

# On the ergodicity of unitary frame flows on Kähler manifolds

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**Abstract.** Let  $(M, g, J)$  be a closed Kähler manifold with negative sectional curvature and complex dimension  $m := \dim_{\mathbb{C}} M \geq 2$ . In this article, we study the *unitary frame flow*, that is, the restriction of the frame flow to the principal  $U(m)$ -bundle  $F_{\mathbb{C}}M$  of unitary frames. We show that if  $m \geq 6$  is even and  $m \neq 28$ , there exists  $\lambda(m) \in (0, 1)$  such that if  $(M, g)$  has negative  $\lambda(m)$ -pinched holomorphic sectional curvature, then the unitary frame flow is ergodic and mixing. The constants  $\lambda(m)$  satisfy  $\lambda(6) = 0.9330\dots$ ,  $\lim_{m \rightarrow +\infty} \lambda(m) = 11/12 = 0.9166\dots$ , and  $m \mapsto \lambda(m)$  is decreasing. This extends to the even-dimensional case the results of Brin and Gromov [On the ergodicity of frame flows. *Invent. Math.* **60**(1) (1980), 1–7] who proved ergodicity of the unitary frame flow on negatively curved compact Kähler manifolds of odd complex dimension.

**Key words:** partially hyperbolic flows, Kähler manifolds, ergodic theory, frame flows, Pestov identity

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## 1. Introduction

1.1. *Ergodicity and mixing of unitary frame flows.* Let  $(M, g, J)$  be a smooth closed (compact, without boundary) Kähler manifold with *negative sectional curvature* and complex dimension  $m \geq 2$ . Let  $SM \rightarrow M$  be the unit tangent bundle and let  $F_{\mathbb{C}}M \rightarrow M$

be the principal  $U(m)$ -bundle of *unitary bases* over  $M$ . A point  $w \in F_{\mathbb{C}}M$  over  $x \in M$  is the data of an orthonormal basis  $(v, \mathbf{e}_2, \dots, \mathbf{e}_m)$  of  $(T_x M, h_x)$ , where  $h_x(\cdot, \cdot) = g_x(\cdot, \cdot) + i g_x(\cdot, J_x \cdot)$  is the canonical Hermitian inner product on the fibres of  $TM$ . Equivalently, we will see  $F_{\mathbb{C}}M$  as a principal  $U(m - 1)$ -bundle over  $SM$  by the projection map  $p : F_{\mathbb{C}}M \rightarrow SM$  defined as  $p(v, \mathbf{e}_2, \dots, \mathbf{e}_m) = v$ .

The *geodesic flow*  $(\varphi_t)_{t \in \mathbb{R}}$  on  $SM$  is defined as  $\varphi_t(v) := \dot{\gamma}_v(t)$ , where  $t \mapsto \gamma_v(t) \in M$  is the geodesic generated by  $v \in SM$ . The *unitary frame flow* on  $F_{\mathbb{C}}M$  is then defined as

$$\Phi_t(v, \mathbf{e}_2, \dots, \mathbf{e}_m) := (\varphi_t(v), P_{\gamma_v(t)} \mathbf{e}_2, \dots, P_{\gamma_v(t)} \mathbf{e}_m),$$

where  $P_{\gamma_v(t)} : T_x M \rightarrow T_{\gamma_v(t)} M$  is the parallel transport along  $\gamma_v$  with respect to the Levi-Civita connection.

Recall that a flow  $(\Phi_t)_{t \in \mathbb{R}}$  on a compact metric space  $\mathcal{M}$  is said to be *ergodic* with respect to an invariant probability measure  $\mu$  if any flow-invariant function  $f \in L^2(\mu)$  is constant. It is said to be *mixing* if for all  $f_1, f_2 \in L^2(\mathcal{M}, \mu)$ ,

$$\lim_{t \rightarrow +\infty} \int_{\mathcal{M}} f_1 \cdot (f_2 \circ \Phi_t) \, d\mu = \int_{\mathcal{M}} f_1 \, d\mu \cdot \int_{\mathcal{M}} f_2 \, d\mu.$$

The geodesic flow  $(\varphi_t)_{t \in \mathbb{R}}$  of any negatively curved compact Riemannian manifold is well known to be ergodic [Ano67, Hop36] with respect to the Liouville measure on  $SM$ . However, for *extensions* of the geodesic flow to principal bundles, ergodicity is a more subtle question since these flows are only partially hyperbolic, not uniformly hyperbolic. The unitary frame flow  $(\Phi_t)_{t \in \mathbb{R}}$  defined above is such an extension of  $(\varphi_t)_{t \in \mathbb{R}}$  to  $F_{\mathbb{C}}M$ ; it preserves a natural flow-invariant smooth measure  $\omega$  induced by the Liouville measure and the Haar measure on the group  $U(m - 1)$ . The purpose of this paper is to investigate ergodicity of  $(\Phi_t)_{t \in \mathbb{R}}$  with respect to  $\omega$ . It was proved by Brin and Gromov [BG80] that this flow is ergodic whenever  $m := \dim_{\mathbb{C}} M$  is odd or  $m = 2$  but the even-dimensional case when  $m \geq 4$  has remained open so far. Our aim is to bring a first positive answer when  $m \geq 6$  is even and  $m \neq 28$ , under some pinching hypothesis for the sectional curvature.

Recall that the *holomorphic sectional curvature* of  $(M, g, J)$  is defined as

$$H(X) := R(X, JX, JX, X)$$

for all unit vectors  $X \in TM$ , where  $R$  is the Riemann curvature tensor of  $(M, g)$ . The manifold is said to be *holomorphically  $\lambda$ -pinched*, for some  $\lambda \in (0, 1]$ , if there exists a constant  $C > 0$  such that

$$-C \leq H(X) \leq -C\lambda$$

for every unit vector  $X$ . The manifold is said to be *strictly holomorphically  $\lambda$ -pinched* if the above inequalities are strict.

To state our main result, we introduce the function  $m \mapsto \lambda(m)$ , defined for even numbers  $m \geq 6$  by

$$\lambda(m) := \frac{308m + 131}{336m + 105}.$$

The function  $m \mapsto \lambda(m)$  is decreasing,  $\lambda(6) = 0.9330\dots$  and  $\lim_{m \rightarrow +\infty} \lambda(m) = 11/12 = 0.9166\dots$

We will prove that the following theorem holds.

**THEOREM 1.1.** *Let  $(M, g, J)$  be a closed connected Kähler manifold of complex dimension  $m \geq 2$ , with negative sectional curvature. The unitary frame flow  $(\Phi_t)_{t \in \mathbb{R}}$  on  $F_{\mathbb{C}}M$  is ergodic and mixing with respect to the smooth measure  $\omega$  if:*

- (i) *the complex dimension  $m$  is odd or  $m = 2$  [BG80];*
- (ii) *the complex dimension  $m \geq 6$  is even,  $m \neq 28$ , and the manifold is strictly holomorphically  $\lambda(m)$ -pinched.*

We will actually show that the unitary frame flow is ergodic if and only if it is mixing. We believe that ergodicity should hold *without any pinching condition* but it is clear from the proofs that our method only works with a pinching condition close to 1. As a comparison, Brin conjectured that the *real frame flow* should be ergodic on negatively curved manifolds with 0.25-pinched real sectional curvature (see [Bri82, Conjecture 2.6]). (In the literature, the word ‘frame flow’ usually refers to what we call here the ‘real frame flow’, that is, the parallel transport of all bases regardless of any almost-complex structure. We added the word ‘unitary’ in the Kähler case and ‘real’ in the Riemannian case to make a distinction.)

In the case of constant holomorphic curvature  $H = -1$  (that is, on compact quotients  $\Gamma \backslash \mathbb{C}H^m$  of the complex hyperbolic space), the ergodicity of unitary frame flow was shown by Howe and Moore [HM79]. In non-constant holomorphic curvature, besides [BG80] in odd complex dimensions and  $m = 2$ , Theorem 1.1 seems to be the first result proving ergodicity of unitary frame flows on negatively curved Kähler manifolds of even complex dimensions  $m \geq 6$ .

As indicated in Theorem 1.1, it also seems that our technique does not apply in complex dimensions  $m = 4$  and  $m = 28$ . The former case is related to the fact that  $S^7$  is parallelizable, whereas the latter case is connected to an open problem in algebraic topology which is to classify reductions of the structure group of the unitary frame bundle  $F_{\mathbb{C}}S^{55}$  over the sphere  $S^{55}$ . More precisely, we are unable to rule out the possible existence of a special  $E_6$ -structure on  $S^{55}$  and this eventually turns out to be problematic to run our argument, see §3 where this is further discussed.

The structure of the argument, described in more detail in §1.2, is somewhat similar to our previous article [CLMS21] proving ergodicity of real frame flows on negatively curved compact Riemannian manifolds of even real dimensions with nearly 0.25-pinched (real) sectional curvature, thus almost answering the long-standing [Bri82, Conjecture 2.6] of Brin mentioned above. Nevertheless, the present article is not a mere adaptation of [CLMS21] as we had to develop new techniques to take into account the specificities of the Kähler setting, see Theorem 3.1, §3.2 or §5.3 for instance. Although it is meant to be self-contained, we encourage the reader to consult [CLMS21, CLMS22, Lef23] as we build here on the framework developed in these articles.

Prior to [CLMS21], the real frame flow was known to be ergodic on negatively curved Riemannian manifolds of odd dimension  $\neq 7$  by Brin and Gromov [BG80, Bri75b], and in even dimensions (and in dimension 7) for manifolds with a pinching close to 1 by Brin and Karcher [BK84] and Burns and Pollicott [BP03].

The real frame flows are historically important examples of *partially hyperbolic flows* studied in the aftermath of Anosov's seminal work on hyperbolic flows [Ano67] by Brin and Pesin [BP74, Bri75b, Bri75a]. The field of partially hyperbolic dynamical systems is now a well-established and active field of dynamical systems, see [HP06] for instance for an introduction to this topic.

Finally, let us mention that, similarly to the real case where ergodicity of the real frame flow was shown to determine the high-energy behaviour of eigenfunctions of Dirac-type operators [JS07], the ergodicity of the unitary frame flow on Kähler manifolds determines the high-energy behaviour of eigenfunctions of Dolbeault Laplacians and Spin<sup>c</sup> Dirac operators [JSZ08].

1.2. *Proof ideas.* Let us summarize the argument which, as mentioned above, is similar to that developed in [CLMS21] and consists of three steps.

- (i) *Hyperbolic dynamics.* Following Brin's ideas [Bri75b] (see also [Lef23] for a more recent approach), the non-ergodicity of the unitary frame flow is described by means of a subgroup  $H \cong \mathrm{U}(m-1)$  called the *transitivity group*, see §2.3. In particular, there exists a smooth flow-invariant principal  $H$ -subbundle  $Q \subset F_{\mathbb{C}}M$  over  $SM$ , such that the restriction of  $(\Phi_t)_{t \in \mathbb{R}}$  to  $Q$  is ergodic.
- (ii) *Algebraic topology.* The group  $H$  thus provides a reduction of the structure group of  $F_{\mathbb{C}}M$  from  $\mathrm{U}(m-1)$  to  $H$ . In particular, restricting to a point  $x_0 \in M$  and identifying  $S_{x_0}M \simeq S^{2m-1}$ , the unitary frame bundle  $F_{\mathbb{C}}S^{2m-1} \rightarrow S^{2m-1}$  must admit a reduction of its structure group to  $H$ . In §3, we classify such reductions and show that for  $m \neq 4, 28$ ,  $H$  must act reducibly on  $\mathbb{C}^{m-1}$ .
- (iii) *Riemannian geometry.* Using the non-Abelian Livšić theorem of [CL22], we then deduce that there exists a smooth flow-invariant complex vector bundle  $\mathcal{V} \rightarrow SM$  which is a subbundle  $\mathcal{V} \subset \pi^*TM$  (where  $\pi : SM \rightarrow M$  is the projection) satisfying certain algebraic properties. In turn, using the twisted Pestov identity (see Lemma 2.1), we rule out the existence of such an object under a certain pinching condition  $\lambda > \lambda(m)$  in §§4 and 5.

1.3. *Structure of the article.* In §2, we recall standard facts from Riemannian and complex geometry, and (partially) hyperbolic dynamical systems needed in the rest of the article. In §3, we study the possible reductions of the structure group of the unitary frame bundle over the sphere, and deduce the existence of non-zero flow-invariant projectors whenever the frame flow is not ergodic. In §4, we derive, using the twisted Pestov identity, an inequality that must be satisfied by such an invariant object. In §5, we complete the proof of Theorem 1.1.

## 2. Preliminaries

2.1. *Riemannian geometry of the unit tangent bundle.* Let  $(M, g)$  be a closed connected Riemannian manifold of real dimension  $n$ . Denote by

$$SM := \{v \in TM \mid |v|_g = 1\}$$

the unit tangent bundle of  $(M, g)$  and by  $\pi : SM \rightarrow M$  the projection map.

2.1.1. *Tangent bundle of SM.* Let  $(\varphi_t)_{t \in \mathbb{R}}$  be the geodesic flow on  $SM$  with generating vector field  $X \in C^\infty(SM, T(SM))$ . The tangent bundle  $T(SM)$  splits as

$$T(SM) = \mathbb{V} \oplus \mathbb{H} \oplus \mathbb{R}X, \tag{2.1}$$

where  $\mathbb{V} := \ker d\pi$  is the *vertical bundle*, and  $\mathbb{H}$  is the *horizontal bundle* defined by means of the Levi-Civita connection, see [Pat99, Ch. 1]. The metric  $g$  induces a canonical metric on  $T(SM)$  called the *Sasaki metric* such that the splitting equation (2.1) is orthogonal.

If  $f \in C^\infty(SM)$  is a smooth function, its gradient  $\nabla f \in C^\infty(SM, T(SM))$  computed with respect to the Sasaki metric splits according to equation (2.1) as

$$\nabla f = \nabla_{\mathbb{V}} f + \nabla_{\mathbb{H}} f + (Xf)X,$$

where  $\nabla_{\mathbb{V}} f \in C^\infty(SM, \mathbb{V})$  is the *vertical gradient* and  $\nabla_{\mathbb{H}} f \in C^\infty(SM, \mathbb{H})$  is the *horizontal gradient*. The  $L^2$ -norm on  $SM$  is defined using the *Liouville measure*  $\mu$  on  $SM$  which is the Riemannian measure induced by the Sasaki metric. Note that the Liouville measure is *invariant* by the geodesic flow.

