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Relaxation of rate-independent evolution problems

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The relaxation of certain time-evolution problems is investigated. As a conceptually simple example, we study elastically deformable bodies that undergo martensitic phase transformations. The movement of the phase boundaries is hindered by dry friction. The fundamental problem is that the phase distribution forms a highly oscillatory microstructure in space. Therefore, it is desirable to derive a coarse-grained system that describes the effective properties. We introduce a concept of relaxation of the evolution system and apply it to the case where only two phases occur and the elastic energy is quadratic. Finally, we present a candidate for the relaxation in the general case.

1. Introduction

We study rate-independent processes $\theta : [0, T] \to \mathcal{P}$, where $(\mathcal{P}, \mathcal{D})$ is a complete metric space, the set of possible states. The word 'process' here is used without reference to its usage in the probabilistic literature. It denotes a function that maps time into some metric space. Rate independence means that reparametrizations of time do not change the evolution equation. Systems of that type arise as limit problems in mechanics when the inertial forces can be neglected and the internal friction generates hysteretic behaviour. In the simplest case, an admissible (defined later) and non-stationary rate-independent process satisfies the ordinary differential equation

$$\frac{\dot{\theta}}{|\dot{\theta}|}(t) = -V'(t, \theta),$$

where V is a convex time-dependent potential. The time derivative occurs only within a strong nonlinearity, hence standard methods can not be used to study existence, uniqueness and the qualitative behaviour of solutions. A detailed derivation of the governing variational inequalities and discussion of mathematical results can be found in [8] and [7]. An important application of the theory of rate-independent processes are materials that can undergo martensitic phase transformations. They constitute the ingredients for shape-memory alloys, where the friction of the phase boundaries is the main source of dissipation and the material behaviour is strongly hysteretic. Classical models for hysteretic behaviour, like the Preissach model, focus on the description of uniaxial systems. The effects of the evolution of the inhomogeneous internal fields lead then to a complicated history dependence. In our approach we track the change of the internal variables, memory operators are not required. The advantage of the spatial approach (as opposed to memory models) is

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that three-dimensional effects can be captured explicitly and methods from elasticity theory apply. On the other hand, the rapidly fluctuating internal fields require a homogenization procedure. In principle, both approaches are equivalent, since the hysteresis operators can be understood as an ensemble of mechanical friction elements coupled with springs.

In many important cases, the existence of admissible processes cannot be expected. The reason is that sequences of approximate solutions might not converge within $\mathcal{P}$. An example where this phenomenon occurs is the martensitic system. There the approximate solutions oscillate rapidly in space and converge only weakly, not strongly. To overcome this difficulty, we relax the problem in the sense that we enlarge the state space $\mathcal{P}$ in such a way that we can ensure the existence of a limit process. The fundamental problem of this approach is that one has to interpret the generalized states appropriately, i.e. the generalized limit process should solve a generalized evolution problem that is consistent with the primordial problem.

The purpose of this paper is to introduce a notion of consistency and show for a special case of the martensitic system that the approach works.

In § 2, the model and the central existence result will be stated briefly. The main ingredients are two functionals, $\mathcal{I} : [0,T] \times \mathcal{P} \to \mathbb{R}_{\geqslant}$ is the potential energy of a state $\theta \in \mathcal{P}$ at time t and $\mathcal{D}(\theta_1,\theta_2)$ (the distance between θ_1 and θ_2) is the minimal dissipation that occurs when the state changes from θ_1 to θ_2. Examples from the theory of martensitic phase transformations that fit into the framework of the rate-independent model will be presented.

In § 3, we will define precisely when an extension $(\mathcal{I},\mathcal{D},\mathcal{P})$ of an unrelaxed problem $(\mathcal{I}^{\mathrm{p}},\mathcal{D}^{\mathrm{p}},\mathcal{P}^{\mathrm{p}})$ constitutes a relaxation. A central requirement is a sequence of approximation operators $S_k : \mathcal{P} \to \mathcal{P}^{\mathrm{p}}$, which realize the values of the relaxed functionals $\mathcal{I}$ and $\mathcal{D}$, i.e.

$$\lim_{k\to\infty} \mathcal{I}^{\mathrm{p}}(S_k(\theta)) = \mathcal{I}(\theta) \quad \text{and} \quad \lim_{k\to\infty} \mathcal{D}^{\mathrm{p}}(S_k(\theta_1), S_k(\theta_2)) = \mathcal{D}(\theta_1,\theta_2).$$

The main result of this paper is the proof that, in simple cases, a heuristically derived model turns out to be a relaxed macroscopic model (theorem 3.6). The proof relies heavily on a well-known result by Kohn in [4], where an explicit formula for the quasiconvex envelope for a stored energy function with two quadratic wells is derived. Another ingredient are H-measures, which were invented by Tartar in [10]. They describe the limit properties of quadratic quantities.

In § 4, we derive a uniqueness result for possible relaxations of the martensitic system (theorem 4.2).

2. The rate-independent model

Let $(\mathcal{P},\mathcal{D})$ be a complete metric state space. We interpret $\mathcal{D}(\theta_1,\theta_2)$ as the minimal dissipation necessary to transform a state θ_1 into θ_2. For every state $\theta \in \mathcal{P}$, the potential energy at time t is denoted as $\mathcal{I}(t,\theta)$. We assume that $\mathcal{I}$ is non-negative, and $\mathcal{I}$ and $\partial\mathcal{I}/\partial t$ are jointly continuous in (t,θ). For a process $\theta : [0,T] \to \mathcal{P}$, we define the total dissipation

$$\mathrm{Var}_{\mathcal{D}}(\theta;0,T) = \sup\biggl\{\sum_{\ell=1}^{N} \mathcal{D}(\theta(t_{\ell-1}),\theta(t_\ell)) \mid 0 = t_0 < \cdots < t_N = T\biggr\}.$$

A state $\theta \in \mathcal{P}$ is denoted as *stable* at time t if, for every $\eta \in \mathcal{P}$, the inequality

$$\mathcal{I}(t,\theta) \leqslant \mathcal{I}(t,\eta) + \mathcal{D}(\theta,\eta) \tag{2.1}$$

is satisfied. A process $\theta : [0,T] \to \mathcal{P}$ satisfies the *energy inequality* if

$$\mathcal{E}(\theta) \leqslant 0, \tag{2.2}$$

where

$$\mathcal{E}(\theta) = \mathcal{I}(T,\theta(T)) - \mathcal{I}(0,\theta(0)) + \mathrm{Var}_{\mathcal{D}}(\theta;0,T) - \int_0^{\mathrm{T}} \partial_t \mathcal{I}(t,\theta(t))\,\mathrm{d}t \tag{2.3}$$

holds.

DEFINITION 2.1. A process $\theta : [0,T] \to \mathcal{P}$ is *admissible* if it is stable for every $t \in (0,T]$ and the energy inequality (2.2) is satisfied.

Note that the energy inequality (2.2) implies that the variation of admissible processes is bounded. Hence admissible processes are continuous except on a countable set.

REMARK 2.2.

(1) It can be shown that if $\mathcal{P} = \mathbb{R}^d$, $\mathcal{D}(\theta_1,\theta_2) = |\theta_2 - \theta_1|_2$ and the potential energy $\mathcal{I}$ is smooth and convex, then every non-stationary admissible process θ satisfies the ordinary differential equation

$$\frac{\dot{\theta}}{|\dot{\theta}|}(t) = -\frac{\partial \mathcal{I}}{\partial \theta}(t,\theta).$$

A collection of related results can be found in [7]. The non-smoothness, which occurs when $\mathcal{P}$ has a boundary, makes a more detailed analysis necessary, but until now this problem has not been resolved in a completely satisfactory way. Since in our central example, the martensitic system, the presence of a boundary is essential, we chose the weak formulation in definition 2.1.

(2) All results that are derived in this work also hold if $\mathcal{D}$ is not symmetric.

(3) The definition of admissible processes is invariant under reparametrizations of time in the following sense. Let $\theta : [0,T] \to \mathcal{P}$ be an admissible process with respect to $(\mathcal{I},\mathcal{D},\mathcal{P})$ and $\varphi : [0,T] \to \mathbb{R}$ a reparametrization of time. Then $\theta \circ \varphi^{-1}$ is an admissible with respect to $(\tilde{\mathcal{I}},\mathcal{D},\mathcal{P})$, where $\tilde{\mathcal{I}}(t,\theta) = \mathcal{I}(\varphi^{-1}(t),\theta)$.

The existence result, which guarantees in the general case that for every initial value $\theta_0 \in \mathcal{P}$ an admissible process θ satisfying $\theta|_{t=0} = \theta_0$ exists, is proven in [8] and [7] for special cases. In the latter paper, a collection of uniqueness results can also be found.

THEOREM 2.3. *Let $X \supset \mathcal{P}$ be a separable Banach space so that*

$$\mathcal{D}(\theta_1,\theta_2) = \|\theta_2 - \theta_1\|.$$

Assume that the following conditions hold.

