LIMIT THEOREMS AND STRUCTURAL PROPERTIES OF THE CAT-AND-MOUSE MARKOV CHAIN AND ITS GENERALISATIONS

SERGEY FOSS,* ****Heriot-Watt University and Sobolev Institute of Mathematics*
TIMOFEI PRASOLOV,** *****Novosibirsk State University*
SEVA SHNEER,* ** *****Heriot-Watt University and Novosibirsk State University*

Abstract

We revisit the so-called cat-and-mouse Markov chain, studied earlier by Litvak and Robert (2012). This is a two-dimensional Markov chain on the lattice $\mathbb{Z}^2$, where the first component (the cat) is a simple random walk and the second component (the mouse) changes when the components meet. We obtain new results for two generalisations of the model. First, in the two-dimensional case we consider far more general jump distributions for the components and obtain a scaling limit for the second component. When we let the first component be a simple random walk again, we further generalise the jump distribution of the second component. Secondly, we consider chains of three and more dimensions, where we investigate structural properties of the model and find a limiting law for the last component.

Keywords: Cat-and-mouse games; multidimensional Markov chain; compound renewal process; regular variation; weak convergence; randomly stopped sums

2020 Mathematics Subject Classification: Primary 60J10
                                         Secondary 60F05

1. Introduction

We analyse the dynamics of a stochastic process with dependent coordinates, commonly referred to as the cat-and-mouse (CM) Markov chain (MC), and of its generalisations. Let $\mathcal{S}$ be a directed graph. Let $\{(C_n, M_n)\}_{n=0}^{\infty}$ denote the CM MC on $\mathcal{S}^2$, defined as follows. We assume that $\{C_n\}_{n=0}^{\infty}$, the location of the cat, is a MC on $\mathcal{S}$ with transition matrix $P = (p(x, y), x, y \in \mathcal{S})$. The second coordinate, the location of the mouse, $\{M_n\}_{n=0}^{\infty}$, has the following dynamics:

- If $M_n \neq C_n$, then $M_{n+1} = M_n$.

- If $M_n = C_n$, then, conditionally on M_n, the random variable (RV) M_{n+1} has distribution $(p(M_n, y), y \in \mathcal{S})$ and is independent of C_{n+1}.

In our model the cat is trying to catch the mouse. The mouse is usually in hiding and not moving, but if the cat hits the same location of the graph, the mouse jumps. The cat does not notice where the mouse jumps to, so it proceeds independently.

The CM MC is an example of a class of models called CM games. CM games are common in game theory. We refer to [9], where the authors showed that a CM game is at the core of many online algorithms and, in particular, may be used in settings considered by [7, 28]. Some special cases of CM games on the plane have been studied by [4]. Two examples of CM games are discussed in [1] in the context of reversible MCs.

There are many related models in applied probability in which time evolution of the process may be represented as a multi-component MC where one of the components has independent dynamics and forms an MC itself (see, e.g., [8, 16–18]). Typically such dependence is modelled using Markov modulation. In this paper we consider the case where the first component is a random walk. Thus, our model can be viewed as a random-walk-modulated random walk. We consider null-recurrent and transient cases where we find proper scaling for the components.

We are mainly motivated by the results of the paper by [26], where the authors analyse scaling properties of the (non-Markovian) sequence $\{M_n\}_{n=0}^{\infty}$ for a specific transition matrix P when $\mathcal{S}$ is either $\mathbb{Z}$, $\mathbb{Z}^2$, or $\mathbb{Z}^+$. Although it deals with a relatively simple case of the CM MC, it clearly illustrates a certain phenomenon related to this type of Markov modulation. We will focus on the case $\mathcal{S} = \mathbb{Z}$. In addition, we assume that the transition matrix P satisfies $p(x, x+1) = p(x, x-1) = 1/2$. It was proven in Theorem 3 of [26] that the convergence in distribution

$$\left\{ \frac{1}{\sqrt[4]{n}} M_{[nt]}, t \geq 0 \right\} \Rightarrow \left\{ B_1(L_{B_2}(t)), t \geq 0 \right\}, \text{ as } n \to \infty,$$

holds, where $B_1(t)$ and $B_2(t)$ are independent standard Brownian motions on $\mathbb{R}$ and $L_{B_2}(t)$ is the local time process of $B_2(t)$ at 0. By convergence in distribution of càdlàg stochastic processes we mean convergence in the $\mathcal{J}_1$-topology (see Appendix A for further explanation).

