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ON THE MODELLING OF IMPERFECT REPAIRS FOR A CONTINUOUSLY MONITORED GAMMA WEAR PROCESS THROUGH AGE REDUCTION

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Abstract

A continuously monitored system is considered, which is subject to accumulating deterioration modelled as a gamma process. The system fails when its degradation level exceeds a limit threshold. At failure, a delayed replacement is performed. To shorten the down period, a condition-based maintenance strategy is applied, with imperfect repair. Mimicking virtual age models used for recurrent events, imperfect repair actions are assumed to lower the system degradation through a first-order arithmetic reduction of age model. Under these assumptions, Markov renewal equations are obtained for several reliability indicators. Numerical examples illustrate the behaviour of the system.

Keywords: Imperfect maintenance; gamma process; ARA1; Markov renewal process; Markov renewal equation; delay time; maintenance policy

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1. Introduction

Most systems suffer a physical degradation before failure. A classical stochastic model to describe a nondecreasing accumulated random degradation is the gamma process. A gamma process is a stochastic process with independent, nonnegative, and gamma-distributed increments with common scale parameter. This process is suitable to model gradual damage monotonically accumulating over time in a sequence of tiny increments, such as wear, fatigue, corrosion, crack growth, etc. [17].

For deteriorating systems, when the degradation level reaches a certain level, the system is no longer able to function satisfactorily. Since it is generally less costly to replace a system before it has failed, maintenance policies based on the system condition are usually proposed, aiming at preventing failures. It has been proved that such maintenance strategies minimize the maintenance cost, improve operational safety, and reduce the quantity and severity of in-service system failures; see, for example [2], [8], and [10]. Condition-based maintenance is based on data collected online through continuous monitoring or inspections. Based on the information data, different maintenance actions are programmed. The condition of the system after a maintenance action depends on the maintenance efficiency considering two extreme cases: a minimal repair, where the condition of the system after the repair is just the same as

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before (as bad as old (ABAO)), and a perfect repair, when the condition of the system after the repair is the same as if it were new (as good as new (AGAN)). Reality lies between these two extreme cases [7]. Since Chaudhuri and Sahu [6] considered the concept of imperfect maintenance, many models have been analysed (see [5] and [14] for a review on imperfect maintenance models).

In the literature, several optimization models for a system subject to an accumulated degradation and under an imperfect maintenance scheme have been proposed. Newby and Barker [12], using the concept of partial repair given by Stadje and Zuckerman [16], described the maintenance process for a system whose state is described using a bivariate stochastic process. Castanier *et al.* [4] proposed a condition-based maintenance model where the effect of the imperfect maintenance is a random function of the observed deterioration of the system. Nicolai *et al.* [13] implemented different imperfect maintenance actions in systems whose degradation is modelled by a nonstationary gamma process. The effect of the maintenance action is twofold: on the one hand, to reduce the system degradation by a random amount and, on the other hand, to modify the structural parameters of the degradation process. The analysis of the model proposed by Nicolai *et al.* [13] is performed assuming that the effect of the imperfect maintenance actions annihilates the overshoot of the gamma process, whereas the present study takes it into account.

The modelling assumptions of the present paper are inspired by [2] and [10], where the reader may find practical justifications for them: a system is considered, subject to a cumulative gradual random deterioration modelled as an homogeneous gamma process. A perfect and continuous monitoring controls the deterioration of the system. The system fails when its degradation level exceeds the threshold L and a signal is immediately sent to the maintenance team. They take τ units of time to arrive on site, and next perform a corrective replacement. Compared to τ, this corrective replacement is short and it is considered as instantaneous. To reduce the system downtime, a preventive maintenance policy is proposed. Under this maintenance strategy, the signal is sent to the maintenance team as soon as the degradation level exceeds a preventive threshold M $(0 < M < L)$. It takes the same delay τ for the maintenance team to arrive, and maintenance actions are assumed to be instantaneous too. A major difference between the present study and [2] and [10] is that all repairs are assumed to be perfect (AGAN) in the quoted papers. Here we consider that it depends on the deterioration level at maintenance times: if the system is found to be failed or too degraded, a perfect corrective or preventive replacement is performed, accordingly. Otherwise, an imperfect repair is applied. Unlike most of the maintenance models that combine degradation processes and imperfect maintenance actions, the maintenance effect is here modelled through a first-order arithmetic reduction of age, mimicking an ARA1 model for recurrent events [7]. The maintenance efficiency is hence controlled through an Euclidian parameter ρ, allowing all situations from perfect (AGAN) to minimal (ABAO) repairs. Within such a setting, the objective of this paper is to analyse the transient behaviour of the system, which is done in the framework of semiregenerative processes with continuous space state.

The paper is structured as follows. In Section 2, the functioning of the initial system is described and the preventive maintenance policy is presented. In Section 3 we develop the mathematical formulation that describes the functioning of the system under the preventive maintenance policy explained in Section 2. Sections 4 and 5 focus on the calculus of different transient reliability measures, which are proved to fulfil Markov renewal equations. In Section 6 we present some numerical results for these reliability measures. Note that, due to the complexity of the Markov renewal equations obtained previously, all numerical computations are here performed through Monte Carlo simulations. Section 7 concludes the paper.

2. Description of the system and of the maintenance strategy

In this section we describe the initial functioning of the system and the introduction of a maintenance strategy to try to improve some performance measures of the system.

2.1. The initial system

A unitary system is considered, with intrinsic deterioration modelled by a gamma process $(X_t)_{t\geq 0}$, where X_t is gamma distributed $\Gamma(\alpha t, \beta)$ with probability distribution function (PDF)

$$f_t(x) = \frac{\beta^{\alpha t}}{\Gamma(\alpha t)} x^{\alpha t - 1} \mathrm{e}^{-\beta x}\, \mathbf{1}_{\mathbb{R}_+}(x),$$

where $\mathbf{1}_{\{\cdot\}}$ denotes the indicator function and $\alpha, \beta > 0$. The cumulative distribution function (CDF) and survival function (SF) of X_t are denoted by F_t and $\bar{F}_t$, respectively. A gamma process is also a Lévy process, with Lévy measure given by

$$\mu(\mathrm{d}s) = \alpha \frac{\mathrm{e}^{-\beta s}}{s}\, \mathbf{1}_{\mathbb{R}_+^*}(s)\, \mathrm{d}s.$$

Recalling that L is the failure threshold, with $L > 0$, the time to failure of the system is the reaching time of level L:

$$\sigma_L = \inf(t > 0\colon X_t > L).$$

At time σ_L, a signal is sent to the maintenance team which arrives at time $\sigma_L + \tau$ and instantaneously replaces the out-of-order system by a new identical system. The system is hence replaced by a new system at time $\sigma_L + \tau$ and the system is unavailable from σ_L up to $\sigma_L + \tau$.

2.2. The preventive maintenance policy

As presented in Section 1, an alert signal is preventively sent to the maintenance team as soon as the system reaches a preventive maintenance level M ($0 \leq M \leq L$), namely at time σ_M. At time $\sigma_M + \tau$, the maintenance team is ready to operate and tries to adjust the system (preventive maintenance action). Just as in an ARA1 model for recurrent events [7], a preventive maintenance (PM) action is considered to remove only some part (ρ per cent) of the age accumulated by the system since the last PM action (or since time $t = 0$), where $\rho \in (0, 1)$. The PM action tends to be perfect when ρ goes to 1 (AGAN repair) and to have no effect when ρ goes to 0 (ABAO repair). In the present situation and because of possible large jumps for a gamma process ($X_{\sigma_M} \in (M, +\infty)$ almost surely), such a PM action may however be insufficient to bring the system back to a lower level than M (details in the following). In that case and according to the previously described PM policy, a second PM action should immediately be planned, which is not coherent. We consequently consider that, in case the system deterioration level remains beyond the PM level M after a PM adjustment, the system is too deteriorated and it is preventively replaced. To sum up, there consequently are three possible actions at maintenance times:

- a corrective replacement (CR) if the system is failed when the maintenance team arrives,
- a single PM action if this PM action brings the system deterioration level below M,
- a PM action and a preventive replacement (PR) if the PM action does not succeed in bringing the system deterioration level below M.

All the maintenance actions are considered to be instantaneous.

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To specifically describe the PM policy, we shall make use of independent copies of $(X_t)_{t\geq 0}$, denoted by $(X_t^{(n)})_{t\geq 0}$ for $n = 1, 2, \dots$. Corresponding reaching times of the thresholds L and M are denoted by $\sigma_L^{(n)}$ and $\sigma_M^{(n)}$, respectively, for $n = 1, 2, \dots$, and we set $(Y_t)_{t\geq 0}$ to be the process describing the evolution of the maintained system.

Let $U_1 = S_1 = \sigma_M^{(1)} + \tau$ be the time of the first maintenance action. We then have $Y_t = X_t^{(1)}$ for all $t < S_1$. At time S_1, different cases are possible.