The *vertical Laplacian*  $\Delta_{\mathbb{V}}$  is then defined as  $\Delta_{\mathbb{V}} := \nabla_{\mathbb{V}}^* \nabla_{\mathbb{V}}$ , where  $\nabla_{\mathbb{V}}^*$  denotes the  $L^2$ -adjoint. Equivalently, given  $f \in C^\infty(SM)$  and  $x \in M$ , denoting the Laplacian of the restriction of  $g_x$  to  $S_x M$  by  $\Delta_{S_x M}$ ,

$$\Delta_{\mathbb{V}} f(v) = \Delta_{S_x M}(f|_{S_x M})(v) \quad \text{for all } v \in S_x M. \tag{2.2}$$

2.1.2. *Fourier decomposition in the fibres.* Since  $\pi : SM \rightarrow M$  is a sphere bundle, any smooth function  $f \in C^\infty(SM)$  can be decomposed into a sum of spherical harmonics on the sphere  $S_x M \simeq S^{n-1}$  above each point  $x \in M$ . In other words, we can write

$$f = \sum_{k=0}^{+\infty} f_k,$$

where  $f_k \in C^\infty(SM)$  is a spherical harmonic of degree  $k \geq 0$ , that is, it satisfies the eigenvalue equation

$$\Delta_{\mathbb{V}} f_k = k(n + k - 2) f_k,$$

where  $\Delta_{\mathbb{V}}$  is the vertical Laplacian on each sphere  $S_x M$  (for  $x \in M$ ) introduced in equation (2.2). The space of spherical harmonics of degree  $k$  over  $M$  defines a vector bundle  $\Omega_k \rightarrow M$  which can be naturally identified with the vector bundle of trace-free symmetric  $k$ -tensors  $S_0^k T M \rightarrow M$  via the map (here we identify  $T^* M$  and  $T M$  by using the metric  $g$ )

$$\pi_k^* : S_0^k T M \xrightarrow{\sim} \Omega_k, \quad \pi_k^* f(v) := f_x(v, \dots, v) \quad \text{for all } v \in S_x M. \tag{2.3}$$

More generally, let  $(E, h) \rightarrow M$  be a Hermitian (or Euclidean) vector bundle over  $M$  equipped with a unitary (or orthogonal) connection  $\nabla^E$ , by which we mean that

$$Yh(e, f) = h(\nabla_Y^E e, f) + h(e, \nabla_Y^E f) \quad \text{for all } e, f \in C^\infty(M, E), \text{ for all } Y \in C^\infty(M, T M).$$

Denoting by  $(\mathcal{E}, \nabla^{\mathcal{E}}) := (\pi^*E, \pi^*\nabla^E)$  the pullback of  $(E, \nabla^E)$  to  $SM$ , any section  $f \in C^\infty(SM, \mathcal{E})$  can be uniquely decomposed into a sum of twisted spherical harmonics over each point  $x \in M$ , that is,

$$f = \sum_{k=0}^{+\infty} f_k, \tag{2.4}$$

where  $f_k \in C^\infty(SM, \mathcal{E})$  is a spherical harmonic of degree  $k$  (with values in  $\mathcal{E}$ ). Note that, with respect to an orthonormal basis  $(\mathbf{e}_\alpha)$  on  $E$  defined locally over  $U \subset M$ , any section  $f \in C^\infty(SM, \mathcal{E})$  can be written as  $f|_U = \sum_\alpha f_\alpha \mathbf{e}_\alpha$ , where  $f_\alpha \in C^\infty(SM|_U)$ . Then, the vertical Laplacian is defined as

$$\Delta_{\nabla}^E f = \sum_{\alpha} (\Delta_{\nabla} f_{\alpha}) \mathbf{e}_{\alpha},$$

where  $\Delta_{\nabla}$  was introduced in equation (2.2). The sections  $f_k \in C^\infty(SM, \mathcal{E})$  then satisfy the eigenvalue equation

$$\Delta_{\nabla}^E f_k = k(n + k - 2) f_k.$$

Equivalently,  $f_k$  is a smooth section of the bundle  $\Omega_k \otimes E$  over  $M$  and this can be identified via equation (2.3) to an element  $S_0^k T M \otimes E$ . We say that a section  $f \in C^\infty(SM, \mathcal{E})$  has *even* (respectively *odd*) Fourier degree if the decomposition of equation (2.4) only involves spherical harmonics of even (respectively odd) degree.

We define the operator  $\mathbf{X} := \nabla_X^{\mathcal{E}}$ , where  $X$  is the geodesic vector field on  $SM$ . This is the infinitesimal generator of the parallel transport of sections of  $E$  along geodesic flow-lines. It has the mapping property

$$\mathbf{X} : C^\infty(M, \Omega_k \otimes E) \rightarrow C^\infty(M, \Omega_{k-1} \otimes E) \oplus C^\infty(M, \Omega_{k+1} \otimes E), \tag{2.5}$$

and therefore splits as a sum  $\mathbf{X} := \mathbf{X}_- + \mathbf{X}_+$ , where each term corresponds to the two summands in equation (2.5).

There is a natural  $L^2$  scalar product on sections  $f, f' \in C^\infty(SM, \mathcal{E})$  given by

$$\langle f, f' \rangle_{L^2} := \int_{SM} h_{\pi(v)}(f(v), f'(v)) \, d\mu,$$

where  $\mu$  is the Liouville measure on  $SM$ , and  $h$  is the Hermitian (or Euclidean) metric on  $E$ .

**2.1.3. Twisted Pestov identity.** This identity will play a fundamental role in our proof of Theorem 1.1. In the non-twisted case, it was first discovered in some particular cases by Mukhometov [Muk75, Muk81] and Amirov [Ami86], and then in its classical shape by Pestov and Sharafutdinov [PS88, Sha94]. Eventually, it was reformulated and described in a general coordinate-free way in [GPSU16, PSU15]. It takes the following form.

**LEMMA 2.1. (Twisted Pestov identity)** *Let  $(M, g)$  be a closed  $n$ -dimensional Riemannian manifold and let  $(E, h)$  be a Hermitian (or Euclidean) vector bundle over  $M$  equipped with*

unitary (or orthogonal) connection  $\nabla^E$ . The following identity holds for all  $k \in \mathbb{Z}_{\geq 0}$  and  $u \in C^\infty(M, \Omega_k \otimes E)$ :

$$\begin{aligned} & \frac{(n+k-2)(n+2k-4)}{n+k-3} \|\mathbf{X}_- u\|_{L^2}^2 - \frac{k(n+2k)}{k+1} \|\mathbf{X}_+ u\|_{L^2}^2 + \|Z(u)\|_{L^2}^2 \\ &= \langle R \nabla_{\nabla}^E u, \nabla_{\nabla}^E u \rangle_{L^2} + \langle \mathcal{F}^E u, \nabla_{\nabla}^E u \rangle_{L^2}, \end{aligned} \tag{2.6}$$

where:

- $Z$  is a first-order differential operator which we do not make explicit;
- the term involving  $R$  takes the form

$$\langle R \nabla_{\nabla}^E u, \nabla_{\nabla}^E u \rangle_{L^2} = \int_M \int_{S_x M} \sum_{\alpha} R(v, \nabla_{\nabla} u_{\alpha}, \nabla_{\nabla} u_{\alpha}, v) |dv| |dx|,$$

where  $R$  is the Riemann curvature tensor,  $|dv|$  is the Lebesgue measure on the sphere  $S_x M$  (induced by  $g_x$ ) and  $|dx|$  is the Riemannian measure,  $u = \sum_{\alpha} u_{\alpha} \mathbf{e}_{\alpha}$  with  $(\mathbf{e}_{\alpha})_{\alpha \in I}$  an orthonormal frame at the point  $x \in M$  of  $E_x$ ;

- the term involving  $\mathcal{F}^E$  takes the form

$$\langle \mathcal{F}^E u, \nabla_{\nabla}^E u \rangle_{L^2} = \int_M \int_{S_x M} \sum_{\alpha} R_E(v, \nabla_{\nabla} u_{\alpha}, u, \mathbf{e}_{\alpha}) |dv| |dx|, \tag{2.7}$$

where  $R_E$  is the curvature tensor of  $E$  and we use the convention:

$$R_E(X, Y, \omega, \eta) := h(R_E(X, Y)\omega, \eta) \quad \text{for all } X, Y \in TM \quad \text{for all } \omega, \eta \in E,$$

and  $h$  is the Euclidean (or Hermitian) metric on the bundle  $E$ .

The operator  $Z$  is not made explicit as we will only need to use  $\|Z(u)\|_{L^2}^2 \geq 0$  in equation (2.6). We refer to [GPSU16, Proposition 3.5] for a proof.

2.2. *Complex geometry.* We use [KN96, Ch. IX] as basic reference for complex geometry.

2.2.1. *Curvature tensors.* Let  $(V, g)$  be a Euclidean vector space of dimension  $n$ . We will usually identify  $V$  with its dual  $V^*$  and  $\Lambda^2 V$  with the space of skew-symmetric endomorphisms of  $V$  using the metric  $g$ . We denote by  $S^p V$  the symmetric  $p$ -tensors on  $V$ ,  $S_0^p V$  the trace-free symmetric tensors and  $\Lambda^p V$  the  $p$ th exterior power. The space  $V^{\otimes 2}$  splits as

$$V^{\otimes 2} = \mathbb{R}g \oplus S_0^2 V \oplus \Lambda^2 V, \tag{2.8}$$

where each summand is invariant under the  $O(n)$ -action, and  $S^2 V = \mathbb{R}g \oplus S_0^2 V$ . The space  $V^{\otimes 2} \simeq \text{End } V$  is equipped with the norm

$$\langle u, v \rangle = \text{Tr}(u^{\top} v), \tag{2.9}$$

where  $\top$  denotes the transpose operator and the decomposition given in equation (2.8) is orthogonal with respect to the metric defined in equation (2.9) so that  $S^2 V$  and  $\Lambda^2 V$  both inherit the metric from equation (2.9).

A curvature tensor  $R$  is an element  $R \in S^2(\Lambda^2 V)$  satisfying the Bianchi identity

$$R(X, Y, Z, W) + R(Z, X, Y, W) + R(Y, Z, X, W) = 0 \quad \text{for all } X, Y, Z, W \in V.$$

The sectional curvature associated to  $R$  is the quadratic map  $\bar{R} : S^2 V \rightarrow \mathbb{R}$  defined by

$$\bar{R}(X, Y) := R(X, Y, Y, X), \quad X, Y \in V.$$

For every  $X, Y \in V$ , we can see  $R(X, Y, \cdot, \cdot)$  as a skew-symmetric endomorphism  $R(X, Y)$  as follows:

$$\langle R(X, Y)Z, W \rangle := R(X, Y, Z, W). \quad (2.10)$$

This skew-symmetric endomorphism extends as a derivation to skew-symmetric endomorphisms of the exterior, symmetric and tensor algebras of  $V$ , denoted respectively by  $R_{\Lambda^p}(X, Y)$ ,  $R_{S^p}(X, Y)$  and  $R_{V^{\otimes p}}(X, Y)$ . In particular, it can be easily checked that

$$R_{V^{\otimes 2}}(X, Y)u = [R(X, Y), u] \quad (2.11)$$

for every  $u \in V^{\otimes 2} = \text{End}(V)$ . The action equation (2.11) is diagonal with respect to the decomposition  $V^{\otimes 2} = S^2 V \oplus \Lambda^2 V$ . For  $X, Y \in V$  and  $\omega, \eta \in \Lambda^p V$ , we set

$$R_{\Lambda^p V}(X, Y, \omega, \eta) := \langle R_{\Lambda^p V}(X, Y)\omega, \eta \rangle,$$

where  $\Lambda^p V$  is equipped with the canonical inner product. We use the analogous notation for  $S^p V$ .

**2.2.2. Curvature and pinching.** If  $(M, g)$  is a Riemannian manifold, we introduce the  $(4, 0)$ -tensor  $g \odot g$  by

$$g \odot g(X, Y, Z, W) := g(X, Z)g(Y, W) - g(X, W)g(Y, Z)$$

for all  $X, Y, Z, W \in TM$ . This is precisely the curvature tensor when  $(M, g)$  is the real hyperbolic space  $\mathbb{H}^n$ , whereas if  $(M, g, J)$  is the complex hyperbolic space  $\mathbb{C}\mathbb{H}^n$ , its curvature tensor  $G$  takes the form:

$$4G(X, Y, Z, W) = g \odot g(X, Y, Z, W) + g \odot g(X, Y, JZ, JW) + 2g(X, JY)g(Z, JW), \quad (2.12)$$

see [KN96, §7, Ch. IX]. Equivalently, equation (2.12) can be rewritten using equation (2.10) as

$$4G(X, Y) = X \wedge Y + JX \wedge JY - 2\langle X, JY \rangle J, \quad (2.13)$$

where  $X \wedge Y$  is the skew-symmetric endomorphism of  $TM$  defined by  $(X \wedge Y)(Z) := g(X, Z)Y - g(Y, Z)X$  for all  $Z \in TM$ .

If  $(M, g, J)$  is any Kähler manifold, the holomorphic sectional curvature is defined for a unit vector  $X \in TM$  by

$$H(X) := \bar{R}(X, JX) = R(X, JX, JX, X).$$



It can be easily checked that the holomorphic curvature of the complex hyperbolic space is  $-1$ , that is,

$$H_{\mathbb{C}\mathbb{H}^m}(X) = G(X, JX, JX, X) = -1$$

for  $|X| = 1$ . By analogy with the real case, we introduce the notion of pinching of the holomorphic curvature (that we recall was also briefly introduced in the introduction).

*Definition 2.2.* (Pinched holomorphic sectional curvature) We say that a Kähler manifold  $(M^{2m}, g, J)$  is negatively holomorphically  $\lambda$ -pinched (for some  $\lambda \in (0, 1]$ ) if there exists a constant  $C > 0$  such that for all unit vectors  $X \in TM$ ,

$$-C \leq H(X) \leq -C\lambda.$$

The manifold is said to be strictly negatively holomorphically  $\lambda$ -pinched if the above inequalities are strict.

Similarly, one can talk about the real (or sectional)  $\delta$ -pinching of the manifold  $(M^{2m}, g)$  by requiring the above inequalities to hold with  $\lambda$  being replaced by  $\delta$ , and  $H(X)$  being replaced by the sectional curvature  $\bar{R}(X, Y)$  (for all pairs of orthogonal unit vectors  $X, Y \in TM$ ). There exist relations between holomorphic and real pinchings, see [Ber60a, Ber60b, BG63].

