Proceedings of the Royal Society of Edinburgh, **134A**, 1013–1022, *2004*

Periodic solutions of higher-dimensional discrete systems

Zhan Zhou, **Jianshe Yu** and **Zhiming Guo**
Department of Applied Mathematics, Guangzhou University,
Guangzhou, Guangdong 510405, People's Republic of China
(zzhou@hnu.net.cn)

(MS received 6 August 2003; accepted 21 April 2004)

Consider the second-order discrete system

$$\Delta^2 X_{n-1} + f(n, X_n) = 0, \quad n \in \mathbb{Z}, \tag{$*$}$$

where $f \in C(\mathbb{R} \times \mathbb{R}^m, \mathbb{R}^m)$, $f(t+M, Z) = f(t, Z)$ for any $(t, Z) \in \mathbb{R} \times \mathbb{R}^m$ and M is a positive integer. By making use of critical-point theory, the existence of M-periodic solutions of $(*)$ is obtained.

1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ be the set of all natural numbers, integers and real numbers, respectively. For $a, b \in \mathbb{Z}$, define $\mathbb{Z}(a) = \{a, a+1, \dots,\}$, $\mathbb{Z}(a, b) = \{a, a+1, \dots, b\}$ when $a \leqslant b$.

Consider the nonlinear second-order discrete system

$$\Delta^2 X_{n-1} + f(n, X_n) = 0, \quad n \in \mathbb{Z}, \tag{1.1}$$

where $m \in \mathbb{N}$, $f = (f_1, f_2, \dots, f_m)^{\mathrm{T}} \in C(\mathbb{R} \times \mathbb{R}^m, \mathbb{R}^m)$, $f(t+M, Z) = f(t, Z)$ for any $(t, Z) \in \mathbb{R} \times \mathbb{R}^m$, M is a positive integer and Δ is the forward difference operator defined by $\Delta X_n = X_{n+1} - X_n$, $\Delta^2 X_n = \Delta(\Delta X_n)$.

Let p be a positive integer. As usual, a solution $\{X_n\}$ of (1.1) is said to be periodic of period p if

$$X_{p+i} = X_i \quad \text{for } i \in \mathbb{Z}. \tag{1.2}$$

In recent years, there has been much progress on the qualitative properties of difference equations, which included results on stability and attractivity [6,12,18,21] and results on oscillation and other topics [1, 9, 10, 16]. Only a few papers discuss the periodic solutions of difference equations [2–4,14,19,20]. As it is known, critical-point theory is an important tool to deal with the existence of periodic solutions of differential equations [11,13,15,17]. The main idea of these papers is to construct a suitable variational structure, so that the critical points of the variational functional correspond to the periodic solutions of the differential equations. It is natural for us to think that critical-point theory may be applied to prove the existence of periodic solutions of difference equations. In fact, by using critical-point theory, Guo and Yu have successfully proved the existence of periodic solutions of (1.1) when $m = 1$ and

$f(t,Z)$ is superlinear in the second variable Z, or when $f(t,Z)$ is sublinear in the second variable Z, in [7] and [8], respectively. Furthermore, [7,8] are the only papers we found which dealt with the problems of periodic solutions of difference equations using critical-point theory. When $f(t,Z)$ is neither superlinear nor sublinear, can we still find the periodic solutions of system (1.1)? The main purpose of this paper is to solve this problem. In fact, our results not only generalize the results in [7], but also improve them. For general background on difference equations, we refer to [1, 5, 10].

Throughout this paper, we suppose that there exists a continuously differential function $F(t,Z) \in C^1(\mathbb{R} \times \mathbb{R}^m, \mathbb{R})$ such that $\nabla_Z F(t,Z) = f(t,Z)$ for any $(t,Z) \in \mathbb{R} \times \mathbb{R}^m$, where $\nabla_Z F(t,Z)$ denotes the gradient of $F(t,Z)$ in Z.

In the following and in the sequel, for any $n \in \mathbb{N}$, $|\cdot|$ will denote the Euclidean norm in $\mathbb{R}^n$, defined by

$$|X| = \left(\sum_{i=1}^{n} X_i^2\right)^{1/2} \quad \text{for any } X = (X_1, X_2, \dots, X_n) \in \mathbb{R}^n.$$

The main results of this paper are as follows.