Case (i): $X_{U_1}^{(1)} > L$. The system failed before S_1. An instantaneous CR takes place at time $S_1 = \sigma_M^{(1)} + \tau$. We then set $Y_{S_1} = 0$.

Case (ii): $X_{U_1}^{(1)} \leq L$. A PM action puts the system back to its deterioration level at time $(1-\rho)U_1$, which is $X_{(1-\rho)U_1}^{(1)}$.

Subcase (ii.1): $X_{(1-\rho)U_1}^{(1)} > M$. The system is considered to be unmaintainable and it is replaced by a new system (PR action) at time S_1; hence, $Y_{S_1} = 0$.

Subcase (ii.2): $X_{(1-\rho)U_1}^{(1)} \leq M$. The system deterioration level after the PM action is $Y_{S_1} = X_{(1-\rho)U_1}^{(1)}$.

Starting from Y_{S_1} after the first maintenance action, the evolution of the system is assumed to be independent of $(Y_t)_{t<S_1}$ and is modelled by $(X_t^{(2)})_{t\geq 0}$ up to the second maintenance action. The reaching time of level M is then

$$\inf(t > S_1 : Y_{S_1} + X_{t-S_1}^{(2)} > M) = S_1 + \sigma_{M-Y_{S_1}}^{(2)}.$$

A second maintenance action is then planned at time $S_2 = S_1 + U_2$, with $U_2 = \sigma_{M-Y_{S_1}}^{(2)} + \tau$.

More generally, assume that $S_1, \dots, S_{n-1}$ and $(Y_t)_{t\leq S_{n-1}}$ to be constructed, with $n \geq 2$. Let $U_n = \sigma_{M-Y_{S_{n-1}}}^{(n)} + \tau$ and $S_n = S_{n-1} + U_n$. We first set $Y_t = Y_{S_{n-1}} + X_{t-S_{n-1}}^{(1)}$ for all $S_{n-1} < t < S_n$, and, consequently, $Y_{S_n^-} = Y_{S_{n-1}} + X_{U_n}^{(n)}$ (almost surely).

Case (a): $Y_{S_n^-} > L$. The system failed before S_n; hence, $Y_{S_n} = 0$.

Case (b): $Y_{S_n^-} \leq L$. A PM action puts the system back to the deterioration level $Y_{S_{n-1}} + X_{(1-\rho)U_n}^{(n)}$.

Subcase (b.1): $Y_{S_{n-1}} + X_{(1-\rho)U_n}^{(n)} > M$. The system is unmaintainable and it is replaced by a new system at time S_n; hence, $Y_{S_n} = 0$.

Subcase (b.2): $Y_{S_{n-1}} + X_{(1-\rho)U_n}^{(n)} < M$. The system deterioration level after the PM action is $Y_{S_n = Y_{S_{n-1}} + X_{(1-\rho)U_n}^{(n)}}$.

A new maintenance action is next planned at time $S_{n+1} = S_n + U_{n+1}$, with $U_{n+1} = \sigma_{M-Y_{S_n}}^{(n+1)} + \tau$. After a maintenance action at time S_n, the future evolution of the maintained system $(Y_t)_{t\geq S_n}$ depends on the past $(Y_t)_{t\leq S_n}$ only through Y_{S_n}, and the process $(Y_t)_{t\geq 0}$ appears as a semiregenerative process with underlying Markov renewal process $(S_n, Y_{S_n})_{n\in\mathbb{N}}$ and interarrival times U_n; see [1]. Note that the sequence $(S_n, (Y_{S_n}, Y_{S_n^-}))_{n\in\mathbb{N}}$ is also a Markov renewal process, which will be used later on for obtaining the Markov renewal equations for both reliability and cost functions.

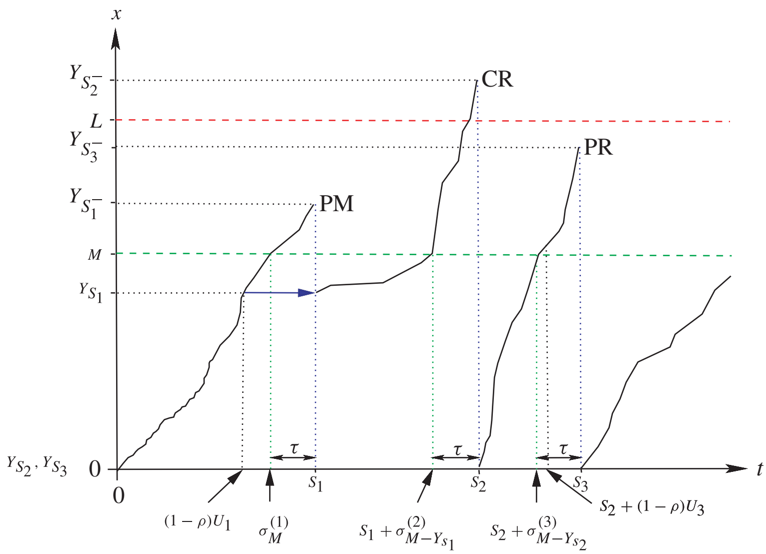

FIGURE 1: The condition-based maintenance policy.

This age-based maintenance policy is illustrated in Figure 1. At the end of the first semicycle, a PM action puts the system back to $Y_{S_1} = X^{(1)}_{(1-\rho)U_1} < M$. At the end of the second semicycle, the system is failed and a CR leads to $Y_{S_2} = 0$. At the end of the third semicycle, a PM action puts the system back to $Y_{S_2} + X^{(3)}_{(1-\rho)U_3} \geq M$ and a PR leads to $Y_{S_3} = 0$.

In case M goes to 0^+, the signal is immediately sent to the maintenance team after a maintenance action. The next maintenance action is hence always performed after the same delay τ. Besides, at each maintenance time the system is either failed or unmaintainable. Maintenance policy is hence reduced to periodic (corrective or preventive) replacements of the system with period τ.

If M tends to L^-, maintenance policy is reduced to perform corrective replacement actions after a delay τ.

Finally, when ρ tends to 0^+, the ABAO maintenance operation leads to a system replacement and, therefore, leads to an AGAN repair.

3. Markov renewal process

The aim of this section is to obtain the kernel of the Markov renewal process

$$(S_n, (Y_{S_n}, Y_{S_n^-}))_{n\in\mathbb{N}},$$

namely the kernel $(q(x, \mathrm{d}s, \mathrm{d}y, \mathrm{d}z))_{x\in[0,M]}$ defined by

$$\begin{aligned} q(x, \mathrm{d}s, \mathrm{d}y, \mathrm{d}z) &= \mathbb{P}(S_1 \in \mathrm{d}s,\ Y_{S_1} \in \mathrm{d}y,\ Y_{S_1^-} \in \mathrm{d}z \mid Y_0 = x) \\ &= \mathbb{P}_x(S_1 \in \mathrm{d}s,\ Y_{S_1} \in \mathrm{d}y,\ Y_{S_1^-} \in \mathrm{d}z) \quad \text{for all } x \in [0, M], \end{aligned}$$

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where $\mathbb{P}_x$ denotes the conditional probability given $Y_0 = x$ (and $\mathbb{E}_x$ denotes the conditional expectation). With this notation, we recall that

$$\begin{aligned}\mathbb{P}(S_n \in \mathrm{d}s,\ Y_{S_n} \in \mathrm{d}y,\ Y_{S_n^-} \in \mathrm{d}z \mid \sigma(S_1, \ldots, S_{n-1}, Y_{S_1}, \ldots, Y_{S_{n-1}}))\\ = \mathbb{P}(S_n \in \mathrm{d}s,\ Y_{S_n} \in \mathrm{d}y,\ Y_{S_n^-} \in \mathrm{d}z \mid Y_{S_{n-1}})\\ = q(Y_{S_{n-1}}, \mathrm{d}s, \mathrm{d}y, \mathrm{d}z)\end{aligned}$$

for all $n \geq 1$, where $\sigma(A)$ stands for the σ-field generated by A, where A is any set of random variables. To obtain the kernel, we first deal with the PDF of $(S_1, X_{(1-\rho)S_1}, X_{S_1})$.

