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EQUILIBRIUM VALUATION OF CURRENCY OPTIONS UNDER A DISCONTINUOUS MODEL WITH CO-JUMPS

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In this paper, we study the equilibrium valuation for currency options in a setting of the two-country Lucas-type economy. Different from the continuous model in Bakshi and Chen [1], we propose a discontinuous model with jump processes. Empirical findings reveal that the jump components in each country's money supply can be decomposed into the simultaneous co-jump component and the country-specific jump component. Each of the jump components is modeled with a Poisson process whose jump intensity follows a mean reversion stochastic process. By solving a partial integro-differential equation (PIDE), we get a closed-form solution to the PIDE for a European call currency option. The numerical results show that the derived option pricing formula is efficient for practical use. Importantly, we find that the co-jump has a significant impact on option price and implied volatility.

Key words: co-jump, currency option, equilibrium valuation, partial integro-differential equation

1. INTRODUCTION

Since Garman and Kohlhagen [15] first derived a pricing formula of a European currency option based on the Black–Sholes model, many researchers have studied currency options under a variety of models such as stochastic volatility model [16], jump-diffusion model with stochastic volatility [6], variance-gamma jump model [10], and stochastic skew model [9]. Although jump models perform better than continuous models empirically, the relationships among the interest rate, exchange rate, and risk premium are always ignored in above-mentioned models. While in equilibrium models, the economic variables are determined with each other. This makes that the equilibrium models better describe the real financial market.

Lucas [19] proposed the two-country economy and assumed that there were only two traded goods, two agents and all the dividends were consumed. Each agent preferred to share the risks and had the same portfolio. By solving the agent's utility maximization problem, Lucas obtained the equilibrium price of a nominal foreign exchange rate and a zero-coupon bond under a continuous model. Later, Bakshi and Chen [1] generalized Lucas's model to the stochastic volatility model. They obtained closed-form solutions for foreign exchange claims following the two-country economy assumption. Then, based on the Bakshi and Chen [1] model, Xing and Yang [22] priced the equilibrium currency option prices under a jump-diffusion model with stochastic volatility. However, the jump parts of the money supplies in the domestic and foreign country were assumed mutually independent and the jump intensities were both constants, which was not very consistent with the real market.

To our knowledge, the model of Santa-Clara and Yan [21] is the first model in which the jump intensity follows explicitly its own stochastic process. Their model does not belong to the affine family of Duffie et al. [13], in that the drifts and the covariance terms are not linear in the state variables. When they calibrated the model to the S&P 500 index option prices from the beginning of 1996 to the end of 2002, they obtained the time series of the implied diffusive volatility and jump intensity. Their empirical results showed that the innovations to the two risk processes were not very correlated with each other, which was consistent with their model assumption that the diffusive volatility and jump intensity followed their own stochastic process. Instead of studying equilibrium stock options pricing in Santa-Clara and Yan [21], Du [12] studied equilibrium currency options pricing under recursive utility. In his consumption-based model, the consumptions in domestic and foreign countries were highly but imperfectly shared disaster with time-varying jump intensities. By using the fast Fourier transform (FFT) method in Carr and Madan [8], he got the numerical prices of equilibrium currency options.

Most of the literature only focuses on the jumps occurred in individual stocks; however, simultaneous jumps called co-jumps among the multiple assets are always overlooked. Unlike the jump risk from a specific country, the co-jump risks come from infrequent events that are highly correlated across a large number of assets. Historically, the financial crisis in Southeast Asia in 1998 can trigger simultaneous jumps in the money supplies of different countries, and similar phenomena can also be observed when the U.S. financial crisis occurred in 2008. This co-jump risk can be regarded as the systemic risk in Das and Uppal [11]. Keddad [17] investigated the degree and the nature of exchange rate co-movements between the Renminbi and a set of seven East Asian currencies. Barunik et al. [5] did empirical analysis to reveal different behaviors of co-jumps during Asian, European, and U.S. trading sessions. They found that co-jumps significantly influenced the correlations in currency markets. The impact of systemic risk is more intense, lasting, and wider. Therefore, this systemic risk should not be ignored and should be taken into consideration for currency option pricing.

Inspired by the mentioned papers, in the two-country economy setting proposed by Lucas [19], we follow the equilibrium method in Bakshi and Chen [1] to price currency options under a discontinuous model with co-jumps and stochastic jump intensities in this paper. We assume that the money supplies in two countries are correlated through a common jump process. The exchange rate is affected by jump risks from the jumps occurred in individual money supply and the co-jump. For the assumption of the distribution of jump size, a normal distribution [20] or double exponential distribution [18] are often used. We assume the distribution of jump size to be arbitrary, which makes our model more general for applications. Under our model, we can obtain a closed-form solution for a European call currency option price by solving a partial integro-differential equation (PIDE). The method of solving the PIDE can refer to Heston [16]. Heston [16] pointed out that the formula for a European option price was associated with the risk-neutralized probabilities, and it was much easier to solve the characteristic functions to get the risk-neutralized probabilities. Later, Bakshi and Chen [1] applied the technique to the stochastic volatility model. Bates [6] applied the technique to the jump-diffusion model with stochastic volatility.

Furthermore, Fu and Yang [14] applied the technique to the Lévy models. Except for the European option, the equilibrium valuation in this paper can also be applied to price some exotic options. However, it is not easy. There are two main difficulties. One difficulty is that the equilibrium pricing method needs to be combined with a macroeconomic theory, while it is hard to construct the macroeconomic background for pricing some equilibrium exotic options. Another difficulty is that the PIDE for the exotic option price under jump models always has no analytical solution. Thus, we have to obtain the exotic option price by numerical methods such as the finite difference method or Monte Carlo simulation.

The article is organized as follows. In Section 2, we perform some empirical examinations and discuss the effects of co-jumps. In Section 3, we determine the dynamic process of the exchange rate and the equilibrium prices for zero-coupon bonds. The expected return of traditional carry trade strategy under our model is also computed. In Section 4, we derive the PIDE for currency option prices and get the equilibrium values of the currency options. In Section 5, we do Monte Carlo simulations to verify the correctness of our formula and do some numerical analysis. The influences of jump factors on option prices and implied volatilities are shown. Finally, we offer concluding remarks in Section 6.

2. EFFECTS OF CO-JUMPS

With the development of global economy, jumps in exchange rates have become more and more frequently, so it is necessary to consider jump processes for currency option pricing. In the context of the two-country economy, we assume that the jumps in exchange rates are due to the changes in monetary supplies. We use the method in Barndorff-Nielsen and Shephard [4] to verify that jumps do exist in the time series of money supplies.

Let $w^*(t)$ be the log-price of an asset. Quadratic variation (QV) is defined on intervals of time length δ.

$$w^*_\delta(t) = w^*(\delta\lfloor t\delta^{-1}\rfloor), \quad t \geq 0.$$

$\lfloor x\rfloor$ is the integer part of x.

$$w_j = w^*(j\delta) - w^*((j-1)\delta), \quad j = 1, 2, \ldots, \lfloor t/\delta\rfloor.$$

The realized quadratic variation process is given by the following equation:

$$[w^*_\delta](t) = \sum_{j=1}^{\lfloor t/\delta\rfloor} w_j^2.$$

The realized bipower variation process (BPV) is given by the following equation:

$$[w^*_\delta]^{[1,1]}(t) = \sum_{j=1}^{\lfloor t/\delta\rfloor - 1} |w_j||w_{j+1}|.$$

Barndorff-Nielsen and Shephard [4] determined a robust estimator by using the multipower variation instead of the bipower variation:

$$[w^*_\delta]^{[1,1,1,1]}(t) = \delta^{-1} \sum_{j=1}^{\lfloor t/\delta\rfloor - 3} |w_j||w_{j+1}||w_{j+2}||w_{j+3}|.$$

The linear jump test statistic variable is given by the following equation:

$$\hat{G} = \frac{\delta^{-1/2}(\mu_1^{-2}[w_\delta^*]^{[1,1]}(t) - [w_\delta^*](t))}{\sqrt{\vartheta\mu_1^{-4}[w_\delta^*]^{[1,1,1,1]}(t)}} \xrightarrow{L} N(0,1),$$

where

$$\mu_1 = \sqrt{2/\pi} \simeq 0.79788, \quad \vartheta = (\pi/4)^2 + \pi - 5 \simeq 0.6090.$$

We downloaded the monthly data of the money supplies M1 in the United States, Australia, and Switzerland from the Federal Reserve Economic Database (FRED). The samples all ranged from January 1980 to December 2018. We use the data of M1 in the United States, Australia, and Switzerland to compute the statistic variable $\hat{G}$. The values of $\hat{G}$ are 2.9967, −25.1773, and −67.2666, respectively.

$$P_1(\hat{G} > 2.9967) = 0.0014,$$
$$P_2(\hat{G} < -25.1773) \approx 0,$$
$$P_3(\hat{G} < -67.2666) \approx 0.$$

Since p-value $P_1 < \alpha$, $P_2 < \alpha$, and $P_3 < \alpha$ at level $\alpha = 0.05$, we would reject the null of a continuous sample path. There is, thus, an evidence that there exist jumps in the time series data of percentage changes in money supplies.

In the most of the existing literature, a single jump process is often used to describe "jumps". However, we think there is always more than one source of jump risk in the money supply. On the one hand, some extreme events can trigger simultaneous jumps in the money supply of different countries. Simultaneous jump called co-jump is regarded as a systemic risk whose impact may be long-lasting and wide-ranging, and therefore cannot be ignored. On the other hand, we believe that the money supply in each country is affected by the country-specific monetary policy risk. Observed from Figures 1–3, the percentage changes of money supplies have jumps in December 2008, which is clearly visible in the figures. The absolute values of percentage changes are 0.0562, 0.0500, and 0.0910 in December 2008 in the United States, Australia, and Switzerland, respectively. According to Caporin et al. [7], the percentage change which is more than six standard deviations in local volatility units can be regarded as an obvious jump. Thus, the simultaneous jump that took place in December 2008 can be regarded as a co-jump, which was caused by the rare financial crisis in the United States in 2008. Next, we would take the co-jump happened in December 2008 as an example to analyze the impacts of the co-jump.