As in the real case, it is a well-known result that negative holomorphic 1-pinching implies that  $(M^{2m}, g, J)$  is holomorphically isometric to a compact quotient  $\Gamma \backslash \mathbb{C}\mathbb{H}^m$ , where  $\Gamma$  is a discrete subgroup of  $\text{Isom}(\mathbb{C}\mathbb{H}^m)$ . In what follows, we will always assume that  $(M^{2m}, g, J)$  is negatively  $\lambda$ -holomorphically pinched and, without loss of generality, we rescale the metric such that  $C = 1$ .

The following lemma proved in [BG63, Proposition 4.2] will be useful.

LEMMA 2.3. (Bishop and Goldberg 1963) *Assume  $(M^{2m}, g, J)$  is a closed Kähler manifold which is negatively holomorphically  $\lambda$ -pinched. Consider unit vectors  $X, Y \in TM$  such that  $g(X, Y) = 0$  and  $g(X, JY) = \cos \theta$ . Then,*

$$-(1 - \frac{3}{4}\lambda \sin^2 \theta) \leq \bar{R}(X, Y) \leq -\frac{1}{4}(3(1 + \cos^2 \theta)\lambda - 2). \tag{2.14}$$

In particular,

$$-1 \leq \bar{R}(X, Y) \leq -\frac{3\lambda - 2}{4}$$

for all orthogonal unit vectors  $X, Y \in TM$ . If  $\lambda \geq 2/3$ , then  $(M^{2m}, g)$  is negatively  $\delta$ -pinched for the sectional curvature with  $\delta = (3\lambda - 2)/4$ .

We now set

$$R = R_0 + \frac{1 + \lambda}{2}G, \tag{2.15}$$

where  $G$  is the curvature tensor defined in equation (2.12). The following holds.

LEMMA 2.4. Assume that  $-1 \leq H(X) \leq -\lambda$  for all unit vectors  $X \in TM$ . Then, for all unit vectors  $X, Y, Z, W \in TM$ ,

$$|R_0(X, Y, Z, W)| \leq \frac{4}{3}(1 - \lambda). \tag{2.16}$$

More generally, for all unit vectors  $X, Y \in TM$ , and for all unit  $\omega, \eta \in \Lambda^p TM$  or  $S^p TM$ ,

$$|(R_0)_{\Lambda^p TM}(X, Y, \omega, \eta)|, |(R_0)_{S^p TM}(X, Y, \omega, \eta)| \leq \frac{4p}{3}(1 - \lambda). \tag{2.17}$$

Proof. Let  $X, Y$  be unit vectors such that  $g(X, Y) = 0$  and  $g(X, JY) = \cos \theta$ . Then,

$$\overline{R_0}(X, Y) = \overline{R}(X, Y) - \frac{1 + \lambda}{2}\overline{G}(X, Y) = \overline{R}(X, Y) + \frac{1 + \lambda}{2}\left(1 - \frac{3}{4}\sin^2 \theta\right).$$

Inserting equation (2.14) in the previous equation,

$$-\frac{1 - \lambda}{2}\left(1 + \frac{3}{4}\sin^2 \theta\right) \leq \overline{R_0}(X, Y) \leq \frac{1 - \lambda}{2}\left(2 - \frac{3}{4}\sin^2 \theta\right).$$

In particular, the previous inequalities yield

$$|\overline{R_0}(X, Y)| \leq 1 - \lambda.$$

Like in the proof of [BK78, Lemma 3.7], the previous inequality then implies equation (2.16) by writing  $R_0(X, Y, Z, W)$  as a sum of terms only involving two vectors in the arguments. The general bound of equation (2.17) follows immediately from equation (2.16) by diagonalizing over  $\mathbb{C}$  the skew-symmetric endomorphism  $R_0(X, Y)$ .  $\square$

2.3. *Isometric extensions of the geodesic flow.* The unitary frame bundle  $\widehat{\pi} : F_{\mathbb{C}M} \rightarrow SM$  is a principal  $U(m - 1)$ -bundle over  $SM$ . Given  $a \in U(m - 1)$ , we denote by  $R_a : F_{\mathbb{C}M} \rightarrow F_{\mathbb{C}M}$  the fibrewise right-action by  $a$ . The unitary frame flow  $(\Phi_t)_{t \in \mathbb{R}}$  is an extension of the geodesic flow to a principal bundle in the sense that it satisfies

$$\pi \circ \Phi_t = \varphi_t \circ \pi, \quad R_a \circ \Phi_t = \Phi_t \circ R_a$$

for all  $t \in \mathbb{R}, a \in U(m - 1)$ . We will denote by  $X_{F_{\mathbb{C}M}}$  its infinitesimal generator.

Initiated by the work of Brin [Bri75b, Bri75a], there is now an established theory describing the ergodic components of such an extension flow  $(\Phi_t)_{t \in \mathbb{R}}$ . This is achieved via the introduction of a closed subgroup  $H \leq U(m - 1)$ , called the *transitivity group*, and defined by means of dynamical holonomies. We refer to [Lef23] for a modern construction of the transitivity group  $H$ . It has the following properties.

THEOREM 2.5. (Brin 1975, [Lef23]) *The following hold:*

- (i) *there exists a natural isomorphism*

$$\text{ev} : \ker_{L^2(\omega)} X_{F_{\mathbb{C}M}} \xrightarrow{\sim} L^2(H \backslash U(m - 1)); \tag{2.18}$$

- (ii) *there exists a principal  $H$ -subbundle  $Q \subset F_{\mathbb{C}M}$  over  $SM$  which is invariant by  $(\Phi_t)_{t \in \mathbb{R}}$  such that  $(\Phi_t|_Q)_{t \in \mathbb{R}}$  is ergodic (with respect to the induced measure on  $Q$ ). In particular, the unitary frame flow is ergodic if and only if  $H = U(m - 1)$ .*

We refer to [Lef23, Corollary 3.10] for further details. Obviously, by the first item, the only possibility for  $\ker_{L^2} X_{F_{\mathbb{C}M}}$  to be reduced to the constants is that  $H = U(m - 1)$ . The isomorphism in equation (2.18) is simply defined by taking an arbitrary point  $z_{\star} \in SM$  and setting for  $f \in \ker_{L^2} X_{F_{\mathbb{C}M}}$ ,

$$\text{ev}(f) := f|_{F_{\mathbb{C}M_{z_{\star}}}}.$$

Such a function turns out to be in  $L^2(F_{\mathbb{C}M_{z_{\star}}}) \simeq L^2(U(m - 1))$  (it is not clear *a priori* that such an evaluation map is well defined, so its definition is also part of Theorem 2.5) and is invariant by the action of  $H$  so it yields an element in  $L^2(H \backslash U(m - 1))$ . The second item in Theorem 2.5 is already a strong topological constraint on the bundle  $F_{\mathbb{C}M}$  and is called a *reduction of the structure group*, see §3 where this is further discussed.

We introduce  $\mathcal{N} \rightarrow SM$ , the *normal bundle*, to be the Euclidean bundle over  $SM$  defined for  $v \in S_x M$  as

$$\mathcal{N}(v) := \text{Span}(v, Jv)^{\perp},$$

where  $\perp$  denotes the orthogonal complement with respect to the Euclidean metric  $g_x$ . Note that  $\mathcal{N}$  is equipped with the complex structure  $J$ . Observe that

$$\pi^*TM = \mathcal{N} \oplus \mathbb{R}v \oplus \mathbb{R}Jv, \tag{2.19}$$

so that  $\mathcal{N}$  can be seen as a subbundle of the pullback bundle  $\pi^*TM$ . Parallel transport with respect to the Levi-Civita connection  $\nabla^{\text{LC}}$  of sections of  $\mathcal{N}$  along geodesic flow-lines is well defined and generated by a first-order differential operator

$$\mathbf{X} := (\pi^*\nabla^{\text{LC}})_X : C^{\infty}(SM, \mathcal{N}) \rightarrow C^{\infty}(SM, \mathcal{N}),$$

which is formally skew-adjoint and commutes with  $J$ .

Other than describing the ergodic components of the unitary frame flow, the group  $H$  allows to construct smooth flow-invariant objects. In what follows, we denote by  $\text{Vect}$  the category of finite-dimensional Euclidean vector spaces and call  $\circ : \text{Vect} \rightarrow \text{Vect}$  an *operation* on this category if  $\circ$  is obtained as a finite composition of the following basic operations: tensor powers  $V^{\otimes m}$  of a vector space  $V$ , symmetric powers  $S^m V$  and exterior powers  $\Lambda^m V$ . Obviously, for any such operation  $\circ$ ,  $\circ(\mathcal{N}) \rightarrow SM$  is a well-defined Euclidean bundle still equipped with an induced generator  $\mathbf{X}$  (for simplicity, we do not introduce any new notation for the generator on this bundle).

The following theorem holds.

**THEOREM 2.6.** (Non-Abelian Livšic theorem, [CL22, Theorem 3.5]) *Let*

$$\circ : \text{Vect} \rightarrow \text{Vect}$$

*be any operation on Vect. Then, there exists an isomorphism*

$$\ker \mathbf{X} \cap C^{\infty}(SM, \circ(\mathcal{N})) \xrightarrow{\sim} \{f \in \circ(\mathbb{R}^{2(m-1)}) \mid hf = f \text{ for all } h \in H\}.$$

The isomorphism map is nothing but evaluation at an arbitrary point of  $SM$  (similarly to Theorem 2.5(i)). In other words, flow-invariant smooth sections of tensor products of the normal bundle correspond exactly to algebraic  $H$ -invariant objects on  $\mathbb{R}^{2(m-1)}$ .

Theorem 2.6 will allow us to generate smooth flow-invariant sections when the unitary frame flow is not ergodic. For instance, if one can show that  $H \leq U(m - 1 - p) \times U(p) \not\cong U(m - 1)$ , that is,  $H$  acts reducibly on  $\mathbb{R}^{2(m-1)} \simeq \mathbb{C}^{m-1}$ , then  $H$  fixes an orthogonal projector  $\pi \in S^2\mathbb{R}^{2(m-1)}$  onto a complex (that is,  $J$ -invariant) space  $V \subset \mathbb{R}^{2(m-1)}$ . In turn, Theorem 2.6 implies that there exists a flow-invariant complex vector bundle  $\mathcal{V} \subset \mathcal{N}$  which is the same as the existence of an orthogonal projector  $\pi_{\mathcal{V}} \in C^\infty(SM, S^2\mathcal{N}) \cap \ker \mathbf{X}$  commuting with  $J$ .

3. Topological reductions and flow-invariant sections

In what follows, we assume that the complex dimension of  $M$  is even and larger than 2, and we write it as  $\dim_{\mathbb{C}} M = m =: 2p + 2$ , with  $p \geq 1$ .

3.1. Topological reductions. By Theorem 2.5, if the unitary frame flow on  $F_{\mathbb{C}}M$  is not ergodic, its transitivity group is a strict subgroup  $H \not\cong U(2p + 1)$  and there exists a strict principal  $H$ -subbundle  $Q \subset F_{\mathbb{C}}M$ . This is known as a reduction of the structure group of  $F_{\mathbb{C}}M$  to  $H$ . Since  $F_{\mathbb{C}}M$  admits a reduction to  $H$ , the same holds true for the restriction of the unitary frame bundle to any sphere  $S_{x_0}M$  for  $x_0 \in M$ . In turn, as  $S_{x_0}M \simeq S^{4p+3}$ , this implies that the unitary frame bundle  $F_{\mathbb{C}}S^{4p+3}$  admits a reduction of its structure group from  $U(2p + 1)$  to  $H$ .

Note that  $U(2p + 2)$  and  $SU(2p + 2)$  act transitively on  $S^{4p+3}$  with isotropy groups  $U(2p + 1)$  and  $SU(2p + 1)$ , respectively, so we can write

$$S^{4p+3} = SU(2p + 2)/SU(2p + 1) = U(2p + 2)/U(2p + 1).$$

The unitary frame bundle  $F_{\mathbb{C}}S^{4p+3}$  can be identified with  $U(2p + 2)$ . Thus, the subgroup  $SU(2p + 2)$  of  $U(2p + 2)$ , seen as a principal  $SU(2p + 1)$ -bundle  $F_{\mathbb{C},SU}S^{4p+3}$  over  $S^{4p+3}$ , is a reduction of  $F_{\mathbb{C}}S^{4p+3}$  to  $SU(2p + 1)$ .

The aim of this section is to examine the possible further reductions of  $F_{\mathbb{C},SU}S^{4p+3}$ . Note that as far as the spheres  $S^{4p+1}$  are concerned (for  $p \geq 1$ ), it was proved in [Leo71] that their special unitary frame bundle  $F_{\mathbb{C},SU}S^{4p+1} \rightarrow S^{4p+1}$  does not admit any reduction.

**THEOREM 3.1.** *Let  $p \geq 1$ . Assume that the principal  $SU(2p + 1)$ -bundle  $F_{\mathbb{C},SU}S^{4p+3}$  over  $S^{4p+3}$  admits a reduction of its structure group to a strict connected subgroup  $H_0 \not\cong SU(2p + 1)$ . Then one of the following holds:*

- (i) either the representation of  $H_0$  on  $\mathbb{C}^{2p+1}$  is reducible;
- (ii) or  $p = 1$ , and  $H_0$  is contained in  $SO(3) \not\cong SU(3)$ ;
- (iii) or  $p = 13$ , and  $H_0$  is contained in  $E_6 \not\cong SU(27)$ .

*Proof.* Let  $\hat{H}_0$  be a maximal strict subgroup of  $SU(2p + 1)$  containing  $H_0$ . Clearly the principal bundle  $P_0 = SU(2p + 2) \rightarrow S^{4p+3}$  also reduces to  $\hat{H}_0$ . Then [Leo71, Theorem 3] applied (with the notation of [Leo71]) to  $G_{2p+1} := SU(2p + 1)$  shows that  $\hat{H}_0$  is a simple Lie group.

If  $\hat{H}_0$  is a classical simple Lie group and  $p \geq 2$ , by [ČC06, Theorem 2.1, (D) and (E)] applied to  $G = \hat{H}_0$ , we immediately obtain that the representation of  $\hat{H}_0$  (and thus also that of  $H$ ) on  $\mathbb{C}^{2p+1}$  is reducible. The above result does not hold for  $p = 1$  (when the corresponding sphere  $S^7$  is parallelizable), but it is easy to check that the only simple Lie group strictly contained in  $SU(3)$  whose representation on  $\mathbb{C}^3$  is irreducible is  $SO(3)$ , embedded in  $SU(3)$  via the complexification of its standard representation on  $\mathbb{R}^3$ . This corresponds to case (ii) of Theorem 3.1.