Proposition 1. *The PDF of* $(S_1, X_{(1-\rho)S_1}, X_{S_1})$ *is* $u^M(t, u, v)$*, where*

- *if* $\tau < t < \tau/\rho$ *and* $M < u < v$,

$$u^M(t, u, v) = f_{\rho t}(v - u) \int_M^{+\infty} f_{\tau-\rho t}(u - w)\bigg(\int_{w-M}^{+\infty} f_{t-\tau}(w - s)\mu(\mathrm{d}s)\bigg)\,\mathrm{d}w, \quad (1)$$

- *if* $t > \tau/\rho$ *and* $u < M < v$,

$$u^M(t, u, v) = f_{(1-\rho)t}(u) \int_{M-u}^{+\infty} f_\tau(v-u-w)\bigg(\int_{w-(M-u)}^{+\infty} f_{\rho t-\tau}(w-s)\mu(\mathrm{d}s)\bigg)\,\mathrm{d}w, \quad (2)$$

- $u^M(t, u, v) = 0$ *elsewhere.*

Proof. Setting φ to be any measurable and nonnegative function, we have to compute

$$\mathbb{E}[\varphi(S_1, X_{(1-\rho)S_1}, X_{S_1})] = \mathbb{E}[\varphi(\sigma_M + \tau, X_{(1-\rho)(\sigma_M+\tau)}, X_{\sigma_M+\tau})].$$

We first divide this expression according to whether $(1 - \rho)(\sigma_M + \tau)$ is greater or smaller than σ_M, or, equivalently, according to whether $(1 - \rho)\tau$ is greater or smaller than $\rho\sigma_M$, and we write

$$\mathbb{E}[\varphi(S_1, X_{(1-\rho)S_1}, X_{S_1})] = I_1(\varphi) + I_2(\varphi),$$

with

$$\begin{aligned}I_1(\varphi) = \mathbb{E}[\varphi(\sigma_M + \tau, X_{(1-\rho)(\sigma_M+\tau)}, X_{\sigma_M+\tau})\,\mathbf{1}_{\{(1-\rho)\tau>\rho\sigma_M\}}],\\ I_2(\varphi) = \mathbb{E}[\varphi(\sigma_M + \tau, X_{(1-\rho)(\sigma_M+\tau)}, X_{\sigma_M+\tau})\,\mathbf{1}_{\{(1-\rho)\tau<\rho\sigma_M\}}].\end{aligned}$$

The first term is equal to:

$$I_1(\varphi) = \sum_{r\geq 0} \mathbf{1}_{\{(1-\rho)\tau>\rho r\}}\,\mathbb{E}[\varphi(r + \tau, X_{(1-\rho)(r+\tau)}, X_{r+\tau})\,\mathbf{1}_{\{X_{r^-}\leq M<X_r\}}]. \quad (3)$$

Setting $\mathcal{F}_u = \sigma(X_s,\ 0 \leq s \leq u)$ for all $u \geq 0$, let us first note that $\{X_{r^-} \leq M < X_r\}$ belongs to $\mathcal{F}_{(1-\rho)(r+\tau)}$ for each r such that $(1-\rho)\tau > \rho r$ (because $(1-\rho)(r+\tau) > r$). By conditioning on $\mathcal{F}_{(1-\rho)(r+\tau)}$, writing $X_{r+\tau} = X_{(1-\rho)(r+\tau)} + (X_{r+\tau} - X_{(1-\rho)(r+\tau)})$, and using the Markov property and the independent and homogeneous increments of $(X_t)_{t\geq 0}$, we obtain

$$\mathbb{E}[\varphi(r + \tau, X_{(1-\rho)(r+\tau)}, X_{r+\tau})\,\mathbf{1}_{\{X_{r^-}\leq M<X_r\}}] = \mathbb{E}[\mathbf{1}_{\{X_{r^-}\leq M<X_r\}}\,g(X_{(1-\rho)(r+\tau)})],$$

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where

$$\begin{aligned} g(x) &= \mathbb{E}[\varphi(r+\tau, x, x + X_{r+\tau} - X_{(1-\rho)(r+\tau)})] \\ &= \mathbb{E}[\varphi(r+\tau, x, x + X_{\rho(r+\tau)})] \\ &= \int_{\mathbb{R}_+} \varphi(r+\tau, x, x+z) f_{\rho(r+\tau)}(z)\,\mathrm{d}z. \end{aligned}$$

This yields

$$\begin{aligned} &\mathbb{E}[\varphi(r+\tau, X_{(1-\rho)(r+\tau)}, X_{r+\tau})\,\mathbf{1}_{\{X_{r^-} \le M < X_r\}}] \\ &\quad = \int_{\mathbb{R}_+} \mathbb{E}[\mathbf{1}_{\{X_{r^-} \le M < X_r\}}\,\varphi(r+\tau, X_{(1-\rho)(r+\tau)}, X_{(1-\rho)(r+\tau)} + z)] f_{\rho(r+\tau)}(z)\,\mathrm{d}z. \end{aligned} \tag{4}$$

Conditioning on $\mathcal{F}_r$ and writing $X_{(1-\rho)(r+\tau)} = X_r + (X_{(1-\rho)(r+\tau)} - X_r)$, we derive

$$\begin{aligned} &\mathbb{E}[\mathbf{1}_{\{X_{r^-} \le M < X_r\}}\,\varphi(r+\tau, X_{(1-\rho)(r+\tau)}, X_{(1-\rho)(r+\tau)} + z)] \\ &\quad = \int_{\mathbb{R}_+} \mathbb{E}[\mathbf{1}_{\{X_{r^-} \le M < X_r\}}\,\varphi(r+\tau, X_r + y, X_r + y + z)] f_{(1-\rho)\tau-\rho r}(y)\,\mathrm{d}y \end{aligned}$$

in the same way, noting that $X_{(1-\rho)(r+\tau)} - X_r$ is identically distributed as $X_{(1-\rho)\tau-\rho r}$. Substituting this expression successively into (4) and next into (3), we obtain

$$\begin{aligned} I_1(\varphi) = \iint_{\mathbb{R}_+^2} \mathrm{d}y\,\mathrm{d}z \sum_{r\ge 0} f_{(1-\rho)\tau-\rho r}(y) f_{\rho(r+\tau)}(z) \\ \times \mathbf{1}_{\{(1-\rho)\tau > \rho r\}}\,\mathbb{E}[\mathbf{1}_{\{X_{r^-} \le M < X_r\}}\,\varphi(r+\tau, X_r + y, X_r + y + z)]. \end{aligned}$$

Following the same arguments as those used in Proposition 2 of [3, p. 76] and setting $\Delta X_r = X_r - X_{r^-}$, we obtain

$$\begin{aligned} I_1(\varphi) &= \iint_{\mathbb{R}_+^2} \mathrm{d}y\,\mathrm{d}z \sum_{r\ge 0} \mathbf{1}_{\{(1-\rho)\tau > \rho r\}}\, f_{(1-\rho)\tau-\rho r}(y) f_{\rho(r+\tau)}(z) \\ &\quad \times \mathbb{E}[\mathbf{1}_{\{X_{r^-} \le M < X_{r^-} + \Delta X_r\}}\,\varphi(r+\tau, X_{r^-} + \Delta X_r + y, X_{r^-} + \Delta X_r + y + z)] \\ &= \int_0^{((1-\rho)/\rho)\tau} \mathrm{d}r \iiint_{\mathbb{R}_+^3} \mathrm{d}y\,\mathrm{d}z\,\mu(\mathrm{d}s)\, f_{(1-\rho)\tau-\rho r}(y) f_{\rho(r+\tau)}(z) \\ &\quad \times \mathbb{E}[\mathbf{1}_{\{X_{r^-} \le M < X_{r^-} + s\}}\,\varphi(r+\tau, X_{r^-} + s + y, X_{r^-} + s + y + z)], \end{aligned}$$

due to the compensation formula. Almost-sure continuity of $(X_r)_{r\ge 0}$ allows us to substitute X_r to X_{r^-} into the previous formula. This yields

$$\begin{aligned} I_1(\varphi) = \int_0^{((1-\rho)/\rho)\tau} \mathrm{d}r \iiiint_{\mathbb{R}_+^4} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mu(\mathrm{d}s)\,\mathbf{1}_{\{x \le M < x+s\}} \\ \times \varphi(r+\tau, x+s+y, x+s+y+z) f_{(1-\rho)\tau-\rho r}(y) f_{\rho(r+\tau)}(z) f_r(x). \end{aligned}$$

By setting $t = r + \tau$, $u = x + s + y$, $v = x + s + y + z$, and $w = x + s$, and keeping s unchanged, we obtain

$$\begin{aligned} I_1(\varphi) = \int_\tau^{\tau/\rho} \mathrm{d}t \iiiint_{\mathbb{R}_+^4} \mathrm{d}x\,\mathrm{d}u\,\mathrm{d}v\,\mu(\mathrm{d}s)\,\mathbf{1}_{\{w-s \le M < w\}} \\ \times \varphi(t, u, v) f_{\tau-\rho t}(u-w) f_{\rho t}(v-u) f_{t-\tau}(w-s). \end{aligned}$$

This gives (1) for $u^M(t, u, v)$ in the case of $\tau < t < \tau/\rho$ (and $M < u < v$). For the second term, we have

$$I_2(\varphi) = \sum_{r \geq 0} \mathbf{1}_{\{(1-\rho)\tau < \rho r\}} \mathbb{E}[\varphi(r + \tau, X_{(1-\rho)(r+\tau)}, X_{r+\tau}) \mathbf{1}_{\{X_{r^-} \leq M < X_r\}}].$$