This result looks natural, since the mouse, observed at the meeting times with the cat, is a simple random walk. The time intervals between meeting times are independent and identically distributed (i.i.d.). They have the same distribution as the time needed for the cat (also a simple random walk) to get from 1 to 0, which has a regularly varying tail with parameter 1/2 (see, e.g., [32]). Thus, the scaling for the location of the mouse is $\sqrt[4]{n} = (n^{1/2})^{1/2}$. Local time $L_{B_2}(t)$ can be interpreted as the scaled duration of time the cat and the mouse spend together.

This leads us to ask to what extent such a phenomenon holds, for what kinds of distributions of jumps, and for which types of Markov modulation. In this paper we show that similar behaviour holds when jump-size distributions of both components have zero mean and finite variance. The behaviour changes slightly when we introduce heavier tails for the jump-size distribution of the mouse. For this case we develop a more general approach based on the work of [23]. In parallel, we introduce additional components while applying an analogue of the aforementioned Markov modulation. Here, through the analysis of dynamical structural properties, we demonstrate a similar phenomenon for additional components.

More specifically, we provide two generalisations of the CM MC introduced above. The first generalisation relates to the jump distribution of the mouse. Given $C_n = M_n$, the RV $M_{n+1} - M_n$ has a general distribution which has a finite first moment and belongs to the strict domain of attraction of a stable law with index $\alpha \in [1, 2]$, with a normalising function $\{b(n)\}_{n=1}^{\infty}$ (note that distributions with a finite second moment belong to the domain of attraction of a normal distribution) and a centring function $n\mathbb{E}(M_{n+1} - M_n)$. We find a weak limit of $\{b^{-1}(\sqrt{n})M_{[nt+1]}\}_{t \geq 0}$, as $n \to \infty$. This model is more challenging than the classical

setting because, when the mouse jumps, the value of this jump and the time until the next jump may be dependent. Also, if the jump distribution of the mouse has an infinite second moment, we cannot use classical results for regenerative jump processes such as Theorem 5.1 from [24]. Next, we consider the case where both components have general distributions with finite second moments. Here our results take into account the approach developed by [33].

In the second generalisation we add more components to the system (we will refer to the objects whose dynamics these components describe as 'agents'), while keeping the chain 'hierarchy'. For instance, the addition of one extra agent (which we refer to as the dog), acting on the cat in the same way as the cat acts on the mouse, slows down the cat and, therefore, also the mouse. We are interested in the effect of this on the scaling properties of the process. Recursive addition of further agents will slow the mouse down further. For the system with three agents we investigate the dynamical structural properties and find a weak limit of $\{n^{-1/8} M_{[nt]}\}_{t \geq 0}$, as $n \to \infty$. The system regenerates when all the agents are at the same point. Therefore, if we find the tail asymptotics of the time intervals between these events, we can split the process into i.i.d. cycles and use a limit theorem for regenerative processes by [24].

We also consider systems with an arbitrary finite number of agents, for which we provide a relatively simple result on weak convergence, for a fixed $t > 0$. The path analysis in this case turns out to be difficult, and we have only partial results. Namely, we have studied the relation between the numbers of jumps of different agents and obtained the limiting law for the last agent. We have not succeeded in finding the asymptotics for the time intervals between regeneration points, which remains an open question.

We note that our paper is devoted to limit theorems for the above-mentioned Markov-modulated MCs. It is worth noting that there are other interesting questions for such models not covered here: stability problems are studied in, e.g., [15, 19, 30], while large deviations problems are studied in, e.g., [2, 3, 14, 20, 22, 27].