It remains to study the case where  $\hat{H}_0$  is (a finite quotient of) one of the five exceptional simple compact Lie groups.

First of all, it suffices to look at complex irreducible representations of the exceptional Lie groups of odd dimension  $2p + 1$ . Moreover, by [ČC06, Proposition 3.1], writing  $4p + 3 = \dim \hat{H}_0 + k + 1$  for some integer  $k$ , there must exist at least  $k$  vector fields on the sphere  $S^{4p+3}$ , so the Radon–Hurwitz number  $\rho(n)$  (defined by the fact that  $\rho(n) - 1$  is the maximal number of linearly independent vector fields on  $S^{n-1}$ ) satisfies

$$\rho(4p + 4) \geq 4p + 3 - \dim \hat{H}_0.$$

For all  $p \geq 3$ , we have  $\rho(4p + 4) \leq 2p + 3$ . Since no exceptional Lie group has an irreducible complex representation of dimension less than 7, it follows that  $2p + 1$  has to be the dimension of an irreducible complex representation of an exceptional Lie group  $\hat{H}_0$ , with

$$7 \leq 2p + 1 \leq \dim \hat{H}_0 + 1. \tag{3.1}$$

Denoting by  $\hat{\mathfrak{h}}$  the Lie algebra of  $\hat{H}_0$ , it turns out that there is no complex odd-dimensional irreducible representation of an exceptional Lie group  $\hat{H}_0$  satisfying equation (3.1) except in the following two cases.

*Case 1.*  $\hat{\mathfrak{h}} = \mathfrak{e}_6$ ,  $\dim \hat{\mathfrak{h}} = 78$ . There are two 27-dimensional irreducible representations of  $\mathfrak{e}_6$  satisfying equation (3.1). This case corresponds to a (theoretical) reduction of the structure group  $SU(27)$  of  $F_{\mathbb{C},SU}S^{55}$  to a subgroup of  $E_6$  (case (iii) of Theorem 3.1).

*Case 2.*  $\hat{\mathfrak{h}} = \mathfrak{g}_2$ ,  $\dim \hat{\mathfrak{h}} = 14$ . The only complex odd-dimensional irreducible representation of  $\mathfrak{g}_2$  satisfying equation (3.1) is the complexification of the real 7-dimensional representation  $\rho_7 : G_2 \rightarrow SO(7)$  given by the embedding  $G_2 \subset SO(7)$  for  $p = 3$ . However, we will show that  $F_{\mathbb{C},SU}S^{15}$  does not admit any reduction to  $G_2$ .

Indeed, if such a reduction  $P_{G_2}$  exists, then the tangent bundle  $TS^{15}$  is isomorphic to the direct sum  $\mathbb{R} \oplus (\mathbb{C} \otimes F_7)$ , where  $F_7 := P_{G_2} \times_{\rho_7} \mathbb{R}^7$  is a real vector bundle of rank 7 over  $S^{15}$ .

Now, to each rank  $k$  real vector bundle  $E$  over  $S^{15}$ , one can associate an element  $\alpha(E)$  in the homotopy group  $\pi_{14}(SO(k))$ , namely, the homotopy class of its clutching function at the equator  $S^{14} \hookrightarrow S^{15}$ . For  $k < l$ , let  $f_{k,l} : SO(k) \rightarrow SO(l)$  be the standard embedding and denote by  $g_{k,l} : \pi_{14}(SO(k)) \rightarrow \pi_{14}(SO(l))$  the group morphisms induced by  $f_{k,l}$  in homotopy. If  $E$  and  $F$  have ranks  $k$  and  $l$ , respectively, clearly

$$\alpha(E \oplus F) = g_{k,k+l}(\alpha(E)) + g_{l,k+l}(\alpha(F)).$$

In our situation, since  $\mathbb{C} \otimes F_7$  is topologically isomorphic to  $F_7 \oplus F_7$ ,  $TS^{15}$  is isomorphic to  $\mathbb{R} \oplus F_7 \oplus F_7$ , so we can write

$$\begin{aligned} \alpha(TS^{15}) &= g_{14,15}(\alpha(F_7 \oplus F_7)) \\ &= g_{14,15}(2g_{7,14}(\alpha(F_7))) = 2g_{14,15}(g_{7,14}(\alpha(F_7))) = 0, \end{aligned}$$

because  $\pi_{14}(\text{SO}(15)) = \mathbb{Z}_2$ . This is a contradiction since the tangent bundle of  $S^{15}$  is non-trivial. Therefore, the case  $\widehat{\mathfrak{h}} = \mathfrak{g}_2$  is impossible, thus finishing the proof.  $\square$

Combined with Theorems 2.5 and 2.6, Theorem 3.1 yields the following corollary.

**COROLLARY 3.2.** *Let  $(M, g, J)$  be a closed connected Kähler manifold with even complex dimension  $m$  and non-ergodic unitary frame flow. Then, if  $m \neq 4, 28$ , there exists a finite cover  $(\widehat{M}, \widehat{g}, \widehat{J})$  of  $(M, g, J)$  and a flow-invariant orthogonal projector  $\pi_{\mathcal{V}} \in C^\infty(S\widehat{M}, S^2\mathcal{N})$  onto a complex subbundle  $\mathcal{V} \subset \mathcal{N}$  of rank  $1 \leq r \leq m/2 - 1$  of even Fourier degree.*

Note that by equation (2.19),  $\mathcal{N}$  is a subbundle of the pullback bundle  $\pi^*TM$  so it makes sense to talk about the decomposition of a section  $f \in C^\infty(SM, S^2\mathcal{N})$  as a sum of spherical harmonics as in equation (2.4). The fact that  $\mathcal{V}$  is complex is equivalent to the commutation relation  $[\pi_{\mathcal{V}}, J] = 0$ .

*Proof of Corollary 3.2.* Up to replacing  $M$  by a finite covering if necessary, we can assume that the transitivity group  $H \leq U(m - 1)$  is connected, see [CLMS21, Lemma 3.3]. In what follows, to keep notation simple, we will still denote this finite cover by  $M$ .

The representation  $\rho : H \rightarrow U(m - 1)$  induces a representation  $\det \rho : H \rightarrow U(1)$  whose image is either  $U(1)$  or  $\{1\}$  (by connectedness of  $H$ ). Following an argument of Brin and Gromov [BG80], we first show that  $(\det \rho)(H) = U(1)$ .

Indeed, assume that  $(\det \rho)(H) = \{1\}$ . As  $(\det \rho)(H)$  is the transitivity group of the frame flow of the complex line bundle  $\Lambda^{m-1,0}\mathcal{N}$ , we get by the non-Abelian Livšic Theorem 2.6 that  $\Lambda^{m-1,0}\mathcal{N}$  is trivial. Now, using that

$$\Lambda^{m-1,0}\mathcal{N} \rightarrow \Lambda^{m,0}\pi^*TM, \quad \omega \mapsto \omega \wedge (v - iJv)$$

is an isomorphism, the triviality of  $\Lambda^{m-1,0}\mathcal{N}$  implies that

$$c_1(\Lambda^{m,0}\pi^*TM) = \pi^*c_1(\Lambda^{m,0}TM) = 0 \in H^2(SM, \mathbb{Z}).$$

However, it can be easily checked using the Gysin sequence [BT82, Proposition 14.33] (and the fact that the dimension of  $M$  is  $n \geq 4$ ) that

$$\pi^* : H^2(M, \mathbb{Z}) \rightarrow H^2(SM, \mathbb{Z})$$

is injective, so  $c_1(\Lambda^{m,0}TM) = 0 = -c_1(K_M)$ , where  $K_M = \Lambda^{m,0}T^*M$  is the canonical line bundle. This is impossible since  $(M, g)$  has negative sectional curvature. Hence,  $(\det \rho)(H) = U(1)$ .

We will now show that the  $H$ -representation  $\rho$  on  $\mathbb{C}^{m-1}$  is reducible. Assume for a contradiction that  $\rho$  is irreducible. By the Schur lemma, the center  $C(H)$  of  $H$  is contained in the set  $U(1)$  of scalar matrices. However, the fact that  $(\det \rho)(H) = U(1)$  shows that  $H$

is not semi-simple, so its centre is at least one-dimensional. We thus obtain the equality  $C(H) = U(1)$ , that is,  $H$  contains the set of scalar matrices.

We now fix an arbitrary point  $x_0 \in M$ , restrict the unitary frame bundle  $F_{\mathbb{C}}M$  to a bundle over  $S_{x_0}M$  and identify  $S_{x_0}M \simeq S^{4p+3}$ . As the structure group of  $F_{\mathbb{C}}M$  reduces to  $H$  by Theorem 2.5, we obtain by restriction to any fibre of  $SM \rightarrow M$  that the structure group of  $F_{\mathbb{C}}S^{4p+3}$  also reduces to  $H$ , that is, there exists a principal  $H$ -bundle  $P_H \subset F_{\mathbb{C}}S^{4p+3}$ . We claim that  $F_{\mathbb{C}}S^{4p+3}$  admits a further reduction to  $H_0 := H \cap \text{SU}(2p + 1)$ . Indeed, the principal  $\text{SU}(2p + 1)$ -bundle  $F_{\mathbb{C},\text{SU}}S^{4p+3}$  is already a reduction of  $F_{\mathbb{C}}S^{4p+3}$  to  $\text{SU}(2p + 1)$  and for every  $v \in S^{4p+3}$ , the intersection of the fibres  $F_{\mathbb{C},\text{SU}}S^{4p+3}(v) \cap P_H(v)$  is non-empty: if  $u \in P_H(v) \subset F_{\mathbb{C}}S^{4p+3}(v)$ , there exists  $z \in U(1)$  such that  $uz \in F_{\mathbb{C},\text{SU}}S^{4p+3}(v)$ , and since  $U(1) \subset H$  (that is,  $H$  contains scalar matrices), we also have  $uz \in P_H(v)$ , so  $uz \in F_{\mathbb{C},\text{SU}}S^{4p+3}(v) \cap P_H(v)$ . It is then straightforward to check that  $F_{\mathbb{C},\text{SU}}S^{4p+3} \cap P_H$  is a principal bundle over  $S^{4p+3}$  with group  $H_0$ .

As  $m \neq 4, 28$  by assumption, we can apply case (i) of Theorem 3.1 to deduce that  $H_0 \not\leq \text{SU}(m - 1)$  acts reducibly on  $\mathbb{C}^{m-1}$ . However, as  $H$  was assumed to act irreducibly on  $\mathbb{C}^{m-1}$ ,  $H_0 = H \cap \text{SU}(m - 1)$  also acts irreducibly on  $\mathbb{C}^{m-1}$  and this is a contradiction. Therefore,  $H$  acts reducibly on  $\mathbb{C}^{m-1}$ .

We can then conclude the following using the non-Abelian Livšic Theorem 2.6: by the remark after Theorem 2.6, there exists a smooth (non-zero) flow-invariant orthogonal projector  $\pi_{\mathcal{V}'} \in C^\infty(SM, S^2\mathcal{N})$  onto a flow-invariant smooth complex bundle  $\mathcal{V}' \subset \mathcal{N}$ . Following [CLMS21, Lemma 3.10], one can find a (possibly different) smooth non-zero flow-invariant orthogonal projector  $\pi_{\mathcal{V}} \in C^\infty(SM, S^2\mathcal{N})$  of even Fourier degree onto a flow-invariant smooth complex bundle  $\mathcal{V} \subset \mathcal{N}$  of complex rank  $1 \leq r \leq m/2 - 1$ .  $\square$

3.2. *Complex normal twisted conformal Killing tensors.* If  $m \neq 4, 28$  and the unitary frame flow is not ergodic, we know by Corollary 3.2 that there exists a flow-invariant orthogonal projector  $\pi_{\mathcal{V}} \in C^\infty(SM, S^2\mathcal{N})$  of even Fourier degree and commuting with  $J$ . The flow-invariance condition is equivalent to  $\mathbf{X}\pi_{\mathcal{V}} = 0$ . A crucial step then is to show that such a flow-invariant section has finite Fourier degree, that is, the decomposition of equation (2.4) only involves a finite number of terms. This is the content of the following lemma.

LEMMA 3.3. *For any operation  $\circ : \text{Vect} \rightarrow \text{Vect}$ , a section  $f \in C^\infty(SM, \circ(\mathcal{N}))$  satisfying  $\mathbf{X}f = 0$  has finite Fourier degree.*

The proof of Lemma 3.3 uses the fact that the sectional curvature of  $(M, g)$  is negative and relies on the twisted Pestov identity in equation (2.6). Lemma 3.3 was first obtained in [GPSU16, Theorem 4.1], see also [CLMS22, Corollary 4.2] for a short self-contained proof. As a consequence, we can decompose

$$\pi_{\mathcal{V}} = u_k + u_{k-2} + \dots + u_2 + u_0, \tag{3.2}$$

where  $k \geq 0$  is even,  $u_i \in C^\infty(M, \Omega_i \otimes S^2TM)$  and  $u_k \neq 0$ . Moreover, since  $J$  has degree 0 (that is, it does not depend on the velocity variable  $v$ ), the commutation relation  $[\pi_{\mathcal{V}}, J] = 0$  yields  $[u_i, J] = 0$  for all  $i \in \{0, \dots, k\}$ .

We now set

$$u := u_k \in C^\infty(M, \Omega_k \otimes S^2TM),$$

the spherical harmonic of highest degree in the decomposition of equation (3.2) of  $\pi_{\mathcal{V}}$ . Using the mapping property in equation (2.5) of  $\mathbf{X}$ , the equation  $\mathbf{X}\pi_{\mathcal{V}} = 0$  then gives  $\mathbf{X}_+u = 0$ . Such a section  $u$  is called a *twisted conformal Killing tensor* in the literature. Moreover, since  $\iota_v\pi_{\mathcal{V}} := \pi_{\mathcal{V}}v = 0$  (because  $\mathcal{V}$  is orthogonal to the span of  $v$  and  $Jv$ ) and  $\iota_v$  has the mapping properties

$$\iota_v : \Omega_k \otimes S^2TM \rightarrow (\Omega_{k-1} \otimes TM) \oplus (\Omega_{k+1} \otimes TM),$$

we obtain that  $\iota_vu \in C^\infty(M, \Omega_{k-1} \otimes TM)$  is of degree  $k - 1$ . The same argument also shows that  $\iota_{Jv}u \in C^\infty(M, \Omega_{k-1} \otimes TM)$ .