Conditioning on $\mathcal{F}_r$ in the expectation and writing $X_{r+\tau} = X_r + (X_{r+\tau} - X_r)$, we obtain

$$\begin{aligned} I_2(\varphi) = &\int_{\mathbb{R}_+} f_\tau(z) \, \mathrm{d}z \\ &\times \sum_{r \geq 0} \mathbf{1}_{\{(1-\rho)\tau < \rho r\}} \mathbb{E}[\varphi(r + \tau, X_{(1-\rho)(r+\tau)}, X_r + z) \mathbf{1}_{\{X_{r^-} \leq M < X_r\}}], \end{aligned}$$

because $(1 - \rho)(r + \tau) < r$. Setting $X_r = X_{r^-} + \Delta X_r$, using the compensation formula and substituting X_{r^-} by X_r in a next step, we obtain

$$\begin{aligned} I_2(\varphi) = &\iint_{\mathbb{R}_+^2} f_\tau(z) \, \mathrm{d}z \, \mu(\mathrm{d}s) \int_{((1-\rho)/\rho)\tau}^{+\infty} \mathrm{d}r \\ &\times \mathbb{E}[\varphi(r + \tau, X_{(1-\rho)(r+\tau)}, X_r + s + z) \mathbf{1}_{\{X_r \leq M < X_r + s\}}]. \end{aligned}$$

Conditioning on $\mathcal{F}_{(1-\rho)(r+\tau)}$, writing $X_r = X_{(1-\rho)(r+\tau)} + (X_r - X_{(1-\rho)(r+\tau)})$, and using the fact that $X_r - X_{(1-\rho)(r+\tau)}$ is identically distributed as $X_{\rho r-(1-\rho)\tau}$, we obtain

$$\begin{aligned} I_2(\varphi) &= \iiiint_{\mathbb{R}_+^4} f_\tau(z) \, \mathrm{d}z \, \mu(\mathrm{d}s) \, \mathrm{d}u \, \mathrm{d}y \int_{((1-\rho)/\rho)\tau}^{+\infty} \mathrm{d}r \\ &\quad \times \varphi(r + \tau, u, u + y + s + z) \mathbf{1}_{\{u+y \leq M < u+y+s\}} f_{(1-\rho)(r+\tau)}(u) f_{\rho r-(1-\rho)\tau}(y) \\ &= \iiiint_{\mathbb{R}_+^4} f_\tau(v - u - w) \, \mathrm{d}w \, \mu(\mathrm{d}s) \, \mathrm{d}u \, \mathrm{d}v \int_{\tau/\rho}^{+\infty} \mathrm{d}t \\ &\quad \times \varphi(t, u, v) \mathbf{1}_{\{w-s \leq M-u < w\}} f_{(1-\rho)t}(u) f_{\rho t-\tau}(w - s), \end{aligned}$$

with $t = r + \tau$, $v = u + y + s + z$, $w = y + s$, and (u, s) unchanged. This yields (2) for $u^M(t, u, v)$ in the case $t > \tau/\rho$ (and $u < M < v$).

Remark 1. Using the fact that the PDF of (σ_M, X_{σ_M}) is

$$f_{(\sigma_M, X_{\sigma_M})}(u, y) = \int_{y-M}^{+\infty} f_u(y - s) \mu(\mathrm{d}s)$$

for all $y > M$ and all $u > 0$ (see [3]), the function $u^M(t, u, v)$ may be written as

$$u^M(t, u, v) = f_{\rho t}(v - u) \int_M^{+\infty} f_{\tau-\rho t}(u - w) f_{(\sigma_M, X_{\sigma_M})}(t - \tau, w) \, \mathrm{d}w$$

if $\tau < t < \tau/\rho$ and $M < u < v$. This corresponds to some kind of intuitive result: roughly speaking, at time $\sigma_M = t - \tau$, the process $(X_r)_{r \geq 0}$ reaches level $w > M$. Next, on the time interval $(t - \tau, (1 - \rho)t]$ with length $\tau - \rho t$, the level is increased by $u - w$ units and the process reaches level u at time $(1 - \rho)t$. Finally, on the time interval $((1 - \rho)t, t]$ with length ρt, the

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level is increased by $v-u$ units and the process reaches level v at time t. In the case of $t > \tau/\rho$ and $u < M < v$, we obtain

$$u^M(t,u,v) = f_{(1-\rho)t}(u)\int_{M-u}^{+\infty} f_\tau(v-u-w)\, f_{(\sigma_{M-u},X_{\sigma_{M-u}})}(\rho t-\tau, w)\,\mathrm{d}w,$$

which may be interpreted in the same way: on the interval $(0,(1-\rho)t]$, the level is increased by u units (with $u < M$). Next, starting from level u, it takes $\rho t-\tau$ time units for the process to exceed level $M-u$ with a level increment of w units in the meantime (and $w > M-u$). At time $(1-\rho)t+\rho t-\tau = t-\tau$, the level is therefore $u+w$. Finally, on the time interval $(t-\tau, t]$ with length τ, the level is increased by $v-u-w$ units and the process reaches level v at time t.

We are now able to provide the kernel of the Markov renewal process $(S_n,(Y_{S_n},Y_{S_n^-}))_{n\in\mathbb{N}}$.

Theorem 1. *The kernel* $(q(x,\mathrm{d}s,\mathrm{d}y,\mathrm{d}z))_{x\in[0,M]}$ *of the Markov renewal process*

$$(S_n,(Y_{S_n},Y_{S_n^-}))_{n\in\mathbb{N}}$$

is provided by

$$\begin{aligned} q(x,\mathrm{d}s,\mathrm{d}y,\mathrm{d}z) &= \mathbf{1}_{\{s>\tau\}}\,\mathbf{1}_{\{y\le M<z\le L\}}\,u^{M-x}(s,y-x,z-x)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}s \\ &\quad + \mathbf{1}_{\{s>\tau\}}\,q_x(s,z)\delta_0(\mathrm{d}y)\,\mathrm{d}z\,\mathrm{d}s \end{aligned} \tag{5}$$

for all $x\in[0,M]$, *where* u^M *is provided by Proposition 1 and*

$$\begin{aligned} q_x(s,z) &= \mathbf{1}_{\{L<z\}}\bigg(\int_0^{z-x} u^{M-x}(s,w,z-x)\,\mathrm{d}w\bigg) \\ &\quad + \mathbf{1}_{\{M<z\le L\}}\bigg(\int_{M-x}^{z-x} u^{M-x}(s,w,z-x)\,\mathrm{d}w\bigg). \end{aligned} \tag{6}$$

The first term on the right-hand side of (5) stands for the PM case, and the two terms on the right-hand side of (6) stand for the CR and PR cases, respectively.

Proof. Given that $Y_0 = x$, we set $S^x = S_1 = \tau+\sigma_{M-x}$. This yields $Y_{S_1^-} = x + X_{S^x}$ and

$$Y_{S_1} = \begin{cases} 0 & \text{if } X_{S^x} > L-x, \\ 0 & \text{if } X_{S^x} \le L-x \text{ and } X_{(1-\rho)S^x} > M-x, \\ x + X_{(1-\rho)S^x} & \text{if } X_{S^x} \le L-x \text{ and } X_{(1-\rho)S^x} \le M-x. \end{cases}$$

For all φ measurable and nonnegative, we therefore have

$$\mathbb{E}_x(\varphi(S_1,Y_{S_1},Y_{S_1^-})) = J_1(x)+J_2(x),$$

with

$$\begin{aligned} J_1(x) &= \mathbb{E}[\varphi(S^x, x+X_{(1-\rho)S^x}, x+X_{S^x})\,\mathbf{1}_{\{X_{S^x}\le L-x,\,X_{(1-\rho)S^x}\le M-x\}}], \\ J_2(x) &= \mathbb{E}[\varphi(S^x,0,x+X_{S^x})(\mathbf{1}_{\{X_{S^x}>L-x\}}+\mathbf{1}_{\{X_{S^x}\le L-x,\,M-x<X_{(1-\rho)S^x}\}})]. \end{aligned}$$

Using Proposition 1 with M substituted by $M-x$, we derive

$$J_1(x) = \iiint_{\mathbb{R}_+^3} \varphi(s, x+u, x+v) u^{M-x}(s, u, v) \mathbf{1}_{\{v \le L-x, u \le M-x\}} \, \mathrm{d}u \, \mathrm{d}v \, \mathrm{d}s$$
$$= \iiint_{\mathbb{R}_+^3} \varphi(s, y, z) \, \mathbf{1}_{\{y \le M < z \le L\}} \, u^{M-x}(s, y-x, z-x) \, \mathrm{d}y \, \mathrm{d}z \, \mathrm{d}s,$$

where $y = x+u$, $z = x+v$, and

$$J_2(x) = \iiint_{\mathbb{R}_+^3} \varphi(s, 0, x+v) u^{M-x}(s, u, v) (\mathbf{1}_{\{L-x<v\}} + \mathbf{1}_{\{v \le L-x, M-x<u\}}) \, \mathrm{d}u \, \mathrm{d}v \, \mathrm{d}s$$
$$= \iint_{\mathbb{R}_+^2} \varphi(s, 0, z) \, \mathbf{1}_{\{L<z\}} \left(\int_x^z u^{M-x}(s, y-x, z-x) \, \mathrm{d}y \right) \mathrm{d}z \, \mathrm{d}s$$
$$+ \iint_{\mathbb{R}_+^2} \varphi(s, 0, z) \, \mathbf{1}_{\{M<z \le L\}} \left(\int_M^z u^{M-x}(s, y-x, z-x) \, \mathrm{d}y \right) \mathrm{d}z \, \mathrm{d}s,$$

which yields the result.

