \end{aligned}$$

for $q \in \mathbb{R}$.

EXAMPLE 2. We will now construct a measure on $[0, 1]$ such that

- (1) $\underline{f} = -\infty$;
- (2) the inverse multifractal formalism for $\underline{\tau}$ and \underline{f} fails; in fact, $-\underline{\tau}(q) < (-\underline{f})^*(q)$ for all q . Hence, the requirement that $\underline{f} \neq -\infty$ in Theorem 2.2 cannot be omitted;
- (3) the inverse multifractal formalism for $\bar{\tau}$ and \bar{f} holds.

We first choose a sequence $(a_n)_{n \in \mathbb{N}_0}$ (where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) of real numbers in $(0, 1)$ such that $a_0 = 1$ and $2a_{n+1} < a_n$ for all n , and

$$u < U$$

where

$$u = \liminf_n \frac{n \log 2}{-\log a_n}, \quad U = \limsup_n \frac{n \log 2}{-\log a_n}.$$

We now construct a measure μ as follows. Firstly, for each positive integer n , we construct inductively a family $\mathcal{I}_n = \{I_{n,1}, \dots, I_{n,2^n}\}$ of 2^n closed disjoint subintervals of $[0, 1]$ with $\text{diam} I_{n,i} = a_n$ for all i as follows.

The start of the induction. We put $I_{0,1} = [0, 1]$.

The inductive step. Assume that the closed disjoint intervals $I_{n,1}, \dots, I_{n,2^n}$ have been constructed with $\text{diam} I_{n,i} = a_n$ for all i . We now construct the intervals $I_{n+1,1}, \dots, I_{n+1,2^{n+1}}$ as follows. Fix $i = 1, \dots, 2^n$. Then $I_{n,i}$ is a closed subinterval of $[0, 1]$ with $\text{diam} I_{n,i} = a_n$. Since $2a_{n+1} < a_n$, it follows that we can choose two pairwise

disjoint subintervals $I_{n+1,2i-1}$ and $I_{n+1,2i}$ of $I_{n,i}$ with $\text{diam}I_{n+1,2i-1} = \text{diam}I_{n+1,2i} = a_{n+1}$ such that the left endpoints of $I_{n+1,2i-1}$ and $I_{n,i}$ coincide, and such that the right endpoints of $I_{n+1,2i}$ and $I_{n,i}$ coincide. This completes the construction of the intervals $I_{n+1,1}, \dots, I_{n+1,2^{n+1}}$.

Now put

$$K = \bigcap_n \bigcup_i I_{n,i}.$$

Finally, we let μ be the unique probability measure supported on K such that

$$\mu(I_{n,i}) = \frac{1}{2^n} \tag{2.8}$$

for all n and all i .

We claim that

$$\underline{\tau}(q) = \min(u(1 - q), U(1 - q)) \tag{2.9}$$

for all q . We will now prove (2.9).

We first prove that $\underline{\tau}(q) \leq \min(u(1 - q), U(1 - q))$. Let $(B(x_i, a_n))_{i \in I}$ be a centred packing of K . Letting \mathcal{L} denote Lebesgue measure in \mathbb{R} , we clearly have

$$\mathcal{L}\left(\bigcup_{i \in I} B(x_i, a_n)\right) = \sum_{i \in I} \mathcal{L}(B(x_i, a_n)) = \sum_{i \in I} 2a_n = 2|I|a_n. \tag{2.10}$$

Next, for a subset E of \mathbb{R} and $r > 0$, let $B(E, r)$ denote the r neighbourhood of E , i.e. $B(E, r) = \{x \in \mathbb{R} \mid \text{dist}(x, E) < r\}$. Since $x_j \in K \subseteq \bigcup_{i=1, \dots, 2^n} I_{n,i}$, we conclude that $\bigcup_{i \in I} B(x_i, a_n) \subseteq \bigcup_{i=1, \dots, 2^n} B(I_{n,i}, a_n)$, whence

$$\mathcal{L}\left(\bigcup_{i \in I} B(x_i, a_n)\right) \leq \sum_{i=1, \dots, 2^n} \mathcal{L}(B(I_{n,i}, a_n)) = \sum_{i=1, \dots, 2^n} 3a_n = 3 \cdot 2^n a_n. \tag{2.11}$$

