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Computer simulation of laser-beam self-focusing in a plasma

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Laser-beam or soliton propagation is best modelled for fast computation using a split-step Fourier method based on an orthogonal transform technique known as the beam-propagation method. The beam-propagation split-step Fourier-transform technique in one and two dimensions for the propagation of a soliton or laser beam respectively in a nonlinear plasma and a split-step Hankel-transform-based algorithm for cylindrical-beam propagation close to circular cross-sectional symmetry and its computational implementation are discussed. Attention is particularly focused on the verification of the paraxial approximations of the soliton or the laser beam using these techniques, after a brief review of the beam-propagation method.

1. Introduction

Many analytical techniques for dealing with self-focusing in a plasma have been based on the approximation of the partial differential equation of the laser beam obtained (Akhamanov et al. 1974; Whitham 1974; Subbarao and Sodha 1984) using linear beam-propagation techniques (see e.g. Yariv 1991). These approximations attempt to reduce the partial differential equation, the quasi-optical equation, to a suitable set of ordinary differential equations, each in a single dimension, because exact solutions cannot always be found in terms of the inverse scattering transform technique (see e.g. Whitham 1974; Novikov et al. 1984).

The beam evolves slowly in its direction of propagation, but the same cannot be assumed in the transverse direction. A convenient way of taking into account the faster transverse variations in general is to use methods based on representation of the beam in a convenient orthogonal space. The choice of the orthogonal transformation space depends on the symmetries and hence the inherent group-theoretical structure and nature of the partial differential equation involved, and finally on computational ease.

A popular choice for an orthogonal space that works reasonably well almost always is the angular spectrum representation in terms of Fourier transformation of the beam, as done by Subbarao (1981), Subbarao and Sodha (1979), Subbarao and Sodha (1984), and others. It is the basis of many computational schemes, which are known as beam-propagation or FFT-based methods, and were first developed by Hasegawa and Tappert (1973), followed

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by Fleck et al. (1976) and Feit and Fleck (1978, 1979), and have been used by many, including Newell and Moloney (1992), Fisher and Bischel (1973) and Korpel et al. (1986). Such simple angular spectrum representations, however, are not always the best choices, since the beam and the nonlinear refractive index generated by it are not well represented by them, and this prompts one to seek other suitable transformation spaces.

Another suitable orthogonal space that has been studied by us for some time now is the momentum space defined by the harmonic-oscillator eigenfunction basis (Subbarao et al. 1983, 1993, 1998; Uma et al. 1993). This could be the Hermite–Gauss transformation or the Laguere–Gauss transformation for the beam. A complete scheme of analysis is possible using any of these non-Fourier-based transformations, as outlined first by Singh et al. (1995), but they are not at present computationally very convenient, since there are no fast techniques like the FFT for such transformations. Beam-propagation techniques in the context of line-spectrum calculations of the Zeeman and Stark effects in the hydrogen atom, developed by Hermann and Fleck (1988) using some harmonic oscillator functions in cylindrical geometry (the Landau functions), can also be useful in this context for laser-beam propagation. We are still in the process of designing these more advanced techniques for beam propagation in nonlinear media using these orthogonal spaces, and hence the present paper will not deal in detail with such methods, although this formulation is contained formally in equation (11) below.

A large number of numerical methods have been used to solve the nonlinear Schrödinger equation (Karpman and Krushkal 1969; Yajima and Outi 1971; Hardin and Tappert 1973; Fisher and Bischel 1975; Ablowitz and Ladik 1976; Greig and Morris 1974; Feit and Fleck 1978, 1979; Fornberg and Whitham 1978; Delfour et al. 1981; Kurki-Suonio et al. 1989; Mori et al. 1988; Cohen et al. 1991; Soto-Crespo and Akhmediev 1993; Esarey et al. 1994). These can be classified into two broad categories: (i) finite-difference methods and (ii) pseudospectral methods.

The speed of transformation into the orthogonal space and back is fast only if the Fourier space is used, together with the FFT. Fourier-transformation-based techniques like the so-called plane-wave representation of the beam are well known (Goodman 1968; Stratton 1971; Marcuse 1972; Hasegawa and Tappert 1973; Subbarao and Sodha 1979, 1984; Subbarao 1981; Subbarao et al. 1983, 1993; Uma 1988; Newell and Jerome 1992; Uma et al. 1993). For soliton propagation, where exact methods (Novikov et al. 1984; Akhmediev and Ankiewicz 1994) are not always possible, this technique is now commonly used in cases where the transformation and inversion work reasonably fast, an FFT-based beam-propagation technique having been developed by Hasegawa and Tappert (1973) (see Hasegawa 1989; Agrawal 1989; Newell and Moloney 1992). The use of such a transformation for analytical purposes has previously been illustrated by us for a nonlinear self-focusing beam (Subbarao 1979, 1981; Subbarao and Sodha 1984).

Taha and Ablowitz (1984) conducted extensive computation in which the split-step Fourier method (Hardin and Tappert 1973; Fisher and Bischel 1973, 1975; Weidman and Herbst 1986) was compared with several finite-difference, pseudospectral and even global methods. In the majority of the numerical experiments, the split-step method has turned out to be superior. The method owes part of its success to the fact that it avoids solving a nonlinear algebraic system of equations at each time level. This high accuracy can be obtained at comparatively low computational cost. Field propagation with split-step FFT is smoother than with its finite-difference counterpart. The split-step FFT method has been applied to a wide variety of optical problems, including wave propagation in the atmosphere and in plasmas (Fleck et al. 1976; Lax et al. 1981; Feit et al. 1982), graded-index fibres (Fleck et al. 1977; Feit and Fleck 1978, 1979; Yevick and Hermansson 1983, 1990), semiconductor lasers (Agrawal 1984a,b; Meissner et al. 1984), unstable resonators (Sziklas and Siegman 1975; Lax et al. 1985) and waveguide couplers (Hermansson et al. 1983; Thylen et al. 1986). In the case of steady-state propagation, where dispersion is replaced by diffraction, it is often referred to as the beam-propagation method (Goldberg et al. 1967; Sziklas and Sieman 1975; Fleck et al. 1976; Feit and Fleck 1978, 1979; Lax et al. 1981, 1985; Hermansson et al. 1983, 1992; Agrawal 1984a,b; Meissner et al. 1984; Thylen et al. 1986; Soto-Crespo et al. 1992; Hewlett et al. 1995; Polodian and Ladouceur 1998; for a review, see Yevick 1994). For solitons in plasmas, see Shapiro and Shivachenko (1984) and Zakharov (1984).

