# LINEARIZATION OF THE PRODUCT OF JACOBI POLYNOMIALS. I

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**1. Introduction.** Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomial of degree $n$, order $(\alpha, \beta)$, $\alpha, \beta > -1$, defined by

$$(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}[(1-x)^{n+\alpha}(1+x)^{n+\beta}]$$

[**9**, p. 67], and let $R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1)$. Then for $n \geqq m$,

$$R_n^{(\alpha,\beta)}(x)R_m^{(\alpha,\beta)}(x) = \sum_{k=n-m}^{n+m} g(k,n,m)R_k^{(\alpha,\beta)}(x),$$

where

$$g(k,n,m) = h(k)\int_{-1}^{1} R_k^{(\alpha,\beta)}(x)R_n^{(\alpha,\beta)}(x)R_m^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta\,dx,$$

$$h(k) = \left(\int_{-1}^{1}[R_k^{(\alpha,\beta)}(x)]^2(1-x)^\alpha(1+x)^\beta\,dx\right)^{-1}.$$

Since $R_n^{(\alpha,\beta)}(1) = 1$, it follows that

$$\sum_k g(k,n,m) = 1. \tag{1}$$

It is known that if $\alpha = \beta \geqq -\frac{1}{2}$ (the ultraspherical case) or if $\alpha = \beta + 1$, then $g(k, n, m) \geqq 0$. See Hsü [**7**] and Hylleraas [**8**]. Hence, for $\alpha = \beta \geqq -\frac{1}{2}$ or $\alpha = \beta + 1$ we have:

$$\sum_k |g(k,n,m)| = 1. \tag{2}$$

This gives a convolution structure to expansions in Jacobi polynomials and permits the $R_n^{(\alpha,\beta)}(x)$ to behave like characters on a compact group (see [**3**]). Consequently, many parts of harmonic analysis, which cannot be extended to orthogonal polynomials in general, can be extended to those Jacobi polynomials for which (2) holds.

Askey [**1**] has extended (2) and conjectured that if $\alpha \geqq \beta$ and $\alpha + \beta + 1 \geqq 0$, then (2) holds. For $\alpha \geqq \beta \geqq -\frac{1}{2}$, Askey and Wainger [**4**] have obtained a weaker result:

$$\sum_k |g(k,n,m)| = O(1) \tag{3}$$

uniformly in $n$ and $m$. Then from the convolution structure given by (3) they obtained a Wiener-Lévy theorem for Jacobi expansions and an analogue

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of the strong Szegö limit theorem for Toeplitz matrices associated with Jacobi polynomials. For $\alpha = \beta$, (2) is one of the main tools used by Askey and Wainger [**5**] to obtain a transplantation theorem for ultraspherical coefficients, from which follows an analogue of the Marcinkiewicz multiplier theorem and an analogue of a theorem of Hardy and Littlewood concerning the Fourier coefficients of even functions, monotonically decreasing in $(0, \pi)$. Additional applications of (2) will be given elsewhere. In this paper we shall prove that the above-mentioned conjecture is correct.

THEOREM. *If* $\alpha \geqq \beta$ *and* $\alpha + \beta + 1 \geqq 0$, *then* $g(k, n, m) \geqq 0$ *for all* $k$, $n$, *and* $m$, *and thus* (2) *holds.*

An important step in our proof is the application of Descartes' rule of signs to part of a recurrence formula for $d(k, n, m)$, a positive multiple of $g(k, n, m)$. In subsequent papers† we shall apply this method to related problems; for instance, to determine those $(\alpha, \beta)$ satisfying $\alpha + \beta + 1 < 0$ for which $g(k, n, m) \geqq 0$.

I wish to thank Professor R. Askey for bringing this problem to my attention.

**2. Proof of the Theorem.** Using

$$P_n^{(\alpha,\beta)}(1) = \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + 1)\Gamma(\alpha + 1)}$$

and the recurrence formula [**6**, p. 169, (11)],

$$\begin{aligned} &2(n + 1)(n + \alpha + \beta + 1)(2n + \alpha + \beta)P_{n+1}^{(\alpha,\beta)}(x) \\ &= (2n + \alpha + \beta + 1)[(2n + \alpha + \beta)(2n + \alpha + \beta + 2)x + \alpha^2 - \beta^2]P_n^{(\alpha,\beta)}(x) \\ &\qquad - 2(n + \alpha)(n + \beta)(2n + \alpha + \beta + 2)P_{n-1}^{(\alpha,\beta)}(x), \end{aligned}$$