Combining (2.10) and (2.11) shows that $2|I|a_n \leq 3 \cdot 2^n a_n$, and so

$$|I| \leq \frac{3}{2} 2^n. \tag{2.12}$$

For each $i \in I$, we can clearly find three intervals $I_{n,j-1}, I_{n,j}, I_{n,j+1}$ such that $K \cap I_{n,j} \subseteq K \cap B(x_i, a_n) \subseteq K \cap (I_{n,j-1} \cup I_{n,j} \cup I_{n,j+1})$. This implies that $\frac{1}{2^n} \leq \mu(B(x_i, a_n)) \leq 3 \frac{1}{2^n}$, and so

$$\mu(B(x_i, a_n))^q \leq c \frac{1}{2^{nq}}, \tag{2.13}$$

where $c = \max(1, 3^q)$.

Combining (2.12) and (2.13) yields

$$\sum_{i \in I} \mu(B(x_i, a_n))^q \leq \sum_{i \in I} c \frac{1}{2^{nq}} = |I|c \frac{1}{2^{nq}} \leq \frac{3c}{2} 2^{n(1-q)}.$$

Taking supremum over all centred packings $(B(x_i, a_n))_{i \in I}$ of K gives $M(a_n; q) \leq 3c2^{n(1-q)}$, from which we conclude that

$$\begin{aligned} \underline{\tau}(q) &\leq \liminf_n \frac{\log M(a_n; q)}{-\log a_n} \leq \liminf_n \left(\frac{\log \frac{3c}{2}}{-\log a_n} + (1-q) \frac{n \log 2}{-\log a_n} \right) \\ &= \liminf_n \left((1-q) \frac{n \log 2}{-\log a_n} \right) = \min(u(1-q), U(1-q)). \end{aligned}$$

Next we prove that $\underline{\tau}(q) \geq \min(u(1-q), U(1-q))$. Let the right endpoint of the interval $I_{n,i}$ be denoted by $x_{n,i}$. Fix $r > 0$ and let n_r denote the unique positive integer satisfying $a_{n_r+1} \leq r < a_{n_r}$. It is clear that $(B(x_{n_r,i}, r))_{i=1, \dots, 2^{n_r}; i \text{ is odd}}$ is a centred packing of K , whence

$$M(r; q) \geq \sum_{\substack{i=1, \dots, 2^{n_r} \\ i \text{ is odd}}} \mu(B(x_{n_r,i}, r))^q. \tag{2.14}$$

It is also clear that $I_{n_r+1, 2i} \subseteq B(x_{n_r,i}, r)$ and $B(x_{n_r,i}, r) \cap K \subseteq (I_{n_r,i} \cup I_{n_r,i+1}) \cap K$. This implies that $\frac{1}{2^{n_r+1}} \leq \mu(B(x_{n_r,i}, r)) \leq 2 \frac{1}{2^{n_r}}$, and so

$$\mu(B(x_{n_r,i}, r))^q \geq c \frac{1}{2^{n_r q}}, \tag{2.15}$$

where $c = \min(2^{-q}, 2^q)$.

Combining (2.14) and (2.15) yields

$$M(r; q) \geq \sum_{\substack{i=1, \dots, 2^{n_r} \\ i \text{ is odd}}} c \frac{1}{2^{n_r q}} = \frac{c}{2} 2^{n_r(1-q)},$$

from which we conclude that

$$\begin{aligned} \underline{\tau}(q) &= \liminf_{r \searrow 0} \frac{\log M(r; q)}{-\log r} \\ &\geq \liminf_{r \searrow 0} \left(\frac{\log \frac{c}{2}}{-\log r} + (1-q) \frac{n_r \log 2}{-\log r} \right). \end{aligned} \tag{2.16}$$

However, since $a_{n_r+1} \leq r < a_{n_r}$, we deduce that

$$\frac{n_r}{n_r+1} \frac{(n_r+1) \log 2}{-\log a_{n_r+1}} \leq \frac{n_r \log 2}{-\log r} \leq \frac{n_r \log 2}{-\log a_{n_r}}. \tag{2.17}$$

Using (2.17) and the fact that $\frac{n_r}{n_r+1} \rightarrow 1$ as $r \searrow 0$, inequality (2.16) now simplifies to

$$\underline{\tau}(q) \geq \liminf_n \left((1-q) \frac{n \log 2}{-\log a_n} \right) = \min(u(1-q), U(1-q)).$$

This completes the proof of (2.9).