The only drawbacks with the split-step FFT are that it needs careful programming skills and non-equidistant grid points cannot usually be employed. To overcome the first problem, we have taken an object-oriented approach to the problem of soliton and laser-beam propagation (temporal and spatial soliton evolution), which to the best of our knowledge has not been done elsewhere (except possibly by Ladouceur and colleagues; see Poladian and Ladouceur 1998). A numerical estimate of the paraxial approximation using the split-step FFT soliton and beam (temporal and spatial) propagator has been obtained. This method has prompted the development of a reasonably accurate semi-analytical scheme that involves the evolution of the Fourier transform in the paraxial regimes with suitable near- and off-paraxial adjustments for better accuracy. The split-step FFT method is described in the next section.

It is important to realize that in all the FFT-based methods one has to keep track of the actual approximations of the nonlinear refractive index, a result that must be stressed in view of our earlier experience (Subbarao and Sodha 1979; Subbarao 1981; Subbarao et al. 1983, 1998a,b) indicating that the type of refractive-index approximation that implicitly results is sensitively dependent on the type of orthogonal transformation of the beam. We return to this point in the final section.

In this paper, an attempt is made to outline the techniques employed to solve the nonlinear Schrödinger equation using beam-propagation computational methods for linear and elliptical/circular cylindrical geometry. The different geometries define different transverse operators and hence different types of transformations. (For spherical geometry, the methods developed by Hermann and Fleck (1988) in the context of atomic wavefunction evolution can be adapted.) The FFT in one and two dimensions works well for the one-transverse-dimensional (soliton) and two-transverse-dimensional (elliptical beam) cases respectively; for a circularly cylindrical beam, we attempt to develop a propagation formulation for nonlinear media in terms of Hankel transformations based on our earlier work (Subbarao and Sodha 1979; Subbarao 1981), adapting it to fast Hankel transformation techniques due to Siegman

(1977) and Lax et al. (1981), making them more suitable for high-performance computing environments. Group-theoretical representations and their use in improving the results are also discussed. The stress throughout is on the techniques rather than on the results, some of which have been reported elsewhere. We return to possible refinements of the technique in the last section.

2. Split-step FFT scheme

We obtain a numerical solution of the normalized scalar nonlinear Schrödinger (NLS) equation using a split-step fast Fourier transform (FFT) method. The normalized NLS equation to be solved is (Whitham 1973; Hasegawa 1979)

$$\frac{\partial E}{\partial t} = i\nabla_\perp^2 E + 2if(|E|^2)\,E. \tag{1}$$

Let us write this equation formally as

$$\partial_t E = (L+N)\,E,$$

where $L \equiv i\nabla_\perp^2$ is the differential operator that accounts for the dispersion and $N \equiv 2if(|E|^2)$ is the nonlinear operator. In general, dispersion (of the soliton pulse or diffraction of the beam) and nonlinearity act together. The split-step Fourier method obtains an approximate solution by assuming that in propagation over a small time interval Δt, the dispersive and nonlinear effects can be assumed to act independently. Propagation from t to $t+\Delta t$ is carried out in two steps. In the first step, $L = 0$ in (1), i.e. only nonlinearity acts. In the second step, $N = 0$ in (1), i.e. only dispersion acts. Mathematically, this can be expressed as

$$E(x, t+\Delta t) \simeq e^{\Delta t L} e^{\Delta t N} E(x,t). \tag{2}$$

The operator $\exp(\Delta t\, L)$ acts in the Fourier domain via the following relation:

$$e^{\Delta t L} E(x,t) = \{F^{-1} e^{\Delta t L(ik)} F\}\, E(x,t). \tag{3}$$

Here F denotes the Fourier-transform operation, and $L(ik)$ is obtained from (1) by replacing the differential operator in L by ik, where k is the wavenumber in the Fourier domain. Since $L(ik)$ is just a number in Fourier space, the evaluation of (3) is straightforward. The use of FFT algorithms (Cooley and Tukey 1965) makes evaluation of (3) relatively fast. The accuracy of the split-step FFT method can be estimated as follows. The exact operator-form solution of (1) is given by

$$E(x, t+\Delta t) = e^{\Delta t(L+N)} E(x,t). \tag{4}$$

On making use of the Baker–Hausdorff formula (Wiess and Maradudin 1962) for two non-commuting operators p and q,

$$e^p e^q = e^{p+q+[p,q]/2+[p-q,[p,q]]/12+\dots}, \tag{5}$$

where $[p,q] = pq-qp$ is the commutator of the two operators, and by comparing (4) and (2), we see that the split-step FFT method ignores the non-commuting nature of the operators L and N. Substituting $p = \Delta t\, L$ and $q = \Delta t\, N$ into (5), we see that the error term results from the single commutator $\frac{1}{2}\Delta t^2[L,N]$. Thus the split-step FFT method is accurate to second order in the step size Δt. The accuracy of the split-step FFT method is improved by adopting a different procedure to propagate the field from t to $t+\Delta t$. In this manner, (2) is replaced by the relation

$$E(x, t+\Delta t) = e^{(\Delta t/2)L}\left(\exp\int_t^{t+\Delta t}\{N(t')+b\}\,dt'\right)e^{(\Delta t/2)L}E(x,t). \tag{6}$$

Now the effect of nonlinearity is included in the middle of the segment rather than at the boundary. Because of the symmetric form of the exponential operators in (6), this method is known as the symmetrized split-step FFT method (Fleck et al. 1976). The integral in the middle exponential operator can be approximated by $\exp(\Delta t N)$ for small Δt when $b = 0$. The advantage of using the symmetrized split-step FFT method is that the error is of third order in the step size Δt, as shown below. Using the relation (5) twice, one can show that

$$e^{p+q} = e^{p/2}e^{s}e^{p/2}, \tag{7}$$

where

$$s = q - \tfrac{1}{24}[p,[p,q]] - \tfrac{1}{12}[q,[q,p]] + \dots,$$

so that the accuracy is improved to third order, i.e. the error is of order $h^3(\cdot)$ where $(\cdot)$ is a higher-order commutator even if we truncate s in the form $s \equiv q$. This can be seen as follows. Substituting

$$E = e^{Lt/2}\Gamma e^{Lt/2} \tag{8}$$

into (1), we get

$$\frac{d\Gamma}{dt} = s(t)\,\Gamma, \tag{9}$$

where

$$s(t) = e^{-Lt/2}N(t)\,e^{Lt/2} + \tfrac{1}{2}L - \tfrac{1}{2}\,\Gamma L \Gamma^{-1}.$$