THEOREM 1.1. *Suppose that $F(t,Z)$ satisfies the following conditions.*

(F_1) *There exists a positive integer $M \geqslant 3$ such that*

$$F(t+M, Z) = F(t,Z) \quad \forall (t,Z) \in \mathbb{R} \times \mathbb{R}^m \text{ and } F(t,Z) \geqslant 0.$$

(F_2) *There exist constants $\delta > 0$, $\alpha \in (0, 1 - \cos(2/M)\pi)$ such that*

$$F(n,Z) \leqslant \alpha |Z|^2 \quad \text{for } n \in \mathbb{N},\ Z \in \mathbb{R}^m \text{ and } |Z| \leqslant \delta.$$

(F_3) *There exist constants $\rho > 0$, $\gamma > 0$, $\beta \in (2, +\infty)$ when M is even or $\beta \in (1 + \cos(1/M)\pi, +\infty)$ when M is odd such that*

$$F(n,Z) \geqslant \beta |Z|^2 - \gamma \quad \text{for } n \in \mathbb{N},\ |Z| \geqslant \rho.$$

Then system (1.1) possesses at least three M-periodic solutions.

By (F_2), we see that $f(n,0) \equiv 0$, and thus $X_n \equiv 0$ is always an M-periodic solution of (1.1), which is called the trivial periodic solution of (1.1). Therefore, we have the following result.

COROLLARY 1.2. *If $F(t,Z)$ satisfies (F_1) to (F_3), then system (1.1) possesses at least two non-trivial M-periodic solutions.*

EXAMPLE 1.3. Assume that $m = 1$ and

$$f(t,Z) = a(Z - \sin Z)(\phi(t) + D),$$

where $a > 2$ when M is even or $a > 2(1 + \cos(1/M)\pi)$ when M is odd, $D > 0$ and $\phi(t)$ is a continuously M-periodic function satisfying $|\phi(t)| < D$. Then

$$F(t,Z) = a(\tfrac{1}{2}Z^2 + \cos Z - 1)(\phi(t) + D)$$

and satisfies all conditions of theorem 1.1. Thus (1.1) has at least two non-trivial M-periodic solutions.

Consider the case where $F(t, Z)$ is superquadratic or $f(t, Z)$ is superlinear. Then we have the following result.

COROLLARY 1.4. *Suppose that $F(t, Z)$ satisfies (F_1) and the following two conditions.*

(F_4) $F(n, Z) = o(|Z|^2)$ *as* $Z \to 0$.

(F_5) *There exist constants $R_1 > 0$, $\alpha_1 > 2$ such that*

$$Z \cdot f(n, Z) \geqslant \alpha_1 F(n, Z) > 0 \quad \text{for } |Z| \geqslant R_1.$$

Then system (1.1) possesses at least three M-periodic solutions.

REMARK 1.5. When $m = 1$, corollary 1.2 reduces to theorem 1.1 in [7].

2. Some basic lemmas

In order to apply critical-point theory to study the existence of periodic solutions of (1.1), we shall state some basic notations and lemmas, which will be used in the proofs of our main results.

Let S be the set of sequences

$$X = (\dots, X_{-n}, \dots, X_{-1}, X_0, X_1, \dots, X_n, \dots) = \{X_n\}_{n=-\infty}^{+\infty},$$

where $X_n = (X_{n,1}, X_{n,2}, \dots, X_{n,m})^{\mathrm{T}} \in \mathbb{R}^m$, m a given positive integer.

For any $X, Y \in S$, $a, b \in \mathbb{R}$, $aX + bY$ is defined by

$$aX + bY := \{aX_n + bY_n\}_{n=-\infty}^{+\infty}.$$

Then S is a vector space.