(i) Lower semicontinuity assumptions for $\mathcal{I}$. *For every* $t \in [0,T]$,

$$\left.\begin{array}{l}\textit{the functional } \mathcal{I}(t,\cdot) \textit{ is weakly lower semicontinuous}\\ \textit{and the mapping } \theta \mapsto \partial_t \mathcal{I}(t,\theta) \textit{ is weakly continuous in } \mathcal{P}.\end{array}\right\} \tag{2.4}$$

(ii) Compactness. *For every* $r > 0$*, the intersection* $\mathcal{P} \cap \{\|x\| \leqslant r\}$ *is a weakly compact subset of* X *and, for every* $t \in [0,T]$*, the set of stable states*

$$\mathcal{S}(t) = \bigcap_{\eta \in \mathcal{P}} \{\theta \in \mathcal{P} \mid \mathcal{I}(t,\theta) \leqslant \mathcal{I}(t,\eta) + \mathcal{D}(\theta,\eta)\} \tag{2.5}$$

is weakly closed.

Then there exists, for every initial state $\theta_0 \in \mathcal{P}$*, an admissible process* $\theta : [0,T] \to \mathcal{P}$*, so that* $\theta|_{t=0} = \theta_0$.

Remark 2.4. The purpose of imbedding $\mathcal{P}$ into a Banach space X is to introduce a Finsler structure $F : \mathcal{P} \times T\mathcal{P} \to [0,\infty)$ that is positively homogeneous (i.e. $F(\theta,\lambda\tau) = \lambda F(\theta,\tau)$ for all $\tau \in T_\theta\mathcal{P}$, $\lambda > 0$) and convex. The problems that we are dealing with in this work are basically only concerned with the length of paths, a Riemannian metric tensor is not needed. For this reason, the framework of Finsler manifolds (see [1, 6]) fits exactly to our requirements. This remark is only for the information of geometrically interested readers. We will not pursue the analogy any further.

The weak closedness of $\mathcal{P}$ and (2.4) is needed to ensure the existence of solutions to the incremental problem,

$$\left.\begin{array}{l}\text{for given } \theta_0 \in \mathcal{P} \text{ and a time discretization } 0 = t_0 < \cdots < t_N = T,\\ \text{find, for } \ell = 1 \ldots N, \text{ states } \theta_\ell \in \mathcal{P} \text{ such that, for fixed } \theta_{\ell-1},\\ \text{the energy sum } \mathcal{I}(t_\ell,\theta_\ell) + \mathcal{D}(\theta_{\ell-1},\theta_\ell) \text{ is minimal.}\end{array}\right\} \tag{IP}$$

The piecewise constant in the time-interpolating process

$$\theta^N(t) := \sum_{\ell=0}^{N-1} \theta_\ell \mathcal{X}_{[t_\ell,t_{\ell+1})}(t) \quad \text{if } t \in [0,T), \quad \theta^N(T) := \theta_N,$$

satisfies the energy inequality. The proof of theorem 2.3 consists of showing that this property is conserved and the stability is acquired as the fineness of the time partition tends to zero.

The variational approach to get existence is closely related to De Giorgi's notion of *minimizing movement* (see [2,3] for an exposition in the context of mean-curvature evolution).

2.1. Our central example: the martensitic system

We will demonstrate that the hysteretic behaviour associated to martensitic phase transformations fits into the framework of rate-independent processes. Let $\Omega \subset \mathbb{R}^d$ be the reference configuration of an elastically deformable body. For every point $x \in \Omega$, the material is in one of n phases. The ith phase is identified with the ith unit vector e_i of $\mathbb{R}^n$. The evolution of the phase distribution within the reference configuration is given by a mapping

$$\theta^{\mathrm{p}} : [0,T] \times \Omega \to P^{\mathrm{p}} := \{e_1, \dots, e_n\}.$$

The superscript p indicates that only pure phase distributions (in contrast to phase mixtures) are considered. The elastic properties of phase e_i are determined by the strain energy density function $W_{e_i} : \mathbb{R}^{d\times d} \to \mathbb{R}_{\geqslant}$. We assume that, for every $e_i \in P$, the function $W_{e_i}(\cdot)$ is quasiconvex, i.e.

$$\forall F \in \mathbb{R}^{d\times d}, \quad \varphi \in C_0^\infty([0,1]^d) : \int_{[0,1]^d} W_{e_i}(F + \nabla\varphi(x))\,\mathrm{d}x \geqslant W(F).$$

The set of kinematically admissible deformations is denoted as $V \subset W^{1,p}(\Omega)$, where $W^{1,p}$ is the Sobolev space consisting of functions whose derivative is integrable to the pth power. The system is driven by time-dependent external forces $g : [0,T] \times \Omega \to \mathbb{R}^d$, which correspond to a continuous linear functional $G \in C^1([0,T], V^*)$. The energy of a phase distribution $\theta^{\mathrm{p}} : \Omega \to \mathcal{P}^{\mathrm{p}} = L^1(\Omega, P^{\mathrm{p}})$ is defined as

$$\mathcal{I}^{\mathrm{p}}(t, \theta^{\mathrm{p}}) = \inf_{u\in V}\biggl(\int_\Omega W_{\theta^{\mathrm{p}}(t,x)}(\nabla u(x))\,\mathrm{d}x - \langle G(t), u\rangle\biggr). \tag{2.6}$$

If appropriate growth conditions are met, the general theory implies that, for fixed θ^{p}, the infimum is in fact a minimum; this follows from the quasiconvexity of the strain energy density functions W_{e_i}. For the dissipation function $\mathcal{D}^{\mathrm{p}}$, we set

$$\mathcal{D}^{\mathrm{p}}(\theta^{\mathrm{p}}, \eta^{\mathrm{p}}) = \sum_{i,j}^{n} \kappa_{ij} \int_\Omega (\theta^{\mathrm{p}}(x)\cdot e_i)(\eta^{\mathrm{p}}(x)\cdot e_j)\,\mathrm{d}x, \tag{2.7}$$

where e_i is the ith unit vector and the coefficients satisfy $\kappa_{ij} = \kappa_{ji} > 0$ if $i \neq j$ and $\kappa_{ii} = 0$.

3. Relaxation of rate-independent evolution problems

Clearly, the existence of theorem 2.3 cannot be applied to the martensitic system, since, by the discreteness of the target space, $\mathcal{P}^{\mathrm{p}}$ is not weakly compact in $L^1(\Omega)$. Therefore, the incremental problem can not be expected to have a solution. In this section we show how the setting can be relaxed in a simple case in order to assure existence of admissible processes. We first state our definition of relaxation (definition 3.1) and state some fundamental connections between alternative concepts (proposition 3.3). Then we present a relaxation for a special case of the martensitic system (theorem 3.6). This is the main result.

In the calculus of variations, the concept of relaxation basically consists of enlarging the search space and extending the energy functions to ensure the existence of generalized solutions. The relation to the original unrelaxed problem is established by showing that generalized states can be approximated by unrelaxed states without increasing the energy. Our approach is an adaption of this concept to rate-independent evolution problems.

Definition 3.1. A rate-independent evolution problem $(\mathcal{I}, \mathcal{D}, \mathcal{P})$ is a relaxation of $(\mathcal{I}^{\mathrm{p}}, \mathcal{D}^{\mathrm{p}}, \mathcal{P}^{\mathrm{p}})$ if the following three conditions are satisfied.

(1) *Extension.* There exists a coarser (weak) topology on $\mathcal{P}$ so that $\mathcal{P}^{\mathrm{p}}$ is dense in $\mathcal{P}$ and $\mathcal{I}$ and $\mathcal{D}$ extend $\mathcal{I}^{\mathrm{p}}$ and $\mathcal{D}^{\mathrm{p}}$.

(3) *Existence of solutions.* For every initial state $\theta_0 \in \mathcal{P}$, a relaxed admissible process satisfying $\theta|_{t=0} = \theta_0$ exists.

(3) *Consistency.* For every relaxed process θ, there exists a recovery sequence $\theta_k^{\mathrm{p}} : [0,T] \to \mathcal{P}^{\mathrm{p}}$ so that

$$\theta_k^{\mathrm{p}}(t) \rightharpoonup \theta(t) \quad \text{for a.e. } t \in [0,T] \quad \text{as } k \to \infty, \tag{3.1}$$

$$\limsup_{k\to\infty} \mathcal{E}^{\mathrm{p}}(\theta_k^{\mathrm{p}}) \leqslant \mathcal{E}(\theta). \tag{3.2}$$

The idea behind the definition can be formulated as follows. If $(\mathcal{I}, \mathcal{D}, \mathcal{P})$ is a relaxation, then for every admissible process θ there exists a sequence of unrelaxed processes $S_k(\theta)$ that acquires stability and dissipativity (the energy inequality) as $k \to \infty$. The last hypothesis of the definition is particularly complicated to check since it requires a specification of how to approximate relaxed processes with non-relaxed processes. Of course, the most natural way to generate a relaxed process is to approximate it pointwise in time, i.e. $\theta_k^p(t)$ does not depend on $\{\theta(s) \mid s \neq t\}$.

Lemma 3.2. *Let $(\mathcal{I}, \mathcal{D}, \mathcal{P})$ be a relaxation of $(\mathcal{I}^{\mathrm{p}}, \mathcal{D}^{\mathrm{p}}, \mathcal{P}^{\mathrm{p}})$ and $S_k : \mathcal{P} \to \mathcal{P}^{\mathrm{p}}$ a sequence of approximation operators so that, for every $\theta, \eta \in \mathcal{P}$, we have that*

$$S_k(\theta) \rightharpoonup \theta \quad \text{as } k \to \infty, \tag{3.3}$$

$$\lim_{k\to\infty} \mathcal{I}^{\mathrm{p}}(S_k(\theta)) = \mathcal{I}(\theta), \tag{3.4}$$

$$\lim_{k\to\infty} \mathcal{D}^{\mathrm{p}}(S_k(\theta), S_k(\eta)) = \mathcal{D}(\theta, \eta).$$

Then there exists a piecewise constant in the time-interpolating recovery sequence.