We finally derive the kernel of the Markov renewal process $(S_n, Y_{S_n})_{n \in \mathbb{N}}$.

Corollary 1. *The kernel* $(\bar{q}(x, \mathrm{d}s, \mathrm{d}y))_{x \in [0,M]}$ *of the Markov renewal process* $(S_n, Y_{S_n})_{n \in \mathbb{N}}$ *is given by*

$$\bar{q}(x, \mathrm{d}s, \mathrm{d}y) = \mathbf{1}_{\{s>\tau\}} \, \mathrm{d}s \{ \mathbf{1}_{\{y \le M\}} \, H_x(s, y) \, \mathrm{d}y + \delta_0(\mathrm{d}y)(I_x(s) + D_x(s)) \}$$

for all $x \in [0, M]$, *where*

$$H_x(s, y) = \int_{M-x}^{L-x} u^{M-x}(s, y-x, v) \, \mathrm{d}v \quad \textit{(PM case)}, \tag{7}$$

$$D_x(s) = \int_{L-x}^{+\infty} \mathrm{d}z \left(\int_0^z u^{M-x}(s, w, z) \, \mathrm{d}w \right) \quad \textit{(CR case)}, \tag{8}$$

$$I_x(s) = \int_{M-x}^{L-x} \mathrm{d}z \left(\int_{M-x}^z u^{M-x}(s, y, z) \, \mathrm{d}y \right) \quad \textit{(PR case)}. \tag{9}$$

Proof. We have

$$\bar{q}(x, \mathrm{d}s, \mathrm{d}y) = \int q(x, \mathrm{d}s, \mathrm{d}y, \mathrm{d}z)$$
$$= \mathbf{1}_{\{s>\tau\}} \, \mathbf{1}_{\{y \le M\}} \, \mathrm{d}s \, \mathrm{d}y \int_M^L u^{M-x}(s, y-x, z-x) \, \mathrm{d}z$$
$$+ \mathbf{1}_{\{s>\tau\}} \, \delta_0(\mathrm{d}y) \, \mathrm{d}s \int_M^{+\infty} q_x(s, z) \, \mathrm{d}z$$
$$= \mathbf{1}_{\{s>\tau\}} \, \mathrm{d}s \left\{ \mathbf{1}_{\{y \le M\}} \, H_x(s, y) \, \mathrm{d}y + \delta_0(\mathrm{d}y) \int_M^{+\infty} q_x(s, z) \, \mathrm{d}z \right\}$$

with

$$\int_M^{+\infty} q_x(s, z) \, \mathrm{d}z = \int_M^L \mathrm{d}z \left(\int_M^z u^{M-x}(s, w-x, z-x) \, \mathrm{d}w \right)$$
$$+ \int_L^{+\infty} \mathrm{d}z \left(\int_x^z u^{M-x}(s, w-x, z-x) \, \mathrm{d}w \right)$$
$$= I_x(s) + D_x(s),$$

and the result holds.

4. The reliability and availability functions

Let $R_x(t)$ be the reliability function of the maintained system at time t, namely the conditional probability that the system has been functioning from time $t = 0$ up to time t without any interruption given that it started from $Y_0 = x$ with $x \in [0, M]$, i.e.

$$R_x(t) = \mathbb{P}_x(T > t),$$

where T is the time to failure of the maintained system and $t \in \mathbb{R}_+$.

As $S_1 = \tau + \sigma_{M-x} > \tau$, let us first remark that, if $t \leq \tau$ then $t < S_1$ and there is no maintenance action on $[0, t]$. In that case, $Y_u = X_u$ on $[0, t]$ and we simply obtain

$$R_x(t) = \mathbb{P}(\sigma_{L-x} > t) = \mathbb{P}(X_t < L - x) = F_t(L - x)$$

for all $t \leq \tau$. We next consider the case in which $t > \tau$.

Theorem 2. *The reliability function fulfils the Markov renewal equation*

$$\begin{aligned} R_x(t) &= G_x(t) + \int_\tau^t \int_0^M R_y(t-s) H_x(s, y)\, \mathrm{d}s\, \mathrm{d}y + \int_\tau^t R_0(t-s) I_x(s)\, \mathrm{d}s \\ &= G_x(t) + \int_\tau^t \int_0^M R_y(t-s) \nu_x(\mathrm{d}s, \mathrm{d}y) \end{aligned}$$

for all $t > \tau$ and $x \in [0, M]$, where

$$G_x(t) = \int_0^{M-x} f_{t-\tau}(y) F_\tau(L - x - y)\, \mathrm{d}y \tag{10}$$

for all $t > \tau$ and $x \in [0, M]$, and

$$\nu_x(\mathrm{d}s, \mathrm{d}y) = [H_x(s, y)\, \mathrm{d}y + I_x(s)\delta_0(\mathrm{d}y)]\, \mathrm{d}s,$$

with H_x and I_x as in (7) and (9).

Proof. Let $t > \tau$. We have

$$R_x(t) = \mathbb{P}_x(T > t, S_1 > t) + \mathbb{P}_x(T > t, S_1 \leq t), \tag{11}$$

with

$$\begin{aligned} \mathbb{P}_x(T > t, S_1 > t) &= \mathbb{P}(X_t \leq L - x, \tau + \sigma_{M-x} > t) \\ &= \mathbb{P}(X_t \leq L - x, X_{t-\tau} \leq M - x) \\ &= \int_0^{M-x} f_{t-\tau}(y) F_\tau(L - x - y)\, \mathrm{d}y \\ &= G_x(t), \end{aligned} \tag{12}$$

by conditioning with respect of $X_{t-\tau}$. We also have

$$\begin{aligned} \mathbb{P}_x(T > t, S_1 \leq t) &= \mathbb{E}_x[\mathbf{1}_{\{S_1 \leq t\}} \mathbf{1}_{\{T > S_1\}} \mathbb{E}_x(\mathbf{1}_{\{T > t\}} \mid \sigma(Y_t, t \leq S_1))] \\ &= \mathbb{E}_x[\mathbf{1}_{\{S_1 \leq t\}} \mathbf{1}_{\{Y_{S_1}^- \leq L\}} R_{Y_{S_1}}(t - S_1)] \\ &= \iint_{[\tau,t]\times[0,M]} R_y(t-s) \int_{z=0}^{z=L} q(x, \mathrm{d}s, \mathrm{d}y, \mathrm{d}z), \end{aligned} \tag{13}$$

where $q(x, \mathrm{d}s, \mathrm{d}y, \mathrm{d}z)$ is given in Theorem 1. Using similar arguments as those used in the proof of Corollary 1, we obtain

$$\int_{z=0}^{z=L} q(x, \mathrm{d}s, \mathrm{d}y, \mathrm{d}z) = \{\mathbf{1}_{\{y \le M\}} H_x(s, y)\,\mathrm{d}y + \delta_0(\mathrm{d}y) I_x(s)\}\,\mathrm{d}s, \tag{14}$$

which yields the result upon substituting (14) into (13) followed by (12) and (13) into (11).

We now deal with the availability function of the maintained system at time t, namely with the conditional probability that the system is working at time t given that it started from $Y_0 = x$, with $x \in [0, M]$:

$$A_x(t) = \mathbb{P}_x(Y_t < L).$$

In the case $t \le \tau (\le S_1)$, both the reliability and availability functions coincide, i.e.

$$A_x(t) = R_x(t) = F_t(L - x)$$

for all $t \le \tau$. In the case $t > \tau$, we may write

$$A_x(t) = \mathbb{P}_x(Y_t < L, S_1 > t) + \mathbb{E}_x[\mathbf{1}_{\{S_1 \le t\}} A_{Y_{S_1}}(t - S_1)]$$

in a similar way to Theorem 2, which yields the following corollary.

Corollary 2. *The availability function fulfils the Markov renewal equation*

$$A_x(t) = G_x(t) + \int_\tau^t \int_0^M A_y(t - s)\bar{q}(x, \mathrm{d}s, \mathrm{d}y)$$

for all $t > \tau$ and all $x \in [0, M]$, where $G_x(t)$ is given by (10) and $\bar{q}$ is given by Corollary 1.