Firstly, co-jumps make the absolute value of correlation between money supplies and short-term volatility increase. The money supplies in different countries all over the world jumped simultaneously in December 2008, which revealed that the co-jump led to a significant increase in correlation between money supplies. In addition, the short-term volatility of the percentage change of the money supplies also increased. Take the United States as an example, the sample standard deviation of the percentage change of the money supply was 0.0184 in 2008, 0.0106 in 2009, and 0.0100 in 2010. While before the occurrence of co-jump, the sample standard deviation was 0.0052 in 2007 and 0.0041 in 2006. This shows that the co-jump can predict the short-term volatility.

Secondly, the shocks caused by the co-jumps among the assets are greater than the jumps in individual assets. The daily data of the CBOE Volatility Index from 2 January 1990 to 31 December 2018 are shown in Figure 4. We see that the close price of the CBOE Volatility Index reached the historical maximum of 80.86 on 20 November 2008. The average

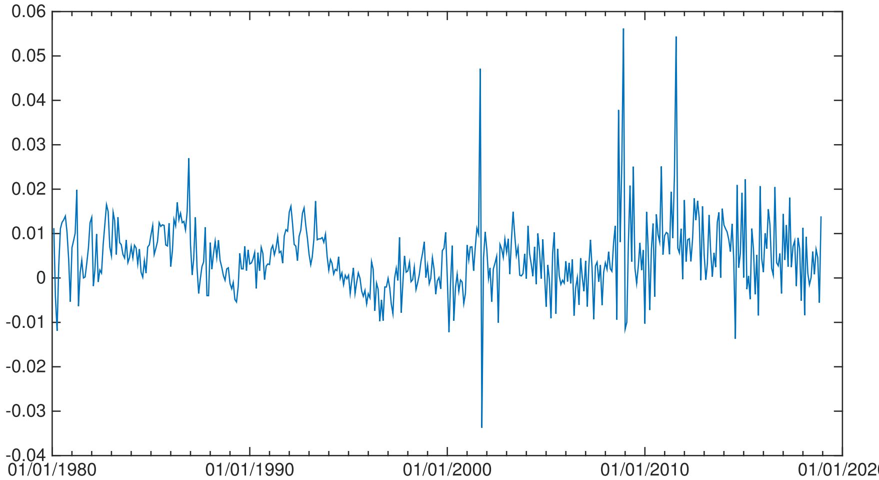

FIGURE 1. Percentage change of the money supply in the United States.

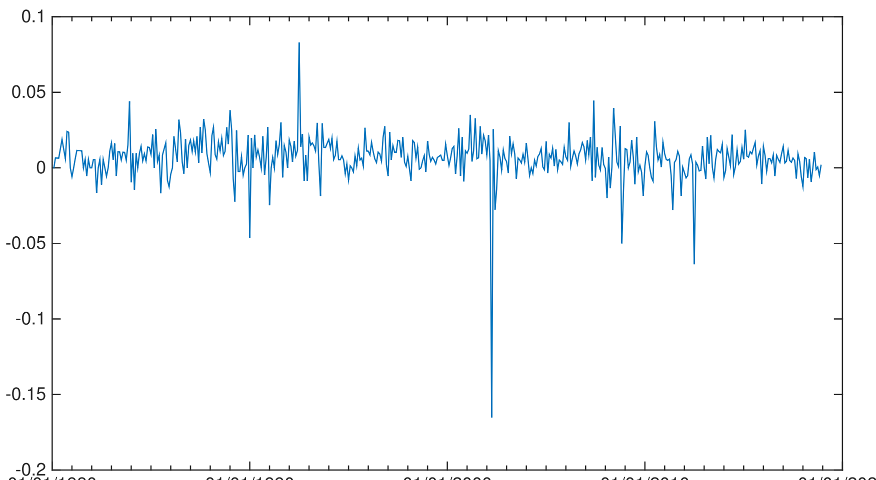

FIGURE 2. Percentage change of the money supply in Australia.

index in November and December 2008 was both above 50. This was a period when the index had remained at a high level since 1990, while the VIX index did not exceed 50 even after "911" event, and so did other jumps in individual assets. These facts show that the shocks caused by the co-jumps are greater than the jumps in individual assets.

Thirdly, the impacts of the co-jumps are more lasting. We use the monthly data of 1-year treasury constant maturity rate in the United States downloaded from FRED for analysis (Figure 5). The rate fell below 0.5 for the first time in December 2008. The rate always remained below 0.5% from December 2008 to November 2015, which meant that the impacts triggered by the co-jump happened in December 2008 lasted for about 7 years. We see that there was no period when the rate remained at a low level for so long since 1980, that is, the effects of co-jumps are more lasting.

In summary, co-jumps make the absolute value of correlation between money supplies and short-term volatility increase. Moreover, the impacts of co-jumps are greater and lasting.

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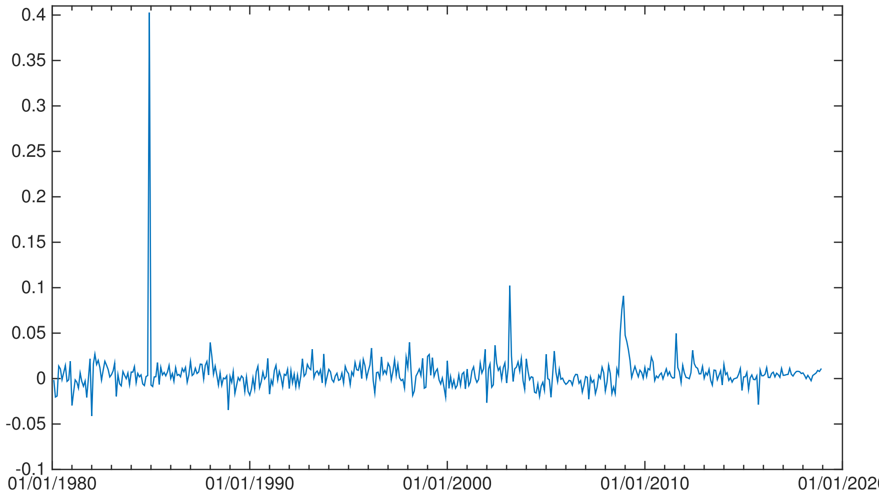

FIGURE 3. Percentage change of the money supply in Switzerland.

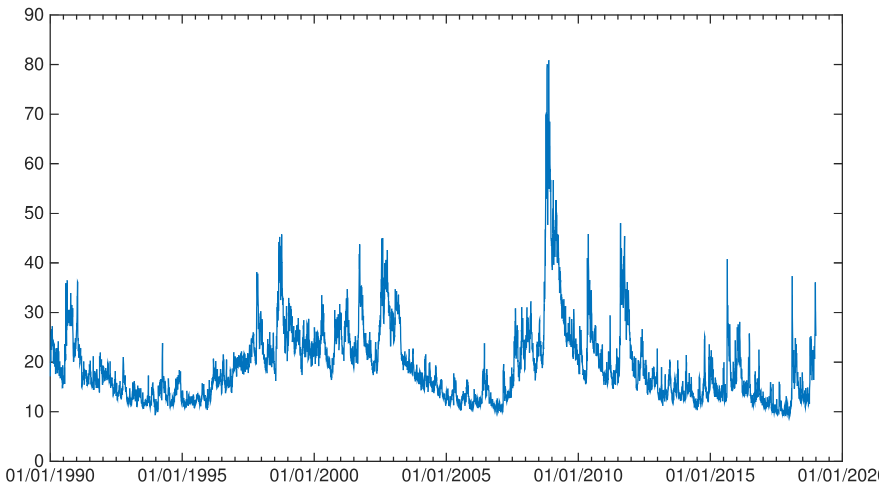

FIGURE 4. The historical close prices of the CBOE Volatility Index.

These characteristics are significantly different from the impacts of the jumps in individual assets, but there are few literatures concerning the pricing of currency options with co-jumps. Based on the above analysis, this paper will discuss the currency options pricing with co-jumps and analyze the impacts of co-jumps on option prices and implied volatilities.

3. THE MODEL

Based on the observations and discussions in Section 2, we believe that the money supply in each country is affected by the co-jump risk and the country-specific monetary policy risk. So, we add a common jump term with stochastic jump intensity into the processes of money supplies in the domestic and foreign country. Besides, we follow the basic assumptions of two-country economy in Bakshi and Chen [1].

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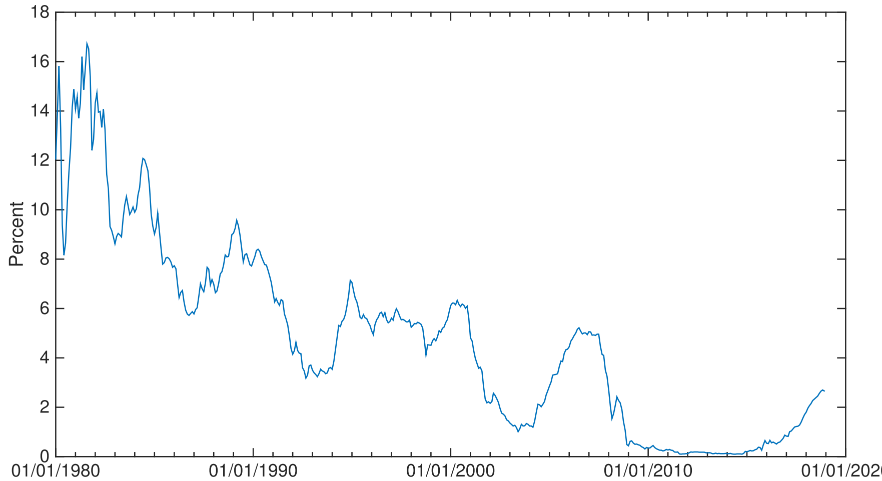

FIGURE **5.** The historical prices of 1-year treasury constant maturity rates in the United States.