Proof. The approximation operators can be extended easily from states to processes. For an arbitrary process θ, we set $\theta_k(t) = S_k(\theta^N(t))$, where θ^N is the piecewise constant interpolate

$$\theta^N(t) = \begin{cases} \theta(t_{\ell-1}), & t \in [t_{\ell-1}, t_\ell),\ t_\ell = \dfrac{\ell}{N} T,\ \ell = 1, \dots, N, \\ \theta(T), & t = T, \end{cases}$$

and N tends slowly enough to infinity as $k \to \infty$ in order to ensure the convergence of the dissipation. Hence we have that

$$\lim_{k\to\infty} \mathrm{Var}_{\mathcal{D}^{\mathrm{p}}}(S_k(\theta^N); 0, T) = \mathrm{Var}_{\mathcal{D}}(\theta; 0, T).$$

For the term

$$\int_{[0,T]} \frac{\partial I}{\partial t}(t, S_k(\theta^N(t)))\, \mathrm{d}t,$$

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we observe that

$$\lim_{k\to\infty}\int_{[0,T]}\frac{\partial I}{\partial t}(t,S_k(\theta^N(t)))\,\mathrm{d}t = \lim_{k\to\infty}\sum_{\ell=0}^{N-1}\int_{[t_\ell,t_{\ell+1})}\frac{\partial I}{\partial t}(t,S_k(\theta(t_\ell)))\,\mathrm{d}t = \lim_{k\to\infty}\sum_{\ell=0}^{N-1}\biggl(\int_{[t_\ell,t_{\ell+1})}\frac{\partial I}{\partial t}(t_\ell,S_k(\theta(t_\ell)))\,\mathrm{d}t + o(|t_{\ell+1}-t_\ell|)\biggr).$$

The error term vanishes as $N\to\infty$ and we are left with the Riemann sum. Since $t\mapsto(\partial I/\partial t)(t,\theta(t))$ has bounded variation, the Riemann sum converges to the correct limit if N tends to infinity slowly enough. The other terms of the energy inequality converge by (3.3), (2.4) and (3.4). □

3.1. Relation between the time continuous relaxation concept and variational approaches

The notion of relaxation outlined in definition 3.1 does not involve minimizing sequences. On the other hand, the construction of admissible processes via minimizing a certain energy demands a generalization of the concept of minimizing sequences. In order to approximate admissible processes, we seek sequences of processes that are piecewise constant in time and satisfy in the limit the two requirements: stability inequality (2.1) and energy inequality (2.2). The approach that is most widely used is to solve an incremental problem (IP). A sufficient condition for the existence of minimizers is that for every $t\in[0,T]$, the functional $(\theta,\eta)\mapsto J(t,\theta,\eta)=\mathcal{I}(t,\theta)+\mathcal{D}(\eta,\theta)$ is lower semicontinuous with respect to a topology that ensures compactness of bounded sequences. For simplicity, this topology will be denoted as a weak topology.

The martensitic system provides a good example where this approach requires relaxation. It is not difficult to find two strain-energy density functions, $W_{e_1}(\cdot)$ and $W_{e_2}(\cdot)$, and a dissipation coefficient $\kappa>0$, so that the homogeneous initial distribution $\theta_0\equiv e_1$ is unstable with respect to a phase mixture, i.e. no minimizing sequence of

$$\theta^{\mathrm{p}}\mapsto\inf_{u\in W^{1,p}([0,1]^d)}\int_{[0,1]^d}W_{\theta^{\mathrm{p}}(x)}(\nabla u)\,\mathrm{d}x+\kappa\cdot\mathrm{meas}\{x\in\Omega\mid\theta^{\mathrm{p}}(x)\neq e_1\}$$

converges strongly in $L^1(\Omega)$. This implies that the weak limit is not a characteristic function. For this reason, Mielke *et al*. developed in [8] a relaxation concept that relies on the structure of the incremental problem. Given a partition $0=t_0<\dots<t_N=T$ and an initial state $\theta_0^{\mathrm{p}}\in\mathcal{P}^{\mathrm{p}}$,

$$\left.\begin{array}{l}\text{find, for every } \varepsilon>0, \text{ almost minimal states } \theta^{\mathrm{p}}_{1,\varepsilon},\dots,\theta^{\mathrm{p}}_{N,\varepsilon}\in\mathcal{P}^{\mathrm{p}}\\ \text{such that}\\ \mathcal{I}^{\mathrm{p}}(t_\ell,\theta^{\mathrm{p}}_{\ell,\varepsilon})+\mathcal{D}^{\mathrm{p}}(\theta^{\mathrm{p}}_{\ell-1,\varepsilon},\theta^{\mathrm{p}}_{\ell,\varepsilon})\\ \qquad\leqslant\varepsilon+\inf\{\mathcal{I}^{\mathrm{p}}(t_\ell,\eta^{\mathrm{p}})+\mathcal{D}^{\mathrm{p}}(\theta^{\mathrm{p}}_{\ell-1,\varepsilon},\eta^{\mathrm{p}}):\eta^{\mathrm{p}}\in\mathcal{P}^{\mathrm{p}}\}.\end{array}\right\}\qquad((\mathrm{AIP})_\varepsilon)$$

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A similar approach has been used by many other authors, mainly in the context of steepest decent dynamics. In [9], an effective evolution model is derived for a certain limit of the Hele-Shaw cell. This is very close to the line of attack we take.

Our aim is now to clarify the relation between our concept of relaxation and the approximate incremental problem. On the one hand, the approximate incremental problem provides very good candidates for the recovery sequence $S_k(\theta)$ without specifying $\mathcal{P}$, i.e. (3.2) is satisfied, only (3.1) is missing. On the other hand, if we have a relaxation, then the recovery sequences are automatically solutions to the approximate incremental problem.

PROPOSITION 3.3. *Let $(\mathcal{I}^{\mathrm{p}}, \mathcal{D}^{\mathrm{p}}, \mathcal{P}^{\mathrm{p}})$ be an unrelaxed rate-independent evolution problem. Then, for every $\theta_0^{\mathrm{p}} \in \mathcal{P}^{\mathrm{p}}$, there exists a solution $(\theta_{\ell,\varepsilon}^{\mathrm{p}})_{\ell=1\dots N}$ to the approximate incremental problem so that*

(1) *$\theta_{\ell,\varepsilon}^{\mathrm{p}}$ is stable at time t_ℓ for $\ell \in \{1, \dots, N\}$;*

(2) *the approximate energy inequality*

$$\mathcal{E}^{\mathrm{p}}\left(\sum_{\ell=1}^{N} \theta_{\ell-1,\varepsilon}^{\mathrm{p}} \mathcal{X}_{[t_{\ell-1},t_\ell)} + \delta(t-T)\theta_{N,\varepsilon}^{\mathrm{p}}\right) \leqslant \varepsilon N \tag{3.5}$$

holds.

Furthermore, if $(\mathcal{I}, \mathcal{D}, \mathcal{P})$ is a relaxation of $(\mathcal{I}^{\mathrm{p}}, \mathcal{D}^{\mathrm{p}}, \mathcal{P}^{\mathrm{p}})$ in the sense of definition 3.1 and, for all $t \in [0, T]$, the functional

$$(\theta, \eta) \mapsto \mathcal{J}(t, \theta, \eta) = \mathcal{I}(t, \theta) + \mathcal{D}(\eta, \theta)$$

is weakly lower semicontinuous, then, for every solution $(\theta_\ell)_{\ell=1,\dots,N}$ to the relaxed incremental problem, there exists a sequence of solutions $(\theta_{\ell,\varepsilon}^{\mathrm{p}})_{\ell=1,\dots,N}$ to the approximate incremental problem (AIP)$_\varepsilon$ that converges weakly to θ_ℓ as ε tends to 0.

