Euro. Jnl of Applied Mathematics (2008), *vol.* 19, *pp.* 635–660.
doi:10.1017/S0956792508007729 Printed in the United Kingdom

Reconstruction in the inverse crack problem by variational methods

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(*Received 6 November 2007; revised 6 June 2008; first published online 1 October 2008*)

We deal with a variational approach to the inverse crack problem, that is the detection and reconstruction of cracks, and other defects, inside a conducting body by performing boundary measurements of current and voltage type. We formulate such an inverse problem in a free-discontinuity problems framework and propose a novel method for the numerical reconstruction of the cracks by the available boundary data. The proposed method is amenable to numerical computations and it is justified by a convergence analysis, as the error on the measurements goes to zero. We further notice that we use the Γ-convergence approximation of the Mumford–Shah functional due to Ambrosio and Tortorelli as the required regularization term.

1 Introduction

Suppose that some defects are present in a homogeneous and isotropic conducting body, which we assume to be contained in Ω, a bounded domain of $\mathbb{R}^N$, $N \geqslant 2$. We assume the defects to be perfectly insulating, but they may have different geometrical properties. For instance we may have, simultaneously, cracks (either interior or surface breaking), cavities or material losses at the boundary. The closed set K denotes the union of the boundaries of these defects, whereas $\tilde{\gamma}$ is a part of the boundary of Ω which is accessible, known and disjoint from K. If a current density $f \in L^2(\tilde{\gamma})$, with zero mean, is applied on $\tilde{\gamma}$, the electrostatic potential $u = u(f,K)$ is the solution to the following Neumann boundary value problem, whose precise formulation will be discussed in Section 3,

$$\begin{cases} \Delta u = 0 & \text{in } \Omega\backslash K, \\ \nabla u \cdot \nu = f & \text{on } \tilde{\gamma}, \\ \nabla u \cdot \nu = 0 & \text{on } \partial(\Omega\backslash K)\backslash\tilde{\gamma}. \end{cases} \tag{1.1}$$

If the defect K is unknown, it is an interesting and challenging problem, with many applications in non-destructive evaluation and medical imaging, how to recover the shape and location of the defect by non-destructive and non-invasive measurements, for example by performing boundary measurements of voltage and current type. Namely, we prescribe one or more currents f and we measure $g = u|_{\gamma}$, where u is the electrostatic potential solving (1.1) and γ is an accessible and known part of $\partial\Omega$. When K is composed of cracks only, this inverse problem is usually referred to as the inverse crack problem. We refer to the review paper [9] and its references for previous results on the inverse crack problem. About uniqueness results, let us recall that one measurement is enough to uniquely determine cavities and material losses at the boundary in any dimension. In two dimensions, two suitably chosen measurements are enough to determine any kind of defects, including cracks. In three dimensions, the uniqueness issue for perfectly insulating cracks is still quite open. In fact, two suitably chosen measurements are enough to identify planar cracks [1]. For general cracks, instead, it is unclear if a finite number of measurements is enough, and in this case how many and which measurements one should take to have unique determination. However, infinitely many measurements are sufficient [12].

In many inverse problems, a two-step procedure is sometimes used, either from a numerical point of view or especially from a theoretical point of view. First a continuation problem from the data is solved, then the information we are looking for is inferred from the properties of the functions obtained by the continuation step. In this paper, we shall pursue such an approach, which in our opinion has the advantage of not requiring strong *a priori* assumptions on the shape and smoothness of the unknown defects. Namely, in the first step, we reconstruct the electrostatic potential u inside Ω from the available Cauchy data. Then, we take the jump set, or discontinuity set, of u, $J(u)$ and, since K is perfectly insulating, it happens that $J(u) \subset K$. If we repeat this procedure, with different, suitably chosen prescribed current densities which guarantee uniqueness for the inverse problem, then we may reconstruct the whole unknown defect. Therefore, we concentrate our attention to the first step, that is the reconstruction of the electrostatic potential u. This corresponds to a unique continuation problem from the Cauchy data for a harmonic function in a domain whose boundary is unknown (and non-smooth). In [18], we have investigated the uniqueness of this continuation and its stability with respect to the errors in the measured Cauchy data. In [19], a careful characterization of the electrostatic potential u to be reconstructed was developed, in the framework of special functions of bounded variation. In the same framework, a variational approach to reconstruction was proposed. In [19], such an approach involved the minimization of functionals, depending on a noise level parameter, given by a relaxed form of the characterization of u, which was stabilized by a regularization term involving the so-called Mumford–Shah functional [17]. It was proven, by Γ-convergence techniques, that the minimizers of those functionals converge, as the noise level approaches zero, to the unknown electrostatic potential u. However, the numerical implementation of the minimization of those functionals presented many difficulties.

The main novelty of the present work is that we establish a new variational formulation, with the same kind of convergence properties of the one in [19], which is amenable to numerical implementation. As a regularization term, we use the Γ-convergence approximation given by Ambrosio and Tortorelli [3, 4], to replace the Mumford–Shah functional. We recall that in their approach the jump set of u is essentially replaced by a new variable v, which should be close to zero near the jump set of u and 1 elsewhere. Therefore, the function v acts as a kind of phase variable. While the substitution of the Mumford–Shah functional with its Ambrosio–Tortorelli approximation is almost classical by now, the corresponding modification to the main part of the functional in [19] is not at all straightforward. Moreover, quite surprisingly, the new formulation turns out to be rather simpler than the old one. In fact, two different terms are now combined into a single one who may also be rewritten in a very simple way. The final result is a functional which is defined on spaces of Sobolev functions, instead of a space of functions of bounded variation, which has good differentiability properties.

More specifically, we assume that the unknown defect K_0 is composed of pieces of C^1 graphs and that u_0 is its corresponding electrostatic potential. Given the noise level ε and noisy Cauchy data $(g_\varepsilon, f_\varepsilon)$, we define the following functional on the two independent variables u and v, both living in suitable Sobolev spaces and with $0 \leqslant v \leqslant 1$:

$$\begin{aligned}\mathscr{F}_\varepsilon(u,v) = \frac{1}{\varepsilon^{q_1}}\left(\int_\Omega ((1-\varepsilon^2)v^2+\varepsilon^2)|\nabla u|^2 - 2\int_{\tilde{\gamma}} f_\varepsilon u + \int_{\tilde{\gamma}} f_\varepsilon \tilde{u}_\varepsilon\right)\\ + \frac{1}{\varepsilon}\int_\gamma |u-g_\varepsilon|^2 + \int_\Omega \left((v^q+\varepsilon^q)|\nabla u|^q + \frac{1}{4\varepsilon}(v-1)^2 + \varepsilon|\nabla v|^2\right).\end{aligned}$$

Here, q is a suitable constant greater than 2, $q_1 = (q-2)/(2q)$; the last term is the Ambrosio–Tortorelli functional, whereas $\tilde{u}_\varepsilon$ is a function, depending on v only, which is the unique solution to

$$\begin{cases}\operatorname{div}(((1-\varepsilon^2)v^2+\varepsilon^2)\nabla\tilde{u}_\varepsilon) = 0 & \text{in } \Omega,\\ ((1-\varepsilon^2)v^2+\varepsilon^2)\nabla\tilde{u}_\varepsilon \cdot \nu = f_\varepsilon & \text{on } \partial\Omega.\end{cases}$$

From the implementation point of view, we make the following remarks. First of all, it involves finding $\tilde{u}_\varepsilon$, that is solving an elliptic problem which might be almost degenerate, wherever v is close to zero, that is near the unknown defect. The second difficulty comes from the implementation of the Ambrosio–Tortorelli functional. However, we notice that this has been extensively studied in the literature. For instance, in [15] March tested it numerically in the context of image segmentation, while Bellettini and Coscia [5] proposed a discrete approximation of it based on finite elements. More recently, Bourdin made use of the Ambrosio–Tortorelli functional in his numerical treatment of image segmentation [6], and of quasi-static evolution of cracks in brittle materials [7]. If a minimizer $(u_\varepsilon, v_\varepsilon)$ to $\mathscr{F}_\varepsilon$ is found (in an approximate way) then u_ε should be an approximation of the looked-for potential u_0, whereas v_ε should provide a good approximation of the looked-for defect K_0. In fact, we expect v_ε to be close to zero near K_0 and 1 elsewhere, therefore it would be enough to threshold v_ε at a small value to find the region where K_0 should be located. Further comments on the properties of the functional and the implementation of the method will be given in Sections 5 and 6.

Our paper is devoted to providing a rigorous justification, by means of a convergence analysis, of the choice of our proposed functional. Namely, we will prove in Theorem 4.6 that, roughly speaking, as $\varepsilon \to 0^+$, quasi-minimizers $(u_\varepsilon, v_\varepsilon)$ of $\mathscr{F}_\varepsilon$ are such that u_ε converges, in a suitable sense, to the looked-for potential u_0. In other words, we define a family of regularized minimum problems whose solutions, the so-called regularized solutions, converge to the looked-for solution as the noise level approaches zero. This is a classical type of results in regularization theory, see for instance reference [13], where the regularized functional depends on a parameter, usually called the regularization coefficient, which in turn depends on the noise level.

The structure of the paper is the following. In Section 2, we present some preliminary results on functions of bounded variation and Γ-convergence. Then we introduce the so-called Mumford–Shah functional [17] and we recall its Γ-convergence approximation by elliptic functionals given by Ambrosio and Tortorelli [3, 4]. In Section 3, we present the direct problem and we state some properties of its solutions. Then, we introduce suitable classes of admissible defects and we recall some of the results developed in [19], namely the characterization of the looked-for electrostatic potential inside a suitable set of functions of bounded variation. In Section 4, we describe the approximating functionals $\mathscr{F}_\varepsilon$, $\varepsilon > 0$, for the reconstruction of the electrostatic potential u_0 by its Cauchy data on the boundary. In Section 5, we discuss some properties of the approximating functionals and we conclude in Section 6 with a description of possible numerical implementations of the method.

2 Preliminaries

Throughout the paper, the integer $N \geqslant 2$ will denote the space dimension. For every $x \in \mathbb{R}^N$, we shall set $x = (x', x_N)$, where $x' \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$. For every $x \in \mathbb{R}^N$ and $r > 0$, we shall denote by $B_r(x)$ the open ball in $\mathbb{R}^N$ centred at x of radius r. Usually we shall write B_r instead of $B_r(0)$. We recall that, for any set $E \subset \mathbb{R}^N$ and any $r > 0$, we denote $B_r(E) = \bigcup_{x \in E} B_r(x)$.

For any non-negative integer k we denote by $\mathscr{H}^k$ the k-dimensional Hausdorff measure. We recall that for Borel subsets of $\mathbb{R}^N$ the N-dimensional Hausdorff measure coincides with $\mathscr{L}^N$, the N-dimensional Lebesgue measure. Furthermore, if $\gamma \subset \mathbb{R}^N$ is a smooth manifold of dimension k, then $\mathscr{H}^k$ restricted to γ coincides with its k-dimensional surface measure. For any Borel $E \subset \mathbb{R}^N$ we let $|E| = \mathscr{L}^N(E)$ and $[E] = \mathscr{H}^{N-1}(E)$.

We recall that a bounded domain $\Omega \subset \mathbb{R}^N$ is said to have a *Lipschitz boundary* if for every $x \in \partial\Omega$ there exist a Lipschitz function $\varphi : \mathbb{R}^{N-1} \to \mathbb{R}$ and a positive constant r such that for any $y \in B_r(x)$ we have, up to a rigid transformation,

$$y \in \Omega \quad \text{if and only if} \quad y_N < \varphi(y').$$

We observe that the boundary of Ω, $\partial\Omega$, has finite $(N-1)$-dimensional Hausdorff measure, that is $[\partial\Omega] < +\infty$.

We say that a function $\varphi : A \to B$, A and B being metric spaces, is *bi-Lipschitz* if it is invertible and φ and $\varphi^{-1} : \varphi(A) \to A$ are both Lipschitz functions. If both the Lipschitz constants of φ and φ^{-1} are bounded by $L > 0$, then we say that φ is *bi-Lipschitz* with constant L.

We recall some basic notation and properties of functions of bounded variation. For more comprehensive treatments see, for instance, [2, 14].

Given an open bounded set $\Omega \subset \mathbb{R}^N$, we denote by $\mathrm{BV}(\Omega)$ the Banach space of *functions of bounded variation*. We recall that $u \in \mathrm{BV}(\Omega)$ if and only if $u \in L^1(\Omega)$ and its distributional derivative Du is a bounded vector measure. We endow $\mathrm{BV}(\Omega)$ with the standard norm as follows. Given $u \in \mathrm{BV}(\Omega)$, we denote by $|Du|$ the total variation of its distributional derivative and we set $\|u\|_{\mathrm{BV}(\Omega)} = \|u\|_{L^1(\Omega)} + |Du|(\Omega)$. We say that a sequence of $\mathrm{BV}(\Omega)$ functions $\{u_j\}_{j=1}^{\infty}$ converges *weakly** in $\mathrm{BV}(\Omega)$ if and only if u_j converges to u

in $L^1(\Omega)$ and Du_j weakly* converges to Du in Ω, that is

$$\lim_j \int_\Omega v \mathrm{d}Du_j = \int_\Omega v \mathrm{d}Du \quad \text{for any } v \in C_0(\Omega). \tag{2.1}$$

We denote by $\mathrm{SBV}(\Omega)$ the space of *special functions of bounded variation*. For any $u \in \mathrm{SBV}(\Omega)$, the density of the absolutely continuous part of Du with respect to $\mathscr{L}^N$ will be denoted by ∇u, the *approximate gradient* of u. The singular part, with respect to $\mathscr{L}^N$, of Du is concentrated on $J(u)$, $J(u)$ being the *approximate discontinuity set* (or *jump set*) of u (see, for instance, [8, Definition 1.57]). We will also use the following definition (see, for instance, [8]). We say that a function $u \in \mathrm{GSBV}(\Omega)$, the space of *generalized functions of bounded variation*, if $u \in L^1(\Omega)$ and for any $T > 0$ its *truncation* $u_T = (-T) \vee (T \wedge u) = \max\{-T, \min\{T, u\}\}$ belongs to $\mathrm{SBV}(\Omega)$. Let us recall that the approximate gradient ∇u of $u \in \mathrm{GSBV}(\Omega)$ is defined almost everywhere and coincides with ∇u_T almost everywhere on $\{u = u_T\}$, and that $J(u) = \bigcup_{T>0} J(u_T)$.