It follows easily from (2.9) that

$$\underline{\tau}^*(\alpha) = -\infty$$

for all α . We deduce from this and Theorem 2.1 that $f(\alpha) \leq \tau^*(\alpha) = -\infty$ for all α , whence

$$f(\alpha) = -\infty$$

for all α . This immediately implies that

$$(-f)^*(q) = \infty$$

for all q . Hence, $-\tau(q) < (-f)^*(q)$ for all q , i.e. $f = -\infty$ and the measure μ fails the inverse multifractal formalism for all q .

Next we show that even though the inverse multifractal formalism for τ and f fails for all q , it holds for $\bar{\tau}$ and \bar{f} . Indeed, it follows from an argument similar to the proof of (2.9) that

$$\bar{\tau}(q) = \max(u(1 - q), U(1 - q)).$$

Also, a standard argument shows that

$$\bar{f}(\alpha) = \begin{cases} -\infty & \text{for } \alpha \in \mathbb{R} \setminus \{u, U\}; \\ u & \text{for } \alpha = u; \\ U & \text{for } \alpha = U. \end{cases}$$

Finally, a straightforward calculation now shows that $-\bar{\tau}(q) = (-\bar{f})^*(q)$ for all q , i.e. the inverse multifractal formalism for $\bar{\tau}$ and \bar{f} holds for all q .

3. Proof of Theorem 2.1. In this section we prove Theorem 2.1.

Proof of Theorem 2.1. We must prove the following two inequalities,

$$f(\alpha) \leq \tau^*(\alpha), \tag{3.1}$$

$$\bar{f}(\alpha) \leq \bar{\tau}^*(\alpha), \tag{3.2}$$

for $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k$.

Proof of (3.1). Fix $\mathbf{q} = (q_1, \dots, q_k) \in \mathbb{R}^k$. We will now prove that $f(\alpha) \leq \langle \alpha | \mathbf{q} \rangle + \tau(\mathbf{q})$. If $f(\alpha) = -\infty$, then this inequality is trivially satisfied. We may therefore assume that $f(\alpha) > -\infty$. As $f(\alpha) < \infty$, this implies that $f(\alpha) \in \mathbb{R}$. Let $\delta > 0$. Since $f(\alpha) - \delta < f(\alpha) = \liminf_{\varepsilon \searrow 0} \liminf_{r \searrow 0} \frac{\log N(\varepsilon, r; \alpha)}{-\log r}$, we can find $\varepsilon_0 > 0$ such that

$$f(\alpha) - \delta < \liminf_{r \searrow 0} \frac{\log N(\varepsilon, r; \alpha)}{-\log r}$$

for all $0 < \varepsilon < \varepsilon_0$. This implies that for each $0 < \varepsilon < \varepsilon_0$ there is a positive real number $r(\varepsilon) > 0$ such that

$$N(\varepsilon, r; \alpha) \geq r^{-f(\alpha)+\delta}$$

for all $0 < r < r(\varepsilon)$.

Next, fix $0 < \varepsilon < \varepsilon_0$ and $0 < r < r(\varepsilon)$. It follows from the definition of $N(\varepsilon, r; \alpha)$ that there is a centred packing $(B(x_i, r))_{i=1}^{N(\varepsilon, r; \alpha)}$ of K with

$$\alpha_j - \varepsilon \leq \frac{\log \mu_j(B(x_i, r))}{\log r} \leq \alpha_j + \varepsilon$$

for all i and all j , i.e.