For any given positive integer M, E_M is defined as a subspace of S by

$$E_M = \{X = \{X_n\} \in S \mid X_{n+M} = X_n,\ n \in \mathbb{Z}\}.$$

E_M can be equipped with norm $\|\cdot\|_{E_M}$ and inner product $\langle\cdot,\cdot\rangle_{E_M}$ as follows,

$$\|X\|_{E_M} := \left(\sum_{n=1}^{M} |X_n|^2\right)^{1/2} \quad \forall X = \{X_n\} \in E_M \tag{2.1}$$

and

$$\langle X, Y\rangle_{E_M} := \sum_{n=1}^{M} X_n \cdot Y_n \quad \forall X = \{X_n\} \in E_M,\ Y = \{Y_n\} \in E_M, \tag{2.2}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^m$ and $X_n \cdot Y_n$ denotes the usual scalar product in $\mathbb{R}^m$.

Define a linear map $L : E_M \to \mathbb{R}^{mM}$ by

$$LX = (X_{1,1}, \dots, X_{M,1}, X_{1,2}, \dots, X_{M,2}, \dots, X_{1,m}, \dots, X_{M,m})^{\mathrm{T}}, \tag{2.3}$$

where $X = \{X_n\}$ and $X_i = (X_{i,1}, X_{i,2}, \dots, X_{i,m})^{\mathrm{T}}$ for $i \in \mathbb{Z}(1, m)$.

It is easy to see that the map L defined in (2.3) is a linear homeomorphism with $\|X\|_{E_M} = |LX|$, and $(E_M, \langle\cdot,\cdot\rangle_{E_M})$ is a Hilbert space, which can be identified with $\mathbb{R}^{mM}$.

Consider the functional J defined on E_M by

$$J(X) = \sum_{n=1}^{M}[\tfrac{1}{2}|\Delta X_n|^2 - F(n, X_n)]. \tag{2.4}$$

In view of $X_{n+M} = X_n \ \forall n \in \mathbb{Z} \ \forall X \in E_M$, equation (2.4) can be rewritten as

$$J(X) = \sum_{n=1}^{M}[(|X_n|^2 - X_n \cdot X_{n+1}) - F(n, X_n)]. \tag{2.5}$$

Since E_M is linearly homeomorphic to $\mathbb{R}^{mM}$, by the continuity of $f(t, Z)$, J can be viewed as a continuously differentiable functional defined on a finite-dimensional Hilbert space. That is, $J \in C^1(E_M, \mathbb{R})$. Furthermore, $J'(X) = 0$ if and only if $\partial J(X)/\partial X_{n,l} = 0$, $n \in \mathbb{Z}(1, M)$, $l \in \mathbb{Z}(1, m)$. If we define $X_0 := X_M$, then

$$\frac{\partial J(X)}{\partial X_{n,l}} = 2X_{n,l} - X_{n+1,l} - X_{n-1,l} - f_l(n, X_n), \quad l \in \mathbb{Z}(1, m), \quad n \in \mathbb{Z}(1, M)$$

or

$$\frac{\partial J(X)}{\partial X_{n,l}} = -[\Delta^2 X_{n-1,l} + f_l(n, X_n)], \quad l \in \mathbb{Z}(1, m), \quad n \in \mathbb{Z}(1, M).$$

Therefore, $X \in E_M$ is a critical point of J, i.e. $J'(X) = 0$ if and only if

$$\Delta^2 X_{n-1,l} + f_l(n, X_n) = 0 \quad \forall n \in \mathbb{Z}(1, M),\ l \in \mathbb{Z}(1, m).$$

That is,

$$\Delta^2 X_{n-1} + f(n, X_n) = 0 \quad \forall n \in \mathbb{Z}(1, M).$$

On the other hand, $\{X_n\} \in E_M$ is M-periodic in n and $f(t, Z)$ is M-periodic in t. So $X \in E_M$ is a critical point of J if and only if $\Delta^2 X_{n-1} + f(n, X_n) = 0 \ \forall n \in \mathbb{Z}$, and X is an M-periodic solution of (1.1). Thus we reduce the problem of finding M-periodic solutions of (1.1) to that of seeking critical points of the functional J in E_M.

Due to the identification of E_M with $\mathbb{R}^{mM}$, we write $J(X)$ as

$$J(X) = \tfrac{1}{2}\langle ALX, LX\rangle - \sum_{n=1}^{M} F(n, X_n), \tag{2.6}$$

where $X = \{X_n\} \in E_M$ and

$$A = \begin{pmatrix} B & & & 0 \\ & B & & \\ & & \ddots & \\ 0 & & & B \end{pmatrix}_{mM\times mM},$$

$$B = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}_{M\times M}$$

and $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^{mM}$.