The special functions of bounded variation satisfy the following compactness and semi-continuity theorem (see, for instance, [2, Theorems 4.7 and 4.8]).

Theorem 2.1 (*SBV compactness and semicontinuity*) *For any fixed p, $1 < p < +\infty$, if $\{u_j\}_{j=1}^\infty$ is a sequence of functions belonging to $SBV(\Omega)$ satisfying for a given constant $C > 0$*

$$\|u_j\|_{L^\infty(\Omega)} \leqslant C \quad \textit{for any } j, \tag{2.2}$$

and

$$\int_\Omega |\nabla u_j|^p + [J(u_j)] \leqslant C \quad \textit{for any } j, \tag{2.3}$$

then we may extract a subsequence, which we relabel $\{u_k\}_{k=1}^\infty$, such that u_k converges weakly in $BV(\Omega)$ to a function $u \in SBV(\Omega)$ and the following lower semi-continuity properties hold*

$$[J(u)] \leqslant \liminf_k [J(u_k)]; \qquad \int_\Omega |\nabla u|^p \leqslant \liminf_k \int_\Omega |\nabla u_k|^p. \tag{2.4}$$

We recall the definition and some basic properties of Γ-convergence. For a more detailed introduction we refer to [10].

Let (X, d) be a metric space. Then a sequence $F_n : X \to [-\infty, +\infty]$, $n \in \mathbb{N}$, Γ-converges as $n \to \infty$ to a function $F : X \to [-\infty, +\infty]$ if for every $x \in X$ we have

$$\text{for every sequence } \{x_n\}_{n\in\mathbb{N}} \text{ converging to } x \text{ we have} \quad F(x) \leqslant \liminf_n F_n(x_n); \tag{2.5}$$

$$\text{there exists a sequence } \{x_n\}_{n\in\mathbb{N}} \text{ converging to } x \text{ such that} \quad F(x) = \lim_n F_n(x_n). \tag{2.6}$$

The function F will be called the Γ-limit of the sequence $\{F_n\}_{n\in\mathbb{N}}$ as $n \to \infty$ with respect to the metric d and we denote it by $F = \Gamma\text{-}\lim_n F_n$.

Let us recall that (2.5) may be rewritten also in the following way:

$$\Gamma\text{-}\liminf_n F_n(x) \geqslant F(x),$$

https://doi.org/10.1017/S0956792508007729 Published online by Cambridge University Press

where

$$\Gamma\text{-}\liminf_n F_n(x) = \inf_{\{x_n \to x\}} \liminf_n F_n(x_n).$$

Moreover, the sequence $x_n \to x$ such that (2.6) holds is usually referred to as the *recovery sequence* for x.

The following theorem, usually known as the Fundamental Theorem of Γ-convergence, illustrates the motivations for the definition of such a kind of convergence.

Theorem 2.2 *Let (X, d) be a metric space and let $F_n : X \to [-\infty, +\infty]$, $n \in \mathbb{N}$, be a sequence of functions defined on X. If there exists a compact set K such that $\inf_K F_n = \inf_X F_n$ for any $n \in \mathbb{N}$ and $F = \Gamma\text{-}\lim_n F_n$, then F admits a minimum over X and we have*

$$\min_X F = \lim_n \inf_X F_n.$$

Furthermore, if $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of points in X which converges to a point $x \in X$ and satisfies $\lim_n F_n(x_n) = \lim_n \inf_X F_n$, then x is a minimum point for F.

The definition of Γ-convergence may be extended in a natural way to families depending on a continuous parameter. For instance we say that the family of functions $\{F_\varepsilon\}_{\varepsilon>0}$ Γ-converges to a function F as $\varepsilon \to 0^+$ if for every sequence of positive ε_n, $n \in \mathbb{N}$, converging to 0 we have $F = \Gamma\text{-}\lim_n F_{\varepsilon_n}$.

Let us define the so-called Mumford–Shah functional, introduced in [17] in the context of image segmentation. Let us fix positive constants b and c. Let $\mathscr{MS} : L^1(\Omega) \to [0, +\infty]$ be given by

$$\mathscr{MS}(u) = b\int_\Omega |\nabla u|^2 + c[J(u)] \quad \text{if } u \in \mathrm{GSBV}(\Omega), \tag{2.7}$$

whereas $\mathscr{MS}(u) = +\infty$ otherwise.

Let us fix q, $1 < q < +\infty$. Let $V : \mathbb{R} \to [0, +\infty)$ be a continuous function such that $V(t) = 0$ if and only if $t = 1$ and let $c_V = \int_0^1 \sqrt{V(t)}\mathrm{d}t$. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a lower semi-continuous non-decreasing function such that $\psi(0) = 0$, $\psi(1) = 1$ and $\psi(t) > 0$ if $t > 0$. For any $\eta > 0$, let us fix $o_\eta \geqslant 0$ such that $\lim_{\eta\to 0^+} o_\eta/\eta^{q-1} = 0$. Finally, we define $\psi_\eta = \psi + o_\eta$.

For instance, we may choose $\psi(t) = t^q$, if $t \geqslant 0$, while $\psi(t) = 0$ if $t < 0$, $V(t) = (t-1)^2/4$, whence $4c_V = 1$, and $o_\eta = \eta^q$.

Then, for any $\eta > 0$, we define the following functional $\mathscr{AT}_q^\eta : L^1(\Omega) \times L^1(\Omega) \to [0, +\infty]$ by

$$\mathscr{AT}_q^\eta(u, v) = \int_\Omega \left(b\psi_\eta(v)|\nabla u|^q + \frac{1}{\eta}V(v) + \eta|\nabla v|^2 \right)$$
$$\text{if } u \in W^{1,q}(\Omega) \text{ and } v \in W^{1,2}(\Omega, [0,1]), \tag{2.8}$$

whereas $\mathscr{AT}_q^\eta(u, v) = +\infty$ otherwise. Here $W^{1,2}(\Omega, [0,1]) = \{v \in W^{1,2}(\Omega) : 0 \leqslant v \leqslant 1$ a.e. in $\Omega\}$. We shall refer to $\mathscr{AT}_q^\eta$ as the Ambrosio–Tortorelli functional.

Let us define the following variant of the Mumford–Shah functional. The main difference is that we replace the exponent 2 with the exponent q, $1 < q < +\infty$. For reasons which

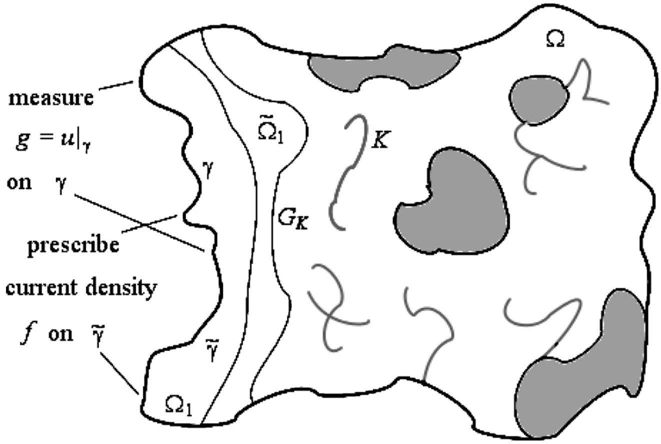

FIGURE 1. Geometric configuration.

will appear evident soon, we also add a formal variable v and we pick $c = 4c_V$. We define the functional $\mathscr{MS}_q : L^1(\Omega) \times L^1(\Omega) \to [0, +\infty]$ by

$$\mathscr{MS}_q(u, v) = b \int_\Omega |\nabla u|^q + 4c_V[J(u)] \quad \text{if } u \in \mathrm{GSBV}(\Omega) \text{ and } v = 1 \text{ a.e. in } \Omega, \tag{2.9}$$

whereas $\mathscr{MS}_q(u, v) = +\infty$ otherwise.

The following important approximation result is due to Ambrosio and Tortorelli, [3, 4] (see also [8]).

Theorem 2.3 *With respect to the metric of* $L^1(\Omega) \times L^1(\Omega)$, *we have*

$$\Gamma\text{-}\lim_{\eta \to 0^+} \mathscr{AT}_q^\eta = \mathscr{MS}_q. \tag{2.10}$$

3 The direct problem and classes of admissible defects

Let Ω, Ω_1 and $\tilde{\Omega}_1$ be three bounded domains contained in $\mathbb{R}^N$, $N \geqslant 2$, with Lipschitz boundaries such that $\Omega_1 \subset \tilde{\Omega}_1 \subset \Omega$ and the following properties are satisfied. First, $\Omega \backslash \overline{\tilde{\Omega}_1}$ is not empty. Then, there exists γ, an open subset of $\partial\Omega$, such that $\overline{\gamma}$ is contained in the interior of $\partial\Omega \cap \partial\Omega_1$ and $\mathrm{dist}(\overline{\Omega}_1, \partial\tilde{\Omega}_1 \cap \Omega) > 0$. Beside γ, we also fix $\tilde{\gamma}$, a closed subset of the interior of $\partial\Omega \cap \partial\Omega_1$. We assume that $\tilde{\gamma}$ has non-empty interior, with respect to the induced topology of $\partial\Omega$.

We assume that Ω, Ω_1, $\tilde{\Omega}_1$, γ and $\tilde{\gamma}$ are fixed throughout the paper. We observe that we shall always drop the dependence of any constant upon N, the space dimension.

Let K be an *admissible defect*, that is K is a compact set contained in $\overline{\Omega}$ such that $\mathrm{dist}(K, \overline{\tilde{\Omega}_1}) > 0$. We denote with G_K the connected component of $\Omega \backslash K$ such that $\tilde{\Omega}_1 \subset G_K$. We observe that $\overline{\gamma} \cup \tilde{\gamma} \subset \partial G_K$. The geometric configuration is illustrated in Figure 1. Here the shaded parts are the other connected components of $\Omega \backslash K$ different from G_K.

Let us fix a number s, $s > 1$ if $N = 2$ or $s \geqslant 2 - (2/N)$ if $N \geqslant 3$, to be chosen later. Let us prescribe $f \in L^s(\partial\Omega)$ such that $\int_{\partial\Omega} f = 0$, $f \not\equiv 0$ and $\mathrm{supp}(f) \subset \tilde{\gamma}$.

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For any bounded open set $D \subset \mathbb{R}^N$, we set $L^{1,2}(D)$ as the following *Deny–Lions space*

$$L^{1,2}(D) = \{u \in L^2_{loc}(D) : \ \nabla u \in L^2(D, \mathbb{R}^N)\}. \tag{3.1}$$

For basic properties of Deny–Lions spaces we refer to [11] and [16]. As a convention, we identify two elements u_1 and u_2 of $L^{1,2}(D)$ whenever $\nabla u_1 = \nabla u_2$ almost everywhere in D. We point out that if D is bounded with Lipschitz boundary then any $v \in L^{1,2}(D)$ belongs to $W^{1,2}(D)$ and, obviously, vice versa. Finally, we notice that the set $\{\nabla u : u \in L^{1,2}(D)\}$ is a closed subspace of $L^2(D, \mathbb{R}^N)$.

Let K be an admissible defect, then there exists a function $u = u(f, K) \in L^{1,2}(\Omega \backslash K)$ such that

$$\int_{\Omega\backslash K} \nabla u \cdot \nabla v = \int_{\tilde{\gamma}} f v \quad \text{for every } v \in L^{1,2}(\Omega\backslash K). \tag{3.2}$$

Such a function is unique in the sense that the gradients of any two solutions to (3.2) coincide almost everywhere in $\Omega\backslash K$. We always take as u the solution satisfying the following two normalization conditions. First,

$$\int_{\gamma} u = 0, \tag{3.3}$$

and, second, since u is constant on any connected component of $\Omega\backslash K$ different from G_K, we pose

$$u = 0 \quad \text{almost everywhere in } \Omega\backslash G_K. \tag{3.4}$$

In such a way, u is defined almost everywhere in Ω and is the unique solution to (3.2)–(3.4).

We wish to remark that (3.2) is the weak formulation of the Neumann type boundary value problem (1.1).

The following regularity properties of u are proved in [19].

Proposition 3.1 *Under the previous assumptions, let $s > N - 1$ and let us fix $f \in L^s(\partial\Omega)$ such that $\int_{\partial\Omega} f = 0$, $f \not\equiv 0$ and* $\operatorname{supp}(f) \subset \tilde{\gamma}$. *Let K be an admissible defect and let u be the solution to* (3.2)–(3.4).