ASSUMPTION 1: *The money supplies M_t, $M_t^*(M_t = 2m_t,\ M_t^* = 2m_t^*)$ in two countries follow jump-diffusion processes with stochastic intensities.*

$$\frac{dm_t}{m_t} = \mu dt - (\lambda_t^{\mathrm{h}} + \lambda_t^{\mathrm{g}})E(e^Z - 1)\,dt + \sigma dB_t + (e^Z - 1)\,(dN_t^{\mathrm{h}} + dN_t^{\mathrm{g}}). \tag{1}$$

Similarly, the process of money supply m_t^* in the foreign country is as follows:

$$\frac{dm_t^*}{m_t^*} = \mu^* dt - (\lambda_t^{\mathrm{f}} + \lambda_t^{\mathrm{g}})E(e^{Z^*} - 1)\,dt + \sigma^* dB_t^* + (e^{Z^*} - 1)(dN_t^{\mathrm{f}} + dN_t^{\mathrm{g}}), \tag{2}$$

where

$$\begin{aligned} d\lambda_t^{\mathrm{h}} &= k(\bar{\lambda}^{\mathrm{h}} - \lambda_t^{\mathrm{h}})\,dt + \sigma_\lambda \sqrt{\lambda_t^{\mathrm{h}}}\,dB_t^{\mathrm{h}}, \\ d\lambda_t^{\mathrm{f}} &= k(\bar{\lambda}^{\mathrm{f}} - \lambda_t^{\mathrm{f}})\,dt + \sigma_\lambda \sqrt{\lambda_t^{\mathrm{f}}}\,dB_t^{\mathrm{f}}, \\ d\lambda_t^{\mathrm{g}} &= k(\bar{\lambda}^{\mathrm{g}} - \lambda_t^{\mathrm{g}})\,dt + \sigma_\lambda \sqrt{\lambda_t^{\mathrm{g}}}\,dB_t^{\mathrm{g}}. \end{aligned} \tag{3}$$

μ is the expected rate of appreciation of home money supply. σ is the volatility of the percentage of home money supply. $e^Z - 1$ is the percentage jump size (conditional on a jump occurring) and can be arbitrarily distributed. Parameters with subscript "" represent the corresponding variables in the foreign country. $\bar{\lambda}^{\mathrm{h}}$, $\bar{\lambda}^{\mathrm{f}}$,* and *$\bar{\lambda}^{\mathrm{g}}$ are the long-term levels of jump intensity factors from the domestic country, foreign country, and global market. k is the mean-reverting rate of jump intensity. B_t and B_t^* have a correlation ρ_m. The standard Brownian motions B_t^{h}, B_t^{f}, and B_t^{g} are mutually independent. In addition, $dB_t \cdot dB_t^{\mathrm{h}} = 0$, $dB_t \cdot dB_t^{\mathrm{f}} = 0$, $dB_t \cdot dB_t^{\mathrm{g}} = 0$, $dB_t^* \cdot dB_t^{\mathrm{h}} = 0$, $dB_t^* \cdot dB_t^{\mathrm{f}} = 0$, and $dB_t^* \cdot dB_t^{\mathrm{g}} = 0$. The Poisson process N_t^{g} can be seen as a global component and N_t^{h}, N_t^{f} be the country-specific component. The Poisson processes N_t^{h}, N_t^{f}, and N_t^{g} are mutually independent. The parameters μ, μ^*, ρ_m, σ, σ^*, k, σ_λ, $\bar{\lambda}^{\mathrm{h}}$, $\bar{\lambda}^{\mathrm{f}}$, $\bar{\lambda}^{\mathrm{g}}$ are all constants.*

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ASSUMPTION 2: *The GDP in home country is $2y_t$ and the GDP in foreign country is $2y_t^*$. They follow the processes*:

$$\frac{dy_t}{y_t} = \mu_y\,dt + \sigma_y\,d\omega_t,$$
$$\frac{dy_t^*}{y_t^*} = \mu_y^*\,dt + \sigma_y^*\,d\omega_y^*.$$

Here, μ_y, σ_y are the instantaneous growth rate and instantaneous volatility of the output in the home country, respectively. μ_y^*, σ_y^* represent the corresponding variables in the foreign country. ω_t, ω_t^*, B_t^{h}, B_t^{f}, B_t^{g} are mutually independent standard Brownian motions.

ASSUMPTION 3: *The agent's life-time problem is given by the following equation*:

$$\max_{c,c^*} E_t \int_t^{\infty} e^{-\rho(s-t)} U(c_s, c_s^*)\,ds, \tag{4}$$

where ρ is the time preference parameter, and U is a utility function. Under the assumption of "perfect pooling" in Lucas [19], *the agents in home and foreign countries prefer to have the same portfolio and the same utility function. Here, we assume that the utility function is a simple logarithmic utility function*:

$$U(c_t, c_t^*) = \gamma \ln c_t + (1-\gamma)\ln c_t^*, \tag{5}$$

where γ is the expenditure share on the domestic good. c, c^* are the consumptions of goods in the home and foreign countries.

If the problem (4) exists a solution under the above three assumptions, an equilibrium price is obtained. Applying the dynamic stochastic programming method, we can obtain the prices of a nominal exchange rate and a zero-coupon bond at time t. For the proof details, see [22].

$$S_t = \frac{1-\gamma}{\gamma}\frac{m_t}{m_t^*}, \tag{6}$$

$$B_{t,\Delta t} = e^{-\rho \Delta t} E_t\left(\frac{m_t}{m_{t+\Delta t}}\right). \tag{7}$$

Apply Ito's lemma with (6), we get the process of a nominal exchange rate which is given by the following equation:

$$\frac{dS_t}{S_t} = \mu_s dt + \sigma dB_t - \sigma^* dB_t^* + (e^{Z}-1)\,(dN_t^h + dN_t^{\mathrm{g}}) + (e^{-Z^*}-1)\,(dN_t^{\mathrm{f}} + dN_t^{\mathrm{g}}), \tag{8}$$

where

$$\mu_s = \mu - \mu^* - (\lambda_t^{\mathrm{h}} + \lambda_t^{\mathrm{g}})E(e^{Z}-1) + (\lambda_t^{\mathrm{f}} + \lambda_t^{\mathrm{g}})E(e^{Z^*}-1) + {\sigma^*}^2 - \sigma\sigma^*\rho_m. \tag{9}$$

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THEOREM 1: *The closed-form solution for the price of the domestic zero-coupon bond* $B(t,\,\tau;\lambda_t^{\mathrm{h}},\,\lambda_t^{\mathrm{g}})$ *is given by the following equation*:

$$B(t,\tau;\lambda_t^{\mathrm{h}},\lambda_t^{\mathrm{g}}) = \exp\left\{\frac{2[E(e^Z-1)+E(e^{-Z}-1)](1-e^{-\zeta_h\tau})}{2\zeta_h-(\zeta_h-k)(1-e^{-\zeta_h\tau})}(\lambda_t^{\mathrm{h}}+\lambda_t^{\mathrm{g}})-(\rho+\mu-\sigma^2)\tau \right.$$
$$\left. -\frac{k(\bar\lambda^{\mathrm{h}}+\bar\lambda^{\mathrm{g}})}{\sigma_\lambda^2}\left[2\ln\left(1-\frac{(\zeta_h-k)(1-e^{-\zeta_h\tau})}{2\zeta_h}\right)+(\zeta_h-k)\tau\right]\right\},$$
$$\zeta_h(\tau)=\sqrt{k^2-2\sigma_\lambda^2[E(e^Z-1)+E(e^{-Z}-1)]}, \tag{10}$$

the nominal interest rate at time t *is given by the following equation:*

$$R_t=\rho+\mu-\sigma^2-[E(e^Z-1)+E(e^{-Z}-1)](\lambda_t^{\mathrm{h}}+\lambda_t^{\mathrm{g}}), \tag{11}$$

The price of the foreign zero-coupon bond $B^*(t,\,\tau;\lambda_t^{\mathrm{f}},\,\lambda_t^{\mathrm{g}})$ *has a similar structure.*

PROOF: The relationship between the zero-coupon bond with maturity $t+\Delta t$ and the interest rate is given as follows:

$$B_{t,\Delta t}\cdot(1+R\Delta t)=1.$$

When $\Delta t\to 0$, we have

$$B_{t,\Delta t}=e^{-R\Delta t}.$$

Combine with formula (7), we have

$$e^{-(R-\rho)\Delta t}=E_t\left(\frac{m_t}{m_{t+\Delta t}}\right). \tag{12}$$

Denote $f(m_t)=1/m_t$, and then apply the Ito's lemma with respect to m_t.

$$df(m_t)=f'(m_{t-})(dm_t)_c+\frac{1}{2}f''(m_{t-})(dm_t)_c^2+[f(m_{t-}\cdot e^Z)-f(m_{t-})](dN_t^{\mathrm{h}}+dN_t^{\mathrm{g}}).$$