The paper is structured as follows. In Section 2 we define our models and formulate our results. In Sections 3 and 4 we analyse the trajectories of the CM MC and *dog-and-cat-and-mouse* (DCM) MC, respectively. This analysis gives the main idea of the proof of our result on scaling properties of the DCM MC (Theorem 3). In Section 5.1, we prove our results on scaling properties of general CM MCs (Theorem 1 and Theorem 2). We shift the time of our process and use characteristic functions to show that the conditions of Theorem 3.1 from [23] hold. In Section 5.2, we prove Theorem 3. We approximate the dynamics of the mouse by considering only the values of the process at times when all agents are at the same point of the integer line, and then use Theorem 5.1 from [24] to obtain the result. In Section 5.3, we prove our result on scaling properties for the system with an arbitrary finite number N of agents. We approximate the Nth component of the system $X^{(N)}$ (which describes the behaviour of the Nth agent) by the component $X^{(N-1)}$, slowed down by an independent renewal process.

The appendix includes definitions and proofs of supplementary results. In Appendix A, we define weak convergence of stochastic processes. Appendix B clarifies the asymptotic closeness of two scaled processes related to Theorem 1. Finally, in Appendix C we provide auxiliary results on randomly stopped sums with positive summands having a regularly varying tail distribution and infinite mean.

Throughout the paper we use the following conventions and notation. For two ultimately positive functions $f(t)$ and $g(t)$ we write $f(t) \sim g(t)$ if $\lim_{t \to \infty} f(t)/g(t) = 1$. For any event A, its *indicator function* $I[A]$ is an RV that takes the value 1 if the event occurs, and the value 0 otherwise. We use the symbol $\Rightarrow$ for the weak convergence of distributions of RVs or vectors, and $\overset{\mathcal{D}}{\Rightarrow}$ for convergence of trajectories of random processes in the $\mathcal{J}_1$-topology (see Appendix A).

Next, $X \stackrel{d}{=} Y$ means that the RVs X and Y are identically distributed. Finally we use the following abbreviations: CM (cat-and-mouse), DCM (dog-and-cat-and-mouse), MC (Markov chain), i.i.d. (independent and identically distributed), RV (random variable), a.s. (almost surely), w.p. (with probability).

2. Models and results

In this section we recall the CM MC on the integers and introduce several of its generalisations.

2.1. 'Standard' cat-and-mouse Markov chain on $\mathbb{Z}$ ($C \rightarrow M$)

Let $\xi = \pm 1$ w.p. $1/2$. Let $\{\xi_n^{(1)}\}_{n=1}^{\infty}$ and $\{\xi_n^{(2)}\}_{n=1}^{\infty}$ be two mutually independent sequences of independent copies of ξ. Given $C_0 = M_0 = 0$, we define the dynamics of CM MC (C_n, M_n) as follows:

$$C_n = C_{n-1} + \xi_n^{(1)},$$

$$M_n = M_{n-1} + \begin{cases} 0, & \text{if } C_{n-1} \neq M_{n-1}, \\ \xi_n^{(2)}, & \text{if } C_{n-1} = M_{n-1}, \end{cases}$$

for $n \geq 1$.

Let $D[[0, \infty), \mathbb{R}]$ denote the space of all right-continuous functions on $[0, \infty)$ having left limits (RCLL or càdlàg functions).

Let $M(nt) = M_{[nt]}$, $t \geq 0$, be a continuous-time stochastic process taking values M_k, $k \geq 0$, for $t \in [k/n, (k+1)/n)$. Clearly, it is piecewise constant and its trajectories belong to $D[[0, \infty), \mathbb{R}]$.

[26] have proved convergence of trajectories

$$\left\{ \frac{1}{\sqrt[4]{n}} M(nt), t \geq 0 \right\} \stackrel{\mathcal{D}}{\Rightarrow} \left\{ B_1(L_{B_2}(t)), t \geq 0 \right\}, \quad \text{as } n \to \infty,$$

where $B_1(t)$ and $B_2(t)$ are independent standard Brownian motions on $\mathbb{R}$ and $L_{B_2}(t)$ is the local time process of $B_2(t)$ at 0.