Then there exists a constant $C_1 > 0$, depending on s, Ω, Ω_1, $\tilde{\Omega}_1$, γ, $\tilde{\gamma}$ only, such that

$$\|\nabla u\|_{L^2(\Omega\backslash K)} \leqslant C_1 \|f\|_{L^s(\partial\Omega)}, \tag{3.5}$$

$$\|u\|_{L^\infty(\Omega)} \leqslant C_1 \|f\|_{L^s(\partial\Omega)}. \tag{3.6}$$

Furthermore, there exists a constant β, $0 < \beta < 1$, depending on s, Ω, Ω_1, $\tilde{\Omega}_1$, γ, $\tilde{\gamma}$ and $\operatorname{dist}(K, \overline{\tilde{\Omega}}_1)$ *only, such that $u \in C^{0,\beta}(\overline{\tilde{\Omega}}_1)$.*

Finally, there exists a constant $r > 2$ and a constant C_2, depending on s, Ω, Ω_1, $\tilde{\Omega}_1$, γ, $\tilde{\gamma}$ and $\operatorname{dist}(K, \overline{\tilde{\Omega}}_1)$ *only, such that $\nabla u \in L^r(\tilde{\Omega}_1, \mathbb{R}^N)$ and*

$$\|\nabla u\|_{L^r(\tilde{\Omega}_1)} \leqslant C_2 \|f\|_{L^s(\partial\Omega)}. \tag{3.7}$$

We remark that, in view of (3.6), u actually belongs to $W^{1,2}(\Omega\backslash K)$. Furthermore, under the additional assumption that $[K] < +\infty$, or equivalently that $[\partial G_K] < +\infty$, we have that u belongs to $\mathrm{SBV}(\Omega)$, its approximate discontinuity set $J(u)$ satisfies $[J(u)\backslash\partial G_K] = 0$

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and, finally, ∇u, the weak derivative of u in $\Omega\backslash K$, coincides almost everywhere in Ω with the approximate gradient of u (see, for instance, [2, Proposition 4.4]).

For what concerns the classes of admissible defects we shall use, let us begin with the following definition.

Definition 3.2 We say that a class $\mathscr{B}$ of subsets of $\mathbb{R}^N$ is *admissible* if there exist constants C and R such that any $K \in \mathscr{B}$ is a non-empty compact set contained in $\overline{B}_R$ such that $[K] \leqslant C$ and $\mathscr{B}$ is compact with respect to the Hausdorff distance.

In the remaining part of this section we shall illustrate some admissible classes. We limit ourselves to the two- or three-dimensional case, however it is not difficult to see how these definitions can be generalized to higher dimensions.

If $N=2$, fixed a positive constant $L \geqslant 1$, we say that Γ is an *L-Lipschitz*, or *L-*$C^{0,1}$, *arc* if, up to a rigid transformation, $\Gamma=\{(x,y)\in\mathbb{R}^2 : -a/2 \leqslant x \leqslant a/2,\ y=\varphi_1(x)\}$, where $L^{-1}\leqslant a \leqslant L$ and $\varphi_1 : \mathbb{R}\to\mathbb{R}$ is a Lipschitz map with Lipschitz constant bounded by L and such that $\varphi_1(0)=0$. For any α, $0\leqslant\alpha\leqslant 1$, we say that Γ is an *L-*$C^{1,\alpha}$ *arc* if φ_1 is $C^{1,\alpha}$ and its $C^{1,\alpha}$ norm is bounded by L. The points $(a/2,\varphi_1(a/2))$ and $(-a/2,\varphi_1(-a/2))$ will be called the *vertices* or *endpoints* of the arc Γ.

Let us now consider the case $N=3$. Let T be the closed equilateral triangle which is contained in the plane $\Pi=\{(x,y,z) \in \mathbb{R}^3 : z=0\}$ with vertices $V_1=(0,1,0)$, $V_2=(-\sqrt{3}/2,-1/2,0)$ and $V_3=(\sqrt{3}/2,-1/2,0)$ and $T' \subset \mathbb{R}^2$ be its projection on the plane Π. Fixed a positive constant $L\geqslant 1$, we call an *L-Lipschitz*, or *L-*$C^{0,1}$, *generalized triangle* a set Γ such that, up to a rigid transformation, $\Gamma=\{(x,y,z)\in\mathbb{R}^3 : (x,y)\in\varphi(T'),\ z=\varphi_1(x,y)\}$, where $\varphi:\mathbb{R}^2\to\mathbb{R}^2$ is a bi-Lipschitz function with constant L such that $\varphi(0)=0$ and $\varphi_1:\mathbb{R}^2\to\mathbb{R}$ is a Lipschitz map with Lipschitz constant bounded by L and such that $\varphi_1(0)=0$. For any α, $0\leqslant\alpha\leqslant 1$, we say that Γ is an *L-*$C^{1,\alpha}$ *generalized triangle* if φ_1 is $C^{1,\alpha}$ and its $C^{1,\alpha}$ norm is bounded by L.

In both cases, the image through φ of any vertex or side of T' will be called a generalized vertex or generalized side of $\varphi(T')$, respectively. The image on the graph of φ_1 of one of the generalized vertices of $\varphi(T')$ will be called a *generalized vertex* of Γ, whereas the image of one of the generalized sides of $\varphi(T')$ will be called a *generalized side* of Γ. We also remark that there exists a constant $L_1>0$, depending on L only, such that we can find $\varphi_2:\mathbb{R}^3\to\mathbb{R}^3$, a bi-Lipschitz function with constant L_1, such that $\Gamma=\varphi_2(T)$.

Definition 3.3 Let us assume that $\Omega\subset B_R\subset\mathbb{R}^N$, with $R\geqslant 1$ and $N=2,3$. For any positive constants $L\geqslant 1$, δ and c, $c<1$, any $k=0,1$ and α, $0\leqslant\alpha\leqslant 1$, such that $k+\alpha\geqslant 1$, we define $\mathscr{B}(N,(k,\alpha),L,\delta,c)$ in the following way. We say that $A\in\mathscr{B}(N,(k,\alpha),L,\delta,c)$ if and only if $A\subset\overline{B}_{2R}$, there exists a positive integer n, depending on A, such that $A=\bigcup_{i=1}^n\Gamma_i$, Γ_i an L-$C^{k,\alpha}$ arc (if $N=2$) or generalized triangle (if $N=3$) for any $i=1,\ldots,n$, such that the following conditions are satisfied:

(i) for any $i,j\in\{1,\ldots,n\}$ with $i\neq j$, we have that either $\Gamma_i\cap\Gamma_j$ is not empty or $\operatorname{dist}(\Gamma_i,\Gamma_j)\geqslant\delta$;

(ii) for any $i,j\in\{1,\ldots,n\}$ with $i\neq j$, if $\Gamma_i\cap\Gamma_j$ is not empty then $\Gamma_i\cap\Gamma_j$ is a common endpoint V if $N=2$ and either a common generalized vertex V or a common

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generalized side γ if $N=3$. Furthermore, in such a case, for any $x \in \Gamma_i$ we have $\mathrm{dist}(x, \Gamma_j) \geqslant c|x-V|$ or $\mathrm{dist}(x, \Gamma_j) \geqslant c\,\mathrm{dist}(x, \gamma)$, respectively.

Let us remark that there exists an integer M, depending on N, R, L, δ and c only, such that for any $A \in \mathscr{B}(N, (k, \alpha), L, \delta, c)$ we have that $n \leqslant M$.

More importantly, we have that any of the classes $\mathscr{B}$ described in Definition 3.3 is non-empty, is composed of non-empty compact sets and is compact with respect to the Hausdorff distance (see for proof the analogous reasonings used to prove Lemma 6.1 of [18]). Finally, if A belongs to any of these classes, then $[A]$ is bounded by a constant depending on the class only. Therefore, any of the classes $\mathscr{B}$ of Definition 3.3 is admissible in the sense of Definition 3.2.

Definition 3.4 For any admissible class $\mathscr{B}$, we shall call $H(\mathscr{B})$ the following subset of $\mathrm{GSBV}(\Omega)$. We say that $u \in \mathrm{GSBV}(\Omega)$ belongs to $H(\mathscr{B})$ if $\nabla u \in L^2(\Omega, \mathbb{R}^N)$, $[J(u) \cap \tilde{\Omega}_1] = 0$ and there exists $A \in \mathscr{B}$, A depending on u, such that $[J(u) \backslash A] = 0$.

Let us recall the following lemma, proved in [19, Lemma 4.3].

Lemma 3.5 *Let $H = H(\mathscr{B})$ for some admissible class $\mathscr{B}$. Let $\{u_j\}_{j=1}^{\infty}$ be a sequence of functions belonging to H satisfying for given constants $C > 0$ and p, $2 \leqslant p < +\infty$,*

$$\|u_j\|_{L^\infty(\Omega)} \leqslant C \quad \text{and} \quad \int_\Omega |\nabla u_j|^p \leqslant C \quad \text{for any } j.$$

Then we may extract a subsequence, which we relabel $\{u_k\}_{k=1}^{\infty}$, such that u_k converges weakly in $BV(\Omega)$ to a function $u \in SBV(\Omega)$ such that $u \in H$. Furthermore,* (2.4) *holds.*

We shall use the following class of admissible defects.

Definition 3.6 For any admissible class $\mathscr{B}$ we call $\mathscr{B}'$ the class of admissible defects K such that $\mathrm{dist}(K, \overline{\tilde{\Omega}_1}) \geqslant \delta$, $\mathscr{H}^{N-2}(K \cap \partial\Omega) < +\infty$ and there exists $A \in \mathscr{B}$ such that $K \subset A$ and $\mathscr{H}^{N-2}(K \cap \overline{A \backslash K}) < +\infty$. Moreover, fixed a constant $s > N-1$, for any q, $2 < q < +\infty$, we call $\mathscr{B}'_q$ the class of admissible defects $K \in \mathscr{B}'$ such that there exists a constant C, depending on K and s, such that for any $f \in L^s(\tilde{\gamma})$, with $\int_{\tilde{\gamma}} f = 0$, we have that $u = u(f, K)$ solution to (1.1) satisfies

$$\|\nabla u\|_{L^q(\Omega)} \leqslant C \|f\|_{L^s(\tilde{\gamma})}. \tag{3.8}$$

First of all, we observe that if $K \in \mathscr{B}'$ and $f \in L^s(\tilde{\gamma})$, with $s > N-1$ and $\int_{\tilde{\gamma}} f = 0$, then $u = u(f, K) \in H(\mathscr{B})$.

We remark that, if $K \in \mathscr{B}'$, and $\mathscr{B}$ is any of the admissible classes of Definition 3.3, then we can find κ, a closed subset of K, such that $\mathscr{H}^{N-2}(\kappa) < +\infty$, and for any $x \in \partial G_K \backslash \kappa$ there exists a Lipschitz function $\varphi : \mathbb{R}^{N-1} \to \mathbb{R}$ and a positive constant r such that we have, up to a rigid transformation, $\partial G_K \cap B_r(x) = \{y = (y', y_N) \in B_r(x) : \ y_N = \varphi(y')\}$.

We remark that, again if $\mathscr{B}$ is any of the admissible classes of Definition 3.3, for any $K \in \mathscr{B}'$, the property of belonging to $\mathscr{B}'_q$, for some $q > 2$, provided $q \leqslant r$, r as in Proposition 3.1, is purely a geometric one, it depends only on the geometric properties of ∂G_K (see the discussion and the examples in [19]).

Throughout the sequel, we assume that Ω, Ω_1, $\tilde{\Omega}_1$, γ and $\tilde{\gamma}$ are fixed. Let us also choose $s > N-1$ and q, $2 < q \leqslant r$, r as is Proposition 3.1. We assume that $\Omega \subset B_R$, with $R \geqslant 1$, and we also fix positive constants δ, L, $L \geqslant 1$, c, $0 < c < 1$, an integer $k = 0, 1$ and α, $0 \leqslant \alpha \leqslant 1$, such that $k + \alpha \geqslant 1$. Let $\mathscr{B}_1$ be the corresponding class described in Definition 3.3.

Let $\mathscr{B}$ be an admissible class, possibly different from $\mathscr{B}_1$, and let $H = H(\mathscr{B})$. Let K_0 be an admissible defect such that $K_0 \in \mathscr{B}'_q$. We notice that K_0 represents our unknown defect.

Let $f_0 \in L^s(\partial\Omega)$ be such that f_0 is not identically equal to zero, $\int_{\partial\Omega} f_0 = 0$ and $\mathrm{supp}(f_0) \subset \tilde{\gamma}$. Let $u_0 = u(f_0, K_0)$ be the solution to (3.2)–(3.4) with f replaced by f_0 and K replaced by K_0. Let $g_0 = u_0|_\gamma$.

We call H_0 the following subset of H.

Definition 3.7 We say that $u \in H_0$ if $u \in H$, $\int_\Omega \nabla u \cdot \nabla w = \int_{\tilde{\gamma}} f_0 w$ for any $w \in W^{1,2}(\Omega)$, $\int_\Omega \nabla u \cdot \nabla(uw_1) = \int_{\tilde{\gamma}} f_0 u w_1$ for any $w_1 \in C_0^\infty(\mathbb{R}^N)$ and $u = g_0$ on γ.

We remark that clearly $u_0 \in H_0$. We define the functional $\mathscr{F}_0 : L^1(\Omega) \to [0, +\infty]$ such that

$$\mathscr{F}_0(u) = b \int_\Omega |\nabla u|^q + 4c_V[J(u)] \quad \text{if } u \in H_0,$$

whereas $\mathscr{F}_0(u) = +\infty$ otherwise. Let us observe that $\mathscr{F}_0(u) < +\infty$.

