On the concept of gravitational force and Gauss's theorem in general relativity

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1. *Introduction.*

Whittaker[1] and Ruse[2] have developed forms of Gauss's theorem in general relativity, their theorems connecting integrals of normal force taken over a closed 2-space V_2 with integrals involving the distribution of matter taken over an open 3-space bounded by V_2. The definition of force employed by them involves the introduction of a normal congruence (with unit tangent vector λ^i), the "force" relative to the congruence being the negative of the first curvature vector of the congruence ($-\delta\lambda^i/\delta s$). This appears at first sight a natural enough definition, because $-\delta\lambda^i/\delta s$ at an event P represents the acceleration relative to the congruence of a free particle travelling along a geodesic tangent to the congruence at P. In order to give physical meaning to this definition of force it is necessary to specify the congruence λ^i physically, and it would seem most natural to choose the congruence of world-lines of flow of the medium. Supposing certain conditions satisfied by this congruence (*cf.* Ruse, *loc. cit.*), the theory of Ruse is applicable, and from this follows a form of Gauss's theorem. But when we examine the situation critically, we realise that the "force" whose normal flux is computed is not the gravitational force at all, but the force arising from stress in the medium, with sense reversed. This is most easily seen by considering the case of a fluid free from stress for which the energy tensor is $T_{ij} = \rho\theta_i\,\theta_j$, where θ^i is a unit vector, defining the world-lines of flow: for such a medium the world-lines of flow are geodesics, and the "force" in the sense of Ruse vanishes if the congruence of reference λ^i consists of the world-lines of flow. But a gravitational field will be there[3]. By

[1] E. T. Whittaker, *Proc. Roy. Soc.* (A) **149** (1935), 384-395.

[2] H. S. Ruse, *Proc. Edin. Math. Soc.* (2) **4** (1935), 144-158.

[3] A form of Gauss's theorem inapplicable to such a fundamental case is not satisfactory. We cannot apply Ruse's form of the theorem to this case with $\lambda^i = \theta^i$, because the special conditions which he imposes on λ^i require $\rho = 0$, as we can see from inspection of his equation (5.7) with $g^i = 0$.

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confining his attention to the statical case, Whittaker has avoided this criticism, because the stress just balances the gravitational force, and in estimating the flux of one we estimate the flux of the other. In regions where no matter is present, there are no world-lines of flow to use as a congruence of reference. But in the statical case it can be shown that the time-lines may be regarded as the lines of flow of an incompressible fluid of vanishing density and pressure, and hence the preceding considerations may be applied there also.

An attempt to obtain an intrinsic form of Gauss's theorem has been made by Temple[1], but his method is unsound, because it is assumed that when the components of a tensor involving the gravitational constant are expanded in powers of that constant, then the coefficients are components of tensors. That is not true[2].

In the present paper it is proposed to examine the concept of gravitational force in general relativity and to develop a form of Gauss's theorem which is intrinsic and general, and does not become inapplicable in important special cases. This theorem is of an infinitesimal character, as indeed we might expect, since almost all theorems in Riemannian space are of that type.

2. *The concept of force.*

As a source of suggestion for the definition of gravitational force in general relativity, let us consider first a question in Newtonian mechanics. Let us suppose that there are particles travelling in all directions in a field of force, the force per unit mass being a vector function of position. We shall refer to the "force per unit mass" simply as the "force." If only *relative* kinematic measurements are permitted, how can the field of force be found? The observation of the acceleration of a particle at Q relative to a particle at P gives us the *difference* between the force at Q and the force at P. *It is impossible by relative observations to determine absolutely the force on any particle*: but if we assign arbitrarily the force at one point, the forces at all other points may be found by relative observations. We shall see that in the general theory of relativity the concept of gravitational force suffers from a similar indeterminacy, but also from a greater indeterminacy because (as we might expect) relative

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[1] G. Temple, *Proc. Roy. Soc.* (A) **154** (1936), 354-363.

[2] *Cf.* J. L. Synge, "A criticism of the method of expansion in powers of the gravitational constant in general relativity," to appear shortly in *Proc. Roy. Soc.* (A).

observations only provide a differential law for the comparison of forces at adjacent events.