we obtain the explicit formula

$$\begin{aligned} (4)\quad &\frac{2(\alpha + 1)}{\alpha + \beta + 2} R_n^{(\alpha,\beta)}(x)R_1^{(\alpha,\beta)}(x) \\ &= \frac{2(n + \alpha + \beta + 1)(n + \alpha + 1)}{(2n + \alpha + \beta + 2)(2n + \alpha + \beta + 1)} R_{n+1}^{(\alpha,\beta)}(x) \\ &+ \frac{\alpha - \beta}{\alpha + \beta + 2}\left[1 - \frac{(\alpha + \beta + 2)(\alpha + \beta)}{(2n + \alpha + \beta + 2)(2n + \alpha + \beta)}\right]R_n^{(\alpha,\beta)}(x) \\ &+ \frac{2n(n + \beta)}{(2n + \alpha + \beta + 1)(2n + \alpha + \beta)} R_{n-1}^{(\alpha,\beta)}(x). \end{aligned}$$

Since $\alpha \geqq \beta > -1$, (4) implies that $g(k, n, 1) \geqq 0$. Hence we may assume that $n \geqq m \geqq 2$. We may also assume that $\alpha + \beta + 1 > 0$, for then the case $\alpha + \beta + 1 = 0$ follows by continuity. Observe that (4) implies that $\alpha \geqq \beta$ is a necessary condition for $g(k, n, m) \geqq 0$.

†*Added in proof.* See, for example, *Linearization of the product of Jacobi polynomials.* II (to appear in Can. J. Math.).

In [**8**] Hylleraas let

$$y_n(z) = F(-n, n+p; q; z), \qquad p+1 > q > 0,$$

and derived the recurrence formula for $c_k = c(k, n, m)$, where $c_k$ is defined by

$$y_n y_m = \sum_{k=n-m}^{n+m} c_k y_k, \qquad n \geqq m.$$

Since

$$P_n^{(\alpha,\beta)}(x) = (-1)^n \binom{n+\beta}{n} F(-n, n+\alpha+\beta+1; \beta+1; (x+1)/2)$$

[**6**, p. 170, (16)] and

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n},$$

it follows that if we let $p = \alpha+\beta+1$, $q = \beta+1$, $z = (x+1)/2$, and $d_k = (-1)^{k+n+m} c_k$, then

$$R_n^{(\alpha,\beta)}(x) R_m^{(\alpha,\beta)}(x) = \sum_{k=n-m}^{n+m} \frac{\binom{k+\alpha}{k}\binom{n+\beta}{n}\binom{m+\beta}{m}}{\binom{k+\beta}{k}\binom{n+\alpha}{n}\binom{m+\alpha}{m}} d_k R_k^{(\alpha,\beta)}(x).$$

Clearly $g_k = g(k, n, m)$ is a (strictly) positive multiple of $d_k$, and thus it suffices to show that $d_k \geqq 0$.

In order to obtain more suitable formulas we also let

$$a = \alpha+\beta+1, \quad b = \alpha - \beta, \quad s = n - m, \quad k = s + j.$$

Note that $a > 0$, $b \geqq 0$, and $s \geqq 0$. From the recurrence formula given by Hylleraas [**8**, (4.13)] for $c_k$ we obtain:

$$\begin{aligned}(5)\quad & \frac{(j+1)(2s+2j+1+a+b)(2n+j+1+a)}{(2s+2j+1+a)} \\ & \qquad \times \frac{(2m-j-1+a)(2s+j+1)}{(2s+2j+2+a)} d_{s+j+1} \\ &= b\left[\frac{(j+1)(2n+j+2a)(2m-j)(2s+j+1)}{(2s+2j+1+a)}\right. \\ & \qquad \left. - \frac{j(2n+j-1+2a)(2m-j+1)(2s+j)}{(2s+2j-1+a)}\right] d_{s+j} \\ & \quad + \frac{(j-1+a)(2s+2j-1+a-b)(2n+j-1+2a)}{(2s+2j-2+a)} \\ & \qquad \times \frac{(2m-j+1)(2s+j-1+a)}{(2s+2j-1+a)} d_{s+j-1}.\end{aligned}$$