2.2. Cat-and-mouse model with a general jump distribution of the mouse ($C \rightarrow M$)

In this subsection we introduce our results for CM MCs with more general distributions of RVs $\xi_n^{(1)}$ and $\xi_n^{(2)}$. We start with the same distribution of $\xi_n^{(1)}$ and generalise the distribution of $\xi_n^{(2)}$. Thus, the cat is a simple random walk and the mouse is a general random walk. We then proceed to generalise the distribution of $\xi_n^{(1)}$ as well.

2.2.1. We continue to assume that the dynamics of the cat is described by a simple random walk on $\mathbb{Z}$. Let $\xi = \pm 1$ w.p. $1/2$. Let $C_0 = 0$, $C_n = C_{n-1} + \xi_n^{(1)}$, where $\xi, \xi_1^{(1)}, \xi_2^{(1)}, \ldots$ are i.i.d. RVs.

Let $M_0 = 0$, $M_n = M_{n-1} + \xi_n^{(2)} I[C_{n-1} = M_{n-1}]$, where $\{\xi_n^{(2)}\}_{n=1}^{\infty}$ are i.i.d. RVs independent of $\{\xi_n^{(1)}\}_{n=1}^{\infty}$. Assume that

$$\mu = \mathbb{E}\xi_1^{(2)} \text{ is finite} \tag{1}$$

and that the distribution of $\xi_1^{(2)}$ belongs to the domain of attraction of a strictly stable distribution, i.e. there exist a function $b(c) > 0$, $c \geq 0$, and an RV $A^{(2)}$ having a strictly stable

distribution with index $\alpha \in [1, 2]$ such that the weak convergence

$$\frac{\sum_{k=1}^{n} \left(\xi_k^{(2)} - \mu\right)}{b(n)} \Rightarrow A^{(2)}, \text{ as } n \to \infty \tag{2}$$

holds. Define

$$\tau(0) = 0 \text{ and } \tau(n) = \inf\{m > \tau(n-1) : C_m = M_m\}, \text{ for } n \geq 1.$$

Given (1), we show that the tail-distribution of $\tau(1)$ is regularly varying with index $1/2$. It follows then that there exists an RV $D^{(2)}$ having a strictly stable distribution with index $1/2$ such that

$$\frac{\tau(n)}{n^2} \Rightarrow D^{(2)}, \text{ as } n \to \infty. \tag{3}$$

In the proof of Theorem 2 we show that in fact there is a weak convergence of two-dimensional random vectors

$$\left(\frac{\sum_{k=1}^{n} \left(\xi_k^{(2)} - \mu\right)}{b(n)}, \frac{\tau(n)}{n^2}\right) \Rightarrow \left(A^{(2)}, D^{(2)}\right), \text{ as } n \to \infty,$$

where the RVs on the right-hand side are independent. Further, let $\left\{\left(A^{(2)}(t), D^{(2)}(t)\right)\right\}_{t \geq 0}$ denote a stochastic process with independent increments such that $\left(A^{(2)}(1), D^{(2)}(1)\right)$ has the same distribution as $\left(A^{(2)}, D^{(2)}\right)$, or, equivalently, the *Lévy process* generated by $\left(A^{(2)}, D^{(2)}\right)$. Let $E^{(2)}(s) = \inf\left\{t \geq 0 : D^{(2)}(t) > s\right\}$, which is a multiple of $L_{B_2}(s)$, the local time at zero of a standard Brownian motion. Thus, the following result is a natural extension of the result by [26]; see remark below.

Theorem 1. *Assume that (1) and (2) hold. Then*

- *if $\mu = 0$, we have*

$$\left\{\frac{M(nt + 1)}{b(\sqrt{n})}, \ t \geq 0\right\} \overset{\mathcal{D}}{\Rightarrow} \left\{A^{(2)}\left(E^{(2)}(t)\right), t \geq 0\right\}, \text{ as } n \to \infty; \tag{4}$$

- *if $\mu \neq 0$, we have*

$$\left\{\frac{M(nt + 1)}{\sqrt{n}}, \ t \geq 0\right\} \overset{\mathcal{D}}{\Rightarrow} \left\{\mu E^{(2)}(t), t \geq 0\right\}, \text{ as } n \to \infty. \tag{5}$$

Remark. If the RVs $\xi_n^{(2)}$ are bounded, then the function $b(n)$ is proportional to $\sqrt{n}$, and it is easy to show that the processes

$$\frac{M(nt + 1)}{\sqrt[4]{n}}, \qquad \frac{M(nt)}{\sqrt[4]{n}}$$

are equivalent (see Appendix B for details). Then the results of Theorem 1 may be reformulated for the scaled process $M(nt)$ in place of $M(nt + 1)$.