Assume that λ is an eigenvalue of B. Since $B - rI$ is positive-definite for $r < 0$ and negative-definite for $r > 4$, where I is the $M \times M$ identity matrix, we see that $\lambda \in [0, 4]$. Assume that $\xi = (\xi_1, \xi_2, \dots, \xi_M)^{\mathrm{T}}$ is an eigenvector associated to λ and define the sequence $\{y_n\}_{n=1}^{\infty}$ as

$$y_i = \xi_i, \quad i = 1, 2, \dots, M, \quad \text{and} \quad y_{n+M} = y_n, \quad n \in \mathbb{Z}.$$

Then $\{y_n\}$ satisfies

$$-y_{n+1} + (2 - \lambda)y_n - y_{n-1} = 0, \quad y_{n+M} = y_n, \quad n \in \mathbb{Z}. \tag{2.7}$$

Since the roots of the equation $-r^2 + (2 - \lambda)r - 1 = 0$ are

$$r_1 = \tfrac{1}{2}(2 - \lambda + \sqrt{4 - (2 - \lambda)^2}\mathrm{i}) \quad \text{and} \quad r_2 = \tfrac{1}{2}(2 - \lambda - \sqrt{4 - (2 - \lambda)^2}\mathrm{i}),$$

set

$$\theta = \arccos \tfrac{1}{2}(2 - \lambda). \tag{2.8}$$

Then

$$y_n = d_1 \cos n\theta + d_2 \sin n\theta = \sqrt{d_1^2 + d_2^2} \cos(n\theta - \theta_0)$$

for some constants d_1, d_2 and θ_0, where $\sqrt{d_1^2 + d_2^2} \neq 0$. By the fact that $y_{n+M} = y_n$, $n \in \mathbb{Z}$, we obtain

$$M\theta = 2k\pi, \quad k = 0, 1, 2, \dots, M - 1.$$

That is,

$$\theta_k = \frac{2k}{M}\pi, \quad k = 0, 1, 2, \dots, M - 1.$$

By (2.8), we see that the eigenvalues of B are

$$\lambda_k = 2\left(1 - \cos \frac{2k}{M}\pi\right), \quad k = 0, 1, 2, \dots, M - 1. \tag{2.9}$$

Thus $\lambda_0 = 0$, $\lambda_1 > 0$, $\lambda_2 > 0, \dots, \lambda_{M-1} > 0$. Therefore,

$$\left.\begin{aligned} \lambda_{\min} &= \min\{\lambda_1, \dots, \lambda_{M-1}\} = 2\left(1 - \cos \frac{2}{M}\pi\right), \\ \lambda_{\max} &= \max\{\lambda_1, \dots, \lambda_{M-1}\} = \begin{cases} 4 & \text{when } M \text{ is even}, \\ 2\left(1 + \cos \dfrac{1}{M}\pi\right) & \text{when } M \text{ is odd}. \end{cases} \end{aligned}\right\} \tag{2.10}$$

Let

$$W = \operatorname{Ker} AL = \{X \in E_M : ALX = 0 \in \mathbb{R}^{mM}\}.$$

Then

$$W = \{X \in E_M \mid X = \{V\},\ V \in \mathbb{R}^m\}.$$

Let Y be the direct orthogonal complement of E_M to W, that is, $E_M = Y \oplus W$.

Assume that H is a real Banach space, $I \in C^1(H, \mathbb{R})$, i.e. I is a continuously Fréchet differentiable functional defined on H. I is said to satisfy the Palais–Smale condition (PS condition) if any sequence $\{u_n\} \subset H$ for which $\{I(u_n)\}$ is bounded and $I'(u_n) \to 0 (n \to \infty)$ possesses a convergent subsequence in H.

Let B_r denote the open ball in H about 0 with radius r and let ∂B_r denote its boundary.