The subscript "c" here represents the part with the absence of jumps. Observe that

$$(dm_t)_c=m_t\left(\frac{dm_t}{m_t}\right)_c,$$

we can rewrite $df(m_t)$ as

$$d\left(\frac{1}{m_t}\right)=-\frac{1}{m_t}\left(\frac{dm_t}{m_t}\right)_c+\frac{1}{m_t}\left(\frac{dm_t}{m_t}\right)_c^2+\left(\frac{1}{m_t\cdot e^Z}-\frac{1}{m_t}\right)(dN_t^{\mathrm{h}}+dN_t^{\mathrm{g}}).$$

The above equation has a discrete time form:

$$\frac{m_t}{m_{t+\Delta t}}=1-\left(\frac{\Delta m_t}{m_t}\right)_c+\left(\frac{\Delta m_t}{m_t}\right)_c^2+\left(e^{-Z}-1\right)\left(\Delta N_t^{\mathrm{h}}+\Delta N_t^{\mathrm{g}}\right)+o(\Delta t). \tag{13}$$

Insert (13) into (12) and take $\Delta t\to 0$, we can get the home interest rate as follows:

$$R_t=\rho+\mu-\sigma^2-[E(e^Z-1)+E(e^{-Z}-1)](\lambda_t^{\mathrm{h}}+\lambda_t^{\mathrm{g}}).$$

The risk premium on any contingent claim is determined by the Euler equation (A3) in Ref. [1] by solving the agent's first-order condition. If the contingent claim is a bond, we have

$$E_t\left[e^{-\rho\Delta t}\frac{m_t}{m_{t+\Delta t}}\left(\frac{\Delta B_t}{B_t}-R_t\Delta t\right)\right]=0.$$

Note that $B(t,\,\tau;\lambda^{\mathrm{h}},\,\lambda^{\mathrm{g}})$ is a function of variable t, λ^{h}, λ^{g}. We apply Ito's lemma to calculate dB_t, then we can get the discredited form ΔB_t. We obtain the partial differential equation (PDE) of the price of $B(t,\,\tau;\lambda^{\mathrm{h}},\,\lambda^{\mathrm{g}})$ when $\Delta t\to 0$

$$-\frac{\partial B}{\partial\tau}+\frac{\partial B}{\partial\lambda^{\mathrm{h}}}k(\bar{\lambda}^{\mathrm{h}}-\lambda^{\mathrm{h}})+\frac{\partial B}{\partial\lambda^{\mathrm{g}}}k(\bar{\lambda}^{\mathrm{g}}-\lambda^{\mathrm{g}})+\frac{1}{2}\frac{\partial^2 B}{\partial{\lambda^{\mathrm{h}}}^2}\sigma_\lambda^2\lambda^{\mathrm{h}}+\frac{1}{2}\frac{\partial^2 B}{\partial{\lambda^{\mathrm{g}}}^2}\sigma_\lambda^2\lambda^{\mathrm{g}}=RB.$$

Solve the above PDE with the terminal condition $B(t,\,0;\lambda^{\mathrm{h}},\,\lambda^{\mathrm{g}})=1$, we can get a closed-form solution of $B(t,\,\tau;\lambda^{\mathrm{h}},\,\lambda^{\mathrm{g}})$ expressed in Formula (10). ■

With the formulas of interest rates at home and abroad, we can get the interest rate differential as follows:

$$\begin{aligned}R_t-R_t^*=(\mu-\mu^*)-(\sigma^2-{\sigma^*}^2)-(\lambda_t^{\mathrm{h}}+\lambda_t^{\mathrm{g}})[E(e^Z-1)+E(e^{-Z}-1)]\\+(\lambda_t^{\mathrm{f}}+\lambda_t^{\mathrm{g}})[E(e^{Z^*}-1)+E(e^{-Z^*}-1)].\end{aligned}$$

We assume $\mu=\mu^*$, $\sigma=\sigma^*$, $Z=Z^*$ for simplicity of calculation, then we have

$$R_t-R_t^*=(\lambda_t^{\mathrm{f}}-\lambda_t^{\mathrm{h}})[E(e^Z-1)+E(e^{-Z}-1)].$$

Since $e^Z-1-Z\geq 0$ for all Z, $R_t-R_t^*$ loads negatively on $\lambda_t^{\mathrm{h}}-\lambda_t^{\mathrm{f}}$. When the jump intensity of money supply at home is higher than abroad, home investors prefer to save due to the precautionary saving, thus the equilibrium home interest rate is lower than its foreign counterpart.

If we assume $\lambda_t^{\mathrm{h}}>\lambda_t^{\mathrm{f}}$, then R_t will be smaller than R_t^* according to the above formula of interest rate differential. Then, a typical currency carry trade is a trading strategy that involves borrowing one unit of the home currency at a low interest rate, converting the borrowed amount into the foreign currency and lending it out at the foreign interest rate, converting the total earnings back to the home currency at last. We calculate the return of the above currency carry trade

$$E[R_t^s]=R_t^*-R_t+E[d\ln S_t/dt]=(\lambda_t^{\mathrm{h}}-\lambda_t^{\mathrm{f}})E(e^{-Z}-1+Z).$$

$E[R_t^s]$ is the expected return from the currency carry trade, and it loads positively on $\lambda_t^{\mathrm{h}}-\lambda_t^{\mathrm{f}}$. The investors have arbitrage opportunities due to the different levels of jump intensities of money supplies in two countries. $\bar{\lambda}^{\mathrm{h}}=\bar{\lambda}^{\mathrm{f}}$ tends to hold during a long time period empirically as in Ref. [12], so $E[R_t^s]$ tends to zero according to the above formula. The observation is consistent with the empirical results in Ref. [3].

4. VALUATION OF CURRENCY OPTIONS

In this section, we will consider the closed-form solution of the equilibrium currency option. The nominal price of a call currency option C is a function of τ, λ^{h}, λ^{f}, λ^{g}, S. And K is the strike price at time T, $\tau=T-t$.

$$C(t,\tau;\lambda^{\mathrm{h}},\lambda^{\mathrm{f}},\lambda^{\mathrm{g}},S)=E_t\left[e^{-\rho\tau}\frac{m_t}{m_T}(S_T-K,0)^+\right]. \qquad \textbf{(14)}$$

THEOREM 2: *Denote $L = \ln S$ and under the conditions of Assumptions 1–3, the value of a currency option satisfies the following PIDE*:

$$
\begin{aligned}
&-\frac{\partial C}{\partial \tau}+\left(\mu_s-\frac{3}{2}\sigma^2-\frac{1}{2}\sigma^{*2}+2\sigma\sigma^*\rho_m\right)\frac{\partial C}{\partial L}+k(\bar{\lambda}^{\mathrm{h}}-\lambda^{\mathrm{h}})\frac{\partial C}{\partial \lambda^{\mathrm{h}}}\\
&+k(\bar{\lambda}^{\mathrm{f}}-\lambda^{\mathrm{f}})\frac{\partial C}{\partial \lambda^{\mathrm{f}}}+k(\bar{\lambda}^{\mathrm{g}}-\lambda^{\mathrm{g}})\frac{\partial C}{\partial \lambda^{\mathrm{g}}}\\
&+\frac{1}{2}(\sigma^2+\sigma^{*2}-2\sigma\sigma^*\rho_m)\frac{\partial^2 C}{\partial L^2}+\frac{1}{2}\sigma_\lambda^2\lambda^{\mathrm{h}}\frac{\partial^2 C}{\partial {\lambda^{\mathrm{h}}}^2}+\frac{1}{2}\sigma_\lambda^2\lambda^{\mathrm{f}}\frac{\partial^2 C}{\partial {\lambda^{\mathrm{f}}}^2}+\frac{1}{2}\sigma_\lambda^2\lambda^{\mathrm{g}}\frac{\partial^2 C}{\partial {\lambda^{\mathrm{g}}}^2}\\
&+(\lambda^{\mathrm{h}}+\lambda^{\mathrm{g}})E[e^{-Z}(C(t,\tau;\lambda^{\mathrm{h}},\lambda^{\mathrm{f}},\lambda^{\mathrm{g}},L+Z)-C(t,\tau;\lambda^{\mathrm{h}},\lambda^{\mathrm{f}},\lambda^{\mathrm{g}},L))]\\
&+\lambda^{\mathrm{f}}E[C(t,\tau;\lambda^{\mathrm{h}},\lambda^{\mathrm{f}},\lambda^{\mathrm{g}},L-Z^*)-C(t,\tau;\lambda^{\mathrm{h}},\lambda^{\mathrm{f}},\lambda^{\mathrm{g}},L)]\\
&+\lambda^{\mathrm{g}}E[e^{-Z}(C(t,\tau;\lambda^{\mathrm{h}},\lambda^{\mathrm{f}},\lambda^{\mathrm{g}},L-Z^*)-C(t,\tau;\lambda^{\mathrm{h}},\lambda^{\mathrm{f}},\lambda^{\mathrm{g}},L))]-RC=0,
\end{aligned}
\qquad \mathbf{(15)}
$$

with the boundary condition: $C(t, 0; \lambda^{\mathrm{h}}, \lambda^{\mathrm{f}}, \lambda^{\mathrm{g}}, L) = (\exp(L_T) - K, 0)^+$.