Iterating these two equations, we have

$$s(t) = N(t) + \frac{1}{2}\left[L, \int_0^t N(t')\,dt' - tN(t)\right]. \tag{10}$$

If $N(t)$ is a constant, the second-order term vanishes, and the error resides in the double commutator. If $N = N_0 + N_1 t$ then we have, to first order,

$$s(t) = N(t) - [L, N]\frac{t^2}{4},$$

and the corrections in Γ will be of order t^3. The above methods mostly developed for FFT could also be used in general for any orthogonal transformation, including the Hermite–Gauss and Laguerre–Gauss transforms that we have discussed earlier, and this generality has been indicated in (6). The choice of b in (6) depends on the type of beam or soliton being studied and the orthogonal transformation. In cases where soliton is of hump type, the appropriate orthogonal transformation is the Hermite–Gauss transform, in which case we should have $b = b_1 - b_2 x^2$ for appropriate values of b_1 and b_2. This value of b reduces the correction in $N(t')$, taking into account the shape of $N(|E|^2)$. The effect on L of the transformation to a suitable orthogonal space is to introduce a simple multiplication factor $\exp(-iK^2\Delta t/2)$. The algorithm for a single propagation step is given by

$$E(t+\Delta t, x) = \mathrm{OT}^{-1}\{e^{-iK^2\Delta t/2}\mathrm{OT}\{e^{iN(|E(t,\mathbf{x})|^2)\Delta t}\mathrm{OT}^{-1}\{e^{-iK^2\Delta t/2}\mathrm{OT}\{E(t,\mathbf{x})\}\}\}\}, \tag{11}$$

where OT is the orthogonal transform and OT^{-1} is the inverse orthogonal transform.

There are no known fast transformation methods for orthogonal spaces other than Fourier space, so that this method is slow, although it is more accurate with an appropriate orthogonal transformation, for example in terms of the Hermite–Gauss transformation (when (1) for a soliton is involved). With a compromise on the accuracy equivalent to the reduction in time-step length, it has, however, been shown by Subbarao (1981) that the orthogonal space involving Fourier transformation is good enough in this case of beam/soliton propagation. We have developed orthogonal transformation in terms of Laguerre–Gauss transformations (Subbarao et al. 1983, 1993, 1998a,b; Uma et al. 1993) for self-focusing of a circular cylindrical beam or Hermite–Gauss double series for self-focusing of an elliptical beam. The choice of b in (11) depends on the type of beam being studied. For a 'solid beam', i.e. one with peak intensity on the propagation axis, the appropriate transform is the Laguerre–Gauss transform when the beam is of circular cylindrical shape and the operator in (11) applies. We then have $b = b_1 - b_2 r^2$ for appropriate values of b_1 and b_2, using a method based on that of Subbarao et al. (1998a). For an axially peaked beam with elliptic cross-section, the appropriate transformation is a double-series Hermite–Gauss transform, which dictates $b = b_1 - b_{21} x^2 - b_{22} y^2$ as the correct choice to be used in conjunction with the operator below (Singh 1996). The split-step FFT technique is applied to the case of temporal soliton and beam propagation (or spatial soliton) as discussed in the next two sections for $b = 0$.

3. Application of the split-step FFT scheme to the case of solitons and self-focusing

In this section, the standard split-step FFT algorithm is applied to the case of a soliton. In this case, the transverse operator is in one dimension only, so that the FFT method can be used directly, as described above. In the second part of this section, this method is applied to the case of self-focusing, when the transverse operator is two-dimensional. A beam that is of elliptical symmetry (the two transverse (x and y) directions are not identical) is better modelled by such a technique. In Sec. 3.3 we go on to construct the Hankel-transformation-based beam-propagation technique from first principles.

3.1. Temporal soliton

In the case of a temporal soliton, $\nabla_\perp^2$ in (1) is given by (Hasegawa and Tappert 1973; Agrawal 1989)

$$\nabla_\perp^2 \equiv \frac{\partial^2}{\partial x^2}. \tag{12}$$

In Fourier space, the algorithm for a single propagation step is modified from (11) to

$$E(t+\Delta t, x) = \mathrm{FFT}^{-1}\{e^{-iK^2\Delta t/2}\mathrm{FFT}\{e^{iN(|E(t,x)|^2)\Delta t}\mathrm{FFT}^{-1}\{e^{-iK^2\Delta t/2}\mathrm{FFT}\{E(t,x)\}\}\}\}, \tag{13}$$

where FFT is the fast Fourier transform and FFT^{-1} is the inverse fast Fourier transform.

We have developed the source code in C++ (on SUN SPARCstation using the GNU C++ Project v2.4 compiler at the IIT Delhi Computer Centre), by taking an object-oriented approach. To the best of our knowledge, this is the first time this approach has been used for soliton-propagation studies. (We have been informed that Poladian and Ladouceur (1998) have also used object-oriented programming techniques.) An acceptable step length for x is 0.00586, while for time the step length is 0.00013. The code will warn the user if the time step is too large for the chosen step size of x by checking the phase shift in the linear as well as in the nonlinear step at each time step. The program can be used for any initial data and any nonlinearity, but we have used Gaussian data for the input beam.

3.2. Beam propagation in a nonlinear self-focusing medium in elliptical cylindrical geometry

In the case of self-focusing of a beam (or spatial soliton formation), the nonlinear Schrödinger equation involved is in two transverse dimensions, and it is also solved using a split-step FFT method in D dimensions, with $D = 2$. The NLS to be solved is (1) with (Singh et al. 1996)

$$\nabla_\perp^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{14}$$

for an elliptical cylindrical beam, where the time variable is the z coordinate measured along the direction of beam propagation (see also Sec. 3.3, where a more complete derivation is given in a slightly different geometry).