5. The expected cost function

Let $c_x(t)$ be the mean cumulated cost on $(0, t]$ given that $Y_0 = x$ with $x \in [0, M]$, that is,

$$c_x(t) = \mathbb{E}_x[C((0, t])],$$

where $C((0, t])$ denotes the maintenance cost in $(0, t]$. We calculate $c_x(t)$, taking into account the following costs for the different maintenance actions: c_{CR}, the CR cost, c_{PR}, the PR cost, c_{PM}, the PM cost, and c_{d}, the downtime cost per unit time.

For $t \le \tau$, again using $Y_u = X_u$ in $[0, t]$, we obtain

$$c_x(t) = c_{\mathrm{d}}\mathbb{E}[(t - \sigma_{L-x})^+] = c_{\mathrm{d}} \int_0^t \mathbb{P}(t - u > \sigma_{L-x})\,\mathrm{d}u = c_{\mathrm{d}} \int_0^t \bar{F}_{t-u}(L - x)\,\mathrm{d}u,$$

where $(t - \sigma_{L-x})^+ = \max(t - \sigma_{L-x}, 0)$ stands for the (possible) downtime on $[0, t]$.

We next consider the case in which $t > \tau$, with

$$c_x(t) = \mathbb{E}_x[C((0, t])\,\mathbf{1}_{\{S_1 > t\}}] + \mathbb{E}_x[C((0, t])\,\mathbf{1}_{\{S_1 \le t\}}]. \tag{15}$$

The first term in (15) is dealt with in the following lemma.

Lemma 1. *For $t > \tau$, we have*

$$\mathbb{E}_x[C((0,t])\,\mathbf{1}_{\{S_1>t\}}] = c_{\mathrm{d}} K_x(t)$$

for all $x \in [0, M]$, with

$$K_x(t) = \int_0^\tau \alpha(t-\tau, t-u, M-x, L-x)\,\mathrm{d}u \quad \textit{for all } x \in [0, M],$$

$$\alpha(t_1, t_2, M, L) = \int_0^M f_{\min(t_1,t_2)}(z)\bar{F}_{|t_1-t_2|}(L-z)\,\mathrm{d}z \quad \textit{for all } t_1, t_2 \geq 0. \tag{16}$$

Proof. Using a similar method to that used in the proof of Proposition 3 of [11], we have

$$\begin{aligned}\mathbb{E}_x[C((0,t])\,\mathbf{1}_{\{S_1>t\}}] &= c_{\mathrm{d}}\mathbb{E}[(t-\sigma_{L-x})^+\,\mathbf{1}_{\{\sigma_{M-x}+\tau>t\}}] \\ &= c_{\mathrm{d}}\mathbb{E}\bigg[\mathbf{1}_{\{\sigma_{M-x}+\tau>t\}}\int_0^{+\infty}\mathbf{1}_{\{(t-\sigma_{L-x})^+>u\}}\,\mathrm{d}u\bigg] \\ &= c_{\mathrm{d}}\int_0^\tau \mathbb{P}[\sigma_{M-x} > t-\tau,\ t-u > \sigma_{L-x}]\,\mathrm{d}u,\end{aligned}$$

because $\sigma_{M-x} > t-\tau$ and $t-u > \sigma_{L-x}$ imply that $u \leq \tau$. Now, for $u \leq \tau$, by conditioning with respect to $\sigma(X_s, s \leq t-\tau)$, we obtain

$$\begin{aligned}\mathbb{P}[\sigma_{M-x} > t-\tau,\ t-u > \sigma_{L-x}] &= \mathbb{P}[X_{t-\tau} < M-x,\ X_{t-u} \geq L-x] \\ &= \int_0^{M-x} f_{t-\tau}(y)\bar{F}_{\tau-u}(L-x-y)\,\mathrm{d}y \\ &= \alpha(t-\tau, t-u, M-x, L-x),\end{aligned}$$

which yields the result.

For the calculus of the second part of (15), we shall need the following technical lemma.

Lemma 2. *For fixed $t > \tau$, in the case $S_1 \leq t$, the conditional expected downtime given $Y_0 = x$ on the first semicycle is*

$$\mathbb{E}_x[(S_1 - \sigma_L)^+\,\mathbf{1}_{\{S_1 \leq t\}}] = W_x(t)$$

for all $x \in [0, M]$, where $W_x(t)$ is given by

$$W_x(t) = W_{1,x}(t)\,\mathbf{1}_{\{t<2\tau\}} + W_{2,x}(t)\,\mathbf{1}_{\{t\geq 2\tau\}}$$

for $t > \tau$ and $x \in [0, M]$, with

$$\begin{aligned}W_{1,x}(t) &= \int_0^{t-\tau}\bar{F}_v(L-x)\,\mathrm{d}v + \int_{t-\tau}^\tau \beta(v, t-\tau, M-x, L-x)\,\mathrm{d}v \\ &\quad + \int_\tau^t\int_0^{M-x} f_{v-\tau}(y)\beta(t-v, \tau, M-x-y, L-x-y)\,\mathrm{d}y\,\mathrm{d}v, \\ W_{2,x}(t) &= \int_0^\tau \bar{F}_v(L-x)\,\mathrm{d}v + \int_\tau^{t-\tau}\alpha(v-\tau, v, M-x, L-x)\,\mathrm{d}v \\ &\quad + \int_{t-\tau}^t \mathrm{d}v\int_0^{M-x} f_{v-\tau}(y)\beta(\tau, t-v, M-x-y, L-x-y)\,\mathrm{d}y,\end{aligned}$$

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where the function $\beta(t_1, t_2, M, L)$ is given by

$$\beta(t_1, t_2, M, L) = \int_M^{+\infty} f_{\min(t_1,t_2)}(z)\bar{F}_{|t_1-t_2|}(L-z)\,\mathrm{d}z \tag{17}$$

for all $t_1, t_2 \geq 0$, and $\alpha(t_1, t_2, M, L)$ is provided by (16).

Proof. We have

$$\begin{aligned}
\mathbb{E}_x[(S_1 - \sigma_L)^+ \mathbf{1}_{\{S_1 \leq t\}}] &= \mathbb{E}[(\sigma_{M-x} + \tau - \sigma_{L-x})^+ \mathbf{1}_{\{\sigma_{M-x}+\tau \leq t\}}] \\
&= \mathbb{E}\bigg[\int_{\mathbb{R}} \mathbf{1}_{\{0<u<\sigma_{M-x}+\tau-\sigma_{L-x}\}} \mathbf{1}_{\{\sigma_{M-x}+\tau\leq t\}}\,\mathrm{d}u\bigg] \\
&= \mathbb{E}\bigg[\int_0^{+\infty} \mathbf{1}_{\{\sigma_{L-x}<v<\sigma_{M-x}+\tau\}} \mathbf{1}_{\{\sigma_{M-x}+\tau\leq t\}}\,\mathrm{d}v\bigg] \\
&= \int_0^t \lambda(v, t, \tau)\,\mathrm{d}v,
\end{aligned}$$

setting $v = \sigma_{M-x} + \tau - u$ and

$$\begin{aligned}
\lambda(v, t, \tau) &= \mathbb{P}[\sigma_{L-x} < v, v - \tau < \sigma_{M-x} \leq t - \tau] \\
&= \mathbb{P}(L - x < X_v, X_{(v-\tau)^+} \leq M - x < X_{t-\tau})
\end{aligned}$$

for all $0 \leq v \leq t$ and all $t > \tau$. We now consider how to compute $\lambda(v, t, \tau)$ for a number of different cases according to the respective ordering of v and $t - \tau$, and of v and τ. Firstly, if $t - \tau < \tau$ then $t < 2\tau$. For $t < 2\tau$, if $v \leq \tau$, we consider the cases $v < t - \tau$ and $v \geq t - \tau$. If $v < t - \tau$ then

$$\lambda(v, t, \tau) = \mathbb{P}(L - x < X_v) = \bar{F}_v(L - x).$$

If $v \geq t - \tau$, we have

$$\begin{aligned}
\lambda(v, t, \tau) &= \mathbb{P}(L - x < X_v,\ M - x < X_{t-\tau}) \\
&= \int_{M-x}^{\infty} f_{t-\tau}(y)\bar{F}_{v-(t-\tau)}(L - x - y)\,\mathrm{d}y \\
&= \beta(t - \tau, v, M - x, L - x),
\end{aligned}$$

where $\beta(t_1, t_2, M, L)$ is given by (17). For $v > \tau$,

$$\begin{aligned}
\lambda(v, t, \tau) &= \mathbb{P}(L - x < X_v,\ X_{v-\tau} < M - x < X_{t-\tau}) \\
&= \int_0^{M-x} f_{v-\tau}(y) \int_{M-x-y}^{\infty} f_{t-v}(w)\bar{F}_{\tau-(t-v)}(L - x - y - w)\,\mathrm{d}y\,\mathrm{d}w \\
&= \int_0^{M-x} f_{v-\tau}(y)\beta(t - v, \tau, M - x - y, L - x - y)\,\mathrm{d}y.
\end{aligned}$$