The fact noticed above is perhaps worthy of emphasis, namely, that *in Newtonian mechanics it is physically impossible to detect the presence of a uniform field of force.* It is only the variations from uniformity that can be detected. The point is easily overlooked if we forget that the particles of any physical frame of reference themselves share in the uniform acceleration (relative to an ideal Newtonian frame of reference) produced by the uniform field of force[1].

Let us now consider the space-time of general relativity, and let us suppose it traversed in all time-like directions by free test-particles whose world-lines are geodesics. When matter is present this requires the annihilation of thin tubes of matter to permit the passage of a free particle—a process analogous to the creation of a small cavity in a gravitating body in the classical theory of attraction. In what way do the world-lines of these particles give evidence of the presence of a gravitational field? Consider two particles whose world-lines L, M are adjacent. If space-time were flat, the normal displacement vector from L and M would vary linearly with proper time measured on L (rectangular Cartesian coordinates being employed). In kinematic language, M would have a constant velocity relative to L.

The gravitational field exhibits itself in the fact that when it is present M no longer has a constant velocity relative to L. If η^i is the infinitesimal displacement vector drawn perpendicular to L from a point A on L to a point B on M, then the acceleration of B relative to A is naturally defined as the vector

$$f^i = \delta^2 \eta^i / \delta s^2, \tag{2.1}$$

where $\delta/\delta s$ is the symbol of absolute differentiation and s the arc-length or proper time measured on L. Now we know[2] that

$$\frac{\delta^2 \eta^i}{\delta s^2} + R^i_{.jkl} \lambda^j \eta^k \lambda^l = 0, \tag{2.2}$$

where $R^i_{.jkl}$ is the curvature tensor and λ^i the unit tangent vector to L. The "excess of the gravitational force at an event B over the gravitational force at an event A" is naturally defined to be the acceleration of B relative to A: hence we have this result:

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[1] Prof. C. Barnes has drawn my attention to Maxwell's remarks regarding this point: J. C. Maxwell, *Matter and Motion* (London, 1894), 85.

[2] *Cf.* J. L. Synge, *Annals of Mathematics* 35 (1934), 705-713, for a simple proof by a method applicable also to the deviation of geodesic null-lines.

I. *If λ^i is an arbitrary time-like unit vector at an event A in space-time with coordinates x^i, then the excess of the gravitational force at an adjacent event B, with coordinates $x^i + \eta^i$, over the gravitational force at A, relative to λ^i, is*

$$f^i = - R^i_{\cdot jkl} \lambda^j \eta^k \lambda^l. \tag{2.3}$$

It is not necessary in stating this result to specify that η^i shall be perpendicular to λ^i, because on account of the skew-symmetry of $R^i_{\cdot jkl}$ the expression in (2.3) is not altered when we add to the components of η^i quantities proportional to λ^i.

In order to compare gravitational forces at points A and B which are not adjacent, we have to introduce a path C joining A to B and a family of unit vectors λ^i along C. If we denote by X^i the gravitational force at points on C, we are not to replace f^i in (2.3) by the ordinary differential dX^i, because that is not a vector, but by the absolute differential $(\delta X^i/\delta u)\, du$, where u is a parameter along C. Hence we arrive at this result:

II. *Given any curve C with unit tangent vector μ^i and a family of unit vectors λ^i defined along C, the gravitational force relative to λ^i varies along C in accordance with the law*

$$\frac{\delta X^i}{\delta s} = - R^i_{\cdot jkl} \lambda^j \mu^k \lambda^l, \tag{2.4}$$

s being the arc-length of C.

If v^i is any unit vector, we shall refer to $X^i v_i$ as *the component of gravitational force in the direction of v^i.*

Multiplication of (2.4) by λ_i gives

$$\lambda_i \frac{\delta X^i}{\delta s} = 0, \tag{2.5}$$

and, if λ^i is propagated parallelly along C,

$$\frac{d}{ds} (X^i \lambda_i) = 0. \tag{2.6}$$

Hence we have this result:

III. *If along any curve C a family of unit vectors λ^i is defined by parallel propagation, then the component in the direction of λ^i of the gravitational force relative to λ^i is constant along C.*

In particular, along a geodesic, with unit tangent vector λ^i, the component in the direction of λ^i of the gravitational force relative to

λ^i is constant along the geodesic. In fact, replacing μ^k by λ^k in (2.4), we see that along a geodesic the gravitational force relative to that geodesic is propagated parallelly.