Following [19], we state these crucial results.

Proposition 3.8 *If $\mathscr{B} = \mathscr{B}_1$ and $k = 1$, then $u_1 \in H_0$ if and only if $u \in H$, $u_1 = u_0$ almost everywhere in G_{K_0} and $\nabla u_1 = \nabla u_0$ almost everywhere in Ω. Furthermore, the functional $\mathscr{F}_0$ admits a minimum point over $L^1(\Omega)$.*

If $\mathscr{B} = \mathscr{B}_1$, $k = 1$ and $\alpha = 1$, then $\mathscr{F}_0(u_0) = \min_{L^1(\Omega)} \mathscr{F}_0$, that is u_0 is a minimizer for $\mathscr{F}_0$. Furthermore, if u_1 is any minimizer of $\mathscr{F}_0$, we have that $u_1 = u_0$ on G_{K_0}, $[J(u_1) \backslash \partial G_{K_0}] = 0$ and u_1 is constant on any connected component of $\Omega \backslash \partial G_{K_0}$ different from G_{K_0}.

4 The reconstruction method

Throughout the sequel, we further assume that K_0 satisfies the following two conditions.

First, we assume that for any $\tilde{K}$ compact subset of $\mathbb{R}^N$, we have $\mathscr{H}^{N-1}(K_0 \cap \tilde{K}) = \mathscr{M}^{N-1}(K_0 \cap \tilde{K})$. Here, for any set S, we denote $\mathscr{M}^{N-1}(S)$ as the $(N-1)$-*dimensional Minkowski content* of S, that is

$$\mathscr{M}^{N-1}(S) = \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} |\{x \in \mathbb{R}^N : \mathrm{dist}(x, S) < \varepsilon\}|,$$

provided the limit exists.

Second, we assume that there exist positive constants $\tilde{\varepsilon}$, $0 < \tilde{c}_1 < \tilde{c}_2 < 1$ and C such that for any ε, $0 < \varepsilon \leqslant \tilde{\varepsilon}$, and any $w \in W^{1,2}(\Omega \cap B_\varepsilon(K_0))$, we can find $\tilde{w} \in W^{1,2}(\Omega \cap B_\varepsilon(K_0) \backslash K_0)$ such that the following two conditions are satisfied. We require that $\tilde{w} = w$ in $\Omega \cap B_\varepsilon(K_0) \backslash \overline{B_{\tilde{c}_2\varepsilon}(K_0)}$ and

$$\|\nabla \tilde{w}\|_{L^2(\Omega \cap B_\varepsilon(K_0) \backslash K_0)} \leqslant C \|\nabla w\|_{L^2(\Omega \cap B_\varepsilon(K_0) \backslash \overline{B_{\tilde{c}_1\varepsilon}(K_0)})}. \tag{4.1}$$

https://doi.org/10.1017/S0956792508007729 Published online by Cambridge University Press

We wish to make some comments on these assumptions. We assume, for the time being, that $K_0 \in \mathscr{B}'$, where $\mathscr{B}$ is one of the admissible classes of Definition 3.3. First, we notice that, in this case, the first assumption described above is always satisfied. In the next proposition we give sufficient conditions for the second assumption to be satisfied.

Proposition 4.1 *Let $K_0 \in \mathscr{B}'$, where $\mathscr{B}$ is one of the admissible classes of Definition* 3.3.

Let us assume that, for any $x_0 \in K_0 \cap \Omega$, there exists $r > 0$, depending on x_0, such that for any U connected component of $(\Omega\backslash K_0) \cap B_r(x_0)$ we can find $r_1 > 0$, an open set U_1, such that $U \cap B_{r_1}(x_0) \subset U_1 \subset U$, and a bijective map $T : U_1 \to (-1,1)^N$ such that the following properties hold. The maps T and T^{-1} are locally Lipschitz and there exists a constant C such that $\|DT\|$ and $\|DT^{-1}\|$ are bounded by C almost everywhere. By the regularity of $Q = (-1,1)^N$, T^{-1} can be actually extended up to the boundary and we have that $T^{-1} : [-1,1]^N \to \mathbb{R}^N$ is a Lipschitz map with Lipschitz constant bounded by C. Furthermore, if we set $\Gamma = [-1,1]^{N-1} \times \{1\}$, we require that $T^{-1}(\Gamma) = \partial U_1 \cap K_0$ and $T^{-1}(y) \in \Omega\backslash K_0$ for any $y \in [-1,1]^N\backslash\Gamma$.

If $x_0 \in K_0 \cap \partial\Omega$, we assume that there exists $r > 0$, depending on x_0, such that for any U connected component of $(\Omega\backslash K_0) \cap B_r(x_0)$ we can find $r_1 > 0$, an open set U_1, such that $U \cap B_{r_1}(x_0) \subset U_1 \subset U$, and a bijective map $T : U_1 \to (0,1) \times (-1,1)^{N-1}$ such that the following properties hold. The maps T and T^{-1} are locally Lipschitz and there exists a constant C such that $\|DT\|$ and $\|DT^{-1}\|$ are bounded by C almost everywhere. By the regularity of $Q_1 = (0,1) \times (-1,1)^{N-1}$, T^{-1} can be actually extended up to the boundary and we have that $T^{-1} : \overline{Q_1} \to \mathbb{R}^N$ is a Lipschitz map with Lipschitz constant bounded by C. Furthermore, if we set $\Gamma_1 = [0,1] \times [-1,1]^{N-2} \times \{1\}$ and $\Gamma_2 = \{0\} \times [-1,1]^{N-1}$, we require that $T^{-1}(\Gamma_1) = \partial U_1 \cap K_0$, $T^{-1}(\Gamma_2) = \partial U_1 \cap \partial\Omega$ and $T^{-1}(y) \in \Omega\backslash K_0$ for any $y \in \overline{Q_1}\backslash(\Gamma_1 \cup \Gamma_2)$.

Then, there exist positive constants $\tilde{\varepsilon}$, $0 < \tilde{c}_1 < \tilde{c}_2 < 1$ and C such that the second assumption is satisfied.

Before sketching the proof, let us observe that the sufficient condition of Proposition 4.1 implies, and it almost coincides with, the assumption of Proposition 4.5 in [19], which guarantees that $K_0 \in \mathscr{B}'_q$ for some $q > 2$. Furthermore, all the examples described in Section 4 of [19], for which Proposition 4.5 in [19] holds, also satisfy the sufficient condition of Proposition 4.1.

Proof of Proposition 4.1 We just present the idea of the proof, leaving the details to the reader. Let us assume that we have a function $w_1 \in W^{1,2}(Q)$ where $Q = (-1,1)^N$. We call $Q' = (-1,1)^{N-1}$ and we also fix a cutoff function $\chi : Q' \to \mathbb{R}$ such that $\chi \in C_0^\infty(Q')$, $0 \leqslant \chi \leqslant 1$, $\chi \equiv 1$ on $(-3/4,3/4)^{N-1}$ and $\chi \equiv 0$ on $Q'\backslash(-7/8,7/8)^{N-1}$. For every $x \in \mathbb{R}^N$, we recall that $x = (x', x_N)$ where $x' \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$. Then, for every ε, $0 < \varepsilon \leqslant 1$, and every $x = (x', x_N) \in Q$, we define

$$\tilde{w}_{1,\varepsilon}(x', x_N) = w_1(x', x_N) + \chi(x')(\hat{w}_{1,\varepsilon}(x', x_N) - w_1(x', x_N)),$$

where

$$\hat{w}_{1,\varepsilon}(x', x_N) = \begin{cases} w_1(x', x_N) & \text{if } -1 < x_N \leqslant 1 - (3/2)\varepsilon, \\ w_1(x', (2/3) - \varepsilon + (x_N/3)) & \text{if } 1 - (3/2)\varepsilon < x_N < 1. \end{cases}$$

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We observe that, by construction, $\tilde{w}_{1,\varepsilon} = w_1$ on $Q' \times (-1, 1-(3/2)\varepsilon]$ and on $(Q' \backslash (-7/8, 7/8)^{N-1}) \times (-1,1)$. Moreover, on $(-3/4, 3/4)^{N-1} \times (1-2\varepsilon, 1)$, $\tilde{w}_{1,\varepsilon}$ depends only on the values of w_1 on the set $(-3/4, 3/4)^{N-1} \times (1-2\varepsilon, 1-\varepsilon)$ and, for some constant C, not depending on ε, we have

$$\|\nabla \tilde{w}_{1,\varepsilon}\|_{L^2((-3/4,3/4)^{N-1}\times(1-2\varepsilon,1))} \leqslant C\|\nabla w_1\|_{L^2((-3/4,3/4)^{N-1}\times(1-2\varepsilon,1-\varepsilon))}.$$

We then construct the function $\tilde{w}$ from the function w in the following way. We construct an open covering of K_0 by suitably chosen neighbourhoods of its points, namely $B_{r_1}(x_0)$, $x_0 \in K_0$. By compactness, we find a subcover made only of a finite number of neighbourhoods. We pick one of these neighbourhoods and we locally modify w inside each component $U \cap B_{r_1}(x_0)$ in the following way. We call $w_1 = w \circ T^{-1}$ and (using a reflection in Γ_2 if we had a neighbourhood of a point in $K_0 \cap \partial\Omega$) we construct $\tilde{w}_{1,\varepsilon}$, and then we consider the function $\tilde{w}_{1,\varepsilon} \circ T$. We have obtained a function with the desired properties in a single neighbourhood. We iteratively proceed with the same construction in any neighbourhood, until we have modified in the required way the function w all over a neighbourhood of K_0, thus constructing $\tilde{w}$. □

Let us fix ε, $0 < \varepsilon \leqslant 1$, then the noisy Cauchy data are given by f_ε and g_ε. Here f_ε belongs to $L^s(\partial\Omega)$ and satisfies $\mathrm{supp}(f_\varepsilon) \subset \tilde{\gamma}$ and $\int_{\partial\Omega} f_\varepsilon = 0$, whereas g_ε belongs to $L^2(\gamma)$ and satisfies $\int_\gamma g_\varepsilon = 0$. We assume that

$$\|f_0 - f_\varepsilon\|_{L^s(\tilde{\gamma})} \leqslant \varepsilon \quad \text{and} \quad \|g_0 - g_\varepsilon\|_{L^2(\gamma)} \leqslant \varepsilon. \tag{4.2}$$

Therefore ε estimates from above the noise level of the measurements.

Let us also fix a positive constant c_1, $0 < c_1 < 1$. For any $a > 0$, we call $H_1(a)$ the space of functions $v \in W^{1,2}(\Omega, [0,1])$ such that $v = 1$ almost everywhere in $\tilde{\Omega}_1$ and for some $A \in \mathscr{B}$ we have $v \geqslant c_1$ almost everywhere in $\Omega \backslash \overline{B_a(A)}$.

For any $0 < \varepsilon \leqslant 1$, let $\eta = \eta(\varepsilon)$ be such that $\lim_{\varepsilon\to 0^+} \eta(\varepsilon) = 0$. Let us also fix, for any $0 < \varepsilon \leqslant 1$, a_ε such that $\lim_{\varepsilon\to 0^+} a_\varepsilon = 0$.

Let $\psi_1 : \mathbb{R} \to \mathbb{R}$ be a lower semi-continuous, non-decreasing function such that $\psi_1(0) = 0$, $\psi_1(1) = 1$ and $\psi_1(c_1) > 0$. Furthermore, we assume that for some constants $C > 0$ and $\tilde{\alpha}$, $0 < \tilde{\alpha} \leqslant 1/2$, $\psi_1^{\tilde{\alpha}}(t) \leqslant C\psi(t)^{1/q}$ for any $t \in [0,1]$.

For instance, if $\psi(t) = t^q$ for any $t \geqslant 0$, and $\psi(t) = 0$ if $t < 0$, then we may choose $\psi_1(t) = t^2$ for any $t \geqslant 0$, and $\psi_1(t) = 0$ if $t < 0$. Here $C = 1$ and $\tilde{\alpha} = 1/2$.

We define $\psi_{1,\eta} = (1 - o_\eta^{1/(\tilde{\alpha}q)})\psi_1 + o_\eta^{1/(\tilde{\alpha}q)}$. Provided $o_\eta < 1$, we have that $\psi_{1,\eta}$ is a lower semi-continuous, non-decreasing function such that $\psi_{1,\eta}(0) = o_\eta^{1/(\tilde{\alpha}q)}$ and $\psi_{1,\eta}(1) = 1$. Furthermore, when $o_\eta^{1/(\tilde{\alpha}q)} < 1/2$, $\psi_{1,\eta}(c_1) > \psi_1(c_1)/2 > 0$ and, for some constant $C > 0$, we have $\psi_{1,\eta}^{\tilde{\alpha}}(t) \leqslant C\psi_\eta^{1/q}(t)$ for any $t \in [0,1]$. In the sequel we shall always assume that $o_\eta^{1/(\tilde{\alpha}q)} < 1/2$. Clearly we always have $\eta = \eta(\varepsilon)$.