Using similarly that  $\iota_v\iota_v\pi_{\mathcal{V}} = \langle \pi_{\mathcal{V}}v, v \rangle = 0$ , a refined algebraic argument allows to show that  $\iota_v\iota_vu = \iota_{Jv}\iota_{Jv}u \in C^\infty(M, \Omega_{k-2})$  is of degree  $k - 2$ , see [CLMS21, Lemma 4.2] for a proof. Furthermore, we have  $\iota_v\iota_{Jv}u = \iota_{Jv}\iota_vu = 0$ , using that  $u$  is symmetric and  $J$  is skew-symmetric, and  $[u, J] = 0$ .

A section  $u \in C^\infty(M, \Omega_k \otimes S^2TM)$  satisfying

$$\mathbf{X}_+u = 0, \quad \iota_vu \text{ has degree } k - 1, \quad \iota_v\iota_vu \text{ has degree } k - 2, \tag{3.3}$$

was called in [CLMS21, §4.1] a *normal twisted conformal Killing tensor*. (The adjective *normal* refers to the conditions on  $\iota_vu$  and  $\iota_v\iota_vu$ .) Here, the section  $u$  satisfies the extra condition  $[u, J] = 0$ . It is thus worth introducing the following terminology.

*Definition 3.4.* A section  $u \in C^\infty(M, \Omega_k \otimes S^2TM)$  satisfying equation (3.3) and  $[u, J] = 0$  is called a *complex normal twisted conformal Killing tensor*.

By Corollary 3.2 and the discussion above, we obtain the following corollary.

**COROLLARY 3.5.** *Let  $(M, g, J)$  be a closed connected Kähler manifold with even complex dimension  $m$  and non-ergodic unitary frame flow. Then, if  $m \neq 4, 28$ , there exists a non-zero complex normal twisted conformal Killing tensor  $u$  of even degree  $k \geq 2$ .*

*Proof.* Corollary 3.5 follows immediately from Corollary 3.2 and the above discussion, except for the point that  $k \geq 2$  which we now prove. If  $k = 0$ , then  $\pi_{\mathcal{V}} = u = u_0$  is of degree 0 and thus  $\pi_{\mathcal{V}}$  can be identified with a section in  $C^\infty(M, S^2TM)$ . However, we must also have  $\iota_v\pi_{\mathcal{V}} = 0$  for all  $v \in TM$  by equation (3.3), so  $\pi_{\mathcal{V}} = 0$ , which contradicts the non-vanishing of  $\pi_{\mathcal{V}}$ . □

The aim of the remaining sections is now to rule out the existence of such a non-zero complex twisted conformal Killing tensor of even degree  $k \geq 2$  under a holomorphic pinching condition  $\lambda > \lambda(m)$ .

#### 4. Bounding the terms in the twisted Pestov identity

Throughout this section,  $(M, g, J)$  is a negatively curved compact Kähler manifold of real dimension  $n = 2m$  with  $\lambda$ -pinched holomorphic curvature, and  $u \in C^\infty(M, \Omega_k \otimes S^2TM)$  is a complex normal twisted conformal Killing tensor of even degree  $k \geq 2$ .



Our aim is to bound from above the terms appearing on the right-hand side of the twisted Pestov identity in equation (2.6), namely, the first term  $\langle R\nabla_{\mathbb{V}}^{S^2TM}u, \nabla_{\mathbb{V}}^{S^2TM}u \rangle_{L^2}$  and the second term  $\langle \mathcal{F}^{S^2TM}u, \nabla_{\mathbb{V}}^{S^2TM}u \rangle_{L^2}$ , and to bound from below the term  $\|\mathbf{X}_-u\|_{L^2}^2$  on the left-hand side. Sometimes, it will be convenient to consider general vector bundles  $E \rightarrow M$  rather than the specific bundle  $S^2TM$ .

To simplify notation, we will drop the volume forms in the integrands. The reader should keep in mind that integrals over spheres  $S_xM$  (for  $x \in M$ ) are always computed with respect to the round measure  $|dv|$  on the sphere, while integrals over  $M$  are computed with respect to the Riemannian measure  $|dx|$  induced by the metric  $g$ . Moreover, we will often work with expressions involving a local orthonormal basis of a vector bundle  $E$ , usually denoted by  $(\mathbf{e}_\alpha)_\alpha$ ; for the simplicity of notation, when we write sums over  $\alpha$ , we will mean that the sums are pointwise (and the basis might change from point to point). We also introduce the following constants:

$$\begin{aligned} \alpha_{n,k} &:= k(n+k-2), & \beta_{n,k} &:= (k(n+k-2)(n-1))^{1/2}, \\ \gamma_{n,k} &:= \frac{(n+k-2)(n+2k-4)k}{(n+k-3)(n+2k-2)(k-1)}, & \delta_{n,k} &:= n+2k-4. \end{aligned}$$

4.1. *Bounding the first term in the right-hand side.* Let  $E \rightarrow M$  be a Euclidean vector bundle equipped with an orthogonal connection  $\nabla^E$ . Then, the following lemma holds.

LEMMA 4.1. *For all  $f \in C^\infty(M, \Omega_k \otimes E)$ ,*

$$\langle R\nabla_{\mathbb{V}}^E f, \nabla_{\mathbb{V}}^E f \rangle_{L^2} \leq -\frac{3\lambda-2}{4}\alpha_{n,k}\|f\|_{L^2}^2 - \frac{3\lambda}{4} \int_M \int_{S_xM} \sum_{\alpha} \langle v, J\nabla_{\mathbb{V}} f_{\alpha} \rangle^2,$$

where we write locally  $f = \sum_{\alpha} f_{\alpha} \mathbf{e}_{\alpha}$  for  $(\mathbf{e}_{\alpha})_{\alpha \in I}$  a local orthonormal basis of  $E$ .

*Proof.* Using the upper bound in equation (2.14) on the sectional curvature from Lemma 2.3,

$$\begin{aligned} \langle R\nabla_{\mathbb{V}}^E f, \nabla_{\mathbb{V}}^E f \rangle_{L^2} &= \int_M \int_{S_xM} \sum_{\alpha} R(v, \nabla_{\mathbb{V}}^E f_{\alpha}, \nabla_{\mathbb{V}}^E f_{\alpha}, v) \\ &\leq -\frac{3\lambda-2}{4}\|\nabla_{\mathbb{V}}^E f\|_{L^2}^2 - \frac{3\lambda}{4} \int_M \int_{S_xM} \sum_{\alpha} \langle v, J\nabla_{\mathbb{V}} f_{\alpha} \rangle^2 \\ &= -\frac{3\lambda-2}{4}\langle \Delta_{\mathbb{V}}^E f, f \rangle_{L^2} - \frac{3\lambda}{4} \int_M \int_{S_xM} \sum_{\alpha} \langle v, J\nabla_{\mathbb{V}} f_{\alpha} \rangle^2 \\ &= -\frac{3\lambda-2}{4}k(n+k-2)\|f\|_{L^2}^2 - \frac{3\lambda}{4} \int_M \int_{S_xM} \sum_{\alpha} \langle v, J\nabla_{\mathbb{V}} f_{\alpha} \rangle^2. \end{aligned}$$

Since  $\alpha_{n,k} := k(n+k-2)$ , this completes the proof. □

4.2. *Bounding the second term on the right-hand side.* Assume now that  $E = \Lambda^p TM$  or  $E = S^p TM$  for some  $p \geq 1$ . Using the decomposition of the Riemannian curvature tensor  $R = R_0 + (1 + \lambda)/2G$  in equation (2.15), we can write the second term on the

right-hand side of the Pestov identity in equation (2.6) as

$$\langle \mathcal{F}^E f, \nabla_{\mathbb{V}}^E f \rangle_{L^2} = \langle \mathcal{F}_0^E f, \nabla_{\mathbb{V}}^E f \rangle_{L^2} + \frac{1 + \lambda}{2} \langle \mathcal{G}^E f, \nabla_{\mathbb{V}}^E f \rangle_{L^2}. \tag{4.1}$$

More precisely,  $\mathcal{F}_0^E$  and  $\mathcal{G}^E$  are defined from  $R_0$  and  $G$ , respectively, by extending the latter to  $E$  as in §2.2.1 and using equation (2.7). We now study separately the two terms in equation (4.1). We start with the following lemma.

LEMMA 4.2. *If  $E = \Lambda^p TM$  or  $E = S^p TM$ , then for all  $f \in C^\infty(M, \Omega_k \otimes E)$ ,*

$$|\langle \mathcal{F}_0^E f, \nabla_{\mathbb{V}}^E f \rangle_{L^2}| \leq \frac{4p}{3} (1 - \lambda) \beta_{n,k} \|f\|_{L^2}^2.$$

Lemma 4.2 will be applied with  $E = TM$  and  $E = S^2 TM$ .

*Proof.* The proof is the same as [CLMS21, Lemma 4.5] by inserting the bound in equation (2.17). □

We now study the second term in equation (4.1).

LEMMA 4.3. *Let  $f \in C^\infty(M, \Omega_k \otimes E)$  such that  $\iota_v f, \iota_{Jv} f$  are of degree  $k - 1$ . Then, the following hold:*

(i) *if  $E = TM$ ,*

$$\begin{aligned} \langle \mathcal{G}^{TM} f, \nabla_{\mathbb{V}}^{TM} f \rangle_{L^2} &= \frac{1}{4} \delta_{n,k} (\|\iota_v f\|_{L^2}^2 + \|\iota_{Jv} f\|_{L^2}^2) + \frac{1}{2} \|f\|_{L^2}^2 \\ &\quad + \frac{1}{2} \int_M \int_{S_x M} \sum_{\alpha} \langle v, J \nabla_{\mathbb{V}} f_{\alpha} \rangle \langle f, J \mathbf{e}_{\alpha} \rangle; \end{aligned}$$

(ii) *if  $E = S^2 TM$  and  $[J, f] = 0$ ,*

$$\langle \mathcal{G}^{S^2 TM} f, \nabla_{\mathbb{V}}^{S^2 TM} f \rangle_{L^2} = \delta_{n,k} \|\iota_v f\|_{L^2}^2 + \|f\|_{L^2}^2.$$

*Proof.* We start with the proof for  $E = TM$ . Note that this equality is an integral equality over  $SM$ . We will actually prove the integral equality over  $S_x M$  for every  $x \in M$ , and then it suffices to integrate over  $x \in M$  to obtain the result. Recall that  $G$  is defined in equation (2.12). Using the expressions in equations (2.7) and (2.13),

$$\begin{aligned} 4 \sum_{\alpha} G(v, \nabla_{\mathbb{V}} f_{\alpha}, f, \mathbf{e}_{\alpha}) &= \sum_{\alpha} (\langle (v \wedge \nabla_{\mathbb{V}} f_{\alpha}) f, \mathbf{e}_{\alpha} \rangle \\ &\quad + \langle (Jv \wedge J \nabla_{\mathbb{V}} f_{\alpha}) f, \mathbf{e}_{\alpha} \rangle + 2 \langle v, J \nabla_{\mathbb{V}} f_{\alpha} \rangle \langle f, J \mathbf{e}_{\alpha} \rangle). \end{aligned} \tag{4.2}$$

The integral over  $S_x M$  of the first term on the right-hand side can be immediately computed using [CLMS21, Lemma 4.6] (in the  $\Lambda^p$  case with  $p = 1$ ; we warn the reader that, in the notation of [CLMS21],  $G$  denotes the curvature tensor of the real hyperbolic space, that is,  $G = g \otimes g$ ). The term  $\sum_{\alpha} \int_{S_x M} \langle v, f \rangle \langle \nabla_{\mathbb{V}} f_{\alpha}, \mathbf{e}_{\alpha} \rangle - \langle v, \mathbf{e}_{\alpha} \rangle \langle \nabla_{\mathbb{V}} f_{\alpha}, f \rangle$  thus corresponds exactly to the term computed in [CLMS21, Equation after (4.14)] with  $p = 1$  since  $\iota_v f$  is of degree  $k - 1$  and yields

$$\begin{aligned} \int_{S_x M} \sum_{\alpha} \langle (v \wedge \nabla_{\mathbb{V}} f_{\alpha}) f, \mathbf{e}_{\alpha} \rangle &= \int_{S_x M} \left( \sum_{\alpha} \langle v, f \rangle \langle \nabla_{\mathbb{V}} f_{\alpha}, \mathbf{e}_{\alpha} \rangle - \langle v, \mathbf{e}_{\alpha} \rangle \langle \nabla_{\mathbb{V}} f_{\alpha}, f \rangle \right) \\ &= (n + 2k - 4) \|\iota_v f\|_{L^2(S_x M)}^2 + \|f\|_{L^2(S_x M)}^2, \end{aligned}$$

where we use the notation

$$\|f\|_{L^2(S_x M)}^2 = \int_{S_x M} g_x(f(v), f(v)), \quad \|\iota_v f\|_{L^2(S_x M)}^2 = \int_{S_x M} |(\iota_v f)(v)|^2.$$

We claim that the integral over  $S_x M$  of the second term in equation (4.2) is equal to  $(n + 2k - 4) \|\iota_{Jv} f\|_{L^2(S_x M)}^2 + \|f\|_{L^2(S_x M)}^2$ . Indeed, observe first that for each  $\alpha$ ,

$$\langle (Jv \wedge J\nabla_{\mathbb{V}} f_{\alpha}) f, \mathbf{e}_{\alpha} \rangle = \langle (v \wedge \nabla_{\mathbb{V}} f_{\alpha}) Jf, J\mathbf{e}_{\alpha} \rangle.$$

Now, since  $f_{\alpha} = \langle f, \mathbf{e}_{\alpha} \rangle = \langle Jf, J\mathbf{e}_{\alpha} \rangle$ , changing the basis  $(\mathbf{e}_{\alpha})$  by  $(J\mathbf{e}_{\alpha})$ , we see that the second term in equation (4.2) is the same as the first term with  $f$  replaced by  $Jf$ . Using  $\iota_v Jf = -\iota_{Jv} f$ , the result now follows from the previous computation. Inserting the previous equality in equation (4.2) gives the desired result (after integration over  $M$ ) since  $\delta_{n,k} = n + 2k - 4$ .