Lemma 2.1 (linking theorem [15]). *Let H be a real Hilbert space, $H = H_1 \oplus H_2$, where H_1 is a finite-dimensional subspace of H. Assume that $J \in C^1(H, \mathbb{R})$ satisfies the PS condition and the following.*

(J$_1$) *There exist constants $a > 0$ and $\rho > 0$ such that $J|_{\partial B_\rho \cap H_2} \geqslant a$.*

(J$_2$) *There exists an $e \in \partial B_1 \cap H_2$ and a constant $R_0 > \rho$ such that $J|_{\partial Q} \leqslant 0$ and $Q \triangleq (\bar{B}_{R_0} \cap H_1) \oplus \{re \mid 0 < r < R_0\}$.*

Then J possesses a critical value $c \geqslant a$, where

$$c = \inf_{h \in \Gamma} \max_{X \in Q} J(h(X))$$

and $\Gamma = \{h \in C(\bar{Q}, H) : h|_{\partial Q} = \mathrm{id}\}$, where id *denotes the identity operator.*

3. Proofs of main results

According to (F$_3$), if we let

$$\gamma_1 = \max\{|F(n, Z) - \beta|Z|^2 + \gamma| : n \in \mathbb{Z},\ |Z| \leqslant \rho\}, \qquad \gamma' = \gamma + \gamma_1,$$

then

$$F(n, Z) \geqslant \beta|Z|^2 - \gamma' \quad \text{for } n \in \mathbb{Z},\ Z \in \mathbb{R}^m. \tag{3.1}$$

To prove theorem 1.1, we need the following lemmas.

Lemma 3.1. *Assume that $F(t, Z)$ satisfies (F$_3$). Then the functional*

$$J(X) = \tfrac{1}{2}\langle ALX, LX\rangle - \sum_{n=1}^{M} F(n, X_n)$$

is bounded from above in E_M.

Proof. For any $X \in E_M$, by (3.1),

$$\begin{aligned} J(X) &\leqslant \tfrac{1}{2}\lambda_{\max}|LX|^2 - \sum_{n=1}^{M} F(n, X_n) \\ &\leqslant \tfrac{1}{2}\lambda_{\max}\|X\|_{E_M}^2 - \sum_{n=1}^{M} (\beta|X_n|^2 - \gamma') \\ &= (\tfrac{1}{2}\lambda_{\max} - \beta)\|X\|_{E_M}^2 + M\gamma'. \end{aligned} \tag{3.2}$$

https://doi.org/10.1017/S0308210500003607 Published online by Cambridge University Press

Since, by (F_3) and (2.10), we see that $\beta > \frac{1}{2}\lambda_{\max}$, then

$$J(X) \leqslant M\gamma'.$$

The proof of lemma 3.1 is complete. □

LEMMA 3.2. *Assume that (F_1) and (F_3) hold. Then the functional J satisfies the PS condition.*

Proof. Let $\{J(X^{(k)})\}$ be a bounded sequence from below, that is, there exists a positive constant c such that

$$-c \leqslant J(X^{(k)}) \quad \forall k \in \mathbb{N}.$$

By the proof of lemma 3.1, it is easy to see that

$$-c \leqslant J(X^{(k)}) \leqslant (\tfrac{1}{2}\lambda_{\max} - \beta)\|X^{(k)}\|_{E_M}^2 + M\gamma',$$

which implies that

$$\|X^{(k)}\|_{E_M}^2 \leqslant (\beta - \tfrac{1}{2}\lambda_{\max})^{-1}(M\gamma' + c).$$

That is, $\{X^{(k)}\}$ is a bounded sequence in the finite-dimensional space E_M. Consequently, it has a convergent subsequence. □

Proof of theorem 1.1. Assumption (F_2) and $F(t, Z) \in C^1(\mathbb{R} \times \mathbb{R}^m, \mathbb{R})$ imply that $F(t, 0) = 0$ and $f(t, 0) = 0$ for any $t \in \mathbb{R}$. Therefore, $\{X_n\}$, where $X_n \equiv 0 \in \mathbb{R}^m$, $n \in \mathbb{Z}$, is a trivial periodic solution of (1.1) with period M. By lemma 3.1, J is bounded from above on E_M. We write $c_0 = \sup_{X \in E_M} J(X)$. There is a sequence $\{X^{(k)}\}_{k=1}^{\infty}$, where $X^{(k)} \in E_M$, such that $c_0 = \lim_{k\to\infty} J(X^{(k)})$. On the other hand, by (3.2), we have

$$J(X) \leqslant (\tfrac{1}{2}\lambda_{\max} - \beta)\|X\|_{E_M}^2 + M\gamma' \quad \text{for } X \in E_M.$$