$$\mu_j(B(x_i, r))^{q_j} \geq r^{\alpha_j q_j + \varepsilon |q_j|}$$

for all i and all j . We conclude from this that

$$\begin{aligned} M(r; \mathbf{q}) &\geq \sum \mu_1(B(x_i, r))^{q_1} \cdots \mu_k(B(x_i, r))^{q_k} \\ &\geq \sum_i r^{\alpha_1 q_1 + \cdots + \alpha_k q_k + \varepsilon(|q_1| + \cdots + |q_k|)} \\ &= r^{\langle \alpha | \mathbf{q} \rangle + \varepsilon \|\mathbf{q}\|_1} N(\varepsilon, r; \alpha) \\ &\geq r^{\langle \alpha | \mathbf{q} \rangle + \varepsilon \|\mathbf{q}\|_1} r^{-\underline{f}(\alpha) + \delta} = r^{\langle \alpha | \mathbf{q} \rangle + \varepsilon \|\mathbf{q}\|_1 - \underline{f}(\alpha) + \delta} \end{aligned}$$

for all $0 < r < r(\varepsilon)$, where $\|\mathbf{q}\|_1 = |q_1| + \cdots + |q_k|$. Taking logarithms and letting r tend to 0, we obtain

$$\underline{f}(\alpha) \leq \langle \alpha | \mathbf{q} \rangle + \varepsilon \|\mathbf{q}\|_1 + \underline{\tau}(\mathbf{q}) + \delta$$

for all $0 < \varepsilon < \varepsilon_0$ and all $\delta > 0$. Finally, letting ε and δ tend to 0 we see that

$$\underline{f}(\alpha) \leq \langle \alpha | \mathbf{q} \rangle + \underline{\tau}(\mathbf{q}).$$

Since $\mathbf{q} \in \mathbb{R}^k$ was arbitrary, this inequality implies that $\underline{f}(\alpha) \leq \inf_{\mathbf{q}} (\langle \alpha | \mathbf{q} \rangle + \underline{\tau}(\mathbf{q})) = \underline{\tau}^*(\alpha)$, which completes the proof of (3.1).

Proof of (3.2). The proof of inequality (3.2) is similar to the proof of (3.1) and is therefore omitted. □

4. Proof of Theorem 2.2. In this section we prove Theorem 2.2. We first prove a small auxiliary lemma.

LEMMA 4.1. *Let $f : \mathbb{R}^k \rightarrow [-\infty, \infty]$ be a function and fix $\mathbf{q} \in \mathbb{R}^k$. The following two statements are equivalent.*

- (1) $(-f)^*(\mathbf{q}) = \infty$.
- (2) $f = -\infty$.

Proof. We clearly have

$$\begin{aligned} &(-f)^*(\mathbf{q}) = \infty \\ \Leftrightarrow &\inf_{\alpha} (\langle \alpha | \mathbf{q} \rangle - f(\alpha)) = \infty \\ \Leftrightarrow &\langle \alpha | \mathbf{q} \rangle - f(\alpha) = \infty \text{ for all } \alpha \\ \Leftrightarrow &f(\alpha) = -\infty \text{ for all } \alpha. \end{aligned}$$

This completes the proof. □

We now turn towards the proof of Theorem 2.2. However, first recall that if $\mathbf{x} = (x_1, \dots, x_k), \mathbf{y} = (y_1, \dots, y_k) \in \mathbb{R}^k$, then we write $\mathbf{x} \leq \mathbf{y}$ if $x_j \leq y_j$ for all j , and we write $\mathbf{x} < \mathbf{y}$ if $x_j < y_j$ for all j . Also recall that we write $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^k$.

Proof of Theorem 2.2. We must prove the following four statements:

Claim 1. For all $\mathbf{q} = (q_1, \dots, q_k) \in \mathbb{R}^k$, we have

$$-\bar{\tau}(\mathbf{q}) \leq (-\bar{f})^*(\mathbf{q}). \tag{4.1}$$

Claim 2. Let $\mathbf{q} = (q_1, \dots, q_k) \in \mathbb{R}^k$. If (i) $\mathbf{0} < \mathbf{q}$ and $\bar{f} \neq -\infty$ or (ii) $A < \infty$, then we have

$$(-\bar{f})^*(\mathbf{q}) \leq -\bar{\tau}(\mathbf{q}). \tag{4.2}$$

Claim 3. For all $\mathbf{q} = (q_1, \dots, q_k) \in \mathbb{R}^k$, we have

$$-\underline{\tau}(\mathbf{q}) \leq (-\underline{f})^*(\mathbf{q}). \tag{4.3}$$

Claim 4. Let $\mathbf{q} = (q_1, \dots, q_k) \in \mathbb{R}^k$. If (i) $\mathbf{0} < \mathbf{q}$ and $\underline{f} \neq -\infty$ or (ii) $A < \infty$, then we have