We now deal with the case  $E = S^2 TM$ . As before, using the expressions in equations (2.7), (2.11) and (2.13),

$$\begin{aligned} 4 \sum_{\alpha} G_{S^2 TM}(v, \nabla_{\mathbb{V}} f_{\alpha}, f, \mathbf{e}_{\alpha}) &= \sum_{\alpha} (\langle [(v \wedge \nabla_{\mathbb{V}} f_{\alpha}), f], \mathbf{e}_{\alpha} \rangle \\ &\quad + \langle [(Jv \wedge J\nabla_{\mathbb{V}} f_{\alpha}), f], \mathbf{e}_{\alpha} \rangle - 2\langle v, J\nabla_{\mathbb{V}} f_{\alpha} \rangle \cdot \langle [J, f], \mathbf{e}_{\alpha} \rangle), \end{aligned} \tag{4.3}$$

where  $(\mathbf{e}_{\alpha})_{\alpha}$  is a local orthonormal basis of  $S^2 TM$ . The integral over  $S_x M$  of the first term can be computed using [CLMS21, Lemma 4.6] (case of symmetric 2-tensors) since  $\iota_v f$  is of degree  $k - 1$  and yields

$$\int_{S_x M} \sum_{\alpha} \langle [(v \wedge \nabla_{\mathbb{V}} f_{\alpha}), f], \mathbf{e}_{\alpha} \rangle = 2(n + 2k - 4) \|\iota_v f\|_{L^2(S_x M)}^2 + 2\|f\|_{L^2(S_x M)}^2. \tag{4.4}$$

The third term vanishes in equation (4.3) since  $[J, f] = 0$  by assumption.

We now claim that the integral over  $S_x M$  of the second term in equation (4.3) is also equal to  $2(n + 2k - 4) \|\iota_v f\|_{L^2(S_x M)}^2 + 2\|f\|_{L^2(S_x M)}^2$ , which will finish the proof. Indeed, observe that  $JX \wedge JY = -J \circ (X \wedge Y) \circ J$  for any  $X, Y \in TM$ , and therefore using also  $[J, f] = 0$ , we get for each  $\alpha$ ,

$$[Jv \wedge J\nabla_{\mathbb{V}} f_{\alpha}, f] = -J \circ [v \wedge \nabla_{\mathbb{V}} f_{\alpha}, f] \circ J.$$

Consequently, we can rewrite

$$\langle [Jv \wedge J\nabla_{\mathbb{V}} f_{\alpha}, f], \mathbf{e}_{\alpha} \rangle = \langle -J \circ [v \wedge \nabla_{\mathbb{V}} f_{\alpha}, f] \circ J, \mathbf{e}_{\alpha} \rangle = \langle [v \wedge \nabla_{\mathbb{V}} f_{\alpha}, f], -J \circ \mathbf{e}_{\alpha} \circ J \rangle.$$

Note that  $(-J \circ \mathbf{e}_{\alpha} \circ J)_{\alpha}$  is also an orthonormal basis of  $S^2 TM$ , and since  $f = -J \circ f \circ J = -\sum_{\alpha} f_{\alpha} J \circ \mathbf{e}_{\alpha} \circ J$ ,

$$f_{\alpha} = \langle f, \mathbf{e}_{\alpha} \rangle = \langle f, -J \circ \mathbf{e}_{\alpha} \circ J \rangle.$$

The claim thus follows from equation (4.4). □

4.3. *Bounding from below the left-hand side.* Going back to the case  $E = S^2TM$ , we now bound from below the term  $\|\mathbf{X}_-u\|_{L^2}^2$  appearing on the left-hand side of the twisted Pestov identity in equation (2.6).

LEMMA 4.4. *Let  $u \in C^\infty(M, \Omega_k \otimes S^2TM)$  be a complex normal twisted conformal Killing tensor in the sense of Definition 3.4. Then, for  $k > 0$ , the following inequality holds:*

$$\begin{aligned} & \frac{(k-1)(n+2k-2)}{k} \|\mathbf{X}_-u\|_{L^2}^2 \\ & \geq \left( \frac{3\lambda-2}{2} \alpha_{n,k-1} - \frac{8(1-\lambda)}{3} \beta_{n,k-1} - \frac{29(1+\lambda)}{48} - \frac{1+\lambda}{4} \delta_{n,k-1} \right) \|\iota_v u\|_{L^2}^2. \end{aligned}$$

The proof of Lemma 4.4 requires an additional step and is postponed to the end of this paragraph.

LEMMA 4.5. *Let  $f \in C^\infty(M, \Omega_k \otimes TM)$  such that  $\iota_v f, \iota_{Jv} f$  are of degree  $k-1$ , and assume  $\lambda \in [2/3, 1]$ . Then, the following holds:*

$$\begin{aligned} \frac{k(n+2k)}{k+1} \|\mathbf{X}_+f\|_{L^2}^2 & \geq \left( \frac{3\lambda-2}{4} \alpha_{n,k} - \frac{4(1-\lambda)}{3} \beta_{n,k} - \frac{29(1+\lambda)}{96} \right) \|f\|_{L^2}^2 \\ & \quad - \frac{1+\lambda}{8} \delta_{n,k} (\|\iota_v f\|_{L^2}^2 + \|\iota_{Jv} f\|_{L^2}^2). \end{aligned}$$

*Proof.* Using the twisted Pestov identity in equation (2.6) with  $E = TM$ , and applying the bounds provided by Lemmas 4.1, 4.2 and 4.3,

$$\begin{aligned} & \frac{k(n+2k)}{k+1} \|\mathbf{X}_+f\|_{L^2}^2 \\ & \geq -\langle R \nabla_{\mathbb{V}}^{TM} f, \nabla_{\mathbb{V}}^{TM} f \rangle_{L^2} - \langle \mathcal{F}^{TM} f, \nabla_{\mathbb{V}}^{TM} f \rangle_{L^2} \\ & \geq \frac{3\lambda-2}{4} \alpha_{n,k} \|f\|_{L^2}^2 + \frac{3\lambda}{4} \int_M \int_{S_x M} \sum_{\alpha} \langle v, J \nabla_{\mathbb{V}} f_{\alpha} \rangle^2 - \frac{4(1-\lambda)}{3} \beta_{n,k} \|f\|_{L^2}^2 \\ & \quad - \frac{1+\lambda}{2} \left( \frac{1}{4} \delta_{n,k} (\|\iota_v f\|_{L^2}^2 + \|\iota_{Jv} f\|_{L^2}^2) + \frac{1}{2} \|f\|_{L^2}^2 \right) \\ & \quad + \frac{1}{2} \int_M \int_{S_x M} \sum_{\alpha} |\langle v, J \nabla_{\mathbb{V}} f_{\alpha} \rangle| |\langle f, J \mathbf{e}_{\alpha} \rangle|. \end{aligned}$$

We now use the estimate

$$\frac{1}{2} \int_M \int_{S_x M} \sum_{\alpha} \langle v, J \nabla_{\mathbb{V}} f_{\alpha} \rangle \langle f, J \mathbf{e}_{\alpha} \rangle \leq \frac{1}{4\varepsilon} \|f\|_{L^2}^2 + \frac{\varepsilon}{4} \int_M \int_{S_x M} \sum_{\alpha} \langle v, J \nabla_{\mathbb{V}} f_{\alpha} \rangle^2,$$

which holds for all  $\varepsilon > 0$ , to deduce that

$$\begin{aligned} \frac{k(n+2k)}{k+1} \|\mathbf{X}_+f\|_{L^2}^2 & \geq \frac{3\lambda-2}{4} \alpha_{n,k} \|f\|_{L^2}^2 + \frac{3\lambda}{4} \int_M \int_{S_x M} \sum_{\alpha} \langle v, J \nabla_{\mathbb{V}} f_{\alpha} \rangle^2 \\ & \quad - \frac{4(1-\lambda)}{3} \beta_{n,k} \|f\|_{L^2}^2 \end{aligned}$$

$$\begin{aligned}
 & -\frac{1+\lambda}{2} \left( \frac{1}{4} \delta_{n,k} (\|\iota_v f\|_{L^2}^2 + \|\iota_{Jv} f\|_{L^2}^2) + \left( \frac{1}{2} + \frac{1}{4\varepsilon} \right) \|f\|_{L^2}^2 \right. \\
 & \left. + \frac{\varepsilon}{4} \int_M \int_{S_x M} \sum_{\alpha} \langle v, J \nabla_{\mathbb{V}} f_{\alpha} \rangle^2 \right).
 \end{aligned}$$

Taking the specific value  $\varepsilon := 6\lambda/(1 + \lambda)$ , we see that the coefficients in front of the term  $\int_M \sum_{\alpha} \int_{S_x M} \langle v, J \nabla_{\mathbb{V}} f_{\alpha} \rangle^2$  cancel out. Moreover, since by assumption  $\lambda \in [2/3, 1]$ , we have the lower bound  $\varepsilon = 6\lambda/(1 + \lambda) \geq 12/5$  and this eventually yields the result.  $\square$

*Remark 4.6.* In the last paragraph of the proof of Lemma 4.5, one could decide to keep using the exact value  $\varepsilon = 6\lambda/(1 + \lambda)$  in the estimate. However, the benefit of doing so would be minor in the final result so, for simplicity, we decided to use the trivial lower bound  $\varepsilon \geq 12/5$ .

We can now prove Lemma 4.4.

*Proof of Lemma 4.4.* Using the equality  $\mathbf{X}(\iota_v u) = \iota_v \mathbf{X}u = \iota_v \mathbf{X}_- u$  and the fact that  $\iota_{Jv} \mathbf{X}_- u = J(\iota_v \mathbf{X}_- u)$ ,

$$\begin{aligned}
 \|\mathbf{X}_- u\|_{L^2}^2 & \geq \|\iota_v \mathbf{X}_- u\|_{L^2}^2 + \|\iota_{Jv} \mathbf{X}_- u\|_{L^2}^2 = 2\|\iota_v \mathbf{X}_- u\|_{L^2}^2 \\
 & = 2\|\mathbf{X}(\iota_v u)\|_{L^2}^2 = 2\|\mathbf{X}_+(\iota_v u)\|_{L^2}^2 + 2\|\mathbf{X}_-(\iota_v u)\|_{L^2}^2 \\
 & \geq 2\|\mathbf{X}_+(\iota_v u)\|_{L^2}^2,
 \end{aligned}$$

where in the second line, we used that  $\iota_v u$  and  $\iota_{Jv} u$  are of degree  $k - 1$  and the mapping property in equation (2.5). By assumption,  $u$  is a complex normal twisted conformal Killing tensor, so this implies that  $\iota_v \iota_{Jv} u = \iota_{Jv} \iota_v u = 0$  and  $\iota_v \iota_v u = \iota_{Jv} \iota_{Jv} u$  is of degree  $k - 2$ . As a consequence, we can apply Lemma 4.5 with  $f := \iota_v u$  (which is of degree  $k - 1$ ). Using the fact that  $\iota_{Jv} f = 0$ , together with the fact that

$$\|\iota_v f\|_{L^2}^2 = \|\iota_v \iota_v u\|_{L^2}^2 \leq \|\iota_v u\|_{L^2}^2,$$

we obtain the announced result.  $\square$

### 5. Pinching estimates

In this section, we prove our main result, Theorem 1.1.

5.1. *Computations.* We start with the following lemma.

LEMMA 5.1. *If  $u \in C^\infty(M, \Omega_k \otimes S^2 T M)$  is a complex normal twisted conformal Killing tensor, the following inequality holds:*

$$B_{n,k}(\lambda) \|u\|_{L^2}^2 + C_{n,k}(\lambda) \|\iota_v u\|_{L^2}^2 \leq 0, \tag{5.1}$$

where

$$B_{n,k}(\lambda) := \frac{3\lambda - 2}{4} \alpha_{n,k} - \frac{8}{3} (1 - \lambda) \beta_{n,k} - \frac{1 + \lambda}{2} \tag{5.2}$$

and

$$C_{n,k}(\lambda) := \gamma_{n,k} \left( \frac{3\lambda - 2}{2} \alpha_{n,k-1} - \frac{8(1 - \lambda)}{3} \beta_{n,k-1} - \frac{29(1 + \lambda)}{48} - \frac{1 + \lambda}{4} \delta_{n,k-1} \right) - \frac{1 + \lambda}{2} \delta_{n,k}. \tag{5.3}$$

*Proof.* Straightforward computation, inserting in the twisted Pestov identity in equation (2.6) the lower bound for  $\|\mathbf{X}_-u\|^2$  (Lemma 4.4), the upper bounds for the terms on the right-hand side (Lemmas 4.1 and 4.2 applied to  $f = u$ , and Lemma 4.3(ii)), as well as using the facts that  $\mathbf{X}_+u = 0$  (by the definition in equation (3.3)) and  $\|Z(u)\|_{L^2}^2 \geq 0$ .  $\square$

Recall that  $n = 2m$  is the real dimension of  $M$ . The end of the proof of Theorem 1.1 then consists in finding a pinching condition  $\lambda > \lambda(m)$  for which the left-hand side of equation (5.1) is non-negative, thus forcing the complex normal twisted conformal Killing tensor  $u$  to be zero, which then contradicts Corollary 3.2. More precisely, we have the following lemma.

LEMMA 5.2. *If the inequalities*

$$B_{n,k}(\lambda) > 0 \quad \text{and} \quad B_{n,k}(\lambda) + \frac{1}{2}C_{n,k}(\lambda) > 0 \tag{5.4}$$

*hold, then equation (5.1) implies  $u \equiv 0$ .*

*Proof.* If  $C_{n,k}(\lambda) \geq 0$ , this is immediate from the first part of equation (5.4). If  $C_{n,k}(\lambda) < 0$ , using that  $\|u\|_{L^2}^2 \geq \|\iota_v u\|_{L^2}^2 + \|\iota_{Jv} u\|_{L^2}^2 = 2\|\iota_v u\|_{L^2}^2$ , we get by equation (5.1):

$$(B_{n,k}(\lambda) + \frac{1}{2}C_{n,k}(\lambda))\|u\|_{L^2}^2 \leq 0,$$

so  $u \equiv 0$  by the second part of equation (5.4).  $\square$

Moreover, the following lemma holds.

LEMMA 5.3. *Assume  $n \geq 4$  and  $k \geq 2$ . One has  $B_{n,k} > 0 \iff \lambda > \lambda_1(n, k)$ , where*

$$\lambda_1(n, k) := \frac{6\alpha_{n,k} + 32\beta_{n,k} + 6}{9\alpha_{n,k} + 32\beta_{n,k} - 6},$$

*and  $B_{n,k} + \frac{1}{2}C_{n,k} > 0 \iff \lambda > \lambda_2(n, k)$ , where*

$$\lambda_2(n, k) := \frac{6\alpha_{n,k} + 32\beta_{n,k} + 6 + \gamma_{n,k}(6\alpha_{n,k-1} + 16\beta_{n,k-1} + 29/8 + 3/2\delta_{n,k-1}) + 3\delta_{n,k}}{9\alpha_{n,k} + 32\beta_{n,k} - 6 + \gamma_{n,k}(9\alpha_{n,k-1} + 16\beta_{n,k-1} - 29/8 - 3/2\delta_{n,k-1}) - 3\delta_{n,k}}.$$

*Proof.* Follows immediately from equations (5.2) and (5.3), as the expressions of  $B_{n,k}$  and  $C_{n,k}$  are affine functions in  $\lambda$ .  $\square$

Before proving Theorem 1.1, we need to study the variations of the sequences  $k \mapsto \lambda_1(n, k)$  and  $k \mapsto \lambda_2(n, k)$ .