Therefore, $J(X) \to -\infty$ as $\|X\|_{E_M}^2 \to \infty$, which implies $\{X^{(k)}\}$ is bounded. Thus $\{X^{(k)}\}$ has a convergent subsequence, denoted by $\{X^{(k_i)}\}$. Let $\bar{X} = \lim_{i\to\infty} X^{(k_i)}$. By the continuity of $J(X)$, it is easy to see that $J(\bar{X}) = c_0$. Clearly, $\bar{X}$ is a critical point of J in E_M.

We claim that $c_0 > 0$. In fact, by assumption (F_2) and the definition of Y, we see that, for any $X \in Y$ with $\|X\|_{E_M} \leqslant \delta$,

$$\begin{aligned} J(X) &= \tfrac{1}{2}\langle ALX, LX\rangle - \sum_{n=1}^{M} F(n, X_n) \\ &\geqslant \tfrac{1}{2}\lambda_{\min}|LX|^2 - \alpha \sum_{n=1}^{M} |X_n|^2 \\ &= (\tfrac{1}{2}\lambda_{\min} - \alpha)\|X\|_{E_M}^2. \end{aligned}$$

Let $\sigma = (\frac{1}{2}\lambda_{\min} - \alpha)\delta^2$. Then

$$J(X) \geqslant \sigma \quad \forall X \in Y \cap \partial B_\delta.$$

https://doi.org/10.1017/S0308210500003607 Published online by Cambridge University Press

Thus we have proved that $c_0 = \sup_{X \in E_M} J(X) \geqslant \sigma > 0$. At the same time, we have also proved that there exist constants $\sigma > 0$ and $\delta > 0$ such that $J|_{\partial B_\delta \cap Y} \geqslant \sigma$. This implies that J satisfies assumption (J_1) of the linking theorem.

Since, for $X \in W$, $ALX = 0$ and, by (F_1),

$$J(X) = \tfrac{1}{2}\langle ALX, LX\rangle - \sum_{n=1}^{M} F(n, X_n) = -\sum_{n=1}^{M} F(n, X_n) \leqslant 0,$$

then $\bar{X} \notin W$ and the critical point $\bar{X}$ of J corresponding to the critical value c_0 is a non-trivial periodic solution of (1.1) with period M.

In order to obtain another non-trivial M-periodic solution of (1.1) different from $\{\bar{X}\}$, we will use the linking theorem. By lemma 3.2, we know that J satisfies the PS condition, and we have also verified that J satisfies condition (J_1) of the linking theorem. In the following, we will prove that J satisfies condition (J_2) of the linking theorem. To this end, let $e \in \partial B_1 \cap Y$. Then, for any $Z \in W$ and $r \in \mathbb{R}$, let $X = re + Z$. We have

$$\begin{aligned} J(X) &= \tfrac{1}{2}\langle AL(re + Z), L(re + Z)\rangle - \sum_{n=1}^{M} F(n, X_n) \\ &= \tfrac{1}{2}\langle AL(re), L(re)\rangle - \sum_{n=1}^{M} F(n, X_n) \\ &\leqslant \tfrac{1}{2}\lambda_{\max}|L(re)|^2 - \beta \sum_{n=1}^{M}(|re_n + Z_n|^2 - \gamma') \\ &= \tfrac{1}{2}\lambda_{\max} r^2 - \beta \sum_{n=1}^{M}(r^2|e_n|^2 + |Z_n|^2 - \gamma') \\ &= (\tfrac{1}{2}\lambda_{\max} - \beta)r^2 - \beta\|Z\|_{E_M}^2 + M\beta\gamma' \\ &\leqslant -\beta\|Z\|_{E_M}^2 + M\beta\gamma'. \end{aligned}$$