Again with a compromise on the accuracy, because of not using orthogonal transformations (to be compensated with the reduction in time-step length), the orthogonal space involves Fourier transformation and is good enough for the cases discussed here. The other steps involved are same as in the case of a temporal soliton, except that now the FFT is two-dimensional (denoted by FFTD) in the transverse direction, i.e. FFT is replaced by FFTD in (13). The algorithm for a single time step can be written as

$$\begin{aligned} E(t+\Delta t, x, y) = {} & \mathrm{FFTD}^{-1}\{e^{-iK^2\Delta t/2}\mathrm{FFTD}\{e^{iN(|E(t,x,y)|^2)\Delta t} \\ & \times \mathrm{FFTD}^{-1}\{e^{-iK^2\Delta t/2}\mathrm{FFTD}\{E(t,x,y)\}\}\}\}, \end{aligned} \tag{15}$$

where, using the operator of (13),

$$K^2 = K_x^2 + K_y^2.$$

The source code for FFTD used is as given in Press et al. (1986). The source code was developed in C++ (on SUN SPARCstation using the GNU C++ Project v2.4 compiler at the IIT Delhi Computer Centre) by taking an object-oriented approach such that the beam object and the FFT object with suitable attributes are used to save considerable memory compared with earlier methods. We have used a 256×256 grid in the transverse two-dimensional space, and the time step is 0.00013. The code will warn the user if the time step is too large for the chosen value of grid size by checking the phase shift in the linear as well as in the nonlinear step at each time step. The program can be used for any initial data, but we have used here a Gaussian beam of fixed amplitude (equal to unity) and of cylindrical symmetry. Numerical accuracy is checked by repeating the simulation for different grid size and time-step size.

The numerical accuracy can also be checked using certain invariants of the NLS.

3.3. Beam propagation in a self-focusing medium with circular cylindrical geometry

We go to some length in formulating the problem in circular cylindrical geometry, giving all the intermediate steps once again for the sake of completeness since they are different in principle from those in the above two subsections and are not available in full elsewhere. For geometries that have some circular symmetry, this method is more suitable than the elliptical-beam method of Sec. 3.2. Our method is based on the cylindrical beam-propagation formulation for nonlinear media in terms of Hankel transformations due to Subbarao and colleagues (Subbarao and Sodha 1979, 1984; Subbarao 1981), adopting fast Hankel transformation techniques due to Siegman (1977), Lax et al. (1981), Agrawal and Lax (1981), Talman (1978) and Magni et al. (1992).

3.3.1. The equations. The electromagnetic wave equation at the original frequency ω obtained after separating the high-frequency oscillation, $\mathbf{E}(r,\theta,z,t) = \mathbf{E}(r,\theta,z)\exp\{i(\omega t - k_z z)\}$ will be of the form

$$2\iota k_z \frac{\partial E}{\partial z} + \left\{\nabla_\perp^2 + k_0^2\left(\epsilon - \frac{k_z^2}{k_0^2}\right)\right\} E = 0, \tag{16}$$

where

$$\nabla_\perp^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \equiv \nabla_r^2 + \nabla_\theta^2$$

for the electric vector $\mathbf{E}$. Here $k_0 = \omega/c$ and c is the speed of light in vacuum. The properties of the medium enter through the constitutive relation that determines ϵ, the time-averaged dielectric constant:

$$\epsilon = \epsilon_L - \epsilon_N(\mathbf{E}\cdot\mathbf{E}^*),$$

where ϵ_L is the linear part and ϵ_N is the nonlinear part; it is assumed that the relaxation time of the nonlinear dielectric constant is small compared with the time period of the wave. A quasi-optical approximation for the rest of the variation of the beam that is valid when

$$\left|\frac{\partial^2 E}{\partial z^2}\right| \ll \left|k_z \frac{\partial E}{\partial z}\right|$$

has also been made in obtaining the above scalar wave equation.

Since

$$E(r,\theta,\eta) = \sum_{n=-M}^{M} E_n(r,\eta)\exp(-in\theta), \tag{17}$$

where

$$E_n(r,\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(r,\theta,\eta)\exp(in\theta)\,d\theta, \tag{17a}$$

when the θ term in the operator $\nabla_\perp^2$ acts on the electric field, it yields a constant n^2 that converts the operator into the one that occurs in the Bessel equation.

Here the azimuthal mode number n, varying between $-M$ and $+M$ determine the order of the Bessel function $J_n(k_\perp r)$. Ideally, $M \to \infty$, but for the practical computation at hand, it is truncated to a finite value depending on the azimuthal detail expected. Introducing the notation

$$\rho = k_\perp r, \qquad Z = \frac{z}{2k_z} k_\perp^2,$$

$$k_\perp^2 = k_0^2 \left(\epsilon_L - \frac{k_z^2}{k_0^2} \right), \qquad L = \nabla_\rho^2 + 1 - \frac{n^2}{\rho^2},$$

$$N(|E|^2) = \frac{k_0^2}{k_\perp^2} \epsilon_N,$$

the scalar wave equation for the electric field components, (2), can be written following Sec. 2, (1), in the form

$$\frac{\partial E_n}{\partial Z} = i\{LE_n + N(|E|^2)\} E_n, \tag{18}$$

Integrating this equation with respect to Z over a small layer of thickness ΔZ, we have

$$E_n(r, Z + \Delta Z) = \exp\left\{ -iL\Delta Z + \int_0^{\Delta Z} N(|E|^2)\, dZ \right\} E_n(r, Z). \tag{19}$$

3.3.2. The split-step method. The exponential operator made up of the sum of the non-commuting operators can be approximated using the Baker–Hausdroff formula, following the split-step rule, as

$$e^{-i(A+B)} \approx e^{-iA/2} e^{-iB} e^{-iA/2}, \tag{20}$$

where $A = L\Delta Z$, L is the operator of the Bessel equation defined above, and $B = \int_0^{\Delta Z} N(|E|^2)\, dZ$ is the inhomogeneity is the refractive index.