$$(-\underline{f})^*(\mathbf{q}) \leq -\underline{\tau}(\mathbf{q}). \tag{4.4}$$

Proof of (4.1). It follows from Theorem 2.1 that $\bar{f}(\boldsymbol{\alpha}) \leq \bar{\tau}^*(\boldsymbol{\alpha}) \leq \langle \boldsymbol{\alpha} | \mathbf{q} \rangle + \bar{\tau}(\mathbf{q})$ for all $\boldsymbol{\alpha}, \mathbf{q} \in \mathbb{R}^k$, whence $-\bar{\tau}(\mathbf{q}) \leq \langle \boldsymbol{\alpha} | \mathbf{q} \rangle - \bar{f}(\boldsymbol{\alpha})$ for all $\boldsymbol{\alpha}, \mathbf{q} \in \mathbb{R}^k$. This clearly implies that $-\bar{\tau}(\mathbf{q}) \leq \inf_{\boldsymbol{\alpha}} (\langle \boldsymbol{\alpha} | \mathbf{q} \rangle - \bar{f}(\boldsymbol{\alpha})) = (-\bar{f})^*(\mathbf{q})$ for all $\mathbf{q} \in \mathbb{R}^k$.

Proof of (4.2). For brevity write

$$t = -(-\bar{f})^*(\mathbf{q}),$$

and note that

$$t = \sup_{\boldsymbol{\alpha}} (-\langle \boldsymbol{\alpha} | \mathbf{q} \rangle + \bar{f}(\boldsymbol{\alpha})).$$

Next we choose $\alpha_0 > 0$ as follows:

If $\mathbf{0} < \mathbf{q}$ and $\bar{f} \neq -\infty$, then $t > -\infty$ (by Lemma 4.1) and $\min_i q_i > 0$, and we can thus choose a positive real number α_0 such that

$$d - \alpha_0 \min_i q_i \leq t. \tag{4.5}$$

If $A < \infty$, then we choose α_0 such that

$$\max_{j=1, \dots, k} \sup_{\substack{x \in K \\ r > 0}} \frac{\log \mu_j(B(x, r))}{\log r} \leq \alpha_0. \tag{4.6}$$

Fix $\delta > 0$.

Momentarily fix $\alpha \in \mathbb{R}^k$. We now choose $d_\alpha \in \mathbb{R}$ as follows: If $\bar{f}(\alpha) > -\infty$, we put

$$d_\alpha = \bar{f}(\alpha) + \delta,$$

and if $\bar{f}(\alpha) = -\infty$, we can clearly choose $d_\alpha \in \mathbb{R}$ such that

$$-t - \delta \leq -d_\alpha + \langle \alpha | \mathbf{q} \rangle.$$

Since $\liminf_{\varepsilon \searrow 0} \limsup_{r \searrow 0} \frac{\log N(\varepsilon, r; \alpha)}{-\log r} = \bar{f}(\alpha) < d_\alpha$, we can find a positive real number $\varepsilon_0(\alpha) > 0$ with $\varepsilon_0(\alpha) \leq \delta$ such that

$$\limsup_{r \searrow 0} \frac{\log N(\varepsilon_0(\alpha), r; \alpha)}{-\log r} < d_\alpha$$

for all n . This implies that there is a positive number $r_0(\alpha) > 0$ such that

$$N(\varepsilon_0(\alpha), r; \alpha) < r^{-d_\alpha} \tag{4.7}$$

for $0 < r < r_0(\alpha)$.

Since $[0, \alpha_0]^k$ is compact and $(B(\alpha, \varepsilon_0(\alpha)))_{\alpha \in [0, \alpha_0]^k}$ is an open cover of $[0, \alpha_0]^k$, we can find finitely many points $\alpha_1, \dots, \alpha_{s_0} \in [0, \alpha_0]^k$ with

$$[0, \alpha_0]^k \subseteq \bigcup_{l=1}^{s_0} B(\alpha_l, \varepsilon_0(\alpha_l)). \tag{4.8}$$

We will write each α_l in coordinate form as $\alpha_l = (\alpha_{l,1}, \dots, \alpha_{l,k})$.