Proof. Fix t and $\theta_0^{\mathrm{p}} \in \mathcal{P}^{\mathrm{p}}$ and let $(\theta_k^{\mathrm{p}})_{k\in\mathbb{N}}$ be a solution of the recursive variational problem

$$\mathcal{I}^{\mathrm{p}}(t, \theta_k^{\mathrm{p}}) + \mathcal{D}^{\mathrm{p}}(\theta_{k-1}^{\mathrm{p}}, \theta_k^{\mathrm{p}}) \leqslant \mathcal{I}^{\mathrm{p}}(t, \theta^{\mathrm{p}}) + \mathcal{D}^{\mathrm{p}}(\theta_{k-1}^{\mathrm{p}}, \theta^{\mathrm{p}}) + 2^{-k}\varepsilon \quad \text{for every } \theta^{\mathrm{p}} \in \mathcal{P}^{\mathrm{p}}. \tag{3.6}$$

Since time does not play a role in the proof, we will omit the t dependence of $\mathcal{I}$ and $\mathcal{I}^{\mathrm{p}}$ in future. Obviously, there always exists θ_k^{p} that satisfies inequality (3.6), since we can use a suitable element of the minimizing sequence of $\mathcal{J}^{\mathrm{p}}(\cdot) = \mathcal{I}^{\mathrm{p}}(\cdot) + \mathcal{D}^{\mathrm{p}}(\theta_{k-1}^{\mathrm{p}}, \cdot)$. Now we show that $(\theta_k^{\mathrm{p}})_{k\in\mathbb{N}}$ is a Cauchy sequence. We test (3.6) with $\theta^{\mathrm{p}} = \theta_{k-1}^{\mathrm{p}}$ and find that $\mathcal{I}^{\mathrm{p}}(\theta_k^{\mathrm{p}})$ is essentially decreasing. From the non-negativity of $\mathcal{I}^{\mathrm{p}}$ it follows that $\mathcal{I}^{\mathrm{p}}(\theta_k^{\mathrm{p}})$ converges as $k \to \infty$. Let $k \leqslant l$, then

$$\begin{aligned} \mathcal{D}^{\mathrm{p}}(\theta_k^{\mathrm{p}}, \theta_l^{\mathrm{p}}) &\leqslant \sum_{i=k}^{l-1} \mathcal{D}^{\mathrm{p}}(\theta_i^{\mathrm{p}}, \theta_{i+1}^{\mathrm{p}}) \\ &= -\mathcal{I}^{\mathrm{p}}(\theta_l^{\mathrm{p}}) + \mathcal{I}^{\mathrm{p}}(\theta_l^{\mathrm{p}}) + \mathcal{D}^{\mathrm{p}}(\theta_{l-1}^{\mathrm{p}}, \theta_l^{\mathrm{p}}) + \sum_{i=k}^{l-2} \mathcal{D}^{\mathrm{p}}(\theta_i^{\mathrm{p}}, \theta_{i+1}^{\mathrm{p}}) \end{aligned}$$

$$\overset{(3.6)}{\leqslant} -\mathcal{I}^{\mathrm{p}}(\theta_l^{\mathrm{p}}) + \mathcal{I}^{\mathrm{p}}(\theta_{l-1}^{\mathrm{p}}) + 2^{-l}\varepsilon + \sum_{i=k}^{l-2} \mathcal{D}^{\mathrm{p}}(\theta_i^{\mathrm{p}}, \theta_{i+1}^{\mathrm{p}})$$

$$\leqslant \cdots \leqslant -\mathcal{I}^{\mathrm{p}}(\theta_l^{\mathrm{p}}) + \mathcal{I}^{\mathrm{p}}(\theta_k^{\mathrm{p}}) + 2^{-k}\varepsilon.$$

Sending k and l to ∞ gives the desired result and the completeness of $\mathcal{P}^{\mathrm{p}}$ induces the existence of a limit $\theta_{*,\varepsilon}^{\mathrm{p}} \in \mathcal{P}^{\mathrm{p}}$. We have to prove now that $\theta_{*,\varepsilon}^{\mathrm{p}}$ is stable, solves the approximate incremental problem and satisfies the approximate energy inequality (3.5). First the stability. We have, for all $\theta^{\mathrm{p}} \in \mathcal{P}^{\mathrm{p}}$, that

$$\begin{aligned}\mathcal{I}^{\mathrm{p}}(\theta_{*,\varepsilon}^{\mathrm{p}}) &= \lim_{k\to\infty}(\mathcal{I}^{\mathrm{p}}(\theta_k^{\mathrm{p}}) + \mathcal{D}^{\mathrm{p}}(\theta_{k-1}^{\mathrm{p}}, \theta_k^{\mathrm{p}})) \\ &\overset{(3.6)}{\leqslant} \lim_{k\to\infty}(\mathcal{I}^{\mathrm{p}}(\theta^{\mathrm{p}}) + \mathcal{D}^{\mathrm{p}}(\theta_k^{\mathrm{p}}, \theta^{\mathrm{p}}) + 2^{-k}\varepsilon) \\ &= \mathcal{I}^{\mathrm{p}}(\theta^{\mathrm{p}}) + \mathcal{D}^{\mathrm{p}}(\theta_{*,\varepsilon}^{\mathrm{p}}, \theta^{\mathrm{p}}).\end{aligned}$$

The second claim is established in a similar fashion,

$$\begin{aligned}\mathcal{I}^{\mathrm{p}}(\theta_{*,\varepsilon}^{\mathrm{p}}) + \mathcal{D}^{\mathrm{p}}(\theta_0^{\mathrm{p}}, \theta_{*,\varepsilon}^{\mathrm{p}}) &= \lim_{k\to\infty}(\mathcal{I}^{\mathrm{p}}(\theta_k^{\mathrm{p}}) + \mathcal{D}^{\mathrm{p}}(\theta_0^{\mathrm{p}}, \theta_k^{\mathrm{p}})) \\ &\leqslant \lim_{k\to\infty}(\mathcal{I}^{\mathrm{p}}(\theta_k^{\mathrm{p}}) + \mathcal{D}^{\mathrm{p}}(\theta_{k-1}^{\mathrm{p}}, \theta_k^{\mathrm{p}}) + \mathcal{D}^{\mathrm{p}}(\theta_0^{\mathrm{p}}, \theta_{k-1}^{\mathrm{p}})) \\ &\overset{(3.6)}{\leqslant} \lim_{k\to\infty}(\mathcal{I}^{\mathrm{p}}(\theta_{k-1}^{\mathrm{p}}) + 2^{-k}\varepsilon + \mathcal{D}^{\mathrm{p}}(\theta_0^{\mathrm{p}}, \theta_{k-1}^{\mathrm{p}})) \\ &\leqslant \cdots \leqslant \mathcal{I}^{\mathrm{p}}(\theta^{\mathrm{p}}) + \mathcal{D}^{\mathrm{p}}(\theta_0^{\mathrm{p}}, \theta^{\mathrm{p}}) + \varepsilon\end{aligned}$$

for every $\theta^{\mathrm{p}} \in \mathcal{P}^{\mathrm{p}}$. Since θ_0^{p} is not restricted by any assumption, we can change t, start the iteration again with $\theta_{*,\varepsilon}^{\mathrm{p}}$ and thus generate an N-tuple $(\theta_{\ell,\varepsilon}^{\mathrm{p}})_{\ell=1\dots N}$ of stable states. We repeat the above arguments inductively and obtain that $(\theta_{\ell,\varepsilon}^{\mathrm{p}})_{\ell=1\dots N}$ is indeed a solution for $(\mathrm{AIP})_\varepsilon$. The approximate energy inequality (3.5) is a direct consequence from the fact that $\theta_{\ell,\varepsilon}^{\mathrm{p}}$ solves the approximate incremental problem. From the definition of $\mathcal{E}^{\mathrm{p}}(\cdot)$, it follows that

$$\begin{aligned}&\mathcal{E}^{\mathrm{p}}\biggl(\sum_{\ell=1}^{N} \theta_{\ell-1}\mathcal{X}_{[t_{\ell-1},t_\ell)} + \delta(t-T)\theta_N^{\mathrm{p}}\biggr) \\ &\qquad = \sum_{\ell=0}^{N-1}(\mathcal{I}^{\mathrm{p}}(t_\ell, \theta_{\ell,\varepsilon}^{\mathrm{p}}) - \mathcal{I}^{\mathrm{p}}(t_\ell, \theta_{\ell-1,\varepsilon}^{\mathrm{p}}) + \mathcal{D}^{\mathrm{p}}(\theta_{\ell-1,\varepsilon}^{\mathrm{p}}, \theta_{\ell,\varepsilon}^{\mathrm{p}})) \\ &\qquad \overset{(\mathrm{AIP})_\varepsilon}{\leqslant} \varepsilon N.\end{aligned}$$

The last claim is that it is possible to construct solutions for the approximate incremental problem from the solutions of the relaxed problem. The assumption that $(\theta_\ell)_{\ell=1\dots N}$ solves the relaxed incremental problem implies that

$$\begin{aligned}\mathcal{J}(\theta_{\ell-1}, \theta_\ell) &\leqslant \inf_{\theta\in\mathcal{P}^{\mathrm{p}}} \mathcal{J}(\theta_{\ell-1}, \theta) \\ &\leqslant \liminf_{k\in\mathbb{N}} \inf_{\theta\in\mathcal{P}^{\mathrm{p}}} \mathcal{J}^{\mathrm{p}}(S_k(\theta_{\ell-1}), \theta^{\mathrm{p}}) \\ &\leqslant \lim_{k\to\infty} \mathcal{J}^{\mathrm{p}}(S_k(\theta_{\ell-1}), S_k(\theta_\ell)) \\ &= \mathcal{J}(\theta_{\ell-1}, \theta_\ell).\end{aligned}$$

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The last identity follows from the assumption that $\mathcal{J}$ is continuous along $(S_k(\theta), S_k(\eta))$. Hence $\mathcal{J}^{\mathrm{p}}(S_k(\theta_{\ell-1}), S_k(\theta_\ell))$ converges to $\inf_{\theta^{\mathrm{p}}\in\mathcal{P}^{\mathrm{p}}} \mathcal{J}^{\mathrm{p}}(S_k(\theta_{\ell-1}), \theta^{\mathrm{p}})$ as $k \to \infty$ and we see that $(S_k(\theta_\ell))$, $\ell = 1, \dots, N$, solves $(\mathrm{AIP})_\varepsilon$ for an appropriate choice of $k(\varepsilon)$. □

3.2. Relaxation of the martensitic system

We now give an example of a system where the relaxation can be carried out successfully. The system is a special case of the martensitic model from § 2. The number of phases has to be limited to two, so that non-relaxed phase distribution functions can be identified with characteristic functions. In order to be able to compute the minimizer to the elastic problem, we have to restrict to linearized elasticity theory, i.e. quadratic stored energy functions $W_{e_1}(\cdot)$ and $W_{e_2}(\cdot)$, and we will even assume that the elasticity tensors $\mathbf{C}$ of both phases coincide. At the present time, it is not clear whether any of the assumptions can be dropped since no counterexample is known.