Since the linear part of the integral is the Bessel operator, the whole problem can be simplified by representing the electric field in the Hankel space, i.e. by taking its Hankel transform. As a result of this simplification, the Bessel operator introduces a constant multiplicative factor, its eigenvalue $k_\perp^2$ being independent of the mode of the Bessel function under consideration. Now using the split-step rule, the fast Hankel transformation (FHT)-based split step operator $\mathrm{SS}_{\mathrm{FHT}_n}$ can be defined by the equation

$$\mathrm{SS}_{\mathrm{FHT}_n}\{E_n(r, 0)\} = E_n(r, \Delta Z), \tag{21}$$

which can be written explicitly as (24) below.

The electric field can be obtained by Fourier (series) inversion of $E_n(r, \Delta Z)$ at the M $(= 2^l)$ values of $\theta_k = 2\pi k/M$, where M is also equal to the number of azimuthal modes selected that are needed to truncate the sum in (17):

$$E(r, \theta_k, \Delta Z) = \mathrm{FFT}^{-1}\{E_n(r, \Delta Z)\} = \mathrm{FFT}^{-1}\{\mathrm{SS}_{\mathrm{FHT}_n}\{FFT\{E(r, \theta_k, 0)\}\}\}. \tag{21a}$$

This completes the job of putting the formulation into an algorithmic form for computing the electric field $E(r, \theta, Z)$ for an arbitrary initial condition at $Z = 0$ and for an arbitrary nonlinear form of $\epsilon_N(|E|^2)$.

3.3.3. Fast Hankel transformation. We follow Siegman (1977) and Lax et al. (1981) for methods to construct fast Hankel transformations. The Hankel transformation (HT) and its inverse are defined through the same equation:

$$\mathrm{HT}_n\{E_n(r,Z)\} = \Phi_n(k_\perp, Z) = \int_0^\infty J_n(k_\perp r)\, F(r)\, r\, dr, \tag{22}$$

where $\Phi(k_\perp, Z)$ is the representation of the electric field in Hankel space; for the inverse-transform case, $E_n(r,0)$ and $\Phi_n(k_\perp, 0)$ are exchanged, as is the integration variable from r to $k_\perp$. One defines new independent variables x and y through

$$r = r_0\, e^{\alpha x}, \qquad k_\perp = k_{\perp 0}\, e^{\alpha y}$$

and new independent variables

$$f_n(x) = rE_n(r), \qquad g_n(y) = k_\perp \Phi_n(k_\perp)$$

to obtain a fast version of Hankel transformation (FHT) that replaces (22) with the equation

$$\mathrm{FHT}_n\{E_n(r,Z)\} = \Phi_n(k_\perp, Z) = \frac{1}{k_\perp} \mathrm{FFT}\{\mathrm{FFT}^{-1}\{j_n(y)\}\, \mathrm{FFT}\{f_n(x)\}\}, \tag{23}$$

where $j_n(x) = J_n(r_0\, k_{\perp 0}\, e^{\alpha x})\, \alpha r_0\, k_{\perp 0}\, e^{\alpha x}$. The electric field can be calculated by reversing the above transformation through an exactly similar algorithm.

The representation of the beam electric field $E(r, \theta, \eta)$ in cylindrical coordinates is made up of two parts. First, the Fourier-mode representation in the periodic θ coordinate is truncated by M modes relevant to the expected azimuthal details, and, secondly, the radial analysis is by Hankel transformation. As a result of going over to the momentum space defined by the Hankel transformation, the Bessel operator $A = L\Delta Z$ (where L is the linear part of the radial electric field wave equation) simply introduces a constant multiplicative factor, its eigenvalue $k_\perp^2$ being independent of the mode of the Bessel function taken. The operator that arises from the nonlinear portion of the dielectric function of the plasma, $\epsilon = \epsilon_L - \epsilon_N(\mathbf{E}\cdot\mathbf{E}^*)$, is

$$B = \int_0^{\Delta Z} N(|E|^2)\, dZ$$

and appears as a sum with the above non-commuting linear operator in an exponent that describes the steady-state beam-envelope evolution of the scalar wave. The split-step method used by us for beam propagation is based on this exponential operator being approximated using the Baker–Hausdorff formula for Lie groups (Hasegawa and Tappert 1973) to split it into two separate consecutive operators. As in Sec. 3.3.2, the fast Hankel transformation (FHT)-based split step operator $\mathrm{SS}_{\mathrm{FHT}_n}$, defined by (21), can be written explicitly as

$$E_n(r, \Delta Z) = \mathrm{FHT}_n^{-1}\{e^{-ik_\perp^2 \Delta Z/2}\mathrm{FHT}_n\{e^{-iN(|E|^2)}\mathrm{FHT}_n^{-1}\{e^{-ik_\perp^2 \Delta Z/2}\mathrm{FHT}_n\{E_n(r,0)\}\}\}\}. \tag{24}$$

Finally, $E(r, \theta_k, \Delta Z) = \mathrm{FFT}^{-1}\{E_n(r, \Delta Z)\}$. A group theoretical representation of the paraxial beam is of particular interest in this paper. The FHT procedure is equivalent to the two-dimensional Fourier transformation, which could be used directly. Depending on the symmetry of the beam, the present method is, however, expected to save computational time.

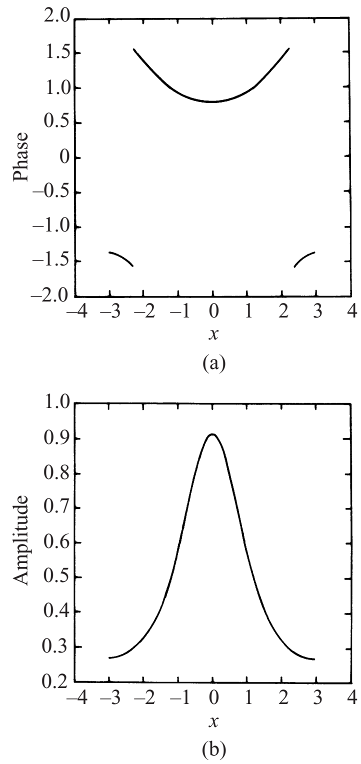

Figure 1. (a) Phase and (b) amplitude of the FFT-propagated soliton with width-wise variable x after 8000 time steps.

4. Results from the FFT propagators

The results from the FFT propagator, which is used to propagate the temporal and spatial solitons (self-focusing) as discussed in the previous two sections, are discussed in the following subsections.