For any r, $1 < r < +\infty$, and any Borel set $E \subset \overline{\Omega}$ we define

$${}_E W^{1,r}(\Omega) = \left\{ w \in W^{1,r}(\Omega) : \int_E w = 0 \right\}.$$

We observe that, by a generalized Poincaré inequality, whenever E has positive

https://doi.org/10.1017/S0956792508007729 Published online by Cambridge University Press

$\mathscr{H}^{N-1}$-measure, then on ${}_E W^{1,r}(\Omega)$ the usual $W^{1,r}(\Omega)$ norm and the norm $\|w\|_{{}_E W^{1,r}(\Omega)} = \|\nabla w\|_{L^r(\Omega)}$ are equivalent. Therefore, we shall set the second one as the natural norm of ${}_E W^{1,r}(\Omega)$.

For any $v \in W^{1,2}(\Omega, [0,1])$ and any $w_1, w_2 \in W^{1,2}(\Omega)$ we define the bilinear form

$$\langle w_1, w_2 \rangle_{v,\eta} = \int_\Omega \psi_{1,\eta}(v) \nabla w_1 \cdot \nabla w_2$$

and we denote the semi-norm

$$|w_1|_{v,\eta} = \langle w_1, w_1 \rangle_{v,\eta}^{1/2} = \left(\int_\Omega \psi_{1,\eta}(v) |\nabla w_1|^2 \right)^{1/2}.$$

We denote, for any $w_1 \in W^{1,2}(\Omega)$,

$$\|w_1\|_{v,\eta} = \left(\|w_1\|^2_{L^2(\Omega)} + |w_1|^2_{v,\eta} \right)^{1/2}.$$

We notice that, if $o_\eta = 0$, then $\| \cdot \|_{v,\eta}$ might not be an equivalent norm to the one of $W^{1,2}(\Omega)$. However, provided $o_\eta > 0$, we have that $\| \cdot \|_{v,\eta}$ is an equivalent norm for $W^{1,2}(\Omega)$, and $\langle \cdot, \cdot \rangle_{v,\eta}$ is a scalar product on ${}_{\tilde{\gamma}} W^{1,2}(\Omega)$ whose corresponding norm, $| \cdot |_{v,\eta}$, is an equivalent norm for ${}_{\tilde{\gamma}} W^{1,2}(\Omega)$.

We finally fix positive constants a_1, a_2, $\tilde{q}$ and $\tilde{\beta}$. For any $0 < \varepsilon \leqslant 1$, let us define $\mathscr{F}_\varepsilon$ as the following functional on $L^1(\Omega) \times L^1(\Omega)$. For any $(u, v) \in L^1(\Omega) \times L^1(\Omega)$ we set

$$\mathscr{F}_\varepsilon(u,v) = \frac{a_1}{\varepsilon^{\tilde{q}}} \sup_{\substack{w \in W^{1,2}(\Omega) \\ |w|_{v,\eta} \leqslant 1}} \left(\int_\Omega \psi_{1,\eta}(v) \nabla u \cdot \nabla w - \int_{\tilde{\gamma}} f_\varepsilon w \right)^2 + \frac{a_2}{\varepsilon^{\tilde{\beta}}} \int_\gamma |u - g_\varepsilon|^2 + \mathscr{A}\mathscr{T}_q^\eta(u,v) \quad \text{if } v \in H_1(a_\varepsilon), \tag{4.3}$$

whereas $\mathscr{F}_\varepsilon(u,v) = +\infty$ otherwise.

Remark 4.2 We observe that, without loss of generality, we may impose on w and u, in the previous definition of $\mathscr{F}_\varepsilon$ (4.3) the following constraints

$$\int_{\tilde{\gamma}} w = 0 \quad \text{and} \quad \int_\gamma u = 0. \tag{4.4}$$

We modify the functional $\mathscr{F}_0$ as follows. We set $\mathscr{F}_0 : L^1(\Omega) \times L^1(\Omega) \to [0, +\infty]$ such that for any $(u, v) \in L^1(\Omega) \times L^1(\Omega)$

$$\mathscr{F}_0(u,v) = \mathscr{M}\mathscr{S}_q(u,v) \quad \text{if } u \in H_0, \tag{4.5}$$

whereas $\mathscr{F}_0(u,v) = +\infty$ otherwise.

Let us concentrate on the following result.

Proposition 4.3 *Under the previous notation and assumptions, let ε_n, $n \in \mathbb{N}$, be a sequence of positive numbers such that $\varepsilon_n \to 0$ as $n \to \infty$. Let $\mathscr{F}_n = \mathscr{F}_{\varepsilon_n}$ and let $(u_n, v_n) \in L^1(\Omega) \times L^1(\Omega)$ be such that, for a constant C, we have*

$$\mathscr{F}_n(u_n, v_n) \leqslant C \quad \text{for any } n \in \mathbb{N}$$

and

$$\|u_n\|_{L^\infty(\Omega)} \leqslant C \quad \text{for any } n \in \mathbb{N}.$$

Then, up to a subsequence, which we do not relabel, we have that (u_n, v_n) converges to (u, v) in $L^1(\Omega) \times L^1(\Omega)$, as $n \to \infty$, where $u \in H_0$ and $v = 1$ almost everywhere in Ω, and

$$\liminf_n \mathscr{F}_n(u_n, v_n) \geqslant \mathscr{F}_0(u, v). \tag{4.6}$$

Proof We immediately observe that $v_n \to v = 1$ in $L^1(\Omega)$. Up to subsequences we may also assume that $v_n \to 1$ almost everywhere in Ω and in $L^p(\Omega)$, for any p, $1 \leqslant p < +\infty$. We may also assume that there exists $\lim_n \mathscr{F}_n(u_n, v_n)$, that is the lim inf is actually a limit.

Let $\eta_n = \eta(\varepsilon_n)$, for any $n \in \mathbb{N}$. With a slight abuse of notation, for the sake of clarity, we shall always drop the subscript and use $\eta = \eta_n = \eta(\varepsilon_n)$, for any $n \in \mathbb{N}$, instead.

We need to prove compactness of the sequence u_n. First of all we notice that, for any function ψ_1 satisfying the above hypothesis, $\psi_{1,\eta}(v_n)\nabla u_n$ and $\psi_{1,\eta}^2(v_n)\nabla u_n$, up to subsequences, converge weakly in $L^q(\Omega, \mathbb{R}^N)$ to V_1 and $V_2 \in L^q(\Omega, \mathbb{R}^N)$, respectively. At this stage, V_1 and V_2 might depend on ψ_1 or on the choice of the subsequence.

We make use of our assumptions on v_n, namely that $v_n \in H_1(a_n)$, where $a_n = a_{\varepsilon_n}$, for any $n \in \mathbb{N}$. Therefore, for any $n \in \mathbb{N}$ we associate to v_n its corresponding set $A_n \in \mathscr{B}$. Hence, again up to subsequences, A_n converges, in the Hausdorff distance, to $A \in \mathscr{B}$. For any positive integer m, let $\Omega_m \subset \Omega$ be an open set with Lipschitz boundary such that, for any $m \in \mathbb{N}$, $\Omega_m \subset \Omega_{m+1}$ and $\Omega \backslash \overline{B_{1/m}(A)} \subset \Omega_m \subset \Omega \backslash \overline{B_{1/(2m)}(A)}$. We have that there exists n_0, depending on m, such that for any $n \geqslant n_0$, $v_n \geqslant c_1$ on Ω_m, thus u_n converges, up to a subsequence, to a function $\tilde{u}_m$ weakly in $W^{1,q}(\Omega_m)$, strongly in $L^p(\Omega_m)$ for any p, $1 \leqslant p < +\infty$, and almost everywhere. By a diagonal argument, we may therefore find a function u, which is defined almost everywhere in $\bigcup_{m=1}^{\infty} \Omega_m$, such that, again up to a subsequence which we do not relabel, $u_n \to u$ almost everywhere in $\bigcup_{m=1}^{\infty} \Omega_m$. Since $\Omega \backslash \bigcup_{m=1}^{\infty} \Omega_m \subset A$, hence $|\Omega \backslash \bigcup_{m=1}^{\infty} \Omega_m| = 0$, u is defined almost everywhere in Ω, $u_n \to u$ almost everywhere in Ω and, by the uniform L^∞ bound, strongly in $L^p(\Omega)$ for any p, $1 \leqslant p < +\infty$.

By Theorem 2.3, we infer that $u \in \mathrm{GSBV}(\Omega)$. Actually, since $\|u\|_{L^\infty}(\Omega) \leqslant C$, we have that $u \in \mathrm{SBV}(\Omega)$. Furthermore, $u \in W^{1,q}(\Omega_m)$ for any m and $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^q(\Omega_m)$ for any m. Therefore, we may also conclude that $\nabla u = V_1 = V_2$ almost everywhere in Ω_m, for any m, and consequently almost everywhere in Ω. So $\nabla u \in L^q(\Omega)$ and, up to a subsequence, $\psi_{1,\eta}(v_n)\nabla u_n$ and $\psi_{1,\eta}^2(v_n)\nabla u_n$ both converge weakly to ∇u in $L^q(\Omega)$.

Recalling that $v = 1$ almost everywhere in Ω, we obtain that

$$\mathscr{M}\mathscr{S}_q(u, v) \leqslant \liminf_n \mathscr{F}_n(u_n, v_n).$$

Hence, what remains to be proved is the fact that $u \in H_0$. We proceed in an analogous way as in the proof of Theorem 5.3 in [19]. Since $v_n \equiv 1$ on $\tilde{\Omega}_1$, we may also assume, without loss of generality, that $u_n \rightharpoonup u$ weakly in $W^{1,q}(\tilde{\Omega}_1)$. Therefore, $u \in W^{1,q}(\tilde{\Omega}_1)$ and, consequently, $[J(u) \cap \tilde{\Omega}_1] = 0$. Moreover, we may assume that $u_n|_{\partial\tilde{\Omega}_1} \to u|_{\partial\tilde{\Omega}_1}$ strongly in $L^q(\partial\tilde{\Omega}_1)$ and, by the uniform L^∞ bound, actually in $L^p(\partial\tilde{\Omega}_1)$ for any p, $1 \leqslant p < +\infty$. Then we immediately obtain that $u = g_0$ on γ. Since $\Omega \backslash \bigcup_{m=1}^{\infty} \Omega_m \subset A \in \mathscr{B}$ and $u \in W^{1,q}(\Omega_m)$ for any m, we conclude that $u \in H$.

Hence, it would be enough to prove that $\int_\Omega \nabla u \cdot \nabla w = \int_{\tilde{\gamma}} f_0 w$ for any $w \in W^{1,2}(\Omega)$ and $\int_\Omega \nabla u \cdot \nabla(uw_1) = \int_{\tilde{\gamma}} f_0 uw_1$ for any $w_1 \in C_0^\infty(\mathbb{R}^N)$.

First of all, without loss of generality, we may assume that both $\psi_{1,\eta}(v_n)\nabla u_n$ and $\psi_{1,\eta}^2(v_n)\nabla u_n$ converge weakly to ∇u in $L^q(\Omega)$. Let us take $w \in W^{1,2}(\Omega)$. We call $f_n = f_{\varepsilon_n}$ and we observe that

$$\left|\int_\Omega \psi_{1,\eta}(v_n)\nabla u_n \cdot \nabla w - \int_{\tilde{\gamma}} f_n w\right| \to 0 \quad \text{as } n \to \infty.$$

As $n \to \infty$, since

$$\int_\Omega \psi_{1,\eta}(v_n)\nabla u_n \cdot \nabla w - \int_{\tilde{\gamma}} f_n w \to \int_\Omega \nabla u \cdot \nabla w - \int_{\tilde{\gamma}} f_0 w,$$

we conclude that

$$\int_\Omega \nabla u \cdot \nabla w = \int_{\tilde{\gamma}} f_0 w \quad \text{for any } w \in W^{1,2}(\Omega).$$

The most difficult term is the non-linear one. Let $w_1 \in C_0^\infty(\mathbb{R}^N)$. We assume that $0 \leqslant w_1 \leqslant 1$ almost everywhere in Ω and that $w_1 = 1$ almost everywhere in Ω_1. We call $\tilde{A}$ the intersection of A with the closure of $\Omega \backslash \overline{\tilde{\Omega}_1}$ and we also assume that w_1 is identically equal to zero in a neighbourhood of $\tilde{A}$. Then we obtain that, first $uw_1 \in W^{1,2}(\Omega)$ and, consequently,

$$\int_\Omega \nabla u \cdot \nabla(uw_1) = \int_{\tilde{\gamma}} f_0 u.$$

Second, $u_n w_1$ belongs to $W^{1,q}(\Omega)$, for any n, and, since for any $n \geqslant n_0$, n_0 depending on w_1, we have that $v_n \geqslant c_1$ whenever $w_1 > 0$, we may conclude that actually $\|u_n w_1\|_{W^{1,q}(\Omega)}$ is uniformly bounded. Therefore, we deduce that

$$\left|\int_\Omega \psi_{1,\eta}(v_n)\nabla u_n \cdot \nabla(u_n w_1) - \int_{\tilde{\gamma}} f_n u_n w_1\right| \to 0 \quad \text{as } n \to \infty. \tag{4.7}$$

Since $w_1 \equiv 1$ on $\tilde{\gamma}$, we have

$$\begin{aligned}\int_\Omega w_1 \psi_{1,\eta}^2(v_n)|\nabla u_n|^2 - \int_{\tilde{\gamma}} f_n u_n &= \int_\Omega w_1(\psi_{1,\eta}^2(v_n) - \psi_{1,\eta}(v_n))|\nabla u_n|^2 \\ &\quad + \int_\Omega \psi_{1,\eta}(v_n)\nabla u_n \cdot \nabla(u_n w_1) - \int_\Omega u_n \psi_{1,\eta}(v_n)\nabla u_n \cdot \nabla w_1 - \int_{\tilde{\gamma}} f_n u_n w_1.\end{aligned}$$