LEMMA 5.4. *For  $n \geq 4$ , the sequences  $k \mapsto \lambda_1(n, k)$  and  $k \mapsto \lambda_2(n, k)$  are decreasing for  $k \geq 2$ .*

*Proof.* It is straightforward to check that both sequences are positive for  $k \geq 2$  and  $n \geq 4$ . Using that  $\alpha_{n,k} = \beta_{n,k}^2/(n - 1)$ , we can write

$$\frac{1}{\lambda_1(n, k)} - 1 = \frac{3\alpha_{n,k} - 12}{6\alpha_{n,k} + 32\beta_{n,k} + 6} = \frac{3 - 12/\alpha_{n,k}}{6 + 32(n - 1)/\beta_{n,k} + 6/\alpha_{n,k}}.$$

Since  $\alpha_{n,k}$  and  $\beta_{n,k}$  are positive and increasing in  $k$ , the numerator in the right-hand term is increasing in  $k$ , whereas the denominator is positive and decreasing in  $k$ . Thus,  $k \mapsto \lambda_1(n, k)$  is decreasing.

Consider now the expression  $\lambda_2(n, k)$ . Again it is easy to check that  $\lambda_2(n, k) > 0$  for  $k \geq 2$  and  $n \geq 4$ , and

$$\begin{aligned} & \frac{1}{\lambda_2(n, k)} - 1 \\ &= \frac{3\alpha_{n,k} - 12 + \gamma_{n,k}(3\alpha_{n,k-1} - 29/4 - 3\delta_{n,k-1}) - 6\delta_{n,k}}{6\alpha_{n,k} + 32\beta_{n,k} + 6 + \gamma_{n,k}(6\alpha_{n,k-1} + 16\beta_{n,k-1} + 29/8 + 3/2\delta_{n,k-1}) + 3\delta_{n,k}} \\ &= \frac{3 - 12/\alpha_{n,k} + \gamma_{n,k}/\alpha_{n,k}(3\alpha_{n,k-1} - 29/4 - 3\delta_{n,k-1}) - 6\delta_{n,k}/\alpha_{n,k}}{6 + 32\beta_{n,k}/\alpha_{n,k} + 6/\alpha_{n,k} + \gamma_{n,k}/\alpha_{n,k}(6\alpha_{n,k-1} + 16\beta_{n,k-1} + 29/8 + 3/2\delta_{n,k-1}) + 3\delta_{n,k}/\alpha_{n,k}}. \end{aligned}$$

Let us denote this last expression by  $E_{n,k}/F_{n,k}$ . We claim that  $E_{n,k}$  is increasing in  $k$  and  $F_{n,k}$  is decreasing in  $k$ .

Using the fact that  $\gamma_{n,k}/\alpha_{n,k} = (n + 2k - 4)/((n + 2k - 2)\alpha_{n,k-1})$ ,

$$\begin{aligned} E_{n,k} &= 3 + 3 \frac{\gamma_{n,k}}{\alpha_{n,k}} \alpha_{n,k-1} - \frac{12 + 6\delta_{n,k}}{\alpha_{n,k}} - \frac{29}{4} \frac{\gamma_{n,k}}{\alpha_{n,k}} - 3 \frac{\gamma_{n,k}}{\alpha_{n,k}} \delta_{n,k-1} \\ &= 6 \frac{n + 2k - 3}{n + 2k - 2} - 6 \frac{n + 2k - 2}{\alpha_{n,k}} - \frac{29}{4} \frac{\gamma_{n,k}}{\alpha_{n,k}} - 3 \frac{\gamma_{n,k}}{\alpha_{n,k}} \delta_{n,k-1} \\ &= 6 - \frac{6}{n + 2k - 2} - \frac{6}{k} - \frac{6}{n + k - 2} - 3 \frac{\gamma_{n,k}}{\alpha_{n,k}} \left( (n + 2k - 4) + \frac{5}{12} \right). \end{aligned}$$

To express  $F_{n,k}$ , we remark that

$$\frac{\gamma_{n,k}}{\alpha_{n,k}} \beta_{n,k-1} = \frac{n + 2k - 4}{n + 2k - 2} \frac{\beta_{n,k-1}}{\alpha_{n,k-1}} = \frac{s_{n,k-1}}{n + 2k - 2},$$

where we denote  $s_{n,k} := (n + 2k - 2)\beta_{n,k}/\alpha_{n,k}$ . A straightforward computation similar to that for  $E_{n,k}$  shows that

$$\begin{aligned} F_{n,k} &= 6 + 3 \frac{n + 2k - 2}{\alpha_{n,k}} + 6 \frac{n + 2k - 4}{n + 2k - 2} + \frac{3}{2} \frac{\gamma_{n,k}}{\alpha_{n,k}} \left( (n + 2k - 4) + \frac{5}{12} \right) \\ &\quad + 32 \frac{s_{n,k}}{n + 2k - 2} + 16 \frac{s_{n,k-1}}{n + 2k - 2}. \end{aligned}$$

To prove our claim, it is thus enough to remark that

$$\begin{aligned} (n + 2k - 4) \frac{\gamma_{n,k}}{\alpha_{n,k}} &= \frac{(n + 2k - 4)^2}{(k - 1)(n + k - 3)(n + 2k - 2)} \\ &= \frac{n - 4}{n - 2} \cdot \frac{1}{k - 1} + \frac{n}{n - 2} \cdot \frac{1}{n + k - 3} + \frac{4}{(k - 1)(n + k - 3)(n + 2k - 2)} \end{aligned}$$

is decreasing in  $k$  (and thus  $\gamma_{n,k}/\alpha_{n,k}$  is decreasing in  $k$  too);

$$s_{n,k} = (n + 2k - 2) \frac{\sqrt{(n - 1)k(n + k - 2)}}{k(n + k - 2)} = \sqrt{(n - 1) \left( 4 + \frac{(n - 2)^2}{k(n + k - 2)} \right)}$$

is decreasing in  $k$ , and

$$\begin{aligned} \frac{n + 2k - 2}{\alpha_{n,k}} + 2 \frac{n + 2k - 4}{n + 2k - 2} &= 2 + \frac{n + 2k - 2}{k(n + k - 2)} - \frac{4}{n + 2k - 2} \\ &= 2 + \frac{(n - 2)^2}{k(n + k - 2)(n + 2k - 2)} \end{aligned}$$

is decreasing in  $k$ . □

5.2. *Proof of ergodicity.* We can now conclude the proof of the ergodicity statement in Theorem 1.1.

*Proof of ergodicity in Theorem 1.1.* We need to show that under the pinching condition  $\lambda > \lambda(m)$  of Theorem 1.1, the unitary frame flow is ergodic. If the frame flow is not ergodic, and  $m \neq 4, 28$ , we know by Corollary 3.2 that there exists a flow-invariant orthogonal projector  $\pi_{\mathcal{V}} \in C^\infty(SM, S^2\mathcal{N})$  of even Fourier degree onto a complex vector bundle  $\mathcal{V} \subset \mathcal{N}$  of rank  $1 \leq r \leq m/2 - 1$ .

By Corollary 3.5 and Lemma 5.1, this yields the existence of a non-zero complex normal twisted conformal Killing tensor  $u \in C^\infty(M, \Omega_k \otimes S^2T^*M)$  of even degree  $k \geq 2$  which satisfies the inequality in equation (5.1). We distinguish two cases.

*Case 1.* If  $k \geq 4$ , Lemmas 5.3 and 5.4 show that if the holomorphic pinching  $\lambda$  satisfies  $\lambda > \max(\lambda_1(n, 4), \lambda_2(n, 4))$ , we then have  $B_{n,k} > 0$  and  $B_{n,k} + \frac{1}{2}C_{n,k} > 0$  which by Lemma 5.2 implies that  $u \equiv 0$ .

*Case 2.* If  $k = 2$ , let  $\pi_{\mathcal{V}}$  be the projector given by Corollary 3.2, whose complex rank  $r$  satisfies  $1 \leq r \leq m/2 - 1$ . By [CLMS21, Lemma 4.2], one can then show that  $\pi_{\mathcal{V}}$  must be of the form

$$\pi_{\mathcal{V}} = \frac{2r}{n} \mathbb{1}_{TM} + u,$$

with  $u \in C^\infty(M, \Omega_2 \otimes S^2T^*M)$  as above. In particular, since  $\mathbb{1}_{TM}$  is parallel, this implies  $\mathbf{X}u = 0$ , that is,  $\mathbf{X}_\pm u = 0$ . Moreover, a quick algebraic computation (see [CLMS21, p. 38]) gives

$$\|\iota_v u\|_{L^2}^2 = \frac{2r}{n(n - 2r)} \|u\|_{L^2}^2, \tag{5.5}$$

and since  $2r \leq m - 2 = n/2 - 2$ ,

$$\|\iota_v u\|_{L^2}^2 \leq \frac{n - 4}{n(n + 4)} \|u\|_{L^2}^2. \tag{5.6}$$

Applying the twisted Pestov identity in equation (2.6) to  $u$  for  $k = 2$ , using that  $\mathbf{X}_\pm u = 0$  (the bound of Lemma 4.4 is therefore useless), and the upper bounds of Lemmas 4.1, 4.2 and 4.3, we find that



$$\left( \frac{3\lambda - 2}{2}n - \frac{8(1 - \lambda)}{3}(2n(n - 1))^{1/2} - \frac{1 + \lambda}{2} \right) \|u\|_{L^2}^2 - \frac{1 + \lambda}{2}n \|t_v u\|_{L^2}^2 \leq 0.$$

Inserting equation (5.5) in the previous inequality, then

$$\left( \frac{3\lambda - 2}{2}n - \frac{8(1 - \lambda)}{3}(2n(n - 1))^{1/2} - \frac{1 + \lambda}{2} \frac{2n}{n + 4} \right) \|u\|_{L^2}^2 \leq 0.$$

This shows that  $u \equiv 0$  when  $k = 2$ , as soon as the holomorphic pinching  $\lambda$  satisfies

$$\lambda \geq \lambda_3(n) := \frac{6n + 16(2n(n - 1))^{1/2} + 6n/(n + 4)}{9n + 16(2n(n - 1))^{1/2} - 6n/(n + 4)}.$$

Summarizing the two cases  $k \geq 4$  and  $k = 2$ , we have obtained that the the unitary frame flow is ergodic as soon as the holomorphic pinching  $\lambda$  satisfies

$$\lambda > \max(\lambda_1(2m, 4), \lambda_2(2m, 4), \lambda_3(2m)) =: \lambda_0(m).$$

Using a formal computing tool, it can be easily checked that  $\lambda_0(m) = \lambda_2(2m)$  for  $m \geq 6$ . However, for practical reasons, we will give a bound which is slightly less accurate, but much easier to compute.

Namely, using the obvious inequality  $((n + 2)(n - 1))^{1/2} < n + 1/2$ , we can write

$$\begin{aligned} \frac{1}{\lambda_1(n, 4)} - 1 &= \frac{3\alpha_{n,4} - 12}{6\alpha_{n,4} + 32\beta_{n,4} + 6} = \frac{12(n + 2) - 12}{24(n + 2) + 32(4(n + 2)(n - 1))^{1/2} + 6} \\ &> \frac{12n + 12}{88n + 86} = \frac{6n + 6}{44n + 43}. \end{aligned}$$

Similarly, since  $16\sqrt{2} < 68/3$ ,

$$\begin{aligned} \frac{1}{\lambda_3(n)} - 1 &= \frac{3n - 12n/(n + 4)}{6n + 16(2n(n - 1))^{1/2} + 6n/(n + 4)} = \frac{3 - 12/(n + 4)}{6 + 16(2(n - 1)/n)^{1/2} + 6/(n + 4)} \\ &> \frac{3 - 12/(n + 4)}{6 + 68/3 + 6/(n + 4)} = \frac{9n}{86n + 362}. \end{aligned}$$

Finally, using the calculation in the proof of Lemma 5.4 together with the obvious inequalities  $64/\sqrt{3} < 37$  and

$$\frac{4}{3} > \gamma_{n,4} = \frac{4(n + 2)(n + 4)}{3(n + 1)(n + 6)} > \frac{4n}{3(n + 1)} \quad \text{for all } n \geq 3,$$

we obtain

$$\begin{aligned} \frac{1}{\lambda_2(n, 4)} - 1 &= \frac{3\alpha_{n,4} - 12 + \gamma_{n,4}(3\alpha_{n,3} - 29/4 - 3\delta_{n,3}) - 6\delta_{n,4}}{6\alpha_{n,4} + 32\beta_{n,4} + 6 + \gamma_{n,4}(6\alpha_{n,3} + 16\beta_{n,3} + 29/8 + 3/2\delta_{n,3}) + 3\delta_{n,4}} \\ &> \frac{12(n + 2) - 12 + 4n/3(n + 1)(9(n + 1) - 29/4 - 3(n + 2)) - 6(n + 4)}{24(n + 2) + 64(n + 1/2) + 6 + 4/3(18(n + 1) + 16\sqrt{3}n + 29/8 + 3/2(n + 2)) + 3(n + 4)} \\ &> \frac{14n - 77/3}{(117 + 64/\sqrt{3})n + 126 + 29/6} > \frac{14n - 26}{154n + 131}. \end{aligned}$$

It is straightforward to check that

$$\frac{6n + 6}{44n + 43} > \frac{9n}{86n + 362} > \frac{14n - 26}{154n + 131} \quad \text{for all } n \geq 0,$$

which implies  $1/\lambda_0(m) - 1 > (14n - 26)/(154n + 131)$ , so eventually

$$\lambda_0(m) < \frac{154n + 131}{168n + 105} = \frac{308m + 131}{336m + 105} =: \lambda(m),$$

thus proving the ergodicity statement of Theorem 1.1. □

We conclude this paragraph by a remark on the remaining cases  $m = 4, 28$ .