PROOF: Applying Ito's lemma with respect to $C(t, \tau; \lambda^{\mathrm{h}}, \lambda^{\mathrm{f}}, \lambda^{\mathrm{g}}, L)$,

$$
\begin{aligned}
dC_t=&\frac{\partial C}{\partial t}dt+\frac{\partial C}{\partial L}\mu_s\,dt+\frac{\partial C}{\partial L}\sigma\,dB_t-\frac{\partial C}{\partial L}\sigma^* dB_t^*+\frac{\partial C}{\partial \lambda^{\mathrm{h}}}k(\bar{\lambda}^{\mathrm{h}}-\lambda^{\mathrm{h}})\,dt+\frac{\partial C}{\partial \lambda^{\mathrm{h}}}\sigma_\lambda\sqrt{\lambda^{\mathrm{h}}}\,dB_t^{\mathrm{h}}\\
&+\frac{\partial C}{\partial \lambda^{\mathrm{f}}}k(\bar{\lambda}^{\mathrm{f}}-\lambda^{\mathrm{f}})\,dt+\frac{\partial C}{\partial \lambda^{\mathrm{f}}}\sigma_\lambda\sqrt{\lambda^{\mathrm{f}}}\,dB_t^{\mathrm{f}}+\frac{\partial C}{\partial \lambda^{\mathrm{g}}}k(\bar{\lambda}^{\mathrm{g}}-\lambda^{\mathrm{g}})\,dt+\frac{\partial C}{\partial \lambda^{\mathrm{g}}}\sigma_\lambda\sqrt{\lambda^{\mathrm{g}}}\,dB_t^{\mathrm{g}}\\
&+\frac{1}{2}(\sigma^2+\sigma^{*2}-2\sigma\sigma^*\rho_m)\left(\frac{\partial^2 C}{\partial L^2}-\frac{\partial C}{\partial L}\right)dt+\frac{1}{2}\frac{\partial^2 C}{\partial {\lambda^{\mathrm{h}}}^2}\sigma_\lambda^2\lambda^{\mathrm{h}}\,dt\\
&+\frac{1}{2}\frac{\partial^2 C}{\partial {\lambda^{\mathrm{f}}}^2}\sigma_\lambda^2\lambda^{\mathrm{f}}\,dt+\frac{1}{2}\frac{\partial^2 C}{\partial {\lambda^{\mathrm{g}}}^2}\sigma_\lambda^2\lambda^{\mathrm{g}}\,dt\\
&+[C(t,\tau;\lambda^{\mathrm{h}},\lambda^{\mathrm{f}},\lambda^{\mathrm{g}},L+Z)-C(t,\tau;\lambda^{\mathrm{h}},\lambda^{\mathrm{f}},\lambda^{\mathrm{g}},L)](dN_t^{\mathrm{h}}+dN_t^{\mathrm{g}})\\
&+[C(t,\tau;\lambda^{\mathrm{h}},\lambda^{\mathrm{f}},\lambda^{\mathrm{g}},L-Z^*)-C(t,\tau;\lambda^{\mathrm{h}},\lambda^{\mathrm{f}},\lambda^{\mathrm{g}},L)](dN_t^{\mathrm{f}}+dN_t^{\mathrm{g}}),
\end{aligned}
$$

then substitute the discredited form ΔC_t into the following equation:

$$
E_t\left[e^{-\rho\Delta t}\frac{m_t}{m_{t+\Delta t}}\left(\frac{\Delta C_t}{C_t}-R_t\Delta t\right)\right]=0,
$$

similar to Theorem 1, we let $\Delta t \to 0$ and can obtain the PIDE of (15). ■

THEOREM 3: *A closed-form solution of Equation (15) is given by the following equation*:

$$
\begin{aligned}
C(t,\tau;\lambda^{\mathrm{h}},\lambda^{\mathrm{f}},\lambda^{\mathrm{g}},L)=&\,e^L B^*(t,\tau;\lambda^{\mathrm{f}},\lambda^{\mathrm{g}})\Pi_1(t,\tau;\lambda^{\mathrm{h}},\lambda^{\mathrm{f}},\lambda^{\mathrm{g}},L)\\
&-KB(t,\tau;\lambda^{\mathrm{h}},\lambda^{\mathrm{g}})\Pi_2(t,\tau;\lambda^{\mathrm{h}},\lambda^{\mathrm{f}},\lambda^{\mathrm{g}},L),
\end{aligned}
\qquad \mathbf{(16)}
$$

where the "risk-neutral" probabilities are given as in Refs. [2, 14]:

$$
\Pi_j(t,\tau;\lambda^{\mathrm{h}},\lambda^{\mathrm{f}},\lambda^{\mathrm{g}},L)=\frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi\ln[K]}f_j(t,\tau,\lambda^{\mathrm{h}},\lambda^{\mathrm{f}},\lambda^{\mathrm{g}},L;\phi)}{i\phi}\right]d\phi,\quad (j=1,2),
\qquad \mathbf{(17)}
$$

here, f_j is the characteristic function of Π_j.

$$\begin{aligned} f_1 = \exp\Bigg\{ & \frac{2[E(e^{i\phi Z}-1) - i\phi E(e^Z - 1)](1-e^{-\xi_h \tau})}{2\xi_h - (\xi_h - k)(1-e^{-\xi_h\tau})}\lambda_t^h \\ & + \frac{2[E(e^{-(1+i\phi)Z^*}-1) + (1+i\phi)E(e^{Z^*}-1)](1-e^{-\xi_f\tau})}{2\xi_f - (\xi_f - k)(1-e^{-\xi_f\tau})}\lambda_t^f \\ & + \frac{\begin{array}{c}2[E(e^{i\phi Z}-1) + E(e^{-(i\phi+1)Z^* - Z} - e^{-Z}) - i\phi E(e^Z - 1) \\ + (1+i\phi)E(e^{Z^*}-1)](1-e^{-\xi_g\tau})\end{array}}{2\xi_g - (\xi_g - k)(1-e^{-\xi_g\tau})}\lambda_t^g \\ & - \frac{k\bar{\lambda}^h}{\sigma_\lambda^2}\left[2\ln\left(1 - \frac{(\xi_h - k)(1-e^{-\xi_h\tau})}{2\xi_h}\right) + (\xi_h - k)\tau\right] \\ & - \frac{k\bar{\lambda}^f}{\sigma_\lambda^2}\left[2\ln\left(1 - \frac{(\xi_f - k)(1-e^{-\xi_f\tau})}{2\xi_f}\right) + (\xi_f - k)\tau\right] \\ & - \frac{k\bar{\lambda}^g}{\sigma_\lambda^2}\left[2\ln\left(1 - \frac{(\xi_g - k)(1-e^{-\xi_g\tau})}{2\xi_g}\right) + (\xi_g - k)\tau\right] \\ & - (\rho + \mu - \sigma^{*2})\tau + (\mu - \mu^*)(1+i\phi)\tau + \sigma\sigma^*\rho_m(\phi^2 - i\phi)\tau \\ & + \frac{1}{2}\sigma^{*2}(3i\phi - \phi^2)\tau - \frac{1}{2}\sigma^2(i\phi + \phi^2)\tau + i\phi L_t - \ln B^*_{t,\tau}\Bigg\}, \end{aligned} \tag{18}$$

$$\begin{aligned} f_2 = \exp\Bigg\{ & \frac{2[E(e^{(i\phi-1)Z}-1) - (i\phi - 1)E(e^Z - 1)](1-e^{-\xi_h^* \tau})}{2\xi_h^* - (\xi_h^* - k)(1-e^{-\xi_h^*\tau})}\lambda_t^h \\ & + \frac{2[E(e^{-i\phi Z^*}-1) + i\phi E(e^{Z^*}-1)](1-e^{-\xi_f^*\tau})}{2\xi_f^* - (\xi_f^* - k)(1-e^{-\xi_f^*\tau})}\lambda_t^f \\ & + \frac{\begin{array}{c}2[E(e^{(i\phi-1)Z}-1) + E(e^{-i\phi Z^* - Z} - e^{-Z}) \\ + (1-i\phi)E(e^Z - 1) + i\phi E(e^{Z^*}-1)](1-e^{-\xi_g^*\tau})\end{array}}{2\xi_g^* - (\xi_g^* - k)(1-e^{-\xi_g^*\tau})}\lambda_t^g \\ & - \frac{k\bar{\lambda}^h}{\sigma_\lambda^2}\left[2\ln\left(1 - \frac{(\xi_h^* - k)(1-e^{-\xi_h^*\tau})}{2\xi_h^*}\right) + (\xi_h^* - k)\tau\right] \\ & - \frac{k\bar{\lambda}^f}{\sigma_\lambda^2}\left[2\ln\left(1 - \frac{(\xi_f^* - k)(1-e^{-\xi_f^*\tau})}{2\xi_f^*}\right) + (\xi_f^* - k)\tau\right] \\ & - \frac{k\bar{\lambda}^g}{\sigma_\lambda^2}\left[2\ln\left(1 - \frac{(\xi_g^* - k)(1-e^{-\xi_g^*\tau})}{2\xi_g^*}\right) + (\xi_g^* - k)\tau\right] \\ & - (\rho + \mu - \sigma^2)\tau + i\phi(\mu - \mu^*)\tau + \sigma\sigma^*\rho_m(\phi^2 + i\phi)\tau \\ & + \frac{1}{2}\sigma^{*2}(i\phi - \phi^2)\tau - \frac{1}{2}\sigma^2(3i\phi + \phi^2)\tau + i\phi L_t - \ln B_{t,\tau}\Bigg\}, \end{aligned} \tag{19}$$

where

$$\xi_h = \{k^2 - 2\sigma_\lambda^2[E(e^{i\phi Z}-1) - i\phi E(e^Z - 1)]\}^{1/2},$$

$$\xi_f = \{k^2 - 2\sigma_\lambda^2[E(e^{-(1+i\phi)Z^*}-1) + (1+i\phi)E(e^{Z^*}-1)]\}^{1/2},$$