We observe that $u_n \to u$ and $\psi_{1,\eta}(v_n) \to 1$ in $L^p(\Omega)$ for any p, $1 \leqslant p < +\infty$, $\psi_{1,\eta}(v_n)\nabla u_n$ converges weakly to ∇u in $L^q(\Omega)$ and $w_1|\nabla u_n|^2$ is uniformly bounded in $L^{q/2}(\Omega)$. Hence, as $n \to \infty$,

$$\int_\Omega w_1(\psi_{1,\eta}^2(v_n) - \psi_{1,\eta}(v_n))|\nabla u_n|^2 \to 0 \tag{4.8}$$

and

$$\int_\Omega u_n \psi_{1,\eta}(v_n)\nabla u_n \cdot \nabla w_1 \to \int_\Omega u\nabla u \cdot \nabla w_1. \tag{4.9}$$

By (4.7), we conclude that

$$\int_\Omega w_1\psi_{1,\eta}^2(v_n)|\nabla u_n|^2 - \int_{\tilde{\gamma}} f_n u_n \;\to\; -\int_\Omega u\nabla u \cdot \nabla w_1.$$

We observe that

$$\int_\Omega w_1|\nabla u|^2 - \int_{\tilde{\gamma}} f_0 u = \int_\Omega \nabla u \cdot \nabla(uw_1) - \int_\Omega u\nabla u \cdot \nabla w_1 - \int_{\tilde{\gamma}} f_0 u = -\int_\Omega u\nabla u \cdot \nabla w_1.$$

Since $\int_{\tilde{\gamma}} f_n u_n$ converges to $\int_{\tilde{\gamma}} f_0 u$, we may conclude that, for any w_1 satisfying the above assumptions, we have

$$\int_\Omega w_1\psi_{1,\eta}^2(v_n)|\nabla u_n|^2 \to \int_\Omega w_1|\nabla u|^2 \quad \text{as } n \to \infty.$$

We know that $\psi_{1,\eta}(v_n)\nabla u_n$ is uniformly bounded in $L^q(\Omega)$. Therefore, we take a sequence of such functions w_1 converging to 1 almost everywhere and we prove that $\psi_{1,\eta}(v_n)\nabla u_n$ converges to ∇u strongly in $L^2(\Omega)$. By a simple interpolation inequality, we also infer that $\psi_{1,\eta}(v_n)\nabla u_n$ converges to ∇u strongly in $L^p(\Omega)$ for any $2 \leqslant p < q$.

Then, if we now take any $w_1 \in C_0^\infty(\mathbb{R}^N)$, we observe that $|u_n w_1|_{v_n,\eta}$ is uniformly bounded with respect to n, therefore

$$\begin{aligned}\int_\Omega \psi_{1,\eta}^2(v_n)\nabla u_n \cdot \nabla(u_n w_1) &= \int_\Omega w_1\psi_{1,\eta}^2(v_n)|\nabla u_n|^2 + \int_\Omega u_n\psi_{1,\eta}^2(v_n)\nabla u_n \cdot \nabla w_1 \\ &\to \int_\Omega w_1|\nabla u|^2 + \int_\Omega u\nabla u \cdot \nabla w_1 = \int_\Omega \nabla u \cdot \nabla(uw_1).\end{aligned}$$

Again using (4.7), the convergence of $\int_{\tilde{\gamma}} f_n u_n w_1$ to $\int_{\tilde{\gamma}} f_0 u w_1$, and similar estimates as previously used, by a simple computation we infer that, for any $w_1 \in C_0^\infty(\mathbb{R}^N)$,

$$\int_\Omega \nabla u \cdot \nabla(uw_1) = \int_{\tilde{\gamma}} f_0 u w_1,$$

thus the proof is concluded. □

In the following theorem we summarize the results obtained so far. We recall that for any Sobolev function v defined on Ω and any constant c, we may define, in a weak sense, the set $\{v < c\}$ which is an open subset of Ω.

Theorem 4.4 *For any $n \in \mathbb{N}$, let $\varepsilon_n > 0$ be such that $\lim_n \varepsilon_n = 0$.*

If $(u_n, v_n) \in L^1(\Omega) \times L^1(\Omega)$ is such that $\mathscr{F}_{\varepsilon_n}(u_n, v_n)$ and $\|u_n\|_{L^\infty(\Omega)}$ are uniformly bounded, then, up to a subsequence, we have that $u_n \to u \in H_0$ strongly in $L^p(\Omega)$ for any p,

https://doi.org/10.1017/S0956792508007729 Published online by Cambridge University Press

$1 \leqslant p < +\infty$. Furthermore, there exist compact sets $\tilde{A} \subset \overline{\Omega}$ and $A \in \mathscr{B}$ such that $\tilde{A} \subset A$ and $[J(u) \backslash \tilde{A}] = 0$ satisfying the following property. For any constant c, $0 < c \leqslant c_1$, the sets $\overline{\{v_n < c\}}$ converge, as $n \to \infty$, to $\tilde{A}$ in the Hausdorff distance.

Finally, if $\mathscr{B} = \mathscr{B}_1$ and $k = 1$, then $u = u_0$ almost everywhere in G_{K_0} and $\nabla u = \nabla u_0$ almost everywhere in Ω.

Proof The theorem immediately follows by combining the results of Propositions 3.8 and 4.3 and by using the fact that $v_n \in H_1(a_{\varepsilon_n})$. □

For any constant $\tilde{\alpha}$, $0 < \tilde{\alpha} \leqslant 1/2$, we set $q_1(\tilde{\alpha}) = 1/(2\tilde{\alpha}) - 1/2 - 1/(2\tilde{\alpha}q)$. We notice that $q_1(\tilde{\alpha}) > 0$ if $0 < \tilde{\alpha} \leqslant 1/2$. We shall set $q_1 = q_1(2) = (q-2)/(2q)$ and we observe that $0 < q_1 < 1/2$. Moreover, $q_1(\tilde{\alpha}) \geqslant q_1$ for any $\tilde{\alpha} \leqslant 1/2$.

What remains to be proved is that the sequence $(u_n, v_n) \in L^1(\Omega) \times L^1(\Omega)$, such that $\mathscr{F}_{\varepsilon_n}(u_n, v_n)$ and $\|u_n\|_{L^\infty(\Omega)}$ are uniformly bounded, exists. Under some further assumptions, we shall prove that there exist a constant C and functions $(u_\varepsilon, v_\varepsilon)$ such that $\mathscr{F}_\varepsilon(u_\varepsilon, v_\varepsilon) \leqslant C$ and $\|u_\varepsilon\|_{L^\infty(\Omega)} \leqslant C$ for any ε, $0 < \varepsilon \leqslant 1$. If we couple this result together with Proposition 4.3, we also obtain a kind of equi-coerciveness for the functionals $\mathscr{F}_\varepsilon$.

Proposition 4.5 *Let $u_0 = u(f_0, K_0)$ and v_0 be identically equal to 1 in Ω. Besides the previous notation and assumptions, let us further assume that the following constants satisfy $0 < \tilde{q} \leqslant 2$, $0 < \tilde{\beta} \leqslant 2$, and that*

$$\limsup_{\varepsilon \to 0^+} \frac{\eta(\varepsilon)^{2q_1}}{\varepsilon^{\tilde{q}}} < +\infty,$$

and, finally, that $a_\varepsilon \geqslant 2\eta(\varepsilon)$.

Then, for any ε, $0 < \varepsilon \leqslant 1$, there exists $(u_\varepsilon, v_\varepsilon) \in L^1(\Omega) \times L^1(\Omega)$ such that $(u_\varepsilon, v_\varepsilon)$ converges to (u_0, v_0) in $L^1(\Omega) \times L^1(\Omega)$ and, for a positive constant C,

$$\mathscr{F}_\varepsilon(u_\varepsilon, v_\varepsilon) \leqslant C \text{ and } \|u_\varepsilon\|_{L^\infty(\Omega)} \leqslant C \quad \text{for any } \varepsilon,\ 0 < \varepsilon \leqslant 1. \tag{4.10}$$

Proof Choose $\xi_\eta > 0$ such that $\xi_\eta/\eta \to 0$ and $o_\eta/\xi_\eta^{q-1} \to 0$ as $\eta \to 0^+$. For instance, provided $o_\eta > 0$ and recalling that $\lim_{\eta \to 0^+} o_\eta/\eta^{q-1} = 0$, we may choose $\xi_\eta = \sqrt{\eta \sqrt[q-1]{o_\eta}}$. Namely, if o_η verifies $0 \leqslant o_\eta \leqslant \eta^q$, we may choose $\xi_\eta = \eta^{1+(1/2(q-1))}$.

We can find $\varepsilon_0 >$ such that $\eta + \xi_\eta < 2\eta \leqslant a_\varepsilon < \delta/4$ for any ε_0, $0 < \varepsilon \leqslant \varepsilon_0$. Then, for any ε, $0 < \varepsilon \leqslant \varepsilon_0$, we define $(u_\varepsilon, v_\varepsilon)$ in the following way. We fix an auxiliary function $\chi : \mathbb{R} \to \mathbb{R}$ such that $\chi \in C^\infty(\mathbb{R})$, $0 \leqslant \chi \leqslant 1$, χ is non-decreasing, $\operatorname{supp}(\chi) \subset (0, +\infty)$ and $\operatorname{supp}(1-\chi) \subset (-\infty, 1)$. We also assume that $\chi(\tilde{c}_1/2) \geqslant c_1$.

For any $x \in \Omega$, we set

$$u_\varepsilon(x) = \chi\left(\frac{\operatorname{dist}(x, K_0)}{\xi_\eta}\right) u_0(x),$$

and

$$v_\varepsilon(x) = \begin{cases} 0 & \text{if } \operatorname{dist}(x, K_0) \leqslant \xi_\eta, \\ \chi\left(\frac{\operatorname{dist}(x, K_0) - \xi_\eta}{\eta}\right) & \text{if } \xi_\eta < \operatorname{dist}(x, K_0) < \xi_\eta + \eta, \\ 1 & \text{if } \operatorname{dist}(x, K_0) \geqslant \xi_\eta + \eta. \end{cases}$$

A simple computation leads us to verify that these functions $(u_\varepsilon, v_\varepsilon)$ satisfy the following properties, for any $0<\varepsilon\leqslant\varepsilon_0$. First, $\|u_\varepsilon\|_{L^\infty(\Omega)}\leqslant\|u_0\|_{L^\infty(\Omega)}$. Second, $u_\varepsilon\in W^{1,q}(\Omega)$ and $v_\varepsilon\in H_1(a_\varepsilon)$. Furthermore, $u_\varepsilon|_\gamma=g_0$. Finally, there exists a constant C such that for any ε, $0<\varepsilon\leqslant\varepsilon_0$, we have then

$$\frac{1}{\varepsilon^{\tilde\beta}}\int_\gamma |u_\varepsilon-g_\varepsilon|^2+\mathscr{A}\mathscr{T}_q^\eta(u_\varepsilon,v_\varepsilon)\leqslant C.$$

Here, we have made use of our assumption on the Minkowski content of K_0. Furthermore, as $\varepsilon\to 0^+$, $(u_\varepsilon,v_\varepsilon)\to(u_0,v_0)$ in $L^1(\Omega)\times L^1(\Omega)$.

Let us then evaluate

$$\sup_{\substack{w\in W^{1,2}(\Omega)\\ |w|_{v_\varepsilon,\eta}\leqslant 1}}\left(\int_\Omega \psi_{1,\eta}(v_\varepsilon)\nabla u_\varepsilon\cdot\nabla w-\int_{\tilde\gamma} f_\varepsilon w\right).$$

We take $w\in W^{1,2}(\Omega)$ and, without loss of generality, we may assume that $\int_{\tilde\gamma} w=0$. We observe that, if $o_\eta=0$, then

$$\int_\Omega \psi_{1,\eta}(v_\varepsilon)\nabla u_\varepsilon\cdot\nabla w=\int_\Omega \psi_{1,\eta}(v_\varepsilon)\nabla u_0\cdot\nabla w.$$

Otherwise, if $o_\eta>0$, we have

$$\int_\Omega \psi_{1,\eta}(v_\varepsilon)\nabla(u_\varepsilon-u_0)\cdot\nabla w=\int_{\Omega\cap B_{\xi_\eta}(K_0)} o_\eta^{1/(\tilde\alpha q)}\nabla(u_\varepsilon-u_0)\cdot\nabla w.$$

The right-hand side may be bounded, in modulus, by the sum of the following two terms

$$\left(\int_{\Omega\cap B_{\xi_\eta}(K_0)} o_\eta^{1/(\tilde\alpha q)}|\nabla u_0|^2\right)^{1/2}|w|_{v_\varepsilon,\eta}\leqslant C o_\eta^{1/(2\tilde\alpha q)}|w|_{v_\varepsilon,\eta}$$

and

$$\left(\int_{\Omega\cap B_{\xi_\eta}(K_0)} o_\eta^{1/(\tilde\alpha q)}|u_0|^2|\nabla\left(\chi(\mathrm{dist}(\cdot,K_0)/\xi_\eta)\right)|^2\right)^{1/2}|w|_{v_\varepsilon,\eta}\leqslant C\frac{o_\eta^{1/(2\tilde\alpha q)}}{\xi_\eta^{1/2}}|w|_{v_\varepsilon,\eta}.$$

Here the constant C does not depend on ε. We further observe that $o_\eta^{1/(2\tilde\alpha q)}/\xi_\eta^{1/2}$ is an infinitesimal of higher order than that of $\xi_\eta^{q_1(\tilde\alpha)}$, thus also than that of $\eta^{q_1(\tilde\alpha)}$.