Finally, fix $0 < r < \min_{l=1, \dots, s_0} r_0(\alpha_l)$ and let $(B(x_i, r))_{i \in I}$ be a centred packing of K . We clearly have

$$\sum_i \mu_1(B(x_i, r))^{q_1} \cdots \mu_k(B(x_i, r))^{q_k} = \Lambda_0(r) + \Pi_0(r), \tag{4.9}$$

where

$$\begin{aligned} \Lambda_0(r) &= \sum_{\substack{i \in I \\ \frac{\log \mu_j(B(x_i, r))}{\log r} \leq \alpha_0 \text{ for all } j}} \mu_1(B(x_i, r))^{q_1} \cdots \mu_k(B(x_i, r))^{q_k}, \\ \Pi_0(r) &= \sum_{\substack{i \in I \\ \alpha_0 < \frac{\log \mu_j(B(x_i, r))}{\log r} \text{ for some } j}} \mu_1(B(x_i, r))^{q_1} \cdots \mu_k(B(x_i, r))^{q_k}. \end{aligned}$$

We first analyse the sum $\Lambda_0(r)$. We have using (4.8)

$$\begin{aligned}
 \Lambda_0(r) &= \sum_{i \in I} \mu_1(B(x_i, r))^{q_1} \cdots \mu_k(B(x_i, r))^{q_k} \\
 &\quad \frac{\log \mu_j(B(x_i, r))}{\log r} \leq \alpha_0 \text{ for all } j \\
 &= \sum_{i \in I} \mu_1(B(x_i, r))^{q_1} \cdots \mu_k(B(x_i, r))^{q_k} \\
 &\quad \left(\frac{\log \mu_j(B(x_i, r))}{\log r} \right)_{j=1, \dots, k} \in [0, \alpha_0]^k \tag{4.10} \\
 &\leq \sum_{l=1}^{s_0} \sum_{i \in I} \mu_1(B(x_i, r))^{q_1} \cdots \mu_k(B(x_i, r))^{q_k} \\
 &\quad \left(\frac{\log \mu_j(B(x_i, r))}{\log r} \right)_{j=1, \dots, k} \in B(\alpha_l, \varepsilon_0(\alpha_l))
 \end{aligned}$$

Next fix i such that $\left(\frac{\log \mu_j(B(x_i, r))}{\log r} \right)_{j=1, \dots, k} \in B(\alpha_l, \varepsilon_0(\alpha_l))$. This implies that $\alpha_{l,j} - \varepsilon_0(\alpha_l) \leq \frac{\log \mu_j(B(x_i, r))}{\log r} \leq \alpha_{l,j} + \varepsilon_0(\alpha_l)$ for all j , whence $\mu_j(B(x_i, r))^{q_j} \leq r^{\alpha_{l,j}q_j - \varepsilon_0(\alpha_l)|q_j|}$ for all j , and so

$$\begin{aligned}
 \mu_1(B(x_i, r))^{q_1} \cdots \mu_k(B(x_i, r))^{q_k} &\leq r^{\alpha_{l,1}q_1 - \varepsilon_0(\alpha_l)|q_1|} \cdots r^{\alpha_{l,k}q_k - \varepsilon_0(\alpha_l)|q_k|} \\
 &= r^{(\alpha_l|\mathbf{q}| - \varepsilon_0(\alpha_l)\|\mathbf{q}\|_1)}.
 \end{aligned}$$

We conclude from this and (4.10) that

$$\begin{aligned}
 \Lambda_0(r) &\leq \sum_{l=1}^{s_0} \sum_{i \in I} r^{(\alpha_l|\mathbf{q}| - \varepsilon_0(\alpha_l)\|\mathbf{q}\|_1)} \\
 &\quad \left(\frac{\log \mu_j(B(x_i, r))}{\log r} \right)_{j=1, \dots, k} \in B(\alpha_l, \varepsilon_0(\alpha_l)) \\
 &\leq \sum_{l=1}^{s_0} N(\varepsilon_0(\alpha_l), r; \alpha_l) r^{(\alpha_l|\mathbf{q}| - \varepsilon_0(\alpha_l)\|\mathbf{q}\|_1)} \\
 &\leq \sum_{l=1}^{s_0} r^{-d_{\alpha_l}} r^{(\alpha_l|\mathbf{q}| - \varepsilon_0(\alpha_l)\|\mathbf{q}\|_1)} \\
 &\leq \sum_{l=1}^{s_0} r^{-d_{\alpha_l}} r^{(\alpha_l|\mathbf{q}| - \delta\|\mathbf{q}\|_1)}.
 \end{aligned}$$