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$$\xi_{\rm g} = \{k^2 - 2\sigma_\lambda^2[E(e^{i\phi Z}-1) + E(e^{-(i\phi+1)Z^*-Z} - e^{-Z}) \\ - i\phi E(e^Z - 1) + (1+i\phi)E(e^{Z^*}-1)]\}^{1/2},$$

$$\xi_{\rm h}^* = \{k^2 - 2\sigma_\lambda^2[E(e^{(i\phi-1)Z}-1) - (i\phi-1)E(e^Z-1)]\}^{1/2},$$

$$\xi_{\rm f}^* = \{k^2 - 2\sigma_\lambda^2[E(e^{-i\phi Z^*}-1) + i\phi E(e^{Z^*}-1)]\}^{1/2},$$

$$\xi_{\rm g}^* = \{k^2 - 2\sigma_\lambda^2[E(e^{(i\phi-1)Z}-1) + E(e^{-i\phi Z^*-Z} - e^{-Z}) \\ + (1-i\phi)E(e^Z-1) + i\phi E(e^{Z^*}-1)]\}^{1/2}.$$

Proof: Similar to [16] and [2], by analogy with the Black–Scholes formula, we choose a solution as the following form (Λ is a set of parameters $\{\lambda^{\rm h}, \lambda^{\rm f}, \lambda^{\rm g}, L\}$),

$$C(t,\tau;\Lambda) = e^L B^*(t,\tau;\lambda^{\rm f},\lambda^{\rm g})\Pi_1(t,\tau;\Lambda) - KB(t,\tau;\lambda^{\rm h},\lambda^{\rm g})\Pi_2(t,\tau;\Lambda), \tag{20}$$

According to the definition of an option, Π_1 and Π_2 are subject to the terminal condition:

$$\Pi_j(t,\tau;\Lambda) = \text{Prob}(L \geq \ln K), \quad j = 1,2. \tag{21}$$

Although it is difficult to compute Π_j directly, we can compute the characteristic function f_j of Π_j. Let $\hat f_1 = B^* f_1$ and $\hat f_2 = B f_2$. First, we insert (20) into (15), then with the F-K theorem, we can get the following two PIDEs of $\hat f_j$:

$$- \frac{\partial \hat f_1}{\partial \tau} + \left(\mu_s - \frac{3}{2}\sigma^2 - \frac{1}{2}\sigma^{*2} + 2\sigma\sigma^*\rho_m\right)\left(\hat f_1 + \frac{\partial \hat f_1}{\partial L}\right) + k(\bar\lambda^{\rm h} - \lambda^{\rm h})\frac{\partial \hat f_1}{\partial \lambda^{\rm h}} \\ + k(\bar\lambda^{\rm f} - \lambda^{\rm f})\frac{\partial \hat f_1}{\partial \lambda^{\rm f}} + k(\bar\lambda^{\rm g} - \lambda^{\rm g})\frac{\partial \hat f_1}{\partial \lambda^{\rm g}} + \frac{1}{2}(\sigma^2 + \sigma^{*2} - 2\sigma\sigma^*\rho_m)\left(\frac{\partial^2 \hat f_1}{\partial L^2} + 2\frac{\partial \hat f_1}{\partial L} + \hat f_1\right) \\ + \frac{1}{2}\sigma_\lambda^2\lambda^{\rm h}\frac{\partial^2 \hat f_1}{\partial {\lambda^{\rm h}}^2} + \frac{1}{2}\sigma_\lambda^2\lambda^{\rm f}\frac{\partial^2 \hat f_1}{\partial {\lambda^{\rm f}}^2} + \frac{1}{2}\sigma_\lambda^2\lambda^{\rm g}\frac{\partial^2 \hat f_1}{\partial {\lambda^{\rm g}}^2} \\ + (\lambda^{\rm h} + \lambda^{\rm g})E[\hat f_1(t,\tau;\lambda^{\rm h},\lambda^{\rm f},\lambda^{\rm g},L+Z) - e^{-Z}\hat f_1(t,\tau;\lambda^{\rm h},\lambda^{\rm f},\lambda^{\rm g},L)] \\ + \lambda^{\rm f}E[e^{-Z^*}\hat f_1(t,\tau;\lambda^{\rm h},\lambda^{\rm f},\lambda^{\rm g},L-Z^*) - \hat f_1(t,\tau;\lambda^{\rm h},\lambda^{\rm f},\lambda^{\rm g},L)] \\ + \lambda^{\rm g}E[e^{-Z-Z^*}\hat f_1(t,\tau;\lambda^{\rm h},\lambda^{\rm f},\lambda^{\rm g},L-Z^*) - e^{-Z}\hat f_1(t,\tau;\lambda^{\rm h},\lambda^{\rm f},\lambda^{\rm g},L)] - R\hat f_1 = 0, \tag{22}$$

with the terminal condition $\hat f_1(t,0,\Lambda;\phi) = E(e^{i\phi L})$, and

$$- \frac{\partial \hat f_2}{\partial \tau} + \left(\mu_s - \frac{3}{2}\sigma^2 - \frac{1}{2}\sigma^{*2} + 2\sigma\sigma^*\rho_m\right)\frac{\partial \hat f_2}{\partial L} + k(\bar\lambda^{\rm h} - \lambda^{\rm h})\frac{\partial \hat f_2}{\partial \lambda^{\rm h}} \\ + k(\bar\lambda^{\rm f} - \lambda^{\rm f})\frac{\partial \hat f_2}{\partial \lambda^{\rm f}} + k(\bar\lambda^{\rm g} - \lambda^{\rm g})\frac{\partial \hat f_2}{\partial \lambda^{\rm g}} \\ + \frac{1}{2}(\sigma^2 + \sigma^{*2} - 2\sigma\sigma^*\rho_m)\frac{\partial^2 \hat f_2}{\partial L^2} + \frac{1}{2}\sigma_\lambda^2\lambda^{\rm h}\frac{\partial^2 \hat f_2}{\partial {\lambda^{\rm h}}^2} + \frac{1}{2}\sigma_\lambda^2\lambda^{\rm f}\frac{\partial^2 \hat f_2}{\partial {\lambda^{\rm f}}^2} + \frac{1}{2}\sigma_\lambda^2\lambda^{\rm g}\frac{\partial^2 \hat f_2}{\partial {\lambda^{\rm g}}^2} \\ + (\lambda^{\rm h} + \lambda^{\rm g})E[e^{-Z}(\hat f_2(t,\tau;\lambda^{\rm h},\lambda^{\rm f},\lambda^{\rm g},L+Z) - \hat f_2(t,\tau;\lambda^{\rm h},\lambda^{\rm f},\lambda^{\rm g},L))] \\ + \lambda^{\rm f}E[\hat f_2(t,\tau;\lambda^{\rm h},\lambda^{\rm f},\lambda^{\rm g},L-Z^*) - \hat f_2(t,\tau;\lambda^{\rm h},\lambda^{\rm f},\lambda^{\rm g},L)] \\ + \lambda^{\rm g}E[e^{-Z}(\hat f_2(t,\tau;\lambda^{\rm h},\lambda^{\rm f},\lambda^{\rm g},L-Z^*) - \hat f_2(t,\tau;\lambda^{\rm h},\lambda^{\rm f},\lambda^{\rm g},L))] - R\hat f_2 = 0, \tag{23}$$

with the terminal condition $\hat f_2(t,0,\Lambda;\phi) = E(e^{i\phi L})$.

Since the coefficients are linear with respect to λ^{h}, λ^{f}, λ^{g}, we follow the idea of [1,16], combine the terminal condition, and choose the form of $\hat{f}_1$:

$$\hat{f}_1(t,\tau,\Lambda;\phi) = \exp[U(\tau) + X_{\mathrm{h}}(\tau)\lambda^{\mathrm{h}} + X_{\mathrm{f}}(\tau)\lambda^{\mathrm{f}} + X_{\mathrm{g}}(\tau)\lambda^{\mathrm{g}} + i\phi L]. \quad \mathbf{(24)}$$

Inserting (24) into (22), we can get four ODEs as follows:

$$\begin{aligned}
&-\frac{\partial X_{\mathrm{h}}(\tau)}{\partial\tau} - kX_{\mathrm{h}}(\tau) + \frac{1}{2}\sigma_\lambda^2 {X_{\mathrm{h}}}^2(\tau) + E[e^{i\phi Z} - 1] - i\phi E[e^{Z} - 1] = 0,\\
&-\frac{\partial X_{\mathrm{f}}(\tau)}{\partial\tau} - kX_{\mathrm{f}}(\tau) + \frac{1}{2}\sigma_\lambda^2 {X_{\mathrm{f}}}^2(\tau) + E[e^{-i\phi Z^* - Z^*} - 1] + (1+i\phi)E[e^{Z^*} - 1] = 0,\\
&-\frac{\partial X_{\mathrm{g}}(\tau)}{\partial\tau} - kX_{\mathrm{g}}(\tau) + \frac{1}{2}\sigma_\lambda^2 {X_{\mathrm{g}}}^2(\tau) + E[e^{i\phi Z} - 1] + E[e^{-i\phi Z^* - Z^* - Z} - e^{-Z}]\\
&\quad - i\phi E[e^{Z} - 1] + (1+i\phi)E[e^{Z^*} - 1] = 0,\\
&-\frac{\partial U(\tau)}{\partial\tau} + k\bar{\lambda}^{\mathrm{h}} X_{\mathrm{h}}(\tau) + k\bar{\lambda}^{\mathrm{f}} X_{\mathrm{f}}(\tau) + k\bar{\lambda}^{\mathrm{g}} X_{\mathrm{g}}(\tau) - \rho - \mu + {\sigma^*}^2 + (\mu - \mu^*)(1+i\phi)\\
&\quad + \sigma\sigma^*\rho_m(\phi^2 - i\phi) + \frac{1}{2}{\sigma^*}^2(3i\phi - \phi^2) - \frac{1}{2}\sigma^2(i\phi + \phi^2) = 0,
\end{aligned}$$

with the initial conditions $X_{\mathrm{h}}(0) = X_{\mathrm{f}}(0) = X_{\mathrm{g}}(0) = U(0) = 0$. $X_{\mathrm{h}}(\tau)$, $X_{\mathrm{f}}(\tau)$, $X_{\mathrm{g}}(\tau)$, $U(\tau)$ have analytic solutions as expressed in (18). Using the same approach, we choose:

$$\hat{f}_2(t,\tau,\Lambda;\phi) = \exp[V(\tau) + Y_{\mathrm{h}}(\tau)\lambda^{\mathrm{h}} + Y_{\mathrm{f}}(\tau)\lambda^{\mathrm{f}} + Y_{\mathrm{g}}(\tau)\lambda^{\mathrm{g}} + i\phi L]. \quad \mathbf{(25)}$$