Therefore, there exists some positive number $R_2 > \delta$ such that, for any $X \in \partial Q$, then $J(X) \leqslant 0$, where $Q = (\bar{B}_{R_2} \cap W) \oplus \{re : 0 < r < R_2\}$. By the linking theorem, J possesses a critical value $c \geqslant \sigma > 0$, where

$$c = \inf_{h \in \Gamma} \max_{u \in Q} J(h(u)) \quad \text{and} \quad \Gamma = \{h \in C(\bar{Q}, E_M) : h|_{\partial \bar{Q}} = \mathrm{id}\}.$$

Let $\tilde{X} \in E_M$ be a critical point corresponding to the critical value c of J, that is, $J(\tilde{X}) = c$. If $\tilde{X} \neq \bar{X}$, then theorem 1.1 holds. Otherwise, $\tilde{X} = \bar{X}$. Then $c_0 = J(\bar{X}) = J(\tilde{X}) = c$, which is the same as

$$\sup_{X \in E_M} J(X) = \inf_{h \in \Gamma} \max_{u \in Q} J(h(u)).$$

Choosing $h = \mathrm{id}$, we have $\sup_{X \in Q} J(X) = c_0$. Because the choice of

$$e \in \partial B_1 \cap Y \in Q = (\bar{B}_{R_2} \cap W) \oplus \{re : 0 < r < R_2\}$$

is arbitrary, we can take $-e \in \partial B_1 \cap Y$. Similarly, there exists a positive number $R_3 > \delta$, for any $X \in \partial Q_1$, $J(X) \leqslant 0$, where $Q_1 = (\bar{B}_{R_3} \cap W) \oplus \{-re : 0 < r < R_3\}$. Again, by the linking theorem, J possesses a critical value $c' \geqslant \sigma > 0$, and

$$c' = \inf_{h \in \Gamma_1} \max_{u \in Q_1} J(h(u))$$

and

$$\Gamma_1 = \{h \in C(\bar{Q}_1, E_M) : h|_{\partial \bar{Q}_1} = \mathrm{id}\}.$$

If $c' \neq c_0$, then the proof is complete. Otherwise, $c' = c_0$, $\sup_{X \in Q_1} J(X) = c_0$. Because $J|_{\partial Q} \leqslant 0$ and $J|_{\partial Q_1} \leqslant 0$, then J attains its maximum at some points in the interior of sets Q and Q_1. But $Q \cap Q_1 \subset W$, and $J(X) \leqslant 0$ for $X \in W$. Thus there is a critical point $X' \in E_M$ satisfying $X' \neq \tilde{X}$ and $J(X') = c' = c_0$.

The proof of theorem 1.1 is now complete. □

Proof of corollary 1.4. We need only to show that (F_2) and (F_3) hold in theorem 1.1. In fact, condition (F_4) clearly implies (F_2). From (F_5), we have

$$\frac{Z}{|Z|} \cdot \frac{\nabla_Z F(n,Z)}{F(n,Z)} \geqslant \frac{\alpha_1}{|Z|} \quad \text{for } n \in \mathbb{Z} \text{ and } |Z| \geqslant R_1.$$

Thus

$$\frac{\mathrm{d} \ln F(n,Z)}{\mathrm{d}|Z|} \geqslant \frac{\alpha_1}{|Z|},$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}|Z|}(\ln F(n,Z) - \alpha_1 \ln |Z|) \geqslant 0 \quad \text{for } n \in \mathbb{Z} \text{ and } |Z| \geqslant R_1. \tag{3.3}$$

Let

$$G = \min\{\ln F(n,Z) - \alpha_1 \ln |Z| : n \in \mathbb{Z} \text{ and } |Z| = R_1\}.$$

By (3.3),

$$\ln F(n,Z) - \alpha_1 \ln |Z| \geqslant G \quad \text{for } n \in \mathbb{Z} \text{ and } |Z| \geqslant R_1.$$