Let us now multiply (2.4) by μ_i. This gives[1]

$$\mu_i \frac{\delta X^i}{\delta s} = -\epsilon(\lambda)\,\epsilon(\mu)\,K, \tag{2.7}$$

where $\epsilon(\lambda)$, $\epsilon(\mu)$ are the indicators of λ^i, μ^i respectively, and K is the Riemannian curvature of the 2-element defined by λ^i, μ^i. If C is a geodesic, then $\delta\mu_i/\delta s = 0$, and (2.7) reads

$$\frac{d}{ds}(X^i \mu_i) = -\epsilon(\lambda)\,\epsilon(\mu)\,K; \tag{2.8}$$

we have the result:

IV. *Along any geodesic C with unit tangent vector μ^i the tangential component of gravitational force relative to λ^i (an arbitrary family of unit vectors assigned along C) varies in accordance with* (2.8).

Let us now investigate the change in gravitational force due to passage round a small circuit. Consider a 2-space $x^i = x^i(u, v)$, and let us proceed round the circuit whose corners are $A_0(u_0, v_0)$, $A_1(u_0 + \Delta u, v_0)$, $A_2(u_0 + \Delta u, v_0 + \Delta v)$, $A_3(u_0, v_0 + \Delta v)$, $A_4(u_0, v_0)$, in order. Let Y^i be an arbitrary vector field, assigned over the 2-space, and let us consider the change produced in the invariant $X^i Y_i$ on passing round the circuit, X^i being calculated relative to an arbitrary field of unit vectors λ^i.

Let us put

$$S^i_{\cdot k} = -R^i_{\cdot jkl}\lambda^j \lambda^l, \tag{2.9}$$

so that the law of propagation of gravitational force is

$$\frac{\delta X^i}{\delta s} = S^i_{\cdot k}\frac{dx^k}{ds}. \tag{2.10}$$

We have then

$$\begin{cases} \dfrac{\delta X^i}{\delta u} = S^i_{\cdot k}\dfrac{\partial x^k}{\partial u} & \text{along } A_0 A_1 \text{ and } A_2 A_3, \\[2ex] \dfrac{\delta X^i}{\delta v} = S^i_{\cdot k}\dfrac{\partial x^k}{\partial v} & \text{along } A_1 A_2 \text{ and } A_3 A_4. \end{cases} \tag{2.11}$$

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[1] *Cf.* L. P. Eisenhart, *Riemannian Geometry* (Princeton, 1926), 113.

Developing along the side $A_4 A_3$ and retaining terms up to the second order in Δu, Δv, we have

$$(2.12)\quad (X^i Y_i)_{A_4} = (X^i Y_i)_{A_3} - \Delta v \left[\frac{d}{dv}(X^i Y_i)\right]_{A_3} + \tfrac{1}{2}(\Delta v)^2 \left[\frac{d^2}{dv^2}(X^i Y_i)\right]_{A_3},$$

where

$$(2.13)\quad \begin{cases} \dfrac{d}{dv}(X^i Y_i) = S^i_{\cdot k}\dfrac{\partial x^k}{\partial v} Y_i + X^i \dfrac{\delta Y_i}{\delta v}, \\ \dfrac{d^2}{dv^2}(X^i Y_i) = \dfrac{\partial}{\partial v}\left(S^i_{\cdot k}\dfrac{\partial x^k}{\partial v} Y_i\right) + S^i_{\cdot k}\dfrac{\partial x^k}{\partial v}\dfrac{\delta Y_i}{\delta v} + X^i \dfrac{\delta^2 Y_i}{\delta v^2}. \end{cases}$$

Developing the terms on the right of (2.12) along $A_3 A_2$, and remembering the order of approximation, we have

$$(2.14)\quad \begin{aligned} (X^i Y_i)_{A_4} = {} & (X^i Y_i)_{A_2} - \Delta u \left[\frac{d}{du}(X^i Y_i)\right]_{A_2} + \tfrac{1}{2}(\Delta u)^2 \left[\frac{d^2}{du^2}(X^i Y_i)\right]_{A_2} \\ & - \Delta v \left\{ \left[S^i_{\cdot k}\frac{\partial x^k}{\partial v} Y_i + X^i \frac{\delta Y_i}{\delta v}\right]_{A_2} - \Delta u \left[\frac{d}{du}\left(S^i_{\cdot k}\frac{\partial x^k}{\partial v} Y_i + X^i \frac{\delta Y_i}{\delta v}\right)\right]_{A_2} \right\} \\ & + \tfrac{1}{2}(\Delta v)^2 \left[\frac{d^2}{dv^2}(X^i Y_i)\right]_{A_2}. \end{aligned}$$