Inserting (24) into (22), we can get four ODEs as follows:

$$\begin{aligned}
&-\frac{\partial Y_{\mathrm{h}}(\tau)}{\partial\tau} - kY_{\mathrm{h}}(\tau) + \frac{1}{2}\sigma_\lambda^2 {Y_{\mathrm{h}}}^2(\tau) + E[e^{i\phi Z - Z} - 1] - (i\phi - 1)E[e^{Z} - 1] = 0,\\
&-\frac{\partial Y_{\mathrm{f}}(\tau)}{\partial\tau} - kY_{\mathrm{f}}(\tau) + \frac{1}{2}\sigma_\lambda^2 {Y_{\mathrm{f}}}^2(\tau) + E[e^{-i\phi Z^*} - 1] + i\phi E[e^{Z^*} - 1] = 0,\\
&-\frac{\partial Y_{\mathrm{g}}(\tau)}{\partial\tau} - kY_{\mathrm{g}}(\tau) + \frac{1}{2}\sigma_\lambda^2 {Y_{\mathrm{g}}}^2(\tau) + E[e^{i\phi Z - Z} - 1] + E[e^{-i\phi Z^* - Z} - e^{-Z}]\\
&\quad + (1 - i\phi)E[e^{Z} - 1] + i\phi E[e^{Z^*} - 1] = 0,\\
&-\frac{\partial V(\tau)}{\partial\tau} + k\bar{\lambda}^{\mathrm{h}} Y_{\mathrm{h}}(\tau) + k\bar{\lambda}^{\mathrm{f}} Y_{\mathrm{f}}(\tau) + k\bar{\lambda}^{\mathrm{g}} Y_{\mathrm{g}}(\tau) - \rho - \mu + \sigma^2 + i\phi(\mu - \mu^*)\\
&\quad + \sigma\sigma^*\rho_m(\phi^2 + i\phi) + \frac{1}{2}{\sigma^*}^2(i\phi - \phi^2) - \frac{1}{2}\sigma^2(3i\phi + \phi^2) = 0,
\end{aligned}$$

with the initial conditions $Y_{\mathrm{h}}(0) = Y_{\mathrm{f}}(0) = Y_{\mathrm{g}}(0) = V(0) = 0$. $Y_{\mathrm{h}}(\tau)$, $Y_{\mathrm{f}}(\tau)$, $Y_{\mathrm{g}}(\tau)$, $V(\tau)$ have analytic solutions as expressed in (18). ■

5. NUMERICAL ANALYSIS

In this section, we consider to do Monte Carlo simulations to verify the correctness of Formula (16). We compute the integrals in "risk-neutral" probabilities in (17) by the Gauss–Kronrod method with the function "quadgk" in matlab (R2014b). Then, in the procedure of the Monte Carlo simulations following, we set the number of time step to be 100 and run 200,000 simulations to compute the call option prices. In the following numerical

TABLE 1. Comparison of European option prices in our model computed with Formula (16) and with the Monte Carlo method. Parameter values are $Z = Z^* = -0.1$, $k = 0.5$, $\sigma_\lambda = 0.5$, $\bar{\lambda}^{\mathrm{h}} = \bar{\lambda}^{\mathrm{f}} = \bar{\lambda}^{\mathrm{g}} = 0.1$, $\lambda_t^{\mathrm{h}} = \lambda_t^{\mathrm{f}} = \lambda_t^{\mathrm{g}} = 0.1$

Strike	Derived from formula		Monte Carlo			Standard
price	Option price	CPU time	Option price	CPU time	% Difference	error
60	39.2126	0.02	39.2445	1916	0.0814%	0.0509
70	30.3750	0.02	30.4087	2026	0.1109%	0.0503
80	22.5793	0.03	22.5998	2062	0.0908%	0.0474
90	16.1392	0.03	16.1152	2059	0.1487%	0.0427
100	11.1385	0.03	11.1425	1971	0.0359%	0.0372
110	7.4596	0.01	7.4540	2002	0.0751%	0.0311
120	4.8715	0.02	4.8705	2055	0.0205%	0.0257
130	3.1156	0.02	3.1185	1993	0.0931%	0.0205
140	1.9580	0.03	1.9560	1995	0.1021%	0.0163

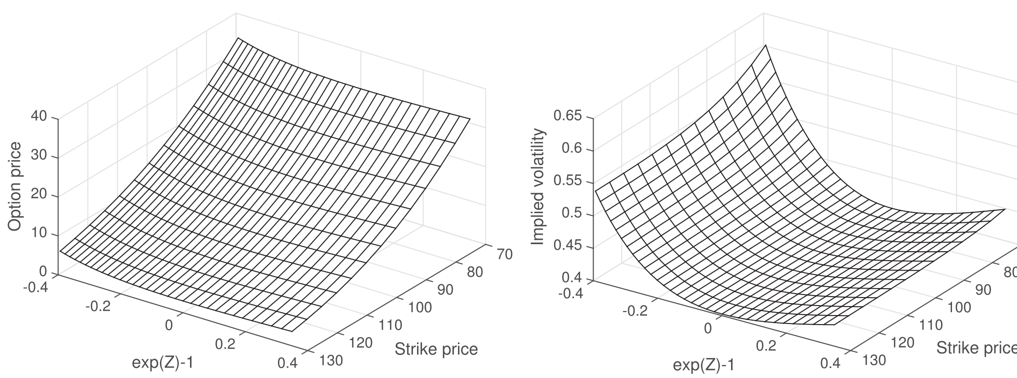

FIGURE 6. The option prices and implied volatilities with different strike price and jump amplitude $e^Z - 1$. Parameter values are $k = 0.5$, $\sigma_\lambda = 0.5$, $\bar{\lambda}^{\mathrm{h}} = \bar{\lambda}^{\mathrm{f}} = \bar{\lambda}^{\mathrm{g}} = 0.1$, $\lambda_t^{\mathrm{h}} = \lambda_t^{\mathrm{f}} = \lambda_t^{\mathrm{g}} = 0.1$.

experiments, the common parameters used are $\rho = 0.05$, $S_t = 100$, $\tau = 0.5$, $\mu = \mu^* = 0.1$, $\sigma = \sigma^* = 0.3$, and $\rho_m = 0.1$. We assume $Z = Z^*$ and denote them to be constants for simplicity of calculation. Table 1 shows the relative percentage differences are all less than 1% and sample standard errors are all less than 0.1; also, the CPU calculation time of the formula is obviously much less than that of the Monte Carlo method. These facts indicate that our derived closed-form solution formula (16) is effective. Next, we will do the numerical analysis using the closed-form solutions for call option prices. The implied volatilities are backed out from the Garman-Kohlhagen [15] formula by substituting the computed option prices into the formula.

In Figure 6, we examine the effects of jump amplitude $e^Z - 1$ on option prices and implied volatilities. The larger jump amplitude means greater potential risk, which leads to a higher option price and implied volatility. Moreover, the effect of jump amplitude on implied volatility is asymmetric. From Figure 6, we see that the impact of downward jump amplitude on implied volatility is significantly greater than that of upward jump amplitude. That is, when the absolute value of jump amplitude is equal, the implied volatility with downward jump is obviously larger than that with upward jump. This reflects that investors are more sensitive to the downward jumps than the upward jumps. Besides, if the jump

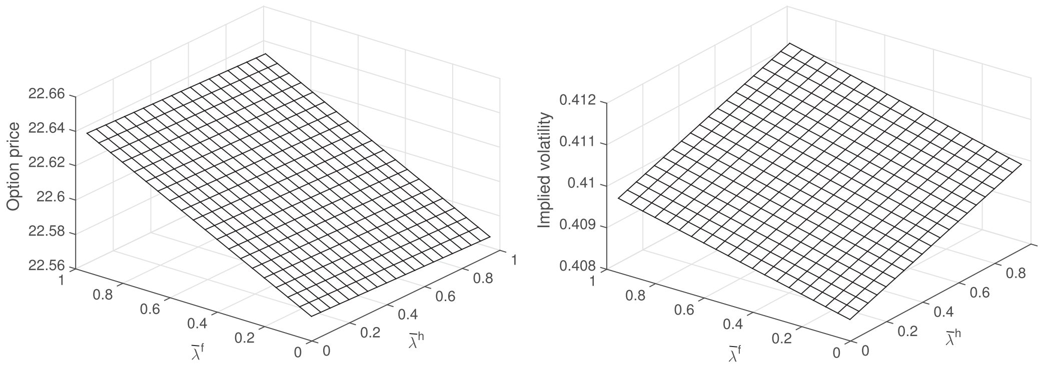

FIGURE **7.** The option prices and implied volatilities with different long-term levels of jump intensities $\bar{\lambda}^{\mathrm{h}}$ and $\bar{\lambda}^{\mathrm{f}}$. Parameter values are $Z = Z^* = -0.1$, $k = 0.5$, $\sigma_\lambda = 0.5$, $\bar{\lambda}^{\mathrm{g}} = 0.1$, $\lambda_t^{\mathrm{h}} = \lambda_t^{\mathrm{f}} = \lambda_t^{\mathrm{g}} = 0.1$.