Hence, for $t < 2\tau$ and $x \in [0, M]$,

$$\begin{aligned}
W_x(t) = {}& \int_0^{t-\tau} \bar{F}_v(L - x)\,\mathrm{d}v + \int_{t-\tau}^{\tau} \beta(v, t - \tau, M - x, L - x)\,\mathrm{d}v \\
&+ \int_\tau^t \int_0^{M-x} f_{v-\tau}(y)\beta(t - v, \tau, M - x - y, L - x - y)\,\mathrm{d}y\,\mathrm{d}v.
\end{aligned}$$

For $t > 2\tau$, we have

$$\lambda(v, t, \tau) = \mathbb{P}(L - x < X_v) = \bar{F}_v(L - x)$$

for $v < \tau$. For $v \geq \tau$, we consider two cases: $v < t - \tau$ and $v \geq t - \tau$. For $v < t - \tau$, we have

$$\begin{aligned}\lambda(v, t, \tau) &= \mathbb{P}(L - x < X_v,\ X_{v-\tau} \leq M - x) \\ &= \int_0^{M-x} f_{v-\tau}(y)\bar{F}_\tau(L - x - y)\,\mathrm{d}y \\ &= \alpha(v - \tau, v, M - x, L - x).\end{aligned}$$

Finally, for $v \geq t - \tau$, we have

$$\begin{aligned}\lambda(v, t, \tau) &= \mathbb{P}(L - x < X_v,\ X_{v-\tau} \leq M - x < X_{t-\tau}) \\ &= \int_0^{M-x} f_{v-\tau}(y)\,\mathrm{d}y \int_{M-x-y}^{\infty} f_{t-v}(w)\bar{F}_{\tau-(t-v)}(L - x - y - w)\,\mathrm{d}w \\ &= \int_0^{M-x} f_{v-\tau}(y)\beta(t - v, \tau, M - x - y, L - x - y)\,\mathrm{d}y.\end{aligned}$$

This provides the result for $t > 2\tau$ and completes the proof.

With Lemmas 1 and 2, the following result holds.

Theorem 3. *The expected cost function at time t with $Y_0 = x$ fulfils the Markov renewal equation*

$$c_x(t) = B_x(t) + \int_\tau^t \int_0^M c_y(t - s)\bar{q}(x, \mathrm{d}s, \mathrm{d}y)$$

with $x \in [0, M]$, where $B_x(t)$ is given by

$$B_x(t) = c_{\mathrm{d}}[K_x(t) + W_x(t)] + c_{\mathrm{CR}} Z_x(t) + (c_{\mathrm{PR}} + c_{\mathrm{PM}})Q_x(t) + c_{\mathrm{PM}} J_x(t)$$

with $K_x(t)$ and $W_x(t)$ provided in Lemmas 1 and 2, and

$$\begin{aligned}Z_x(t) &= \int_\tau^t D_x(s)\,\mathrm{d}s, \\ Q_x(t) &= \int_\tau^t I_x(s)\,\mathrm{d}s, \\ J_x(t) &= \int_\tau^t \int_0^M H_x(s, y)\,\mathrm{d}s\,\mathrm{d}y,\end{aligned}$$

where $H_x(s, y)$, $D_x(s)$, and $I_x(s)$ are defined in (7)–(9).

Proof. Starting from (15), we have

$$c_x(t) = \mathbb{E}_x[C((0, t])\,\mathbf{1}_{\{S_1 > t\}}] + \mathbb{E}_x[C((0, S_1])\,\mathbf{1}_{\{S_1 \leq t\}}] + \mathbb{E}_x[C((S_1, t])\,\mathbf{1}_{\{S_1 \leq t\}}], \tag{18}$$

where the first term on the right-hand side has been computed in Lemma 1. The second term on the right-hand side is

$$\mathbb{E}_x[C((0, S_1])\,\mathbf{1}_{\{S_1 \leq t\}}] = c_{\mathrm{d}} W_x(t) + c_{\mathrm{CR}} Z_x(t) + (c_{\mathrm{PM}} + c_{\mathrm{PR}})Q_x(t) + c_{\mathrm{PM}} J_x(t),$$

where $W_x(t)$ is provided in Lemma 2 and

$$\begin{aligned}
Z_x(t) &= \mathbb{P}_x(S_1 \le t, Y_{S_1^-} > L),\\
Q_x(t) &= \mathbb{E}_x[\mathbf{1}_{\{S_1 \le t\}}\, \mathbf{1}_{\{Y_{S_1^-} \le L\}}\, \mathbf{1}_{\{Y_{S_1} > M\}}],\\
J_x(t) &= \mathbb{E}_x[\mathbf{1}_{\{S_1 \le t\}}\, \mathbf{1}_{\{Y_{S_1^-} \le L\}}\, \mathbf{1}_{\{Y_{S_1} \le M\}}].
\end{aligned}$$

Owing to Corollary 1, we obtain

$$\begin{aligned}
Z_x(t) &= \iiint_{[\tau,t]\times[0,M]\times[L,+\infty[} q(x, \mathrm{d}s, \mathrm{d}y, \mathrm{d}z) = \int_\tau^t D_x(s)\,\mathrm{d}s,\\
Q_x(t) &= \iiint_{[\tau,t]\times[M,+\infty[\times[0,L]} q(x\mathrm{d}s, \mathrm{d}y, \mathrm{d}z) = \int_\tau^t I_x(s)\,\mathrm{d}s,\\
J_x(t) &= \iiint_{[\tau,t]\times[0,M]\times[0,L]} q(x, \mathrm{d}s, \mathrm{d}y, \mathrm{d}z) = \int_\tau^t \int_0^M H_x(s, y)\,\mathrm{d}s\,\mathrm{d}y.
\end{aligned}$$

For the last term on the right-hand side of (18), by conditioning on $\sigma(Y_{S_1}, S_1)$, we obtain

$$\begin{aligned}
\mathbb{E}_x[C((S_1, t])\,\mathbf{1}_{\{S_1 \le t\}}] &= \mathbb{E}_x[c_{Y_{S_1}}(t - S_1)\,\mathbf{1}_{\{S_1 \le t\}}]\\
&= \iint_{\mathbb{R}_+^2} c_y(t-s)\,\mathbf{1}_{\{s \le t\}}\,\bar{q}(x, \mathrm{d}s, \mathrm{d}y),
\end{aligned}$$

which completes this proof.

6. Numerical examples

In order to illustrate the analytical results, we consider several numerical examples. To make the numerical assessments, a possibility might have been to follow [9] and use some integration scheme for integral equations with singular kernels [15] for example to solve the Markov renewal equations developed in the paper. Unfortunately, owing to the complexity of our Markov kernel, this has not been possible. That is why the numerical computations have finally been performed through Monte Carlo (MC) simulations. To shorten the large computing times induced by our intricate model, the parallel computer EMPIRE of the Universidad de Extremadura has been used.

For each of the following examples, the parameters of the gamma process measuring the intrinsic deterioration of the system are $\alpha = 1.5$ and $\beta = 3$. The system is assumed to be new at time 0, that is, $Y_0 = 0$. The failure threshold is $L = 10$. The induced approximated expected time to exceed level 10 is $\mathbb{E}(\sigma_L) \simeq 20.37$ time units. The maintenance efficiency is provided by $\rho = 0.5$. The costs associated with the different maintenance actions are $c_{\mathrm{CR}} = 100$ monetary units, $c_{\mathrm{PR}} = 60$ monetary units, $c_{\mathrm{PM}} = 5$ monetary units, and $c_u = 2$ monetary units per time unit. For the delay time τ, different values are considered in the following.

In the first case, we take $\tau = 5$ time units. In Figure 2 we plot the expected cost at time $t = 150$ versus the preventive threshold M. The data have been obtained using MC simulation for 50 values of M ranging from 0 to 10, with 4000 realizations in each point.

Now, using Figure 2, we can find a value of M that minimizes $c_{0,M}(150)$, that is, find some M_{opt} such that

$$c_{0,M_{\mathrm{opt}}}(150) = \inf\{c_{0,M}(150),\ 0 \le M \le 10\},$$

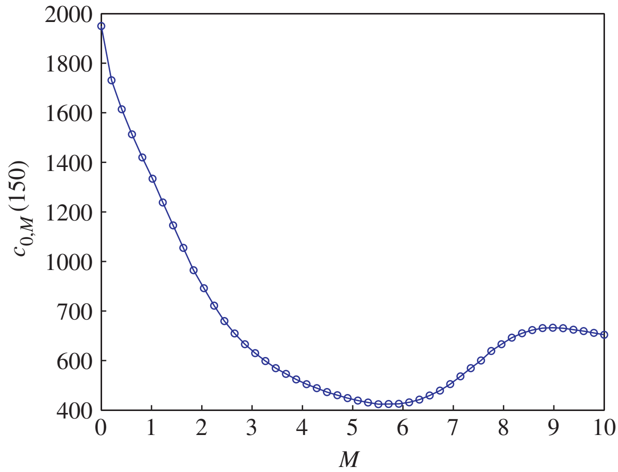

FIGURE 2: Expected cost at time $t = 150$ versus the preventive threshold M.