The relaxed state space $\mathcal{P}$ is now $L^1(\Omega, [0,1])$. For the Banach space X, we take $L^1(\Omega, \mathbb{R})$. For a displacement gradient $F = \nabla u$, we denote by $E = \frac{1}{2}(F + F^{\mathrm{T}})$ the linearized strain tensor. Let $\mathbf{C} \in \mathrm{Lin}(\mathbb{R}^{d\times d}, \mathbb{R}^{d\times d})$ be a convex elasticity tensor (i.e. $E(F){:}\mathbf{C}[E(F)] \geqslant cE(F){:}E(F)$ for some $c > 0$). For both the potential energy and the dissipation functional, we need to extend the integrand from pure phase distributions to phase mixtures. The energy functions of the pure phases are

$$W_{e_i}(F) = \tfrac{1}{2}\mathbf{C}[E - A_i]{:}(E - A_i) \tag{3.7}$$

for two stress-free strains $A_i \in \mathbb{R}^{d\times d}$. We set

$$\mathbb{W}(\theta, E) = \tfrac{1}{2}\mathbf{C}[E-(1-\theta)A_1 - \theta A_2]{:}(E-(1-\theta)A_1 - \theta A_2) + \tfrac{1}{2}\gamma\theta(1-\theta), \tag{3.8}$$

where

$$\gamma = \min\biggl\{(A_2 - A_1){:}\mathbf{C}[A_2 - A_1] - \frac{(S{:}\mathbf{C}[A_2 - A_1])^2}{S{:}\mathbf{C}[S]} : S = \omega\otimes v + v\otimes\omega,\ \omega, v \in \mathbb{R}^d\biggr\}. \tag{3.9}$$

This formula is due to Kohn [4]. We denote by ω^* the vector of unit length that realizes the minimum. Since $(1-\theta)\theta$ vanishes if $\theta \in \{0,1\}$, it is clear that the model is indeed an extension of the model with only pure phase distributions. According to definition 2.7, the unrelaxed dissipation functional reads

$$\mathcal{D}^{\mathrm{p}}(\theta_1^{\mathrm{p}}, \theta_2^{\mathrm{p}}) = \kappa \int_\Omega \{(1 - \theta_2^{\mathrm{p}}(x))\theta_1^{\mathrm{p}}(x) + (1 - \theta_1^{\mathrm{p}}(x))\theta_2^{\mathrm{p}}(x)\}\,\mathrm{d}x.$$

We define the relaxed dissipation as

$$\mathcal{D}(\theta_1, \theta_2) = \kappa \int_\Omega |\theta_1(x) - \theta_2(x)|\,\mathrm{d}x. \tag{3.10}$$

The second requirement is that for every initial state an admissible process exists. For the sake of completeness, we state the existence result for the relaxed system. The proof can be found in [8].

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^d$, $d \in \{1,2,3\}$, be a bounded Lipschitz domain, $\Gamma \subset \partial\Omega$, so that $\mathcal{H}^{d-1}(\Gamma) > 0$, $\mathcal{P} = L^1(\Omega,[0,1]) \subset X = L^1(\Omega,\mathbb{R})$, $\mathbf{C} \in \mathrm{Lin}(\mathbb{R}^{d\times d},\mathbb{R}^{d\times d})$ a convex elasticity tensor, $\kappa > 0$, $G \in C^1([0,T],V^*)$ and*

$$\mathcal{I}(t,\theta) = \min_{u\in V} \int_\Omega \mathbb{W}(\theta, E(u))\,\mathrm{d}x + \langle G(t), u\rangle,$$
$$\mathcal{D}(\theta_1,\theta_2) = \kappa \int_\Omega |\theta_2(x) - \theta_1(x)|\,\mathrm{d}x,$$

where $V = u_0 + W^{1,2}_\Gamma(\Omega)$ for some function $u_0 \in W^{1,2}(\Omega)$,

$$W^{1,2}_\Gamma(\Omega) = \{u \in W^{1,2}(\Omega) : u|_\Gamma = 0\}$$

and $\mathbb{W}$ is given by (3.8). If $\Gamma = \partial\Omega$, then for every initial value $\theta_0 \in \mathcal{P}$ there exist an admissible process $\theta \in BV([0,T],\mathcal{P})$ in the sense of definition 2.1. The process θ can be chosen left-hand side continuous except at $t = 0$.

At the present time we are not able to show existence in the case that $\Gamma \subsetneq \partial\Omega$, i.e. if, at some part of the boundary, Neumann conditions are prescribed. However, for the remainder of the analysis, this restriction is not required. Therefore, we state theorem 3.5 in a slightly more general way without implying that the extended model is a relaxation of the original model.

To fulfil the second requirement of definition 3.1, we have to specify a sequence of approximation operators. We define a two-scale approximation function

$$f : [0,1]\times\mathbb{R} \to \{0,1\} : f(\theta,s) = \begin{cases} 1, & s \in [0,\theta] \bmod 1, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 3.5. *Let $\omega^* \in \mathbb{R}^d$ be the optimal wave-vector defined in (3.9). For a relaxed state $\theta \in \mathcal{P}$, set $S_k(\theta)(x) = f(\theta(x), kx\cdot\omega^*)$. Assume the hypotheses of theorem 3.4, except that now $\Gamma \neq \partial\Omega$ is possible. Then, for every $\theta, \theta_1, \theta_2 \in \mathcal{P}$,*

$$S_k(\theta) \rightharpoonup \theta \text{ as } k \to \infty,$$
$$\lim_{k\to\infty} \mathcal{D}^{\mathrm{p}}(S_k(\theta_1), S_k(\theta_2)) = \mathcal{D}(\theta_1,\theta_2), \tag{3.11}$$
$$\lim_{k\to\infty} \mathcal{I}^{\mathrm{p}}(S_k(\theta)) = \mathcal{I}(\theta) \tag{3.12}$$

holds. Furthermore, there exists a constant g_0 and continuous mapping $g : [0,T] \to L^2(\Omega)$ such that

$$\frac{\partial I}{\partial t}(t,\theta) = g_0 + \int_\Omega g(t)\theta\,\mathrm{d}x$$

holds.

Theorem 3.6 (Main result). *Assume the hypotheses of theorem 3.4, including $\Gamma = \partial\Omega$. Then the extended model is a relaxation of the model without phase mixtures.*

Proof of theorem 3.6. Since on $\mathcal{P} = L^1(\Omega,[0,1])$ the topologies of $L^1(\Omega)$ and $L^2(\Omega)$ are equivalent, it follows that $\partial I/\partial t$ is continuous on $[0,T]\times\mathcal{P}$. The claim is now a consequence of lemma 3.2 and theorems 3.4 and 3.5. □

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Proof of theorem 3.5. Clearly, we have that $S_k(\theta) \overset{k\to\infty}{\rightharpoonup} \theta$ in $L^2(\Omega)$. Due to the quadratic nature of $\mathcal{D}^{\mathrm{p}}$ and $\mathcal{I}^{\mathrm{p}}$, we can use Fourier analysis to compute the limit energies. By the fundamental theorem for H-measures [10, theorem 1.1], there exists a matrix-valued Radon measure $\mu^{ij} \in \mathcal{M}(\mathbb{R}^d \times S^{d-1})$, $i, j \in \{1, 2\}$, and a subsequence (not relabelled), so that, for every function $\phi_1, \phi_2 \in C_c(\Omega)$ and every function $\Psi \in C(S^{d-1})$, one has

$$\lim_{k\to\infty} \int_{\Omega} [\mathcal{F}((S_k(\theta_i) - \theta_i)\phi_1)(\xi)][\mathcal{F}((S_k(\theta_j) - \theta_j)\phi_2)(\xi)]^* \Psi\left(\frac{\xi}{|\xi|}\right) \mathrm{d}\xi = \langle \mu^{ij}, \phi_1\phi_2^* \otimes \Psi \rangle,$$

where $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is the Fourier transform. For the sequence

$$(S_k(\theta_1) - \theta_1, S_k(\theta_2) - \theta_2),$$

the H-measure can be computed explicitly. Using [10, example 2.1], one can deduce immediately that there exists a 2×2 Hermitian matrix M so that

$$\mu^{ij} = \tfrac{1}{2} M_{ij}(x)(\delta(\cdot - \omega^*) + \delta(\cdot + \omega^*)) \tag{3.13}$$

and

$$M_{ii}(x) = \int_0^1 (f(\theta_i(x), s) - \theta_i(x))^2 \,\mathrm{d}s = \theta_i(x)(1 - \theta_i(x)) \quad \text{for } i \in \{1, 2\}. \tag{3.14}$$

For the off-diagonal elements of M, we have the formula

$$M_{12}(x) = \frac{1}{2} \lim_{\rho\to 0} \lim_{k\to\infty} \frac{1}{\mathrm{vol}(B(\rho, x))} \int_{B(\rho,x)} (S_k(\theta_1(x)) - \theta_1(x))(S_k(\theta_2(x)) - \theta_2(x)) \,\mathrm{d}x$$

for every Lebesgue-point of (θ_1, θ_2). By $B(\rho, x)$, we denote the ball centred at $x \in \Omega$ with radius ρ. The H-measure μ^{ij} encodes enough information so that both limits (in $\mathcal{I}$ and $\mathcal{D}$) can be computed. This observation is conceptionally important, since it demonstrates that the right configuration space for that particular rate-independent evolution problem is the set of H-measures, not just the set of concentrations. This remark concerns only possible future applications, where more complex configurations spaces might be necessary to find relaxed formulations.