2.2.2.   Assume now that both $\xi_1^{(1)}$ and $\xi_1^{(2)}$ have general distributions on the integer lattice. The main difference for the mouse is that we now need to assume that the second moment of $\xi_1^{(2)}$ is finite. Our main result in this setting is that replacing the simple random walk with a general random walk does not change the scaling if we assume aperiodicity and finite second moments for the increments. By aperiodicity (strong aperiodicity in the sense of [32]) we mean here that

$$G.C.D.\{k\colon \mathbb{P}(\xi_1^{(1)} = k) > 0\} = 1,$$

where 'G.C.D.' stands for 'greatest common divisor'.

Theorem 2. *Assume that* $\mathbb{E}\xi^{(1)} = 0$, $0 < \mathbf{Var}\xi_1^{(1)} < \infty$ *and* $\xi_1^{(1)}$ *has an aperiodic distribution.* *Assume* $0 < \mathbf{Var}\xi_1^{(2)} < \infty$ *and, therefore, (2) holds with* $b(n) = \sqrt{n\mathbf{Var}\xi_1^{(2)}}$ *and a standard normal RV* $A^{(2)}$. *Then the statements (4) and (5) of Theorem 1 continue to hold, with* $b(\sqrt{n}) = n^{1/4}\sqrt{\mathbf{Var}\xi_1^{(2)}}$ *in (4).*

2.3. Dog-and-cat-and-mouse model ($D \to C \to M$)

In this subsection we present a generalisation of the CM MC to the case of three dimensions. Let $\xi = \pm 1$ w.p. 1/2. Let $\{\xi_n^{(1)}\}_{n=1}^{\infty}$, $\{\xi_n^{(2)}\}_{n=1}^{\infty}$, and $\{\xi_n^{(3)}\}_{n=1}^{\infty}$ be mutually independent sequences of independent copies of ξ. Given $D_0 = C_0 = M_0 = 0$, we can define the dynamics of the DCM MC $\{(D_n, C_n, M_n)_n\}_{n=1}^{\infty}$ as follows: for $n \geq 1$,

$$D_n = D_{n-1} + \xi_n^{(1)},$$

$$C_n = C_{n-1} + \begin{cases} 0, & \text{if } D_{n-1} \neq C_{n-1}, \\ \xi_n^{(2)}, & \text{if } D_{n-1} = C_{n-1}, \end{cases}$$

$$M_n = M_{n-1} + \begin{cases} 0, & \text{if } C_{n-1} \neq M_{n-1}, \\ \xi_n^{(3)}, & \text{if } C_{n-1} = M_{n-1}. \end{cases}$$

Let $T^{(3)}(0) = 0$ and $T^{(3)}(k) = \min\{n > T^{(3)}(k-1)\colon D_n = C_n = M_n\}$, for $k \geq 1$. We show that the tail-distribution of $T^{(3)}(1)$ is regularly varying with index 1/4. Further, we show that there exists a positive RV $D^{(3)}$ with a stable distribution and Laplace transform $\exp\left(-s^{1/4}\right)$ such that

$$\frac{T^{(3)}(k)}{2^7 k^4} \Rightarrow D^{(3)}, \text{ as } k \to \infty. \tag{6}$$

Let $\{D^{(3)}(t)\}_{t \geq 0}$ be a Lévy process generated by $D^{(3)}$ and $E^{(3)}(s) = \inf\{t \geq 0\colon D^{(3)}(t) > s\}$.