We observe that

$$\begin{aligned}\int_\Omega \psi_{1,\eta}(v_\varepsilon)\nabla u_0\cdot\nabla w&=\int_\Omega \nabla u_0\cdot\nabla w-\int_\Omega(1-\psi_{1,\eta}(v_\varepsilon))\nabla u_0\cdot\nabla w\\&=\int_{\tilde\gamma} f_0 w-\int_\Omega(1-\psi_{1,\eta}(v_\varepsilon))\nabla u_0\cdot\nabla w.\end{aligned}$$

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Therefore, using the previous estimates and the hypotheses on the noisy Cauchy data, we obtain

$$\left|\int_\Omega \psi_{1,\eta}(v_\varepsilon)\nabla u_\varepsilon\cdot\nabla w-\int_{\tilde\gamma} f_\varepsilon w\right|\leqslant C\eta^{q_1(\tilde\alpha)}|w|_{v_\varepsilon,\eta}+\left|\int_{\tilde\gamma}(f_0-f_\varepsilon)w\right|+\left|\int_\Omega(1-\psi_{1,\eta}(v_\varepsilon))\nabla u_0\cdot\nabla w\right|$$
$$\leqslant C\eta^{q_1(\tilde\alpha)}|w|_{v_\varepsilon,\eta}+C\varepsilon|w|_{v_\varepsilon,\eta}+\left|\int_\Omega(1-\psi_{1,\eta}(v_\varepsilon))\nabla u_0\cdot\nabla w\right|,$$

where again C does not depend on ε.

The final term to be estimated may be bounded by

$$\left|\int_{\Omega\cap B_{\xi_\eta+\eta}(K_0)}\nabla u_0\cdot\nabla w\right|+\left|\int_{\Omega\cap B_{\xi_\eta+\eta}(K_0)}\psi_{1,\eta}(v_\varepsilon)\nabla u_0\cdot\nabla w\right|.$$

We observe that

$$\left|\int_{\Omega\cap B_{\xi_\eta+\eta}(K_0)}\psi_{1,\eta}(v_\varepsilon)\nabla u_0\cdot\nabla w\right|\leqslant\left(\int_{\Omega\cap B_{\xi_\eta+\eta}(K_0)}\psi_{1,\eta}(v_\varepsilon)|\nabla u_0|^2\right)^{1/2}|w|_{v_\varepsilon,\eta}.$$

Since $|\nabla u_0|^2$ belongs to $L^{q/2}(\Omega)$, we infer that

$$\left(\int_{\Omega\cap B_{\xi_\eta+\eta}(K_0)}\psi_{1,\eta}(v_\varepsilon)|\nabla u_0|^2\right)^{1/2}\leqslant\|\nabla u_0\|_{L^q(\Omega)}|\Omega\cap B_{\xi_\eta+\eta}(K_0)|^{(q-2)/(2q)}.$$

By our assumption on the Minkowski content of K_0, we may find a constant C independent of ε such that

$$\left(\int_{\Omega\cap B_{\xi_\eta+\eta}(K_0)}\psi_{1,\eta}(v_\varepsilon)|\nabla u_0|^2\right)^{1/2}\leqslant C\eta^{q_1}.$$

Now we use the second assumption on K_0 that we have imposed at the beginning of the section. Without loss of generality, we may assume that ε_0 is such that for any ε, $0<\varepsilon\leqslant\varepsilon_0$, $\eta(\varepsilon)\leqslant\tilde\varepsilon/2$ and $\xi_\eta\leqslant\frac{\tilde c_1}{2(1-\tilde c_1)}\eta$. Therefore, if $t\geqslant\tilde c_1(\xi_\eta+\eta)$, then $t\geqslant\xi_\eta+(\tilde c_1/2)\eta$. Let $\tilde w$ be the function constructed from w, as in the beginning of the section, in the set $\Omega\cap B_{\xi_\eta+\eta}(K_0)$. We notice that

$$\int_{\Omega\cap B_{\xi_\eta+\eta}(K_0)}\nabla u_0\cdot\nabla w=\int_{\Omega\cap B_{\xi_\eta+\eta}(K_0)}\nabla u_0\cdot\nabla\tilde w$$

and, by our hypotheses on χ and (4.1), we have that

$$\|\nabla\tilde w\|_{L^2(\Omega\cap B_{\xi_\eta+\eta}(K_0)\backslash K_0)}\leqslant C\|\nabla w\|_{L^2(\Omega\cap B_{\xi_\eta+\eta}(K_0)\backslash\overline{B_{\tilde c_1(\xi_\eta+\eta)}(K_0)})}\leqslant C_1|w|_{v_\varepsilon,\eta}.$$

Here, as usual, C and C_1 do not depend on ε. Using the fact that $\nabla u_0\in L^q(\Omega)$ and with an analogous computation as before, we obtain that, for a constant C independent of ε,

$$\left|\int_{\Omega\cap B_{\xi_\eta+\eta}(K_0)}\nabla u_0\cdot\nabla w\right|\leqslant C\eta^{q_1}|w|_{v_\varepsilon,\eta}.$$

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We conclude that there exists a constant C such that, for any ε, $0<\varepsilon\leqslant\varepsilon_0$, and any $w\in W^{1,2}(\Omega)$, we have

$$\left|\int_\Omega \psi_{1,\eta}(v_\varepsilon)\nabla u_\varepsilon\cdot\nabla w-\int_{\tilde\gamma} f_\varepsilon w\right|\leqslant C(\eta^{q_1}+\varepsilon)|w|_{v_\varepsilon,\eta}.$$

Thus, the proof is concluded. □

Let us note that the two assumptions on K_0 prescribed at the beginning of this section have been used only in the proof of Proposition 4.5. Let us also note that we have not proven a Γ-convergence result, in order to prove that $\mathscr{F}_\varepsilon$ Γ-converges to $\mathscr{F}_0$ as $\varepsilon\to 0^+$, we should find a recovery sequence for any $u\in H_0$ such that $\mathscr{M}\mathscr{S}_q(u,1)$ is finite. First of all, we should impose further restrictions on the constants involved. Furthermore, the following difficulty should be tackled. Namely, even when we have a good characterization of H_0, that is when $\mathscr{B}=\mathscr{B}_1$ and $k=1$, any $u\in H_0$ is only a piecewise constant function on any connected component of $\Omega\backslash K_0$ different from G_{K_0}. Therefore, the jump set of u might be rather complicated and, at least in dimension three and higher, we might even have $\mathscr{H}^{N-1}(J(u)\cap K_0\cap\Omega)<\mathscr{H}^{N-1}(K_0\cap\Omega)$, unless $\alpha=1$ (see Remark 5.7 in [19]). Since we are already in the position to prove a convergence result for quasi-minimizers (see Theorem 4.6) we believe that obtaining a Γ-convergence result is not worth the effort and all the technicalities needed to prove it.

We now state the main result of the paper, which is easily obtained by combining Theorem 4.4 and Proposition 4.5.

Theorem 4.6 *Let $u_0=u(f_0,K_0)$. Let us assume that the assumptions of Proposition 4.5 are satisfied. We also assume that $\mathscr{B}=\mathscr{B}_1$ and $k=1$.*

Then there exists a constant E_0, depending on s, Ω, Ω_1, $\tilde\Omega_1$, γ, $\tilde\gamma$ only, such that for any $E\geqslant E_0$ the following holds.

For any $n\in\mathbb{N}$, let $\varepsilon_n>0$ be such that $\lim_n\varepsilon_n=0$ and let

$$m_n=\inf\{\mathscr{F}_{\varepsilon_n}(u,v):\ (u,v)\in L^1(\Omega)\times L^1(\Omega)\ \textit{and}\ \|u\|_{L^\infty(\Omega)}\leqslant E\}.$$

Then if $(u_n,v_n)\in L^1(\Omega)\times L^1(\Omega)$ is such that $\|u_n\|_{L^\infty(\Omega)}\leqslant E$ and

$$\lim_n(\mathscr{F}_{\varepsilon_n}(u_n,v_n)-m_n)=0,$$

we have that, up to a subsequence, $u_n\to u\in H_0$ strongly in $L^p(\Omega)$ for any p, $1\leqslant p<+\infty$, where $u=u_0$ almost everywhere in G_{K_0} and $\nabla u=\nabla u_0$ almost everywhere in Ω.

Furthermore, there exist compact sets $\tilde A\subset\overline{\Omega}$ and $A\in\mathscr{B}$ such that $\tilde A\subset A$ and $[J(u)\backslash\tilde A]=0$ satisfying the following property. For any constant c, $0<c\leqslant c_1$, the sets $\overline{\{v_n<c\}}$ converge, as $n\to\infty$, to $\tilde A$ in the Hausdorff distance.

5 Properties of the functional $\mathscr{F}_\varepsilon$

In this section, we investigate some properties of the functional $\mathscr{F}_\varepsilon$. For this purpose, we shall use a different, but completely equivalent, formulation of the functional.

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Throughout this section, we shall fix a constant ε, $0<\varepsilon\leqslant 1$, and we shall assume that o_η, $\eta=\eta(\varepsilon)$, satisfies

$$0<o_\eta<\eta^{q-1}.$$

Without loss of generality, we also assume that ψ, V and ψ_1 are bounded all over $\mathbb{R}$. We define ψ_η and $\psi_{1,\eta}$ as in the previous sections and, again without loss of generality, we assume that ψ and ψ_1 are such that $\psi_\eta\geqslant o_\eta/2$ and $\psi_{1,\eta}\geqslant o_\eta^{1/(\tilde{\alpha}q)}/2$, respectively, all over $\mathbb{R}$.

Let us call $W(\Omega)=\{\tilde{v}\in W^{1,2}(\Omega)\cap L^\infty(\Omega) : \tilde{v}=0 \text{ a.e. in } \tilde{\Omega}_1\}$ with norm $\|\tilde{v}\|_{W(\Omega)}=\|\tilde{v}\|_{L^\infty(\Omega)}+\|\nabla\tilde{v}\|_{L^2(\Omega)}$. To any $\tilde{v}\in W(\Omega)$ we associate the function $v=1-\tilde{v}$. We remark that $v\in W^{1,2}(\Omega)\cap L^\infty(\Omega)$ and $v=1$ almost everywhere in $\tilde{\Omega}_1$. Furthermore, provided $0\leqslant\tilde{v}\leqslant 1$ almost everywhere in Ω, we also have $0\leqslant v\leqslant 1$ almost everywhere in Ω.

Since $\psi_{1,\eta}$ is a strictly positive, bounded function, for any $\tilde{v}\in W(\Omega)$, the space ${}_{\tilde{\gamma}}W^{1,2}(\Omega)$ may be equipped with the scalar product $\langle\cdot,\cdot\rangle_{v,\eta}$ whose corresponding norm $|\cdot|_{v,\eta}$ is equivalent to the usual one of ${}_{\tilde{\gamma}}W^{1,2}(\Omega)$. We also recall that ${}_{\gamma}W^{1,q}(\Omega)$ will be equipped with the equivalent norm $\|u\|_{{}_{\gamma}W^{1,q}(\Omega)}=\|\nabla u\|_{L^q(\Omega)}$ for any $u\in{}_{\gamma}W^{1,q}(\Omega)$.

We define $\tilde{\mathscr{F}}_\varepsilon : {}_{\gamma}W^{1,q}(\Omega)\times W(\Omega)\to\mathbb{R}$ as follows. For any $(u,\tilde{v})\in{}_{\gamma}W^{1,q}(\Omega)\times W(\Omega)$, recalling that $v=1-\tilde{v}$, we set

$$\tilde{\mathscr{F}}_\varepsilon(u,\tilde{v})=\frac{a_1}{\varepsilon^{\tilde{q}}}\sup_{\substack{w\in{}_{\tilde{\gamma}}W^{1,2}(\Omega)\\ |w|_{v,\eta}\leqslant 1}}\left(\int_\Omega\psi_{1,\eta}(v)\nabla u\cdot\nabla w-\int_{\tilde{\gamma}}f_\varepsilon w\right)^2 + \frac{a_2}{\varepsilon^{\tilde{\beta}}}\int_\gamma|u-g_\varepsilon|^2+\int_\Omega\left(b\psi_\eta(v)|\nabla u|^q+\frac{1}{\eta}V(v)+\eta|\nabla v|^2\right). \quad (5.1)$$

Let us observe that, if $0\leqslant\tilde{v}\leqslant 1$ almost everywhere in Ω and $v=(1-\tilde{v})\in H_1(a_\varepsilon)$, then $\tilde{\mathscr{F}}_\varepsilon(u,\tilde{v})=\mathscr{F}_\varepsilon(u,v)$. We recall that $\mathscr{F}_\varepsilon(u,v)$ is finite only if $u\in W^{1,q}(\Omega)$ and $v\in H^1(a_\varepsilon)$ and that, if $u\in{}_{\gamma}W^{1,q}(\Omega)$, then $\mathscr{F}_\varepsilon(u,v)\leqslant\mathscr{F}_\varepsilon(u+C,v)$ for any $C\in\mathbb{R}$.