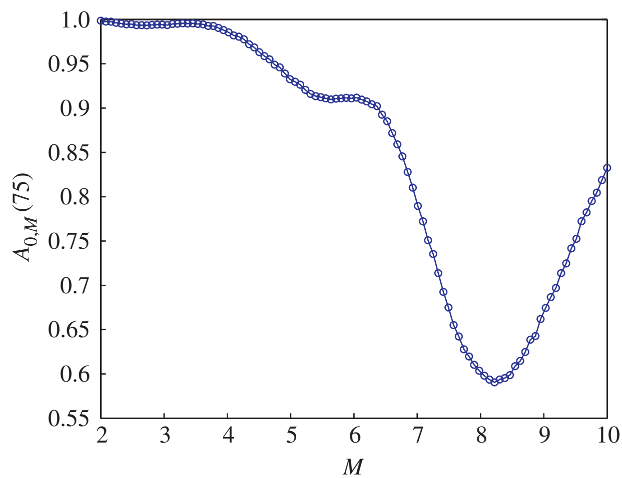

FIGURE 3: Availability at time $t = 75$ versus the preventive threshold M.

where $c_{0,M}(150)$ denotes the expected cost at time $t = 150$ for each value of M, starting from $Y_0 = 0$. By inspection, the expected cost $c_{0,M}(150)$ just presents a unique minimum and it is reached for $M_{\text{opt}} \simeq 5.5102$ with an expected cost of 423.9 monetary units.

Taking $\tau = 10$ time units, in Figure 3 we plot the values of the availability $A_{0,M}(75)$ at time $t = 75$ versus the preventive threshold M and in Figure 4 we plot the values of the expected cost $c_{0,M}(75)$ at time $t = 75$ versus the preventive threshold M. The data in these figures have been obtained using MC simulation for 100 values from 2 to 10, and 40 000 realizations in each point. Based on Figure 3, we can see that the availability at time $t = 75$ reaches its minimum at $M^* \simeq 8.2222$, with $A_{0,M^*}(75) \simeq 0.5905$. Hence, for any value of M, the probability that the system is working at time $t = 75$ exceeds or is equal to 59.05%. Based on Figure 4, we can see that the cost function $c_{0,M}(75)$ reaches its minimum at $M^* \simeq 4.4242$, with $c_{0,M^*}(75) \simeq 316.0753$ monetary units. Also, using both Figures 3 and 4, it is possible to

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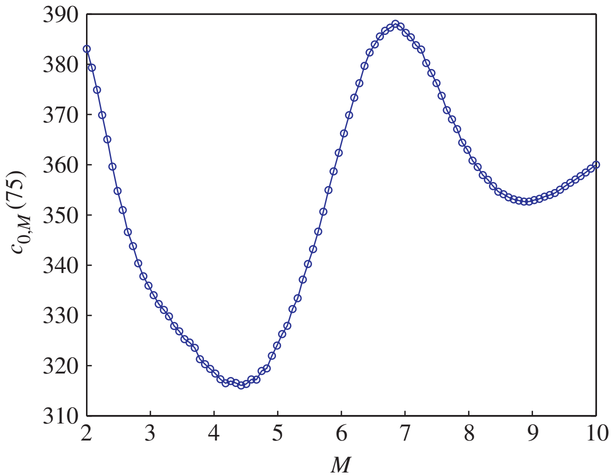

FIGURE 4: Expected cost at time $t = 75$ versus the preventive threshold M.

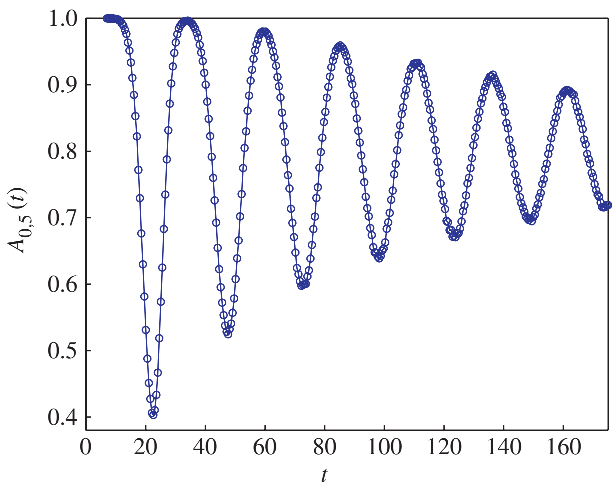

FIGURE 5: Transient availability versus time t given $M = 5$.

find some optimal M^{**} minimizing the cost $c_{0,M}(75)$ under some availability constraint such as $A_{0,M}(75) \geq 0.99$ for example. This provides $M^{**} \simeq 3.8586$, with $c_{0,M^{**}}(75) \simeq 320.2977$ monetary units and $A_{0,M^{**}}(75) = 0.9905$. Symmetrically, it is also possible to optimize the availability function under some cost constraint.

In Figure 5 we plot the transient availability versus time for $\tau = 15$ and $M = 5$. The data in this figure has been obtained using MC simulation for 350 values from 0 to 175 and 40 000 realizations in each point. As can be observed in Figure 5, the availability function shows some alternating decreasing and increasing periods with respect to time, which can be explained by the following: at the beginning, there is no maintenance action and the availability function decreases with time t until the first maintenance action at time S_1 is more likely to have been performed, namely until the probability that $t > S_1$ becomes larger. Indeed, we observe that

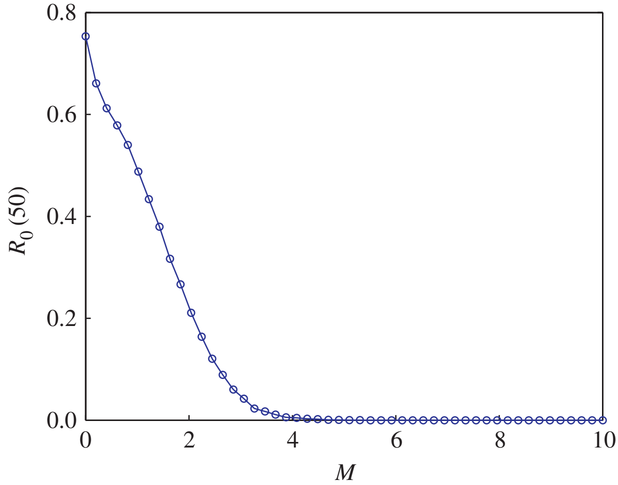

FIGURE 6: Transient reliability versus M at time $t = 50$.

$A_{0,5}(t)$ decreases up to $t \simeq 22.5645$, to be compared with

$$\mathbb{E}(S_1) = \mathbb{E}(\sigma_M) + \tau \simeq 25.3111$$

time units (and $\mathbb{E}(\sigma_L) \simeq 20.3912$ time units). After reaching its first minimum at $t \simeq 22.5645$, the availability function increases along with the probability that a first maintenance action has already been performed at time t. After a while, the probability that the system fails increases with the distance between t and the (nearly almost surely past) first maintenance action, which leads to a decreasing period, and a second minimum at $t \simeq 47.6361$, and so on. Note that the randomness of the maintenance times induces some attenuation in the decreasingness and increasingness over time.

In Figure 6 we plot the transient reliability versus the degradation level M at time $t = 50$ for $\tau = 15$. The data in this figure has been obtained using 50 points from 0 to 10 and 20 000 realizations in each point. As we can check, the transient reliability is here decreasing against the preventive threshold M. This means that the shorter the preventive threshold M, the larger the reliability. In this way, if the point is to maximize the reliability at time $t = 50$ with respect to M, it is best to take $M = 0$, namely perform periodic replacements. Though this seems challenging to prove from Theorem 2, it seems to be coherent with intuition, because a smaller M should involve more frequent replacements.

7. Conclusions and future extensions

In this work we analysed the reliability of a system subject to a continuous degradation modelled as a gamma process with imperfect delay repair. The functioning of the system was described through a semiregenerative process, and we showed that some transient reliability measures fulfil Markov renewal equations. Numerical examples of these reliability measures were shown. For these numerical examples, we used Monte Carlo simulation of the process due to the complexity of the Markov renewal equations, mainly caused by the overshoot of the gamma process and by the imperfect repair nature that reduces the system age. It would be interesting to compare this model (age-based repair) with a similar model in which the imperfect

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repair would reduce the system degradation itself instead of the system age, leading to some kind of virtual degradation. This should be part of a future work.

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