4.1. Results on soliton propagation

The FFT propagator described above has been used to propagate an initial Gaussian beam (or soliton) $E(x,0) = E_0\, e^{-x^2/x_0^2}$ to give $E(x,t)$ at finite time $t = 8000h$ ($h = t = 1.25 \times 10^{-4}$). Figures 1(a) and (b) show the variation of the phase and the amplitude with the width-wise variable x. It is clear that a parabolic approximation is valid for the phase from the general appearance of Fig. 1(a). The result for the intensity is that we can approximate the field by a modified Gaussian in the paraxial region. A parallelization scheme saves computational time even when off-paraxial corrections are involved. The off-paraxial contributions are best taken into account using a semi-analytical method based on FFT by writing the field form as in Subbarao (1981):

$$E(x,t) = \int_{-\infty}^{\infty} e^{i\alpha x} e^{\beta(\alpha,t)}\, d\alpha \tag{25}$$

with $\beta(\alpha, t=0) = -1/4x_0^2\,\alpha^2 + \text{const}$ as the initial condition, where x_0 is the beam width of the original Gaussian soliton. x_0 is the complex initial chirp if

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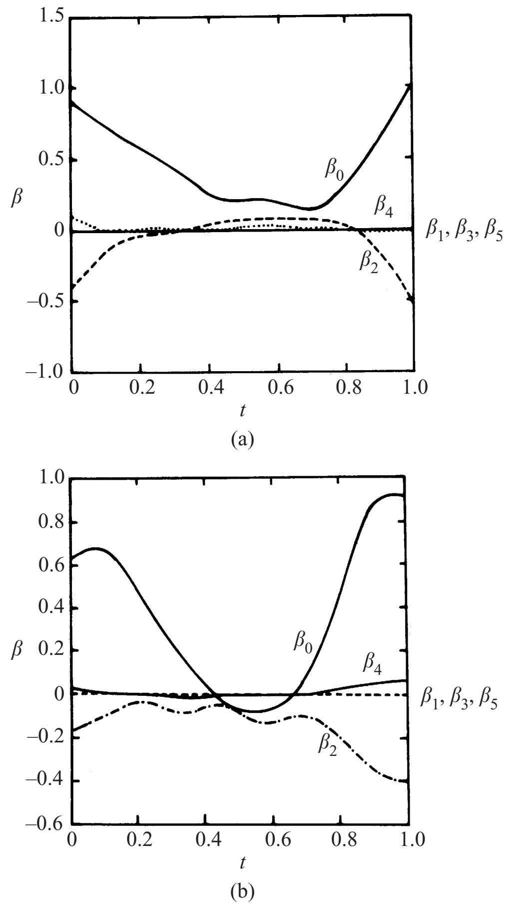

Figure 2. Variation of (a) the real parts and (b) the imaginary parts of the coefficients β_0, β_2, β_4 and β_6 with time t (on this scale, $\beta_6 = 0$). (See the discussion after (26).)

present. It is seen that $B(\alpha, t) = e^{\beta(\alpha,\, t)}$ is the Fourier inverse of $E(x, t)$. For $t = 0$,

$$B(\alpha, 0) = e^{\beta(\alpha,\, 0)} = \frac{1}{2\pi^{1/2}} E_0\, x_0\, e^{-\alpha^2 x_0^2/4},$$

given through the Fourier inversion formula, so that

$$\beta(\alpha, 0) = -\frac{1}{4}\alpha^2 x_0^2 + \ln \frac{E_0\, x_0}{2\pi^{1/2}}. \qquad (26)$$

We have been able to implement a split-step beam/soliton propagator to obtain the FFT approximation in (13) with the above form for the final FFT of the field. The field with the complex wave propagator $e^{\beta(\alpha,\, t)}$ with $\beta(\alpha, t) = \beta_0(t) + \beta_2(t)\,\alpha^2 + \beta_4(t)\,\alpha^4 + \beta_6(t)\,\alpha^6$ is a reasonably good fit for the numerical result. This can be seen from Fig. 2, where the coefficients β_0, β_2, β_4 and β_6 have been obtained from the computed FFT field. The various values of β_i ($i = 1, 2, 3, \ldots$) have been plotted by extrapolating the parameters from the FFT profile at 10 values of $n\Delta t$ (obtained by the split-step beam propagation method) using Mathematica 2.0 and then extracting the β curves shown in Fig. 2 through

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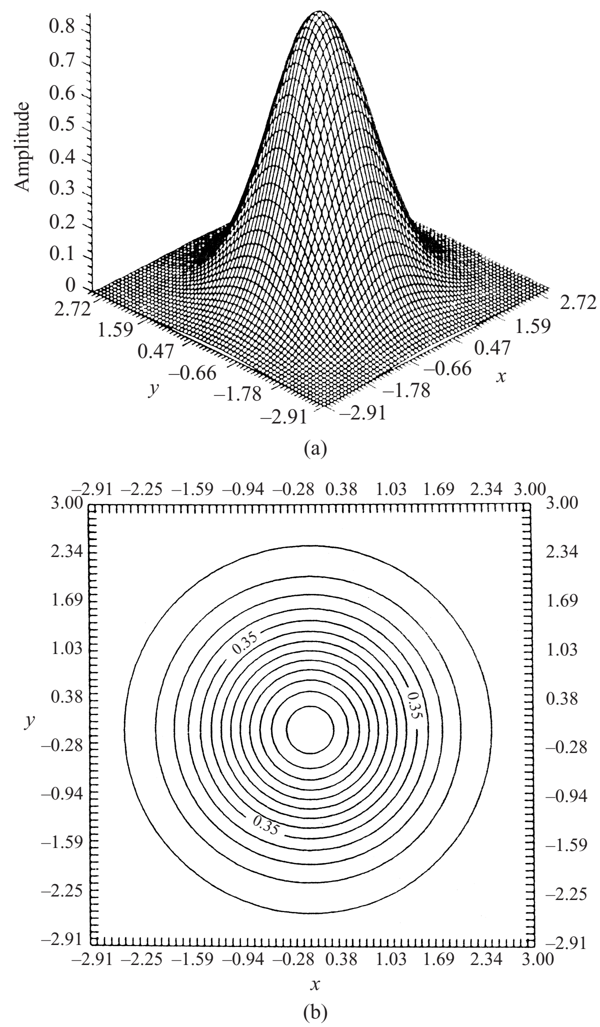


Figure 3. Initially Gaussian beam: (a) variation of amplitude with width-wise variables x and y at $t = 0$; (b) contour map for the same.

spline fitting. It can be seen that $\beta_6 \approx 0$ is always a good approximation for both the real part (Fig. 2a) and the imaginary part (Fig. 2b) of β_6 in comparison with the other βs. The coefficients up to β_4 are good approximations to the non-paraxial corrections according to Fig. 2. This is valid for a whole range of nonlinear field strengths. The odd powers of α in $\nu(\alpha, t)$ do not occur when the field is symmetric about an axis (the t or z axis). Also, when $\beta_4 = 0$, the Gaussian beam/soliton preserves its form. The propagator is essentially the Fourier transform of the field at arbitrary time t (the variable t in the beam-propagation problem is to be interpreted as the distance along the beam axis when self-focusing is being considered).