Similarly if we develop along $A_0 A_1$ and $A_1 A_2$, we obtain for $(X^i Y_i)_{A_0}$ an expression as in (2.14), but with u and v interchanged. Subtraction gives

$$(2.15)\quad \begin{aligned} (X^i Y_i)_{A_4} - (X^i Y_i)_{A_0} = \Delta u\, \Delta v \Bigg\{ & \left[\frac{d}{du}\left(S^i_{\cdot k}\frac{\partial x^k}{\partial v} Y_i + X^i \frac{\delta Y_i}{\delta v}\right)\right]_{A_2} \\ & - \left[\frac{d}{dv}\left(S^i_{\cdot k}\frac{\partial x^k}{\partial u} Y_i + X^i \frac{\delta Y_i}{\delta u}\right)\right]_{A_2} \Bigg\}. \end{aligned}$$

In view of the order of approximation, we may replace A_2 by A_0 on the right. Some terms cancel, and we obtain

$$(2.16)\quad \begin{aligned} (X^i Y_i)_{A_4} - (X^i Y_i)_{A_0} = \Delta u\, \Delta v \Bigg[& \left(\frac{\delta S^i_{\cdot k}}{\delta u}\frac{\partial x^k}{\partial v} - \frac{\delta S^i_{\cdot k}}{\delta v}\frac{\partial x^k}{\partial u}\right) Y_i \\ & + \left(\frac{\delta^2 Y_i}{\delta u\, \delta v} - \frac{\delta^2 Y_i}{\delta v\, \delta u}\right) X^i \Bigg]_{A_0}. \end{aligned}$$

Now

$$(2.17)\quad \frac{\delta^2 Y_i}{\delta u\, \delta v} - \frac{\delta^2 Y_i}{\delta v\, \delta u} = R_{ijkl}\, Y^j \frac{\partial x^k}{\partial u}\frac{\partial x^l}{\partial v},$$

and hence, since Y_i is an arbitrary vector, *the infinitesimal increment* ΔX_i *in the gravitational force on passing round an infinitesimal circuit is*

$$(2.18)\quad \Delta X_i = \Delta u\, \Delta v \left(\frac{\delta S_{ik}}{\delta u}\frac{\partial x^k}{\partial v} - \frac{\delta S_{ik}}{\delta v}\frac{\partial x^k}{\partial u} - R_{ijkl} X^j \frac{\partial x^k}{\partial u}\frac{\partial x^l}{\partial v}\right),$$

the last part of the expression being, of course, that due to parallel propagation.

Substituting for S_{ik} from (2.9), and making use of Bianchi's identity

$$R_{ijkl,m} + R_{ijlm,k} + R_{ijmk,l} = 0, \tag{2.19}$$

we obtain the following result:

V. *On passing round an infinitesimal space-like circuit of area* ΔS, *the covariant component of gravitational force relative to a field of unit vectors* λ^i *receives an increment* ΔX_i *given by*

$$\begin{aligned}\Delta X_i/\Delta S = &- R_{ijkl}\, X^j \mu^k \nu^l + R_{ijkl,m}\, \lambda^j \mu^k \nu^l \lambda^m \\ &+ R_{ijkl}\, (\lambda^j_{,m} \lambda^l + \lambda^l_{,m} \lambda^j)\, (\mu^k \nu^m - \mu^m \nu^k),\end{aligned} \tag{2.20}$$

where μ^i, ν^i *are any two orthogonal unit vectors in* ΔS, *the sense of description being* $\mu^i \to \nu^i$.

3. *Gauss's theorem.*

Let A be any point in space-time, and let a time-like unit vector λ^i be arbitrarily assigned at A. Let us draw the geodesics through A perpendicular to λ^i, and in the 3-space so formed take a closed 2-space V_2, enclosing a portion V_3 of the 3-space, including the point A. Let us draw the congruence of geodesics normal to V_3; let their unit tangent vectors be λ^i, a field of unit vectors being in this way uniquely defined in space-time adjacent to A, λ^i at A having been arbitrarily assigned. Now, the gravitational force X^i being arbitrarily chosen at A, let the field of gravitational force in space-time adjacent to A be calculated by propagation in accordance with (2.4) along the geodesics emanating from A, using the vector-field λ^i defined above.