Theorem 3. *We have* $\mathbb{E}M\left(T^{(3)}(1)\right) = 0$, $\sigma^2 = \mathbf{Var}M\left(T^{(3)}(1)\right) = 2$, *and*

$$\left\{\frac{M(nt)}{2^{-7/8} n^{1/8} \sigma}, t \geq 0\right\} \overset{\mathcal{D}}{\Rightarrow} \left\{B(E^{(3)}(t)), t \geq 0\right\}, \text{ as } n \to \infty,$$

where B(t) is a standard Brownian motion, independent of $E^{(3)}(t)$.

2.4. Linear hierarchical chains $\left(X^{(1)} \to X^{(2)} \to \ldots \to X^{(N)}\right)$ of length N

In this subsection we consider a generalisation of the CM MC to the case of N dimensions. Owing to the complexity of sample paths for $N > 3$, it does not seem to be possible to prove an analogue of (3) and (6). For this setting, we prove the convergence for every fixed $t > 0$.

Let $\xi = \pm 1$ w.p. $1/2$. Let $\left\{\{\xi_n^{(j)}\}_{n=1}^{\infty}\right\}_{j=1}^{N}$ be mutually independent sequences of independent copies of ξ. Assume $X_0^{(1)} = \ldots = X_0^{(N)} = 0$. Then MC $\left(X_n^{(1)}, \ldots, X_n^{(N)}\right)$ is defined as follows:

$$X_n^{(1)} = X_{n-1}^{(1)} + \xi_n^{(1)},$$

$$X_n^{(j)} = X_{n-1}^{(j)} + \begin{cases} 0, & \text{if } X_{n-1}^{(j-1)} \neq X_{n-1}^{(j)}, \\ \xi_n^{(j)}, & \text{if } X_{n-1}^{(j-1)} = X_{n-1}^{(j)}, \end{cases}$$

for $j \in \{2, \ldots, N\}$ and for $n \geq 1$.

For the result below we need the following distribution. Let G_α be the distribution function of a one-sided strictly stable law satisfying the condition $x^\alpha(1 - G_\alpha(x)) \to (2 - \alpha)/\alpha$, as $x \to \infty$.

Theorem 4. *Let $\{\zeta_i\}_{i=1}^{\infty}$ be i.i.d. RVs satisfying $\mathbb{P}\{\zeta_i \geq y\} = G_{1/2}(9/y^2)$. Let ψ be an RV with a standard normal distribution and independent of $\{\zeta_i\}_{i=1}^{\infty}$. Then, for any fixed $t > 0$, we have*

$$\frac{X_{[nt]}^{(N)}}{n^{1/2^N}} \Rightarrow t^{1/2^N} \psi \prod_{i=1}^{N} \sqrt{\sqrt{\frac{\pi}{2}} \zeta_i}, \quad \text{as } n \to \infty.$$

3. Trajectories of the 'standard' cat-and-mouse model

In order to prove Theorems 1–3, we first revisit the 'standard' CM model and highlight a number of properties that are of use in the analysis of the more general CM and DCM models discussed here.

We assume that $C_0 = M_0 = 0$. Let $V_n = |C_n - M_n|$, for $n \geq 0$. We can write $M_{n+1} = M_n + \xi_{n+1}^{(2)} I[V_n = 0]$, for $n \geq 1$. Then

$$V_{n+1} = |C_{n+1} - M_{n+1}| = \left| C_n - M_n + \xi_{n+1}^{(1)} - \xi_{n+1}^{(2)} I[V_n = 0] \right|.$$

We can further observe that

$$V_{n+1} = \begin{cases} \left| \xi_{n+1}^{(1)} - \xi_{n+1}^{(2)} \right| \stackrel{d}{=} 1 + \xi_{n+1}^{(1)}, & \text{if } V_n = 0, \\ \left| C_n - M_n + \xi_{n+1}^{(1)} \right| \stackrel{d}{=} V_n + \xi_{n+1}^{(1)}, & \text{if } V_n \neq 0. \end{cases}$$

Thus, V_n forms a MC. Let $p_i(j) = \mathbb{P}\{V_{n+1} = j | V_n = i\}$, for $i, j \geq 0$. Note that $p_0(j) = p_1(j)$ for any j.