The paraxial results along with the non-paraxial β_4 contribution shown in the figures need fewer computations compared with the full non-paraxial results, which would need enormous computational time. A typical field form with the β_4 term present in (x, t) space can be expressed in terms of parabolic cylindrical

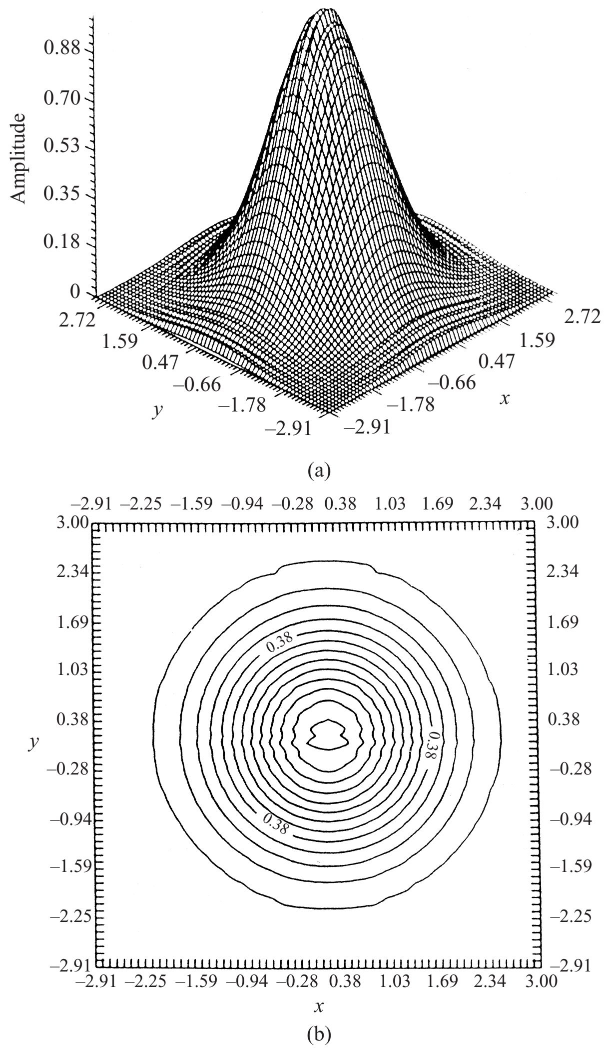


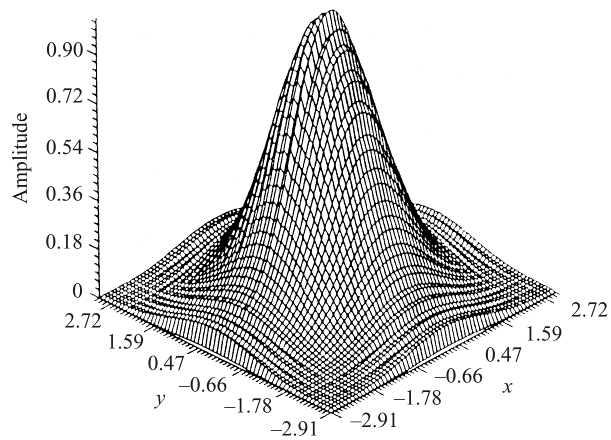
Figure 4. Gaussian beam after evolution for $t = 3000\Delta t$ ($\Delta t = 1.11 \times 10^{-4}$): (a) variation of amplitude with width-wise variables x and y; (b) contour map for the same.

functions (Subbarao 1981), which is a highly accurate analytical field form. The FFT technique allows us to use this highly accurate method repeatedly and rapidly.

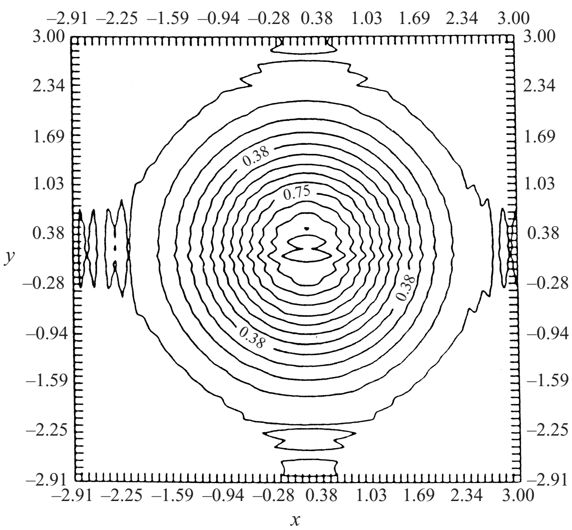
This computation vindicates our earlier paraxial analysis (see Subbarao et al. 1998), and allows us to take non-paraxial terms up to β_4 to improve the results. A full treatment in this direction needs to be considered in the future.