Let n^i be the unit normal vector to V_3 and m^i the unit normal vector to V_2 lying in V_3, but drawn out from the interior. By the generalised Green-Stokes theorem[1] we have then

$$\int_{(2)} (X^i \lambda^j - X^j \lambda^i)\, m_i n_j\, dS = - \int_{(3)} (X^i \lambda^j - X^j \lambda^i)_{,j}\, n_i\, d\sigma, \tag{3.1}$$

these integrals being taken over V_2 and V_3 respectively, dS and $d\sigma$ being positive elements of 3-volume and area. Use has been made of the fact that n^i is time-like and m^i space-like, their indicators being

[1] For this form, see J. L. Synge, "Integral electromagnetic theorems in general relativity" (*Proc. Roy. Soc.* A, 157 (1936), 434-443), equation (2.25).

therefore -1 and $+1$ for the signature $+++-$ in space-time. But over V_3 we have $n^i = \lambda^i$, and thus (3.1) gives

$$\int_{(2)} X^i m_i \, dS = \int_{(3)} (X^i \lambda^j - X^j \lambda^i)_{,j} \lambda_i \, d\sigma \tag{3.2}$$

$$= \int_{(3)} (X^i \lambda_i \lambda^j_{,j} + X_{i,j} \lambda^i \lambda^j + X^j_{,j}) \, d\sigma,$$

since $\lambda^i \lambda_i = -1$, $\lambda^i_{,j} \lambda_i = 0$ from the unit character of λ^i.

Now if $p^i_{(a)}$, q^i $(a = 1, 2, 3)$ is an orthogonal tetrad of unit vectors, q^i being time-like, then the contravariant components of the fundamental tensor are[1]

$$g^{ij} = p^i_{(a)} p^j_{(a)} - q^i q^j, \tag{3.3}$$

and so for any vector field Z^i in space-time

$$Z^i_{,i} = g^{ij} Z_{i,j} = Z_{i,j} p^i_{(a)} p^j_{(a)} - Z_{i,j} q^i q^j. \tag{3.4}$$

Along any one of the geodesics forming V_3 we have

$$\lambda^i \mu_i = 0, \quad \delta\mu_i/\delta s = 0, \tag{3.5}$$

where μ^i is the unit tangent vector to the geodesic. Hence

$$\lambda_{i,j} \mu^i \mu^j = 0, \tag{3.6}$$

throughout V_3, and in particular at A for every μ^i in V_3. Also at A we have

$$\lambda_{i,j} \lambda^i = 0, \tag{3.7}$$

and hence, replacing Z in (3.4) by λ, and taking $q^i = \lambda^i$, we have at A

$$\lambda^i_{,i} = 0. \tag{3.8}$$

Also from (2.4) we have at A

$$X_{i,j} \lambda^i \lambda^j = 0. \tag{3.9}$$

Hence the first two parts of the integrand of the triple integral in (3.2) vanish at A, and hence since the left hand side of (3.2) obviously represents the normal flux of force outward across V_2, we may state this result:

VI. *If N denotes the outward normal flux of gravitational force across a closed 2-space whose interior is composed of geodesics drawn from a point A perpendicular to a time-like unit vector λ^i at A, then*

$$\lim_{\sigma \to 0} N/\sigma = X^i_{,i}, \tag{3.10}$$

[1] L. P. Eisenhart, *Riemannian Geometry* (Princeton, 1926), 96.

the divergence of the force-field at A, σ being the 3-volume contained in the 2-space, and the force-field being defined by propagation according to (2.4) *along geodesics and calculated relative to the geodesic congruence normal to the interior of the 2-space.*

Substituting X for Z in (3.4), we have at A

$$X^i_{,i} = X_{i,j}\,\mu^i_{(a)}\,\mu^j_{(a)} - X_{i,j}\,\lambda^i\lambda^j, \tag{3.11}$$

where $\mu^i_{(a)}$ is a triad of mutually orthogonal unit vectors in V_3. Hence by (2.4) and (3.9),

$$X^i_{,i} = -\,R_{ijkl}\,\mu^i_{(a)}\,\lambda^j\,\mu^k_{(a)}\,\lambda^l, \tag{3.12}$$

or[1]

$$X^i_{,i} = R_{ij}\,\lambda^i\lambda^j, \tag{3.13}$$

where R_{ij} is the Ricci tensor. *The divergence of gravitational force is equal to the mean curvature of space-time for the direction* λ^i.