If $\bar{f}(\alpha_l) > -\infty$, then $d_{\alpha_l} = \bar{f}(\alpha_l) + \delta$, whence

$$r^{-d_{\alpha_l}} r^{(\alpha_l|\mathbf{q}| - \delta\|\mathbf{q}\|_1)} = r^{-(-(\alpha_l|\mathbf{q}| + \bar{f}(\alpha_l)) - \delta)\|\mathbf{q}\|_1 - \delta} \leq r^{-t - \delta\|\mathbf{q}\|_1 - \delta}.$$

On the other hand, if $\bar{f}(\alpha_l) = -\infty$, then d_{α_l} is chosen such that $-t - \delta \leq -d_{\alpha_l} + \langle \alpha_l | \mathbf{q} \rangle$, whence

$$r^{-d_{\alpha_l}} r^{(\alpha_l|\mathbf{q}| - \delta\|\mathbf{q}\|_1)} = r^{-d_{\alpha_l} + (\alpha_l|\mathbf{q}| - \delta\|\mathbf{q}\|_1)} \leq r^{-t - \delta\|\mathbf{q}\|_1 - \delta}.$$

Hence

$$\begin{aligned} \Lambda_0(r) &\leq \sum_{l=1}^{s_0} r^{-l-\delta\|\mathbf{q}\|_1-\delta} \\ &= s_0 r^{-t-\delta\|\mathbf{q}\|_1-\delta}. \end{aligned} \tag{4.11}$$

Next we analyse the sum $\Pi_0(r)$. We divide the analysis into two cases depending on whether $\mathbf{0} < \mathbf{q}$ and $\bar{f} \neq -\infty$, or $A < \infty$. We first assume that $\mathbf{0} < \mathbf{q}$ and $\bar{f} \neq -\infty$. In this case $q_j > 0$, whence $\mu_j(B(x_i, r))^{q_j} \leq 1$, and so

$$\begin{aligned} \Pi_0(r) &= \sum_{i \in I} \mu_1(B(x_i, r))^{q_1} \cdots \mu_k(B(x_i, r))^{q_k} \\ &\quad \alpha_0 < \frac{\log \mu_j(B(x_i, r))}{\log r} \text{ for some } j \\ &= \sum_{j=1}^k \sum_{i \in I} \mu_1(B(x_i, r))^{q_1} \cdots \mu_k(B(x_i, r))^{q_k} \\ &\quad \alpha_0 < \frac{\log \mu_j(B(x_i, r))}{\log r} \\ &\leq \sum_{j=1}^k \sum_{i \in I} \mu_j(B(x_i, r))^{q_j} \\ &\quad \alpha_0 < \frac{\log \mu_j(B(x_i, r))}{\log r} \end{aligned} \tag{4.12}$$

Next fix j and i such that $\alpha_0 < \frac{\log \mu_j(B(x_i, r))}{\log r}$. This implies that $\mu_j(B(x_i, r)) \leq r^{\alpha_0}$, and so

$$\mu_j(B(x_i, r))^{q_j} \leq r^{\alpha_0 q_j},$$

where we again have used the fact that we are assuming that $\mathbf{0} < \mathbf{q}$, and so, in particular, $q_j > 0$. We conclude from this and (4.12) that

$$\begin{aligned} \Pi_0(r) &\leq \sum_{j=1}^k \sum_{i \in I} r^{\alpha_0 q_j} \\ &\quad \alpha_0 < \frac{\log \mu_j(B(x_i, r))}{\log r} \\ &\leq \sum_{j=1}^k \sum_{i \in I} r^{\alpha_0 q_j} \\ &\leq k r^{\alpha_0 \min_i q_i} |I|. \end{aligned} \tag{4.13}$$