4.2. Beam propagation in cylindrical geometry for self-focusing

The FFTD propagator, the two-dimensional version of the FFT described in Sec. 3.2, has been used to propagate an initial Gaussian beam (Gaussian in the radial direction) to give $E(x, y, t)$ at infinite time t. The evolution of the amplitude with the width-wise variables x and y is shown in Fig. 3. We can



(a)



(b)

Figure 5. Gaussian beam after evolution for $t = 6000\Delta t$ ($\Delta t = 1.11\times10^{-4}$): (a) variation of amplitude with width-wise variables x and y; (b) contour map for the same.

approximate the field by a modified Gaussian, as in Sec. 4.1. Apart from intensity peaking, other details, like a four-island structure, develop in the non-paraxial beam profile, as seen in Figs 3–6, and might be characteristic of the type of mesh used. If a circular mesh, as in the Hankel transformation, is used then we hope to reduce these peculiar four-cornered beam distortions. The contour plots and beam profiles shown in these figures have been produced using standard graphics packages on a SUN SPARCstation. For larger propagation distances, non-paraxial contributions to the field appear and must be taken into account in a suitable theory over and above what has already been done. Because of the large number of FFT computations in the two-dimensional FFT used here (about $n\times n$ computations, with $n = 64$ or 256), it is advisable to use parallel or compressed algorithms in this context. We have developed a preliminary version of the former of these (Singh et al. 1996; Singh

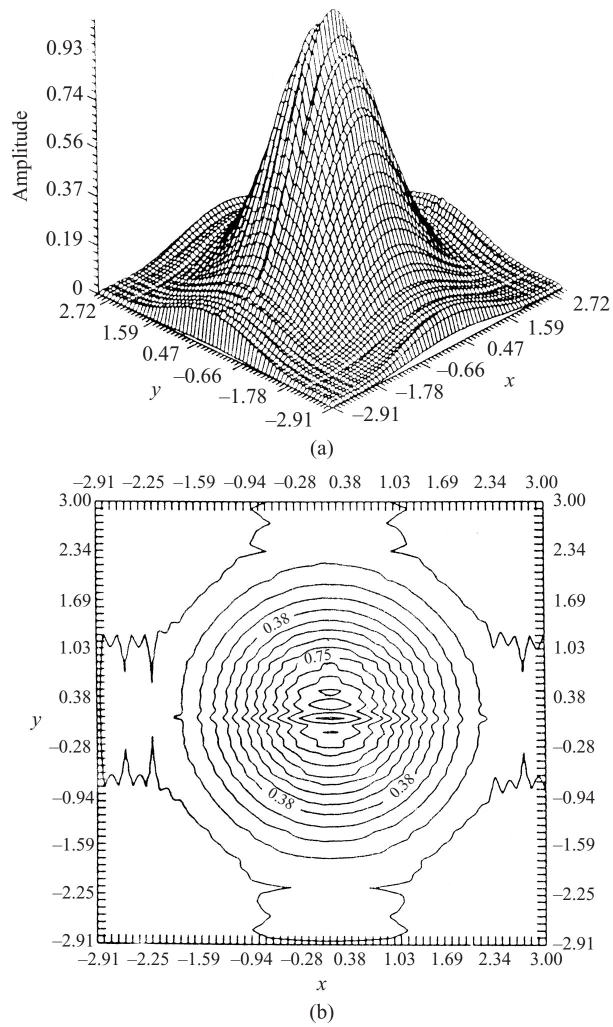


Figure 6. Gaussian beam after evolution for $t = 9000\Delta t$ ($\Delta t = 1.11 \times 10^{-4}$): (a) variation of amplitude with width-wise variables x and y; (b) contour map for the same.

1996) using well-known parallelization techniques of the FFT (Hockney and Jessope 1988; Kumar et al. 1994). As we have discussed earlier, the α space is Fourier space in the case of a soliton. It will be Hankel space (Subbarao and Sodha 1979; Subbarao 1981) in the case of a circular cylindrical beam, to which these results can easily be extended, particularly when cylindrical symmetry is maintained and the two perpendicular directions in the cross-section are uncoupled (unlike in the case of, for example, an elliptical cross-section beam). The figures do not show the evolution of $\beta(\alpha, t)$ for an elliptical cross-section beam, for which a suitable quartic expression in double Fourier space has yet to be identified. Long-distance characteristics are currently being investigated using our Hankel transformation code. For complete cylindrical symmetry, the results of Lax et al. (1981) hold, while, when the θ harmonics also evolve, it is necessary to separate them from numerical errors.

5. Discussion

The FFT-based split-step propagator (Sec. 2) for a soliton (Sec. 3.1) and an FFTD (two-dimensional FFT)-based propagator (Sec. 3.2) or a fast Hankel transform (FHT)-based propagator (Sec. 3.3) for a cylindrical laser beam have been presented. Although the soliton results are satisfactory, it is possible that the Hankel-transform technique outlined here will give improved results for the self-focusing beam propagator with respect to computational errors and symmetries in the errors. For a large class of useful problems, the possibility of constructing a more efficient paraxial FFT propagator using this technique has been presented (Sec. 4.1). Apart from refinements needed in these techniques, it has been pointed out that other orthogonal transformations involving harmonic-oscillator modal representations may prove to be more accurate (see (11)) if fast computational algorithms can be constructed for them. Parallelization can be applied to each of the techniques, and has already been explored by us, among others (Singh et al. 1996; Masoudi and Arnold 1995), but more effort is needed.

A number of other refinements have been tried in order to improve the beam-propagation code, and the techniques developed by us should be considered in this context. A combination of finite-difference and beam-propagation techniques using the Padé approximation of the propagating operator or otherwise has been developed by Yavick (1994). This method allows one to change the step length during propagation. In addition, boundary conditions can be included more flexibly, even for an absorbing medium; periodic boundary conditions are replaced by boundary conditions allowing for radiation leakage (Malkin 1993; Akhmediev 1994; Subbarao et al. 1998a) by suitable reflection laws to account for energy leakage or energy absorption from the beam-propagation region. Generalizations of the beam-propagation method by going beyond the quasi-optical equation and including wide-angle propagation have also been incorporated by many authors (e.g. Hermansson et al. 1983, 1992; Ratowsky and Fleck 1992; see Yevick 1994). The work described here can be generalized to include vector waves instead of the scalar-wave approximation adopted here, in which, by using the fact that the inhomogeneities in the medium are not appreciable over the wavelength of the laser in the quasi-optical equation, we have effectively been able to suppress information about the slow polarization evolution of the beam. Many authors have tried to include, in Cartesian coordinates, the polarization of the waves in the beam-propagation method (e.g. Thylen and Yevick 1982; Okoshi and Kitazawa 1990; Liu et al. 1993; Akhmediev and Soto-Crespo 1994; Kriezis and Papagiannakis 1997; Poladian and Ladouceur 1998). Such vector-wave approaches are particularly challenging with regard to resonance regions in the plasma and the consideration of geometries other than Cartesian.

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