*Remark 5.5.* In the case  $m = 28$ , one can check (using the LiE program for instance) that  $E_6$  fixes an element of  $S^3\mathbb{C}^{27}$ . As we saw in the proof of Corollary 3.2, the transitivity group is *never* semi-simple as it always contains the scalar matrices: Theorem 3.1 therefore does not exclude that subgroups of  $E_6 \times U(1) \not\subseteq U(27)$  occur as transitivity groups in complex dimension  $m = 28$ . In this case, the transitivity group would fix an orthogonal projector of  $\text{End}(S^3\mathbb{C}^{27})$ . Nevertheless, the argument given below involving the twisted Pestov identity does not carry over to  $\text{End}(S^3\mathbb{C}^{27})$  because this vector space involves tensorial powers of too high degree. In other words, the argument would result in a pinching condition  $\lambda > \lambda(28)$  for some  $\lambda(28) > 1$  so the statement would be empty. A similar remark holds for the case  $m = 4$ . This should be compared with [CLMS21, Theorem 3.8] where we only deal with elements of  $\mathfrak{o}(\mathcal{N})$  with  $\mathfrak{o} \in \{\text{id}, S^2, \Lambda^2, \Lambda^3\}$ . The worse pinching estimate in [CLMS21, Theorem 1.2] comes from the exterior power  $\Lambda^3\mathcal{N}$ .

5.3. *Proof of mixing.* It now remains to show the mixing property of Theorem 1.1, which is implied by the following result.

PROPOSITION 5.6. *The unitary frame flow on a negatively curved Kähler manifold  $(M, g)$  is ergodic if and only if it is mixing.*

*Proof.* Mixing implies ergodicity so it remains to show that ergodicity implies mixing in this setting. By [Lef23, Proof of Lemma 3.7], it suffices to show that the equation

$$X_{F_{\mathbb{C}}M}u = i\lambda u, \quad \lambda \in \mathbb{R} \setminus \{0\}, \quad u \in L^2(F_{\mathbb{C}}M) \tag{5.7}$$

implies that  $u \equiv 0$ . Let  $u \in L^2(F_{\mathbb{C}}M)$  be a solution to equation (5.7). Since  $F_{\mathbb{C}}M \rightarrow SM$  is a principal  $U(m - 1)$ -bundle over  $SM$ , the space  $L^2(F_{\mathbb{C}}M)$  splits as

$$L^2(F_{\mathbb{C}}M) = \bigoplus_{k=0}^{+\infty} L^2(SM, \Theta_k), \tag{5.8}$$

where  $\Theta_k$  is the vector bundle over  $SM$  whose fibre over  $v \in SM$  is the eigenspace of the Casimir operator of  $U(m - 1)$  acting on functions of the fibre  $(F_{\mathbb{C}}M)_v$ , associated to the eigenvalue  $\mu_k \geq 0$  (and  $\mu_k \neq \mu_j$  for  $k \neq j$ ). Since  $X_{F_{\mathbb{C}}M}$  preserves the splitting equation (5.8) (because the frame flow  $(\Phi_t)_{t \in \mathbb{R}}$  commutes with the right-action of  $U(m - 1)$ ), we deduce that for each  $k \geq 0$ , the component  $u_k \in L^2(SM, \Theta_k)$  of some function  $u = \sum_k u_k \in \bigoplus_{k=0}^{+\infty} L^2(SM, \Theta_k)$  satisfying equation (5.7) must satisfy

$$X_{F_{\mathbb{C}}M}u_k = i\lambda u_k.$$

Since  $\Theta_0 = \mathbb{C}$  is trivial over  $SM$ , the equation  $X_{F_{\mathbb{C}M}}u_0 = i\lambda u_0$  reads  $Xu_0 = i\lambda u_0$ , where  $u_0 \in L^2(SM)$  and  $X$  is the geodesic vector field. However, the geodesic flow is mixing as  $(M, g)$  has negative curvature, so we deduce that  $u_0 = 0$ .

For  $k \neq 0$ , [Lef23, Proposition 2.7] based on microlocal analysis shows that  $u_k \in L^2(SM, \Theta_k)$  is actually smooth, that is,  $u_k \in C^\infty(SM, \Theta_k)$ . Moreover,  $X_{F_{\mathbb{C}M}}|u_k|^2 = 0$ , so using the ergodicity assumption on the frame flow  $(\Phi_t)_{t \in \mathbb{R}}$ , we get that  $|u_k|$  is constant on  $F_{\mathbb{C}M}$ . If  $u_k \neq 0$ , then up to rescaling, we obtain that  $u_k : F_{\mathbb{C}M} \rightarrow U(1)$  is a well-defined smooth map. To simplify the notation, we will drop the index  $k$  from now on and simply write  $u$  instead of  $u_k$ , and  $\mu$  instead of  $\mu_k$ .

Denote by  $P$  the unit circle bundle of the complex line bundle  $\Lambda^{m-1,0}\mathcal{N} \rightarrow SM$ . Now, observe that there is a natural surjective bundle map  $\psi : F_{\mathbb{C}M} \rightarrow P$  given by

$$\psi : (v, \mathbf{e}_2, \dots, \mathbf{e}_m) \mapsto \left( v, \frac{1}{\sqrt{2}}(\mathbf{e}_2 - iJ \mathbf{e}_2) \wedge \dots \wedge \frac{1}{\sqrt{2}}(\mathbf{e}_m - iJ \mathbf{e}_m) \right).$$

There is a natural unitary parallel transport along geodesic flow-lines of sections of  $\Lambda^{m-1,0}\mathcal{N}$ , so there is a flow  $(\Phi_t^P)_{t \in \mathbb{R}}$  on  $P$  with generator  $X_P$  extending the geodesic flow  $(\varphi_t)_{t \in \mathbb{R}}$  as in §2.3, that is, writing  $\pi : P \rightarrow SM$  for the projection map, one has  $\pi \circ \Phi_t^P = \varphi_t \circ \pi$  for all  $t \in \mathbb{R}$ . Moreover,  $\psi$  intertwines the frame flow on  $F_{\mathbb{C}M}$  and the flow on  $P$ , that is

$$\Phi_t^P \circ \psi = \psi \circ \Phi_t. \tag{5.9}$$

We claim that the following holds.

CLAIM 5.7. *There exists a smooth function  $w \in C^\infty(P)$  such that  $u = \psi^*w$  and  $X_P w = i\lambda w$ .*

Note that, once we know that  $u = \psi^*w$  for some function  $w$ , the relation  $X_P w = i\lambda w$  is immediate using equation (5.9) and  $X_{F_{\mathbb{C}M}}u = i\lambda u$ .

*Proof.* We show that  $u = \psi^*w$  for some  $w \in C^\infty(P)$ . We fix an arbitrary point  $v_0 \in SM$  and  $w_0 \in (F_{\mathbb{C}M})_{v_0}$ . There is then a commutative diagram

$$\begin{CD} (F_{\mathbb{C}M})_{v_0} @>\psi>> P_{v_0} \\ @V \cong VV @VV \cong V \\ U(m-1) @>\det>> U(1) \end{CD} \tag{5.10}$$

where the downward arrows are isometries. Hence, by restricting  $u$  to the fibre  $(F_{\mathbb{C}M})_{v_0}$  and identifying isometrically  $(F_{\mathbb{C}M})_{v_0} \simeq U(m-1)$ , we get a map  $F := u(v_0)$  such that

$$F : F_{\mathbb{C}M} \simeq U(m-1) \rightarrow U(1),$$

and  $\Delta_{U(m-1)}F = \mu F$  with  $\mu \neq 0$  since  $u$  takes values in  $\Theta_k$  for  $k \neq 0$ . In other words,  $F$  is an eigenfunction of the Casimir operator on  $U(m-1)$  associated to the eigenvalue  $\mu \neq 0$  and of constant modulus.

Now, recall that  $U(m-1)$  is a split group extension of the circle group  $U(1)$  by the special unitary group  $SU(m-1)$ , that is,  $\det : U(m-1) \rightarrow U(1)$  is a fibre bundle with

fibres isometric to  $SU(m - 1)$ . We claim that such a function  $F$  must then necessarily be constant on the  $SU(m - 1)$ -fibres of the bundle  $U(m - 1)$ , that is,  $F = \det^* f$  for some  $f \in C^\infty(U(1))$ ,  $\mu = j^2$  for some integer  $j \geq 0$  and  $f$  is an eigenfunction of  $\Delta_{U(1)}$  with eigenvalue  $j^2$ .

Indeed, since  $U(m - 1) = SU(m - 1) \times_{\mathbb{Z}_{m-1}} U(1)$ , there is a  $\mathbb{Z}_{m-1}$ -bundle map  $p : SU(m - 1) \times U(1) \rightarrow U(m - 1)$ ,  $p(w, z) = wz$ , and this map is locally a Riemannian isometry. (The  $\mathbb{Z}_{m-1}$ -action on  $SU(m - 1) \times U(1)$  is simply given by  $(w, z) \mapsto (w\omega^{-k}, \omega^k z)$  for  $(w, z) \in SU(m - 1) \times U(1)$ ,  $k \in \{0, \dots, m - 2\}$  and  $\omega := e^{2i\pi/(m-1)}$ .) As a consequence, we get that  $\Delta_{SU(m-1) \times U(1)} p^* F = \mu p^* F$ . As  $SU(m - 1) \times U(1)$  is a Riemannian product, its eigenfunctions are obtained as sums of products of eigenfunctions on each factor and the eigenvalues are sums of the eigenvalues on each factor. Hence, we can write

$$p^* F(w, z) = \sum_{j=1}^N a_j(w) b_j(z) = \sum_{j=1}^N a_j(w) z^{k_j}$$

for some finite number  $N > 0$ , where  $(w, z) \in SU(m - 1) \times U(1)$ ,  $k_j \in \mathbb{Z}$  and  $k_{j_1} \neq k_{j_2}$  for  $j_1 \neq j_2$ ,  $a_j \neq 0$  is an eigenfunction of  $\Delta_{SU(m-1)}$  associated to the eigenvalue  $\lambda_j$  and  $\lambda_j + k_j^2 = \mu > 0$ . However, we also know that  $p^* F$  has constant modulus (equal to 1). Freezing an arbitrary point  $w \in SU(m - 1)$  and looking at  $|p^* F|^2(w, z)$  as a function of  $z \in \mathbb{S}^1$ , we easily get that  $N$  must be equal to 1, that is,  $p^* F(w, z) = a(w) z^k$  for some  $k \in \mathbb{Z}$ . Hence,  $a \in C^\infty(SU(m - 1))$  satisfies  $|a| = 1$  and  $\Delta_{SU(m-1)} a = \lambda a$  for some  $\lambda \geq 0$  (with  $k^2 + \lambda = \mu$ ).

We claim that  $\lambda = 0$ , that is,  $a$  is constant, and  $F$  is thus constant along the  $SU(m - 1)$ -fibres. Indeed, if  $\lambda \neq 0$ , using

$$\begin{aligned} 0 &= \Delta_{SU(m-1)} |a|^2 = (\Delta_{SU(m-1)} a) \bar{a} + a (\Delta_{SU(m-1)} \bar{a}) - 2 \nabla a \cdot \nabla \bar{a} \\ &= 2\lambda - 2|\nabla a|^2, \end{aligned}$$

we get that  $|\nabla a|$  is a non-zero constant. Since  $SU(m - 1)$  is simply connected, the map  $a$  lifts to the universal cover of  $U(1)$ , so there exists a map  $\theta : SU(m - 1) \rightarrow \mathbb{R}$  such that  $a = e^{i\theta}$ . At a critical point of  $\theta$ , we thus get  $\nabla a = 0$ , which is absurd.

As a consequence,  $\lambda = 0$  and this shows that  $F$  is constant along  $SU(m - 1)$ -fibres of the bundle map  $\det : U(m - 1) \rightarrow U(1)$ . In turn, as equation (5.10) commutes, we get that  $u$  is constant on the preimages of the map  $\psi : F_{\mathbb{C}} M \rightarrow P$  so  $u = \psi^* w$  for some smooth  $w \in C^\infty(P)$ . □

We thus have just shown that if there is a non-zero solution to equation (5.7), then there is also a non-zero solution to

$$X_P w = i\lambda w, \quad \lambda \in \mathbb{R} \setminus \{0\}, w \in C^\infty(P). \tag{5.11}$$

Hence, it remains to show that equation (5.11) has no non-zero solutions, and we will actually show it for  $w \in L^2(P)$ .

Assume that  $w \in L^2(P)$  is a solution to equation (5.11). As above, using that  $P$  is a principal  $U(1)$ -bundle, we can decompose

$$L^2(P) = \bigoplus_{k \in \mathbb{Z}} L^2(SM, L_k), \quad (5.12)$$

where the sections of  $L_k$  are given by functions  $f \in L^2(P)$  satisfying  $Vf = ikf$ , where  $V$  is the infinitesimal generator of the  $U(1)$ -action on the fibres of  $P$ . Observe that  $L_k = L^{\otimes k}$  for  $k \in \mathbb{Z}$  where  $L := L_1$  and  $L_1 \simeq \Lambda^{m-1,0}\mathcal{N}$ . Now, using the splitting equation (5.12), the equation  $X_P w = i\lambda w$  reads  $X_P w_k = i\lambda w_k$  for all  $k \in \mathbb{Z}$ , where  $w_k \in L^2(SM, L_k)$  (similarly to equation (5.8) and the subsequent argument). By [Lef23, Proposition 2.7], we also have that  $w_k$  is smooth, that is,  $w_k \in C^\infty(SM, L_k)$ . Moreover,  $X|w_k|^2 = 0$ , where  $|w_k|^2(v) := \int_{P_v} |w_k(v, u)|^2 du$  and  $u \in P_v$  is the variable in the  $U(1)$ -fibre, so  $|w_k|$  is constant by ergodicity of the flow  $(\varphi_t)_{t \in \mathbb{R}}$ . If  $k = 0$ ,  $L_0 = \mathbb{C}$  is trivial over  $SM$  so the mixing of the geodesic flow  $(\varphi_t)_{t \in \mathbb{R}}$  then implies that  $w_0 \equiv 0$ . If  $k \neq 0$  and  $w_k \neq 0$ , then we obtain a smooth nowhere vanishing section  $w_k \in C^\infty(SM, L_k)$ , so  $L_k$  is trivial. However, in turn, this implies that the first Chern class of  $L_1 = \Lambda^{m-1,0}\mathcal{N}$  is torsion. From the proof of Corollary 3.2, we obtain that the canonical bundle  $K_M$  of  $M$  has torsion first Chern class so the image of  $c_1(K_M)$  in  $H^2(M, \mathbb{R})$  vanishes. This is a contradiction since this image is equal to  $[-1/(2\pi)\rho]$  where  $\rho$  is the Ricci form of  $M$ , which is not exact since  $M$  has negative definite Ricci tensor. Hence, any solution to equation (5.11) is trivial, and thus, so is any solution to equation (5.7). This finishes the proof of Proposition 5.6.  $\square$

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