We provide a different and simpler formulation of the most difficult term of the functional, namely the term including the sup. Let us define, for any $\tilde{v}\in W(\Omega)$, the function $\tilde{u}_\varepsilon=\tilde{u}_\varepsilon(\tilde{v})$ which is the solution to the following boundary value problem. We require that $\tilde{u}_\varepsilon\in{}_{\tilde{\gamma}}W^{1,2}(\Omega)$ and that

$$\langle\tilde{u}_\varepsilon,w\rangle_{v,\eta}=\int_{\tilde{\gamma}}f_\varepsilon w \quad \text{for any } w\in{}_{\tilde{\gamma}}W^{1,2}(\Omega). \quad (5.2)$$

We observe that (5.2) admits a unique solution and that it is the weak formulation of the following boundary value problem:

$$\begin{cases}\operatorname{div}(\psi_{1,\eta}(v)\nabla\tilde{u}_\varepsilon)=0 & \text{in } \Omega,\\ \psi_{1,\eta}(v)\nabla\tilde{u}_\varepsilon\cdot\nu=f_\varepsilon & \text{on } \partial\Omega,\end{cases} \quad (5.3)$$

where as usual $v=1-\tilde{v}$.

https://doi.org/10.1017/S0956792508007729 Published online by Cambridge University Press

Therefore, we conclude that

$$\sup_{\substack{w\in_{\tilde{\gamma}}W^{1,2}(\Omega)\\ |w|_{v,\eta}\leqslant 1}}\left(\int_\Omega \psi_{1,\eta}(v)\nabla u\cdot\nabla w-\int_{\tilde{\gamma}} f_\varepsilon w\right)^2=|u-\tilde{u}_\varepsilon|^2_{v,\eta}.$$

Furthermore, we may also observe that

$$|u-\tilde{u}_\varepsilon|^2_{v,\eta}=\int_\Omega \psi_{1,\eta}(v)|\nabla u|^2-2\int_\Omega \psi_{1,\eta}(v)\nabla\tilde{u}_\varepsilon\cdot\nabla u+\int_\Omega \psi_{1,\eta}(v)|\nabla\tilde{u}_\varepsilon|^2$$

and, finally,

$$|u-\tilde{u}_\varepsilon|^2_{v,\eta}=\int_\Omega \psi_{1,\eta}(v)|\nabla u|^2-2\int_{\tilde{\gamma}} f_\varepsilon u+\int_{\tilde{\gamma}} f_\varepsilon\tilde{u}_\varepsilon.$$

Let us state the following proposition which may be immediately proved by the direct method.

Proposition 5.1 *Under the further assumption that ψ_1 is continuous all over* $\mathbb{R}$*, the following problems admit a solution.*

(i) $\min\tilde{\mathscr{F}}_\varepsilon$ *on* ${}_\gamma W^{1,q}(\Omega)\times W(\Omega)$*, with the constraint* $0\leqslant\tilde{v}\leqslant 1$.

(ii) $\min\tilde{\mathscr{F}}_\varepsilon$ *on* ${}_\gamma W^{1,q}(\Omega)\times W(\Omega)$*, with constraints* $0\leqslant\tilde{v}\leqslant 1$ *and* $v\in H^1(a_\varepsilon)$.

(iii) $\min\tilde{\mathscr{F}}_\varepsilon$ *on* ${}_\gamma W^{1,q}(\Omega)\times W(\Omega)$*, with constraints* $0\leqslant\tilde{v}\leqslant 1$ *and* $\|u\|_{L^\infty(\Omega)}\leqslant E$*, for any* $E\geqslant E_0$*,* E_0 *as in Theorem* 4.6.

(iv) $\min\tilde{\mathscr{F}}_\varepsilon$ *on* ${}_\gamma W^{1,q}(\Omega)\times W(\Omega)$*, with constraints* $0\leqslant\tilde{v}\leqslant 1$*,* $v\in H^1(a_\varepsilon)$ *and* $\|u\|_{L^\infty(\Omega)}\leqslant E$*, for any* $E\geqslant E_0$*,* E_0 *as before.*

Let us remark that the existence of a solution to Problem (ii) is clearly equivalent to the fact that $\mathscr{F}_\varepsilon$ admits a minimum over $L^1(\Omega)\times L^1(\Omega)$, whereas solving Problem (iv) is equivalent to finding a minimum of $\mathscr{F}_\varepsilon$ over $L^1(\Omega)\times L^1(\Omega)$ with the constraint $\|u\|_{L^\infty(\Omega)}\leqslant E$.

Finally, we investigate the differentiability properties of $\tilde{\mathscr{F}}_\varepsilon$. Let us now assume that, furthermore, the functions ψ, V and ψ_1 are actually of class C^1 and such that their derivatives are bounded all over $\mathbb{R}$. For any $(u_0,\tilde{v}_0)\in{}_\gamma W^{1,q}(\Omega)\times W(\Omega)$, $\tilde{\mathscr{F}}_\varepsilon$ is differentiable in $(u_0,\tilde{v}_0)$, with respect to the ${}_\gamma W^{1,q}(\Omega)\times W(\Omega)$ norm. Let $D\tilde{\mathscr{F}}_\varepsilon(u_0,\tilde{v}_0):{}_\gamma W^{1,q}(\Omega)\times W(\Omega)\to\mathbb{R}$ be the differential in $(u_0,\tilde{v}_0)$. Then, for any $(u,\tilde{v})\in{}_\gamma W^{1,q}(\Omega)\times W(\Omega)$, we shall compute $D\tilde{\mathscr{F}}_\varepsilon(u_0,\tilde{v}_0)[(u,\tilde{v})]$.

For any $\tilde{v}_0$ and any $\tilde{v}$ in $W(\Omega)$, we call $D\tilde{u}_\varepsilon(\tilde{v}_0)[\tilde{v}]$ the solution to the following problem

$$\begin{cases}\operatorname{div}(\psi_{1,\eta}(v_0)\nabla(D\tilde{u}_\varepsilon(\tilde{v}_0)[\tilde{v}]))=\operatorname{div}(\psi'_{1,\eta}(v_0)\tilde{v}\nabla(\tilde{u}_\varepsilon(\tilde{v}_0))) & \text{in } \Omega,\\ \psi_{1,\eta}(v_0)\nabla(D\tilde{u}_\varepsilon(\tilde{v}_0)[\tilde{v}])\cdot\nu=0 & \text{on } \partial\Omega.\end{cases}\tag{5.4}$$

Here, obviously, $v_0=1-\tilde{v}_0$. We recall that the weak formulation of (5.4) is looking for a function $D\tilde{u}_\varepsilon(\tilde{v}_0)[\tilde{v}]\in{}_{\tilde{\gamma}}W^{1,2}(\Omega)$ such that

$$\langle D\tilde{u}_\varepsilon(\tilde{v}_0)[\tilde{v}],w\rangle_{v_0,\eta}=\int_\Omega \psi'_{1,\eta}(v_0)\tilde{v}\nabla(\tilde{u}_\varepsilon(\tilde{v}_0))\cdot\nabla w\quad\text{for any } w\in{}_{\tilde{\gamma}}W^{1,2}(\Omega).$$

Then, straightforward computations lead to

$$
\begin{aligned}
D\tilde{\mathscr{F}}_\varepsilon(u_0,\tilde{v}_0)[(u,\tilde{v})] &= \frac{a_1}{\varepsilon^{\tilde{q}}}\int_\Omega \left(2\psi_{1,\eta}(v_0)\nabla u_0\cdot\nabla u - \psi'_{1,\eta}(v_0)|\nabla u_0|^2\tilde{v}\right)\\
&+\frac{a_1}{\varepsilon^{\tilde{q}}}\int_{\tilde{\gamma}}(f_\varepsilon D\tilde{u}_\varepsilon(\tilde{v}_0)[\tilde{v}]-2f_\varepsilon u)+\frac{2a_2}{\varepsilon^{\tilde{\beta}}}\int_\gamma (u_0-g_\varepsilon)u\\
&+b\int_\Omega\left(q\psi_\eta(v_0)|\nabla u_0|^{q-2}\nabla u_0\cdot\nabla u-\psi'_\eta(v_0)|\nabla u_0|^q\tilde{v}\right)\\
&+\frac{1}{\eta}\int_\Omega(-V'(v_0)\tilde{v})+2\eta\int_\Omega\nabla\tilde{v}_0\cdot\nabla\tilde{v}. \qquad (5.5)
\end{aligned}
$$

6 Conclusion

We conclude with the following remarks on the implementation of the method. Our aim would be to find a minimizer of the functional $\mathscr{F}_\varepsilon$ under the additional constraint $\|u\|_{L^\infty(\Omega)}\leqslant E$, with a sufficiently large E. We have proved that a sequence of such minimizers converges to the looked-for electrostatic potential u_0. Thus we would like to solve Problem (iv) in Proposition 5.1. However, the constraint $(1-\tilde{v})\in H_1(a_\varepsilon)$ is quite difficult to implement from a numerical point of view, even if we only require that a_ε is an infinitesimal greater than $2\eta(\varepsilon)$. For computational purposes, we believe that this constraint might be dropped. The convergence result would not hold any more, however there are several reasons for hoping that the constraint might be overlooked in the numerical computations. First of all, the Ambrosio–Tortorelli term, the higher integrability of ∇u, and the L^∞ bound, might provide enough stabilization by themselves, in order to obtain a reasonably good approximate solution. Moreover, in order to keep the severely ill-posedness of the problem at bay, one should avoid a too fine discretization of the problem. If the discretization is not too fine, then the constraint should play a much lesser role. Therefore, we suggest that one should try to solve Problem (iii) of Proposition 5.1. One might also try to drop even the L^∞ bound on u, and solve Problem (i), instead. However, we think this should not be done because it might weaken in a considerable way the stability of the reconstruction. In fact, we recall that even when we treat the continuation problem from Cauchy data of a harmonic function on a given domain, then usually a uniform L^∞ bound is required in order to guarantee stability.

Let us point out that the functional to be minimized is non-convex. Therefore, one might reach a local minimum instead of a global one. By our convergence analysis, Theorem 4.4, we have shown that this might not cause any matter. In fact, if ε_n, $n\in\mathbb{N}$, is such that $\varepsilon_n\to 0^+$ as $n\to\infty$, then it would be enough to find (u_n,v_n) such that $\|u_n\|_{L^\infty(\Omega)}$ and $\mathscr{F}_{\varepsilon_n}(u_n,v_n)$ remain uniformly bounded. Thus, there might be no need to find the absolute minimizers, local minimizers might suffice. Furthermore, the same analysis suggests that one should test the numerical solution on the functionals corresponding to a finite sequence of positive, decreasing numbers ε_n.

We also note that our method provide us a quite simple way of detecting the jump set of u_0, which is our ultimate goal, by simply thresholding the function v at a suitable small positive parameter. Furthermore, we observe that the choices of ψ, V, ψ_1 and $\tilde{\alpha}$, of the parameters a_1, a_2 and b, of the constants $\tilde{q}$ and $\tilde{\beta}$ are quite arbitrary. Also, we have

https://doi.org/10.1017/S0956792508007729 Published online by Cambridge University Press

a lot of freedom in choosing the parameters $\eta(\varepsilon)$ and o_η (and, if this is the case, a_ε as well). Careful choices of these data might help us to improve the reconstruction. It might be convenient to choose these functions smooth, for instance $\psi(t)=t^q$ and $\psi_1(t)=t^2$ for any $t\in[0,1]$. Or maybe other choices might lead to better results, for example we might take $\psi(t)=t$ and $\psi_1(t)=t^{2/q}$ for any $t\in[0,1]$, where we lose the differentiability of ψ_1 at 0. Particularly interesting is the choice of the parameter η with respect to ε, that is the answer to the following question by numerical experiments: what is the best parameter of the Ambrosio–Tortorelli approximation with respect to the noise level of the Cauchy data?

Finally, we wish to point out another possible interpretation of our functional. Roughly speaking, we look for functions u and $\tilde{v}$ such that u approximately solves problem (5.3), that is the problem solved by $\tilde{u}_\varepsilon(\tilde{v})$, its boundary values on γ are close to g_ε and u and $\tilde{v}$ are subject to the regularization due to the Ambrosio–Tortorelli functional. It might be reasonable, from a numerical point of view, to replace in the functional $\tilde{\mathscr{F}}_\varepsilon$ the independent variable u with the dependent variable $\tilde{u}_\varepsilon(\tilde{v})$, thus obtaining a functional depending only on the variable $\tilde{v}$, namely $\hat{\tilde{\mathscr{F}}}_\varepsilon(\tilde{v})=\tilde{\mathscr{F}}_\varepsilon(\tilde{u}_\varepsilon(\tilde{v}),\tilde{v})$, that is

$$\hat{\tilde{\mathscr{F}}}_\varepsilon(\tilde{v})=\frac{a_2}{\varepsilon^{\tilde{\beta}}}\int_\gamma|\tilde{u}_\varepsilon(\tilde{v})-g_\varepsilon|^2+\int_\Omega\left(b\psi_\eta(v)|\nabla\tilde{u}_\varepsilon(\tilde{v})|^q+\frac{1}{\eta}V(v)+\eta|\nabla v|^2\right).$$

Here $\tilde{v}\in W(\Omega)$ and $0\leqslant\tilde{v}\leqslant 1$ almost everywhere in Ω. We might also add a constraint on the $L^\infty(\Omega)$ norm of $\tilde{u}_\varepsilon(\tilde{v})$.

Acknowledgements

This work is partly supported by MiUR under grant no. 2006014115 and by GNAMPA under 2007 Project 'Problemi inversi nelle equazioni alle derivate parziali per le applicazioni'. The author wishes to thank the referees for their useful comments.

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