However, since K is compact (and therefore, in particular, bounded) and $(B(x_i, r))_{i \in I}$ is a centred packing of K , there is a constant c such that $|I| \leq cr^{-d}$. Using (4.13), we conclude from this and the choice of α_0 (cf. (4.5)) that

$$\Pi_0(r) \leq ck r^{-d+\alpha_0 \min_i q_i} \leq ck r^{-t} \leq ck r^{-t-\delta\|\mathbf{q}\|_1-\delta}. \tag{4.14}$$

Next we assume that $A < \infty$. We conclude immediately from this and the choice of α_0 (cf. (4.6)) that $\Pi_0(r) = 0$, whence

$$\Pi_0(r) = 0 \leq ck r^{-t-\delta\|\mathbf{q}\|_1-\delta}. \quad (4.15)$$

Combining (4.11), (4.14) and (4.15) gives

$$\begin{aligned} \sum_i \mu_1(B(x_i, r))^{q_1} \cdots \mu_k(B(x_i, r))^{q_k} &= \Lambda_0(r) + \Pi_0(r) \\ &\leq (s_0 + kc) r^{-t-\delta\|\mathbf{q}\|_1-\delta}. \end{aligned}$$

Since the packing $(B(x_i, r))_{i \in I}$ was arbitrary, this implies that $M(r, \mathbf{q}) \leq (s_0 + kc) r^{-t-\delta\|\mathbf{q}\|_1-\delta}$ for all $0 < r < \min_{l=1, \dots, s_0} r_0(\alpha_l)$, whence

$$\frac{\log M(r, \mathbf{q})}{-\log r} \leq t + \delta\|\mathbf{q}\|_1 + \delta + \frac{\log(s_0 + kc)}{-\log r}$$

for all $0 < r < \min_{l=1, \dots, s_0} r_0(\alpha_l)$. Letting $r \searrow 0$ and letting $\delta \searrow$ now gives $\bar{\tau}(\mathbf{q}) \leq t$. This proves (4.2).

Proofs of (4.3) and (4.4). The proofs of inequalities (4.3) and (4.4) are similar to the proofs of (4.1) and (4.2), respectively, and are therefore omitted. \square

ACKNOWLEDGEMENT. I thank an anonymous referee for many detailed and useful comments and remarks. In particular, I thank the referee for pointing out that the proofs of inequalities (4.1) and (4.3) follow immediately from Theorem 2.1.

REFERENCES

1. R. Badii and A. Politi, *Complexity: Hierarchical structures and scaling in physics*, Cambridge Nonlinear Science Series Vol. 6 (Cambridge University Press, Cambridge, England 1997).
2. C. Beck and F. Schlögl, *Thermodynamics of chaotic systems: An introduction*, Cambridge Nonlinear Science Series Vol. 4 (Cambridge University Press, Cambridge, England 1993).
3. K. J. Falconer, *Techniques in fractal geometry* (John Wiley and Sons, Chichester 1997).
4. T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia and B. J. Shraiman, Fractal measures and their singularities: The characterization of strange sets, *Phys. Rev. A* **33** (1986), 1141–1151.
5. H. Hentschel and I. Procaccia, The infinite number of generalized dimensions of fractals and strange attractors, *Physica* **8D** (1983), 435–444.
6. K.-S. Lau, L^p -spectrum and multifractal formalism, *Fractal geometry and stochastics* (Finsterbergen, 1994), 55–90, *Prog. Probab.*, 37, Birkhäuser, Basel, 1995.
7. L. Olsen, Multifractal geometry, in *Proceeding, Fractal Geometry and Stochastics II*, Greifswald, Germany, August 1998. *Progress in Probability*, Vol. 46 (Editors Bandt, C., Graf, S. and Zähle, M.), 1–37. Birkhäuser Verlag, 2000
8. L. Olsen, Mixed divergence points of self-similar measures, *Indiana Univ. Math. J.* **52** (2003), 1343–1372.
9. L. Olsen, Mixed generalized dimensions of self-similar measures, *J. Math. Anal. Appl.* **306** (2005), 516–539.
10. A. Rényi, Some fundamental questions of information theory, *Magyar Tud. Akad. Mat. Fiz. Oszt. Közl.* **10** (1960), 251–282.
11. A. Rényi, On measures of entropy and information, in *Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability*, 1960 (University of California Press, Berkeley, 1961), 547–561.