Using Lusin's theorem we choose, for a given number $\delta > 0$, a compact subset $\Omega_\delta \subset \Omega$, so that $\mathrm{meas}(\Omega \setminus \Omega_\delta) < \delta$ and $(\theta_1, \theta_2)|_{\Omega_\delta}$ is continuous. From the compactness of Ω_δ, we obtain the existence of a number $\varepsilon(\rho) > 0$, $\lim_{\rho\to 0} \varepsilon(\rho) = 0$, so that $|\theta_i(x) - \theta_i(y)| \leqslant \varepsilon$ holds for all $i \in \{1, 2\}$, $y \in B(\rho, x) \cap \Omega_\delta$. We recall that the sequence $(S_k(\theta_1), S_k(\theta_2))$ is bounded in $L^\infty(\Omega)$ and hence no concentration can occur. This implies that

$$\begin{aligned} M_{12}(x) = \frac{1}{2} \lim_{\rho\to 0} \lim_{k\to\infty} & \frac{1}{\mathrm{vol}(B(\rho, x))} \\ & \times \int_{B(\rho,x)} (S_k(\theta_1(y)) - \theta_1(y))(S_k(\theta_2(y)) - \theta_2(y)) \,\mathrm{d}y \\ = \frac{1}{2} \lim_{\delta\to 0} \lim_{\rho\to 0} \lim_{k\to\infty} & \frac{1}{\mathrm{vol}(B(\rho, x))} \\ & \times \int_{B(\rho,x)\cap\Omega_\delta} S_k(\theta_1(y)) S_k(\theta_2(y)) \,\mathrm{d}y - \theta_1(x)\theta_2(x) \end{aligned}$$

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$$= \frac{1}{2} \lim_{\delta\to\infty} \lim_{\rho\to 0} \lim_{k\to\infty} \frac{1}{\operatorname{vol}(B(\rho,x))} \times \int_{B(\rho,x)\cap\Omega_\delta} f(\theta_1(x) + \underbrace{\theta_1(y)-\theta_1(x)}_{=\mathcal{O}(\varepsilon)}, k\omega^*\cdot y) \cdot f(\theta_2(x) + \underbrace{\theta_2(y)-\theta_2(x)}_{=\mathcal{O}(\varepsilon)}, k\omega^*\cdot y)\,\mathrm{d}y - \theta_1(x)\theta_2(x)$$
$$= \tfrac{1}{2}(\min\{\theta_1(x),\theta_2(x)\} - \theta_1(x)\theta_2(x)).$$

After these computations, we can derive identity (3.11) from the fact that

$$\lim_{k\to\infty} \mathcal{D}^{\mathrm{p}}(S_k(\theta_1), S_k(\theta_2)) = \lim_{k\to\infty} \kappa \int_\Omega \{(1-S_k(\theta_1(x)))S_k(\theta_2(x)) + (1-S_k(\theta_2(x)))S_k(\theta_1(x))\}\,\mathrm{d}x$$
$$= \kappa \int_\Omega (\theta_1(x)+\theta_2(x) - 4M_{12}(x) - 2\theta_1(x)\theta_2(x))\,\mathrm{d}x$$
$$= \kappa \int_\Omega |\theta_1(x)-\theta_2(x)|\,\mathrm{d}x.$$

In the martensitic model, the most complicated functional is the potential energy, since the energy density is determined by a non-local operator. Due to the simple quadratic nature, the H-measure can again be used to evaluate $\lim_{k\to\infty}\mathcal{I}^{\mathrm{p}}(S_k(\theta))$. In principle, two approaches are available to get rid of the influence of the boundary conditions. The first method consists of partitioning the reference configuration Ω in little cubes so that θ is approximately constant on every cube. By careful estimation of the errors, one can show that $\mathcal{I}^{\mathrm{p}}(S_k(\theta))$ realizes $\mathcal{I}(\theta)$ as $k\to\infty$. This method can be applied to very general stored energy functions; we will use it in § 4 to deduce explicit representations for the relaxed formulation.

On the other hand, one can exploit the quadratic structure of the minimization problem more directly by proving that the effect of the boundary conditions can be simulated by a compact operator. We will use the second strategy to establish (3.12).

Regrouping the terms in the potential energy and using the weak continuity of linear terms, we find that

$$\lim_{k\to\infty} \mathcal{I}^{\mathrm{p}}(S_k(\theta)) = \frac{1}{2} \lim_{k\to\infty} \int_\Omega (E(u_k-u) - (A_2-A_1)(S_k(\theta)-\theta)) : \mathbf{C}[E(u_k-u) - (A_2-A_1)(S_k(\theta)-\theta)]\,\mathrm{d}x$$
$$+ \frac{1}{2}\int_\Omega (E(u) - (1-\theta)A_1 - \theta A_2) : \mathbf{C}[E(u) - (1-\theta)A_1 - \theta A_2]\,\mathrm{d}x - \langle G, u\rangle$$
$$= I_1 + I_2,$$

where u and $v_k := u_k - u$ are determined by the weak Euler–Lagrange equations

$$\int_\Omega (\mathbf{C}[E(u) - (A_2-A_1)\theta]) : E(\psi)\,\mathrm{d}x = 0 \quad \forall \psi\in H^1(\Omega),\quad \psi|_\Gamma = 0,\quad u|_\Gamma = u_0|_\Gamma, \tag{3.15}$$

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$$\int_{\Omega} (\mathbf{C}[E(v_k) - \rho_k]) : E(\psi) \, \mathrm{d}x = 0 \quad \forall \psi \in H^1(\Omega), \quad \psi|_{\Gamma} = 0, \quad v_k|_{\Gamma} = 0, \tag{3.16}$$

and $\rho_k = (A_2 - A_1)(S_k(\theta) - \theta)$. Note that the existence of solutions follows from variational principles; Korn's inequality guarantees that the associated energy is coercive in H^1. We are left with the task to compute I_1. The influence of the boundary can be removed by applying the following well-known result.

LEMMA 3.7. *Let $\phi \in C_c^{\infty}(\Omega)$ be an arbitrary test function, and $\tilde{v}_k$ be a weak solution of*

$$\operatorname{div}(\mathbf{C}[E(\tilde{v}_k)]) = \operatorname{div}(\mathbf{C}[\rho_k]\phi), \quad x \in \mathbb{R}^d, \tag{3.17}$$

where $\rho_k \rightharpoonup 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$ as $k \to \infty$ and $v_k \in H^1(\Omega)$ is determined by (3.16). Then $E(\tilde{v}_k) \in L^2(\mathbb{R}^d)$ and

$$\lim_{k \to \infty} \|(E(v_k) - \rho_k)\phi - (E(\tilde{v}_k) - \rho_k \phi)\|_{L^2(\mathbb{R}^d)} = 0$$

holds.

Proof. The integrability of $E(\tilde{v}_k)$ follows from the representation via the Fourier symbol (the acoustic tensor). One cannot expect that $\tilde{v}_k \in L^2(\mathbb{R}^d)$ holds, since the average inside the divergence on the right-hand side of (3.17) might be non-zero. We normalize the expression by adding a constant times the characteristic function of Ω. Let

$$c_k = \frac{1}{\operatorname{vol}(\Omega)} \int_{\mathbb{R}^d} \mathbf{C}[\rho_k]\phi \, \mathrm{d}x$$

and

$$\operatorname{div}(\mathbf{C}[E(\tilde{w}_k)]) = \operatorname{div}(\mathbf{C}[\rho_k]\phi - c_k \mathcal{X}_{\Omega}), \quad x \in \mathbb{R}^d, \tag{3.18}$$

$$\int_{\Omega} (\mathbf{C}[E(w_k) - (\rho_k - c_k \mathcal{X}_{\Omega})]) : E(\psi) \, \mathrm{d}x = 0 \qquad \forall \psi \in H^1(\Omega), \quad \psi|_{\Gamma} = 0, \quad w_k|_{\Gamma} = 0. \tag{3.19}$$