Let

$$U^{(2)}(0) = 0, \qquad U^{(2)}(k) = \min\left\{n > U^{(2)}(k-1) : V_n \in \{0, 1\}\right\}.$$

Since $p_0(j) = p_1(j)$ for any j, we have that the RVs $\left\{U^{(2)}(k) - U^{(2)}(k-1)\right\}_{k=1}^{\infty}$ are i.i.d., and the RV $\left(U^{(2)}(k) - U^{(2)}(k-1)\right)$ does not depend on $V_{U^{(2)}(k-1)}$, for $k \geq 1$. From the Markov property

we have

$$V_{U^{(2)}(k)+1} \stackrel{d}{=} 1 + \xi_1^{(1)} = \begin{cases} 0, & \text{w.p. } \frac{1}{2}, \\ 2, & \text{w.p. } \frac{1}{2}. \end{cases}$$

Thus, after each time instant $U^{(2)}(k)$ the cat and the mouse jump with equal probabilities either to the same point or to two different points at distance 2 from each other. In the latter case, $V_{U^{(2)}(k+1)} = 1$, since the cat's jumps are 1 or -1. For the cat, let $\tau_m^{(1)} = \min \{n: \sum_{k=1}^n \xi_k^{(1)} = m\}$ denote the hitting time of the state m. Then

$$U^{(2)}(1) \stackrel{d}{=} 1 + \begin{cases} 0, & \text{w.p. } \frac{1}{2}, \\ \tau_1^{(1)}, & \text{w.p. } \frac{1}{2}. \end{cases}$$

The tail asymptotics for $\tau_1^{(1)}$ are known: $\mathbb{P}\{\tau_1^{(1)} > n\} \sim \sqrt{2/(\pi n)}$, as $n \to \infty$ (see, e.g., Section III.2 in [12] for a related result). Since $\tau_1^{(1)}$ has a distribution with a regularly varying tail, for any $m = 2, 3, \ldots$ we have

$$\mathbb{P}\left\{\tau_m^{(1)} > n\right\} \sim m\mathbb{P}\left\{\tau_1^{(1)} > n\right\} \sim \sqrt{2m^2/(\pi n)}, \text{ as } n \to \infty.$$

4. Trajectories in the dog-and-cat-and-mouse model

In this section we study structural properties of the DCM MC on $\mathbb{Z}$. We describe the main idea of the analysis, which may be of independent interest, as we believe it may be applied to other multi-component MCs.

As before, let $\{T^{(3)}(n)\}_{n=0}^\infty$ be the meeting-time instants, when all three agents meet at a certain point of $\mathbb{Z}$, and let $\{J_k^{(3)}\}_{k=1}^\infty = \{T^{(3)}(k) - T^{(3)}(k-1)\}_{k=1}^\infty$ be the times between such events. Let $M_{T^{(3)}(n)}$, $n = 0, 1, \ldots$, be the locations of the mouse (and, therefore, the common locations of the agents) at the embedded epochs $T^{(3)}(n)$, and let $\{Y_k^{(3)}\}_{k=1}^\infty = \{M_{T^{(3)}(k)} - M_{T^{(3)}(k-1)}\}_{k=1}^\infty$ be the corresponding jump sizes between the embedded epochs. Because of time homogeneity, random vectors $\{(Y_k^{(3)}, J_k^{(3)})\}$ are i.i.d.

Let $N(t) = \max \{n: T^{(3)}(n) = \sum_{k=1}^n J_k^{(3)} \le t\}$, for $t \ge 0$. Let $S_0 = 0$ and $S_n = \sum_{k=1}^n Y_k^{(3)}$. We show that the statement of Theorem 3 holds if we replace M_n with a continuous-time process

$$\widetilde{M}(t) = S_{N(t)} = \sum_{k=1}^{N(t)} Y_k, \text{ for } t \ge 0.$$

The process $\widetilde{M}(t)$ is a so-called *coupled continuous-time random walk* (see [5]), and we use Theorem 5.1 from [24] to obtain its scaling properties.