So far the field equations have not been employed. These read[2], with the cosmological constant zero,

$$R_{ij} = -\,\kappa\,(T_{ij} - \tfrac{1}{2}g_{ij}\,T), \qquad \kappa = 8\pi G/c^4. \tag{3.14}$$

Hence

$$X^i_{,i} = -\,\kappa\,(T_{ij}\lambda^i\lambda^j + \tfrac{1}{2}T). \tag{3.15}$$

The following result is immediate:

VII. *At a point of space-time unoccupied by matter the divergence of the gravitational force vanishes, the force being calculated as in VI relative to an arbitrary time-like unit vector* λ^i; *also* $\lim_{\sigma \to 0} N/\sigma = 0$.

When matter is present the result (3.15) suffers (as far as intrinsic character is concerned) from the fact that the vector λ^i at A is arbitrary. We shall now make it truly intrinsic.

The equations

$$T^i_{.j}\,\lambda^j + \Theta\,\lambda^i = 0, \tag{3.16}$$

in which Θ is a root of the determinantal equation

$$|\,T^i_{.j} + \Theta\,\delta^i_j\,| = 0, \tag{3.17}$$

determine an orthogonal tetrad of principal directions. That which

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[1] *Cf.* L. P. Eisenhart, *Riemannian Geometry* (Princeton 1926), 113.

[2] It is possible to adopt two definitions of the energy tensor, differing by a factor c^2. That here employed reduces for a stream of unstressed matter to $T_{ij} = \rho\theta_i\,\theta_j$, where θ^i is a unit vector and ρ is energy-density, not mass-density. This form is to be preferred, because in general relativity energy should be regarded as the more primitive concept, from which mass is a convenient conventional derivative.

is time-like gives the direction of the world-line of flow of matter, whereas those which are space-like give the directions of principal stress. If $\Theta_{(a)}$ are the roots of (3.17) corresponding to the principal directions of stress, and $\Theta_{(4)}$ that corresponding to the world-line of flow, then $\Theta_{(a)}$ are the principal stresses and $\Theta_{(4)}$ is the proper density of energy[1].

Let us choose λ^i (the unit vector with respect to which the gravitational force is calculated) in the direction of the world-line of flow at A. Then

$$\begin{cases} T^i_{.j}\lambda^j + \rho\lambda^i = 0, \qquad T_{ij}\lambda^i\lambda^j = \rho, \\ T = T^i_{.i} = -\rho - \Sigma, \end{cases} \tag{3.18}$$

where ρ is the proper density of energy and Σ is the sum of the three principal stresses. Hence by (3.15) we have this result:

VIII. GAUSS'S THEOREM: *At a point of space-time the divergence of gravitational force, calculated with respect to a world-line of flow of matter as in VI, is given by*

$$X^i_{,i} = -\tfrac{1}{2}\kappa(\rho - \Sigma), \tag{3.19}$$

where ρ is the proper density of energy and Σ the sum of the three principal stresses, and the limit of the ratio of normal outward flux of gravitational force to included mass is[2]

$$\lim_{\sigma \to 0} \frac{Nc^2}{\rho\sigma} = -\frac{4\pi G}{c^2}\left(1 - \frac{\Sigma}{\rho}\right). \tag{3.20}$$

For a perfect fluid $\Sigma = -3p$, where p is the pressure.

It is interesting to note that for a region occupied by radiation only we have $\Sigma = -\rho$, and hence the limit of the ratio of normal outward flux of gravitational force to included energy is

$$\lim_{\sigma \to 0} \frac{N}{\rho\sigma} = -\frac{8\pi G}{c^4}. \tag{3.21}$$

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[1] *Cf.* J. L. Synge, *Trans. Roy. Soc. Canada*, Sect. III, **28** (1934), 163, where however a factor c^2 enters because there the concept of mass was taken as fundamental.

[2] The factor c^2 is present in the denominator on the right hand side of (3.20) because we have used proper time s instead of the usual time in our definition of acceleration, so that our X^i is the usual force (comparable to that of Newtonian mechanics) divided by c^2.