Then, by the weak convergence of ρ_k, we have that $\lim_{k \to \infty} \|c_k \mathcal{X}_{\Omega}\|_{L^2(\mathbb{R}^d)} = 0$. The fact that the Fourier transform of $\mathbf{C}[\rho_k]\phi - c_k \mathcal{X}_{\Omega}$ is smooth by the boundedness of Ω implies that there exists a solution $\tilde{w}_k \in L^2(\mathbb{R}^d)$. Since $E(v_k)$ and $E(\tilde{v}_k)$ depend strongly continuous in L^2 on the right-hand side, it follows that

$$\begin{aligned} &\lim_{k \to \infty} \|(E(v_k) - \rho_k)\phi - (E(\tilde{v}_k) - \rho_k \phi)\|_{L^2(\mathbb{R}^d)} \\ &\qquad = \lim_{k \to \infty} \|(E(w_k) - \rho_k)\phi - (E(\tilde{w}_k) - \rho_k \phi)\|_{L^2(\mathbb{R}^d)} \end{aligned}$$

holds. Next, we see that

$$(E(w_k) - \rho_k)\phi - (E(\tilde{w}_k) - \rho_k \phi) = E(w_k \phi - \tilde{w}_k) - \tfrac{1}{2}(\nabla \phi \otimes w_k + (\nabla \phi \otimes w_k)^{\mathrm{T}}).$$

The last term converges weakly to 0 in $H^1(\Omega)$ and, by the compact embedding, strongly in $L^2(\Omega)$. Denote $\eta_k = \phi w_k - \tilde{w}_k$. It follows from the same argument that

η_k converges strongly to 0 in $L^2_{\rm loc}(\mathbb{R}^d)$. We find that

$$\begin{aligned}
&\operatorname{div}(\mathbf{C}[E(\eta_k)]) \\
&\quad = \operatorname{div}(\mathbf{C}[E(w_k\phi)]) - \operatorname{div}(\mathbf{C}([E(\tilde{w}_k)])) \\
&\quad = \phi \operatorname{div}(\mathbf{C}[E(w_k)]) - \operatorname{div}(\mathbf{C}[E(\tilde{w}_k)]) + \operatorname{div}(\mathbf{C}[w_k \otimes \nabla\phi + (w_k \otimes \nabla\phi)^{\mathrm{T}}]) \\
&\quad = \phi \operatorname{div}(\mathbf{C}[\rho_k - c_k\mathcal{X}_\Omega]) - \operatorname{div}(\mathbf{C}[\rho_k\phi - c_k\mathcal{X}_\Omega]) \\
&\qquad\qquad + \operatorname{div}(\mathbf{C}[w_k \otimes \nabla\phi + (w_k \otimes \nabla\phi)^{\mathrm{T}}]) \quad \text{by (3.18) and (3.19)} \\
&\quad = -\nabla\phi \cdot \rho_k + \operatorname{div}(\mathbf{C}[\nabla\phi \otimes w_k + (\nabla\phi \otimes w_k)^{\mathrm{T}}]) \\
&\quad \rightharpoonup 0
\end{aligned}$$

in $L^2(\Omega)$ as $k \to \infty$ since $w_k \rightharpoonup 0$ in $H^1(\Omega)$ and $\rho_k \rightharpoonup 0$ in $L^2(\Omega)$. The claim follows now from the coercivity of the elasticity tensor $\mathbf{C}$,

$$\begin{aligned}
\lim_{k\to\infty} \int_{\mathbb{R}^d} E(\eta_k){:}E(\eta_k)\,\mathrm{d}x &\leqslant \lim_{k\to\infty} C \int_{\mathbb{R}^d} E(\eta_k){:}\mathbf{C}[E(\eta_k)]\,\mathrm{d}x \\
&= -\lim_{k\to\infty} C \int_{\mathbb{R}^d} \eta_k \cdot \operatorname{div}(\mathbf{C}[E(\eta_k)])\,\mathrm{d}x \\
&= 0,
\end{aligned}$$

by the strong convergence of η_k and the boundedness of $\operatorname{div}(\mathbf{C}[E(\eta_k)])$ in L^2. This concludes the proof of the lemma. □

Since we are only interested in the weak limit of the integrand, we can first apply Plancherel's formula, the fundamental theorem of H-measures, and finally formulae (3.13) and (3.14),

$$\begin{aligned}
&\frac{1}{2}\lim_{k\to\infty} \int_\Omega (E(v_k) - \rho_k)\phi{:}\mathbf{C}[(E(v_k) - \rho_k)\phi]\,\mathrm{d}x \\
&\qquad = \frac{1}{2}\lim_{k\to\infty} \int_{\mathbb{R}^d} (\mathcal{F}(\rho_k\phi)){:}(\mathcal{F}(\rho_k\phi))^* \Psi\left(\frac{\omega}{|\omega|}\right)\mathrm{d}\omega \\
&\qquad = \tfrac{1}{2}\langle \mu, \phi^2 \otimes \Psi\rangle \\
&\qquad = \tfrac{1}{2}\Psi(\omega^*),
\end{aligned}$$

where Ψ is the symbol that is associated to the differential operator and μ (the component μ^{11} in (3.13)) is the H-measure of $E(v_k) - \rho_k$. It is quite easy to see that

$$\Psi(\omega^*) = \gamma, \tag{3.20}$$

where γ is defined in (3.9). For the convenience of the reader, we provide a short proof.

It follows from (3.16) that the symbol of the operator $\rho \mapsto v(\rho)$ is given by $A^{-1}(\omega)\mathbf{C}[\hat{\rho}(\omega)]\omega$, where the acoustic tensor $A(\omega) \in \mathbb{R}^{d\times d}_{\rm sym}$ is determined by the equation

$$a \cdot A(\omega)a = (a \wedge \omega){:}\mathbf{C}[a \wedge \omega] \quad \text{for all } a \in \mathbb{R}^d.$$

The notation $a \wedge \omega$ is a shorthand for the symmetrized rank-1 matrix $\frac{1}{2}(a\otimes\omega + \omega\otimes a)$. But, on the other hand, $\hat{v}(\omega)$ is a minimizer of $\Psi_1(\omega, v) = \frac{1}{2}(\hat{v}\wedge\omega - \hat{\rho}){:}\mathbf{C}[\hat{v}\wedge\omega - \hat{\rho}]$,

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which follows from the Euler–Lagrange equation,

$$\begin{aligned} h \wedge \omega{:}\mathbf{C}[\hat{v} \wedge \omega - \hat{\rho}] = 0 \quad \forall h \quad &\Leftrightarrow \quad h \cdot A(\omega)\hat{v} - h \wedge \omega{:}\mathbf{C}[\hat{\rho}] = 0 \quad \forall h \\ &\Leftrightarrow \quad h \cdot A(\omega)\hat{v} - h \cdot \mathbf{C}[\hat{\rho}]\omega = 0 \quad \forall h \\ &\Leftrightarrow \quad \hat{v} = A^{-1}(\omega)\mathbf{C}[\hat{\rho}]\omega. \end{aligned}$$

The last step, which establishes (3.20), consists of showing that the functions

$$\Psi_1(y) \quad \text{and} \quad \Psi_2(y) = \frac{1}{2}\left(\hat{\rho}{:}\mathbf{C}[\hat{\rho}] - \frac{(y{:}\mathbf{C}[\hat{\rho}])^2}{y{:}\mathbf{C}[y]}\right), \quad y \in S = \{\omega \wedge \hat{v} \mid \omega, \hat{v} \in \mathbb{R}^d\}$$

agree on their minima. But this is a direct consequence of the fact that the minimizers of Ψ_1 are nothing else but a projection of $\hat{\rho}$ on the set S.

Before we can conclude that the H-measure indeed gives the value of I_1, we have to exclude concentrations towards the boundary. Let $\eta \in C^\infty(\Omega)$ be an arbitrary test function, which we do not require to satisfy any boundary condition. Using the weak Euler–Lagrange equation (3.16) again, we see that

$$\begin{aligned} &\int_\Omega E(v_k){:}\mathbf{C}[E(v_k)]\eta \,\mathrm{d}x \\ &\qquad = \int_\Omega E(v_k){:}\mathbf{C}[\rho_k]\eta \,\mathrm{d}x - \int_\Omega \tfrac{1}{2}(v_k \otimes \nabla\eta + (v_k \otimes \nabla\eta)^{\mathrm{T}}){:}\mathbf{C}[E(v_k) - \rho_k] \,\mathrm{d}x \\ &\qquad = I_1 + I_2. \end{aligned}$$

The integrand of I_2 is a product of a sequence of functions that converges weakly to 0 in L^2, and is therefore bounded, and a sequence that converges strongly to 0. Hence I_2 converges to 0 as k tends to ∞. The integrand of I_1 is a product of a fixed continuous function with a sequence of functions that is uniformly bounded in L^2. The uniform integrability of the second term implies that I_1 goes to 0 as the support of η becomes small. Hence $E(v_k){:}\mathbf{C}[E(v_k)]$ does not concentrate towards the boundary of Ω.

From the preceding calculations, it follows that there exists a continuous map $L : L^2(\Omega, \mathbb{R}) \to H^1(\Omega, \mathbb{R}^d)$ so that $u(\theta) = u_0 + L\theta$. We obtain the representation of $\partial I/\partial t$ by applying the adjoint of L to $\dot{G}$. □

4. Necessary conditions for relaxations of the martensitic system

At the end of this paper we apply known lower semicontinuity results to derive formulae for possible relaxations of the martensitic system. In particular, we present a naive candidate for a relaxation of the general martensitic system that is based only on the phase portions. Thus all information about the geometry of the microstructure is neglected. Until now, we are unable to check any of the requirements of definition 3.1. However, it can be shown that if a relaxation only based on phase portions exists, it has to coincide with our postulated model.