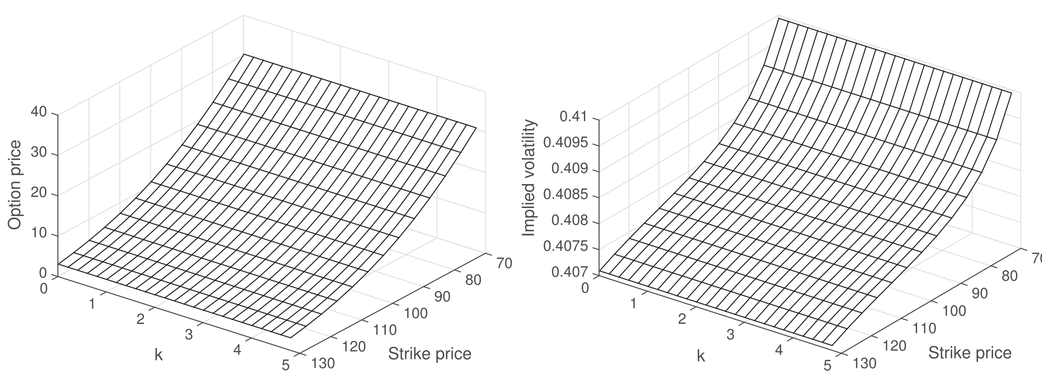

FIGURE **8.** The option prices and implied volatilities with different strike price and the mean-reverting rate of jump intensity k. Parameter values are $Z = Z^* = -0.1$, $\sigma_\lambda = 0.5$, $\bar{\lambda}^{\mathrm{h}} = \bar{\lambda}^{\mathrm{f}} = \bar{\lambda}^{\mathrm{g}} = 0.1$, $\lambda_t^{\mathrm{h}} = \lambda_t^{\mathrm{f}} = \lambda_t^{\mathrm{g}} = 0.1$.

amplitudes $e^Z - 1$ and $e^{Z^*} - 1$ are zero (i.e., $Z = Z^* = 0$), there will be no jumps and our model will become a continuous model. When there are no jumps, we find that the implied volatilities are almost constants by calculation, which verifies the endogenesis of our model.

In Figure 7, we show the effects of the long-term level of jump intensity of money supply in domestic and foreign countries on the option prices and implied volatilities. When $\bar{\lambda}^{\mathrm{h}}$ remains unchanged, the option price will increase when $\bar{\lambda}^{\mathrm{f}}$ increases. On the contrary, when $\bar{\lambda}^{\mathrm{f}}$ remains unchanged, the option prices will decrease when $\bar{\lambda}^{\mathrm{h}}$ increases. Intuitively, when $\bar{\lambda}^{\mathrm{h}}$ becomes larger, it will reduce the expected return of investing the currency option denominated in domestic currency, and so the option price will decline. However, when $\bar{\lambda}^{\mathrm{f}}$ increases, investors prefer to buy currency options denominated in domestic currency to avoid risks from foreign countries. For the implied volatility, the increase in $\bar{\lambda}^{\mathrm{h}}$ or $\bar{\lambda}^{\mathrm{f}}$ means a larger potential risk, so the implied volatility will rise.

In Figure 8, we show the effects of the mean-reverting rate of jump intensity k on option price and implied volatility. And in Figure 9, we show the effects of volatility of jump intensity σ_λ on option price and implied volatility. It can be seen from the two figures that the effects of both the mean-reverting rate k and the volatility of jump intensity σ_λ

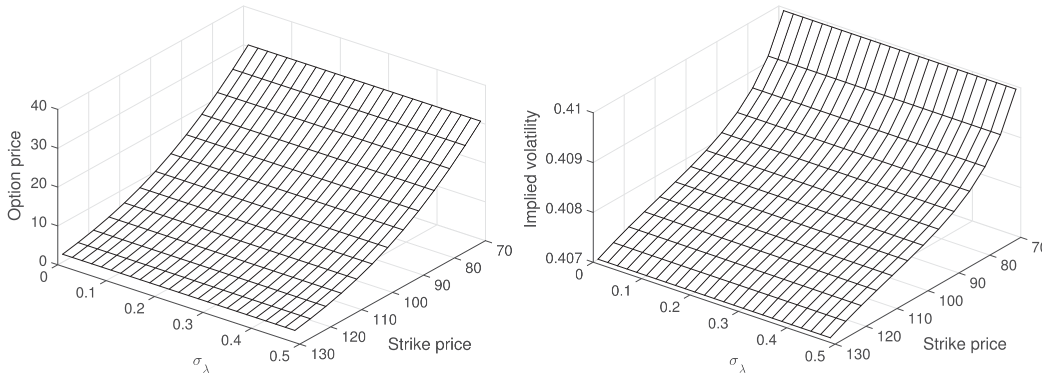

FIGURE 9. The option prices and implied volatilities with different strike price and volatility of jump intensity σ_λ. Parameter values are $Z = Z^* = -0.1$, $k = 0.5$, $\bar{\lambda}^{\mathrm{h}} = \bar{\lambda}^{\mathrm{f}} = \bar{\lambda}^{\mathrm{g}} = 0.1$, $\lambda_t^{\mathrm{h}} = \lambda_t^{\mathrm{f}} = \lambda_t^{\mathrm{g}} = 0.1$.

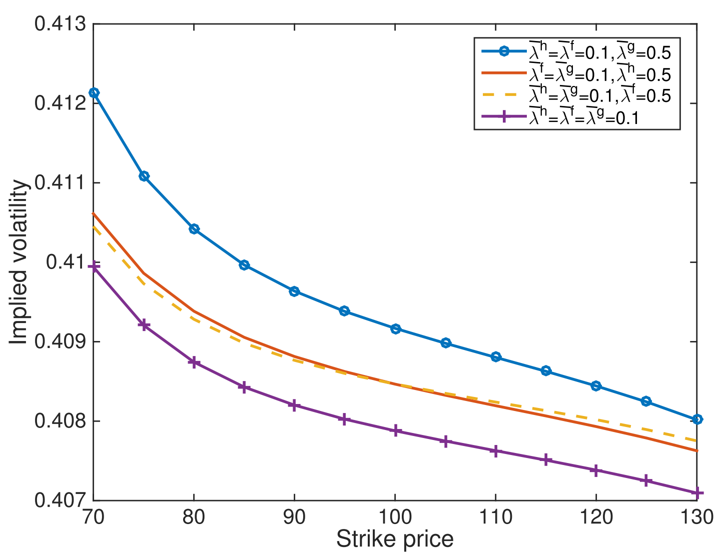

FIGURE 10. The impacts of $\bar{\lambda}^{\mathrm{h}}$, $\bar{\lambda}^{\mathrm{f}}$, and $\bar{\lambda}^{\mathrm{g}}$ on implied volatilities. Parameter values are $K = 80$, $Z = Z^* = -0.1$, $\sigma = 0.5$, $k = 0.5$, $\sigma_\lambda = 0.5$, $\lambda_t^{\mathrm{h}} = \lambda_t^{\mathrm{f}} = \lambda_t^{\mathrm{g}} = 0.1$.

are not very significant compared with the jump amplitude and the long-term level of jump intensity.

Next, we will firstly show the impact of long-term co-jump intensity level $\bar{\lambda}^{\mathrm{g}}$ on implied volatility if $\bar{\lambda}^{\mathrm{h}} = \bar{\lambda}^{\mathrm{f}}$. We find the implied volatility increases obviously when $\bar{\lambda}^{\mathrm{g}}$ rises from 0.1 to 0.5 as seen from the "+" curve and the "o" curve in Figure 10. Secondly, we compare the effects of different long-term levels of jump intensities in two countries on implied volatility. The increase of implied volatility caused by the rise of $\bar{\lambda}^{\mathrm{h}}$ or $\bar{\lambda}^{\mathrm{f}}$ is not as large as that caused by the rise of $\bar{\lambda}^{\mathrm{g}}$, which can be observed from the "-" curve, the "- -" curve, and the "o" curve in Figure 10. This indicates that $\bar{\lambda}^{\mathrm{g}}$ has a more significant impact on the implied volatility than $\bar{\lambda}^{\mathrm{h}}$ and $\bar{\lambda}^{\mathrm{f}}$. Thirdly, as seen from the "-" curve and the "- -" curve, we see that the implied volatilities are closed to each other with a single increase of $\bar{\lambda}^{\mathrm{h}}$ or $\bar{\lambda}^{\mathrm{f}}$ when $\bar{\lambda}^{\mathrm{g}}$ remains unchanged, especially when the options are at-the-money. Thus, it can

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be inferred that if the systemic risk is uncontrollable, we can stabilize our monetary policy as much as possible, which is still very meaningful for stabilizing the financial derivatives market.

6. CONCLUSION

In the setting of Lucas-type two-country economy, we price the equilibrium currency options under a discontinuous model with co-jumps. The jump parts of the money supplies in two countries can be correlated through a common jump process with stochastic jump intensity. Besides, the money supply in each country is also affected by the jump risk come from their own country. Our model has advantages in distinguishing which country the jumps in exchange rate come from. In the numerical analysis part, we do Monte Carlo simulations to verify the correctness of our closed-form solutions and discuss the influences of jump factors from home or abroad on implied volatility. The closed-form solutions help us greatly improve the computational efficiency. We find that the co-jump has a significant impact on option price and implied volatility. Moreover, our model can be extended with more general jump processes such as Lévy processes. We will concentrate ourselves on this topic in our future research.

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