From [8, (3.3) and (3.8)] we obtain:

$$d_{n+m} = \frac{\binom{2n+a-1}{n}\binom{2m+a-1}{m}\binom{n+m+(a-b-1)/2}{n+m}}{\binom{n+(a-b-1)/2}{n}\binom{m+(a-b-1)/2}{m}\binom{2n+2m+a-1}{n+m}}$$

and

$$d_{n-m} = \frac{\binom{2m+a-1}{m}\binom{n}{m}\binom{n+(a+b-1)/2}{m}}{\binom{m+(a-b-1)/2}{m}\binom{2m}{m}\binom{2n+a}{2m}}.$$

Since $a > 0$, $\alpha = (a+b-1)/2 > -1$ and $\beta = (a-b-1)/2 > -1$, it follows that $d_{n+m} > 0$ and $d_{n-m} > 0$. Setting $j = 0$ in (5) and using $d_{s-1} = 0$, we see that

$$d_{s+1} = \frac{4bm(n+a)(2s+2+a)}{(2s+1+a+b)(2n+1+a)(2m-1+a)} d_s \geqq 0.$$

For $j \geqq 1$ we let $J = j - 1$ and write the coefficient of $d_{s+j}$ in (5) in the form

$$\text{coef}(d_{s+j}) = \frac{bF(J)}{(2s+2J+3+a)(2s+2J+1+a)}, \tag{6}$$

where (recall that $s = n - m$)

$$\begin{aligned} F(J) &= (J+2)(J+2n+2a+1)(2m-J-1)(J+2s+2)(2J+2s+a+1) \\ &\quad - (J+1)(J+2n+2a)(2m-J)(J+2s+1)(2J+2s+a+3) \\ &= -6J^4 - 12[2s+a+2]J^3 + 2[-16s^2 - 4(4a+9)s + 4m(n+a) \\ &\quad - 3a^2 - 19a - 17]J^2 + 2[-8s^3 - 4(3a+8)s^2 + 2\{4m(n+a) - 2a^2 \\ &\quad - 17a - 17\}s + 4m(n+a)(a+2) - 7a^2 - 19a - 10]J \\ &\quad + 4[4(n+a+1)(m-1)s^2 + 2(2n+an+a^2+3a+2)(m-1)s \\ &\quad + (n+3an+3a+1)(m-1) + am + a^2(3m-2)]. \end{aligned}$$

Notice that the coefficients of $J^4$ and $J^3$ are negative and the constant term is positive. Denoting the coefficient of $J^k$ in $F(J)$ by $\text{coef}(J^k)$ and recalling that $n \geqq m \geqq 2$, we obtain

$$\begin{aligned} \text{coef}(J) - 2s\,\text{coef}(J^2) &= 48s^3 + 40(a+2)s^2 + 4a(a+2)s \\ &\quad + 2a(8m+4mn-19) + 2a^2(4m-7) + 4(4mn-5) > 0. \end{aligned} \tag{7}$$

If $\text{coef}(J^2) \leqq 0$, then it is obvious that $F(J)$ has only one variation of sign. If $\text{coef}(J^2) > 0$, then by (7), $\text{coef}(J) > 0$ and thus again $F(J)$ has only one variation of sign. Consequently, by Descartes' rule, $F(J)$ has exactly one positive root (temporarily considering $J$ as a real variable), and hence there exists a positive integer $J_0$ depending on $n$, $m$, and $a$ such that $F(J) \geqq 0$,

$J = 0, 1, \ldots, J_0 - 1$, and $F(J) < 0$, $J = J_0, J_0 + 1, \ldots$ . Therefore, by (6), $\operatorname{coef}(d_{s+j}) \geqq 0, j = 1, 2, \ldots, J_0$, and $\operatorname{coef}(d_{s+j}) \leqq 0, j = J_0 + 1, J_0 + 2, \ldots$ . In (5) it is clear that $\operatorname{coef}(d_{s+j+1}) > 0, j = 1, 2, \ldots, 2m - 1$, and $\operatorname{coef}(d_{s+j-1}) > 0$, $j = 1, 2, \ldots, 2m$.

If $J_0 < 2m$, then by successive applications of (5) we obtain $d_{s+j+1} \geqq 0$, $j = 1, 2, \ldots, J_0$, and (transposing the term $d_{s+j}$ to the other side of the equal sign and using $d_{s+2m+1} = 0$) $d_{s+j-1} \geqq 0, j = 2m, 2m - 1, 2m - 2, \ldots, J_0 + 1$. Similarly, if $J_0 \geqq 2m$, then $d_{s+j+1} \geqq 0, j = 1, 2, \ldots, 2m - 1$. In either case,

$$d_{s+j} \geqq 0, \qquad j = 2, 3, \ldots, 2m - 1,$$

which completes the proof.

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