DEFINITION 4.1. Let $c \in P = \operatorname{conv}(P^{\mathrm{p}})$ (the unit simplex in $\mathbb{R}^n$) and, for every $i \in \{1, \dots, n\}$, the positive function $W_{e_i} : \mathbb{R}^{d\times d} \to \mathbb{R}_\geqslant$ be quasiconvex and satisfy the growth condition

$$\alpha|F|^p - C \leqslant W_{e_i}(F) \leqslant C(1 + |F|^p) \tag{4.1}$$

for positive constants α, C and $p > 1$. The function

$$\mathbb{W}(\theta, F) = \inf\left\{\int_{[0,1]^d} W_{\eta^{\mathrm{p}}(x)}(F + \nabla\phi(x))\,\mathrm{d}x \mid \eta^{\mathrm{p}} : [0,1]^d \to P^{\mathrm{p}}, \int_{[0,1]^d} \eta^{\mathrm{p}}(x)\,\mathrm{d}x = \theta, \phi \in W_0^{1,\infty}([0,1]^d)\right\}$$

is the relaxation of $(W_{e_i})_{i=1\dots N}$ at given phase portions.

A central assumption in the existence theorem 2.3 is that the mappings $\theta \mapsto \mathcal{I}(\theta)$ and $(\theta,\eta) \mapsto \mathcal{D}(\theta,\eta)$ are weakly lower semicontinuous in $\mathcal{P}$ and $\mathcal{P} \times \mathcal{P}$. There can be only one relaxation of the martensitic system, so that $\mathcal{I}$ and $\mathcal{D}$ satisfy this lower semicontinuity property in $L^1(\Omega, P)$ and $L^1(\Omega, P \times P)$.

THEOREM 4.2. *Let $(\mathcal{I}, \mathcal{D}, \mathcal{P})$ be a relaxation of $(\mathcal{I}^{\mathrm{p}}, \mathcal{D}^{\mathrm{p}}, \mathcal{P}^{\mathrm{p}})$, so that the functionals $\theta \mapsto \mathcal{I}(\theta)$ and $(\theta,\eta) \mapsto \mathcal{D}(\eta,\theta)$ are weakly lower semicontinuous in $L^1(\Omega, P)$ and $L^1(\Omega, P \times P)$. Then the relaxed quantities are given by the following formulae,*

$$\mathcal{P} = L^1(\Omega, P), \tag{4.2}$$

$$\mathcal{I}(\theta) = \inf_{u \in V} \int_\Omega \mathbb{W}(\theta, \nabla u)\,\mathrm{d}x - \langle G, u\rangle, \tag{4.3}$$

$$\mathcal{D}(\theta,\eta) = \int_\Omega D(\eta(x), \theta(x))\,\mathrm{d}x, \tag{4.4}$$

where

$$D(a,c) = \inf\left\{\sum_{j,i=1}^{n} m_{ji}\kappa_{j\to i} : m_{ji} \geqslant 0,\ \sum_{i=1}^{n} m_{ji} = e_j \cdot a,\ \sum_{j=1}^{n} m_{ji} = e_i \cdot c\right\}.$$

Obviously, the relaxed model given by (4.2)–(4.4) agrees with the non-relaxed model if only pure phase distributions are taken into account. Two difficulties prevent us from concluding that the extended model is a relaxation of the model without phase mixtures. Firstly, we do not know whether admissible processes exist. The source of the difficulty is that the weak closedness of the stable set $\mathcal{S}(t)$ (see (2.5)) is not clear. The second difficulty is to show that the relaxed model can indeed be approximated with the non-relaxed model. Here the problem lies in the fact that, typically, the approximation of the relaxed stored energy function requires the choice of very special geometries of the microstructure of the phase distribution. It is not clear whether the approximating microstructures for two different distributions of phase portions can be chosen in such a way that the dissipation is indeed minimal, as the definition of $\mathcal{D}$ suggests. For example, if the optimal microstructure is given by a simple laminate, where the lamination direction ω^* depends non-trivially on θ, then the dissipation functional (4.4) clearly underestimates the true dissipation.

Proof. We only have to show that $\mathcal{I}(\cdot)$ and $\mathcal{D}(\cdot,\cdot)$ are the weak lower semicontinuous envelopes of $\mathcal{I}^{\mathrm{p}}(\cdot)$ and $\mathcal{D}^{\mathrm{p}}(\cdot,\cdot)$ in $L^1(\Omega, P)$ and $L^1(\Omega, P) \times L^1(\Omega, P)$. The first step is to prove that $\mathcal{I}$ and $\mathcal{D}$ are weakly lower semicontinuous. For general functions

$\mathbb{W}$, the functional

$$(u,\theta) \mapsto \int_\Omega \mathbb{W}(\theta, u, \nabla u)\,\mathrm{d}x$$

is discussed in detail in [5]. In particular, theorem 4.17 covers exactly our case except that here the stored energy function takes the value $+\infty$, i.e.

$$W(F,\theta) = \begin{cases} W_{e_i}(F) & \text{if } \theta = e_i \text{ for some } i \in \{1,\dots,N\}, \\ \infty & \text{otherwise.} \end{cases}$$

One can easily check that the theorem can be generalized to this case without change.

The functional

$$(\eta,\theta) \mapsto \int_\Omega D(\eta,\theta)\,\mathrm{d}x$$

is lower semicontinuous since D is convex by standard arguments. Consider now $\theta_i, \eta_i \in P$ and let $m_{jk}^{(i)}$ be the minimizers in the definition of $D(\theta_i, \eta_i)$ for $i = 1, 2$, that is,

$$m_{jk}^{(i)} \geqslant 0, \quad \sum_{l=1}^n m_{jl}^{(i)} = e_j \cdot \theta_i \quad \text{and} \quad \sum_{j=1}^n m_{jl}^{(i)} = e_l \cdot \eta_i.$$

Then, for $\lambda \in [0,1]$, the convex combination $m_{jl}^{(3)} = \lambda m_{jl}^{(1)} + (1-\lambda) m_{jl}^{(2)}$ forms an admissible set in the definition of $D(\lambda\theta_1 + (1-\lambda)\theta_2, \lambda\eta_1 + (1-\lambda)\eta_2)$, which gives the convexity result after separating the minimum into the weighted sum of two minima.

The second step consists of showing that the values of $\mathcal{I}$ and $\mathcal{D}$ can actually be achieved by non-relaxed sequences. This is clear from the definition for $\mathcal{I}$ and we will demonstrate it for $\mathcal{D}$.

Choose an arbitrary pair of relaxed states $\eta, \theta \in \mathcal{P}$ and fix $\varepsilon > 0$. Find, analogously to the proof of theorem 3.5, a compact set K_ε, so that $\operatorname{meas}(\Omega \setminus K_\varepsilon) \leqslant \varepsilon$ and $(\eta,\theta)|_{K_\varepsilon}$ are continuous. Next we cover Ω with a countable disjoint collection of cubes $Q_\ell \subset \Omega$, so that $|\theta(x) - \theta(y)| + |\eta(x) - \eta(y)| \leqslant \varepsilon$ for all $x, y \in Q_\ell \cap K_\varepsilon$. Set

$$\theta_\varepsilon = \sum_\ell \theta_\ell \mathcal{X}_{Q_\ell} \quad \text{and} \quad \eta_\varepsilon = \sum_\ell \eta_\ell \mathcal{X}_{Q_\ell},$$

where $\theta_\ell = \theta(x_\ell)$, $\eta_\ell = \eta(x_\ell)$ and

$$x_\ell \in \begin{cases} Q_\ell \cap K_\varepsilon & \text{if } Q_\ell \cap K_\varepsilon \neq \{\}, \\ \Omega & \text{otherwise.} \end{cases}$$

Using the definition of D, we find numbers $m_{ij\ell} \geqslant 0$ so that

$$\sum_i m_{ij\ell} = e_j \cdot \eta_\ell \quad \text{and} \quad \sum_j m_{ij\ell} = e_i \cdot \theta_\ell.$$

We split Q_ℓ into n^2 segments $Q_{ij\ell}$ of size $\operatorname{meas}(Q_\ell) \cdot m_{ij\ell}$ and set

$$\theta_\varepsilon^{\mathrm{p}}(x) = \sum_{i,j,\ell} e_i \cdot \mathcal{X}_{Q_{ij\ell}}(x), \quad \eta_\varepsilon^{\mathrm{p}}(x) = \sum_{i,j,\ell} e_j \cdot \mathcal{X}_{Q_{ij\ell}}(x), \quad x \in \Omega.$$

From the definition of D, it follows that

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_\Omega D(\eta^{\mathrm{p}}_\varepsilon(x),\theta^{\mathrm{p}}_\varepsilon(x))\,\mathrm{d}x &= \lim_{\varepsilon\to 0}\int_\Omega D(\theta_\varepsilon(x),\eta_\varepsilon(x))\,\mathrm{d}x \\ &= \lim_{\varepsilon\to 0}\int_{K_\varepsilon} D(\theta_\varepsilon(x),\eta_\varepsilon(x))\,\mathrm{d}x \\ &= \lim_{\varepsilon\to 0}\int_{K_\varepsilon} D(\theta(x),\eta(x))\,\mathrm{d}x \\ &= \int_\Omega D(\theta(x),\eta(x))\,\mathrm{d}x.\end{aligned}$$

This implies the last claim. □

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