That is,

$$F(n,Z) \geqslant a_1 |Z|^{\alpha_1} \quad \text{for } n \in \mathbb{Z} \text{ and } |Z| \geqslant R_1,$$

where $a_1 = e^G$. Let $\rho_1 \geqslant R_1$ satisfying $a_1 \rho_1^{\alpha_1 - 2} > 2$. Then, for $n \in \mathbb{Z}$ and $|Z| \geqslant \rho_1$,

$$|F(n,Z)| \geqslant a_1 |Z|^{\alpha_1 - 2} |Z|^2 \geqslant a_1 {\rho_1}^{\alpha_1 - 2} |Z|^2.$$

Let $\beta = a_1 \rho_1^{\alpha_1 - 2}$. Then (F_3) holds. The proof of corollary 1.4 is complete. □

Acknowledgments

This work is supported by the Excellent Young Teachers Program of MOE, People's Republic of China, and the Research Fund for the Doctoral Program of Higher Education of the People's Republic of China (grant no. 20020532014).

References

1 R. P. Agarwal. *Difference equations and inequalities: theory, methods and applications* (New York: Marcel Dekker, 1992).

2 R. P. Agarwal and J. Popenda. Periodic solutions of first order linear difference equations. *Math. Comput. Modelling* **22** (1995), 11–19.

3 R. P. Agarwal and E. Yu. Romanenko. Stable periodic solutions of difference equations. *Appl. Math. Lett.* **11** (1998), 81–84.

4 R. P. Agarwal and W. Zhang. Periodic solutions of difference equations with general periodicity. *Comput. Math. Appl.* **42** (2001), 719–727.

5 S. N. Elaydi. *An introduction to difference equations* (Springer, 1999).

6 L. H. Erbe, H. Xia and J. S. Yu. Global stability of a linear nonautonomous delay difference equations. *J. Diff. Eqns Applic.* **1** (1995), 151–161.

7 Z. M. Guo and J. S. Yu. Existence of periodic and subharmonic solutions for second-order superlinear difference equations. *Sci. China* A **46** (2003), 506–515.

8 Z. M. Guo and J. S. Yu. The existence of periodic and subharmonic solutions to subquadratic second-order difference equations. *J. Lond. Math. Soc.* **68** (2003), 419–430.

9 I. Györi and G. Ladas. *Oscillation theory of delay differential equations with applications* (Oxford University Press, 1991).

10 V. L. Kocic and G. Ladas. *Global behavior of nonlinear difference equations of higher order with applications* (Dordrecht: Kluwer Academic, 1993).

11 J. Q. Liu and Z. Q. Wang. Remarks on subharmonics with minimal periods of Hamiltonian systems. *Nonlin. Analysis* **7** (1993), 803–821.

12 H. Matsunaga, T. Hara and S. Sakata. Global attractivity for a nonlinear difference equation with variable delay. *Comput. Math. Appl.* **41** (2001), 543–551.

13 J. Mawhin and M. Willem. *Critical point theory and Hamiltonian systems* (Springer, 1989).

14 P. Y. H. Pang and R. P. Agarwal. On periodicity of difference equations of a general type. *J. Diff. Eqns Applic.* **2** (1996), 271–287.

15 P. H. Rabinowitz. *Minimax methods in critical point theory with applications to differential equations*. Conference Board of the Mathematical Sciences, Regional Conference Series in Applied Mathematics, vol. 65 (Providence, RI: American Mathematical Society, 1986).

16 X. H. Tang and J. S. Yu. Oscillation of nonlinear delay difference equations. *J. Math. Analysis Applic.* **249** (2000), 476–490.

17 Y. T. Xu and Z. M. Guo. Applications of a Z_p index theory to periodic solutions for a class of functional differential equations. *J. Math. Analysis Applic.* **257** (2001), 189–205.

18 J. S. Yu. Asymptotic stability for a linear difference equation with variable delay. *Comput. Math. Appl.* **36** (1998), 203–210.

19 Z. Zhou. Periodic orbits on discrete dynamical systems. *Comput. Math. Appl.* **45** (2003), 1155–1161.

20 Z. Zhou and J. Wu. Attractive periodic orbits in nonlinear delayed discrete-time neural networks with delayed feedback. *J. Diff. Eqns Applic.* **8** (2002), 467–483.

21 Z. Zhou and Q. Zhang. Uniform stability of nonlinear difference systems. *J. Math. Analysis Applic.* **225** (1998), 486–500.

(*Issued 29 October 2004*)