4.1. Distribution of the random variable $J_1^{(3)}$

We assume that $D_0 = C_0 = M_0 = 0$. Let $V_n = (V_{n1}, V_{n2}) = (|D_n - C_n|, |C_n - M_n|)$. Then the following recursion holds:

$$(D_{n+1}, C_{n+1}, M_{n+1}) = \left(D_n + \xi_{n+1}^{(1)}, C_n + \xi_{n+1}^{(2)}I[V_{n1} = 0], M_n + \xi_{n+1}^{(3)}I[V_{n2} = 0]\right).$$

Note further that

$$
V_{n+1} \stackrel{d}{=}
\begin{cases}
\left(1 + \xi_{n+1}^{(1)}, 1 + \xi_{n+1}^{(2)}\right), & \text{if } V_{n1} = V_{n2} = 0, \\[2mm]
\left(1 + \xi_{n+1}^{(1)}, V_{n2} + \xi_{n+1}^{(2)}\right), & \text{if } V_{n1} = 0 \text{ and } V_{n2} \neq 0, \\[2mm]
\left(V_{n1} + \xi_{n+1}^{(1)}, 1\right), & \text{if } V_{n1} \neq 0 \text{ and } V_{n2} = 0, \\[2mm]
\left(V_{n1} + \xi_{n+1}^{(1)}, V_{n2}\right), & \text{if } V_{n1} \neq 0 \text{ and } V_{n2} \neq 0.
\end{cases}
\tag{7}
$$

Thus, $\{V_n\}_{n=0}^{\infty}$ is a MC. Let $p_{ij}(k, l) = \mathbb{P}\{V_{n+1} = (k, l) | V_n = (i, j)\}$, for $i, j, k, l \geq 0$. Note that $p_{00}(k, l) = p_{01}(k, l)$ for any $k, l \geq 0$.

Let

$$
U^{(3)}(0) = 0 \quad \text{and} \quad U^{(3)}(k) = \min\left\{n > U^{(3)}(k-1) : V_n \in \{(0, 0), (0, 1)\}\right\}.
$$

Since $p_{00}(m, l) = p_{01}(m, l)$ for any m, l, we have that the RVs $\left\{U^{(3)}(k) - U^{(3)}(k-1)\right\}_{k=1}^{\infty}$ are i.i.d., and the RV $\left(U^{(3)}(k) - U^{(3)}(k-1)\right)$ does not depend on $V_{U^{(3)}(k-1)}$, for $k \geq 1$.

In other words, each time $n = U^{(3)}(k)$, the DCM process visits either a state on the 'main diagonal' $D_n = C_n = M_n$ or an auxiliary state $D_n = C_n = M_n \pm 1$. To find the tail asymptotics for $T_1^{(3)}$, we find tail asymptotics for $U^{(3)}(1)$ and then use the natural link between time instants $T^{(3)}(1)$ and $\left\{U^{(3)}(k)\right\}_{k=1}^{\infty}$.

Lemma 1. *Let $V_0 \in \{(0, 0), (0, 1)\}$. Then we have*

$$
\mathbb{P}\left\{U^{(3)}(1) > n\right\} \sim \frac{1}{2^{1/4}\Gamma(3/4)n^{1/4}}, \quad \text{as } n \to \infty.
$$

Further, $U^{(3)}(1) = 1$ if and only if $V_{U^{(3)}(1)} = (0, 0)$.

Proof. Let $V_0 = (0, 0)$. It is apparent from the first line of Equation (7) that

$$
\mathbb{P}\{V_1 = (0, 0)\} = \mathbb{P}\{V_1 = (2, 0)\} = \mathbb{P}\{V_1 = (0, 2)\} = \mathbb{P}\{V_1 = (2, 2)\} = \frac{1}{4}.
$$

Since $p_{00}(k, l) = p_{01}(k, l)$, the RV V_1 has the same distribution given $V_0 = (0, 1)$.

Let $V_1 = (0, 2)$ (Figure 1c). From the second and the fourth lines of Equation (7) we know that $|V_{(k+1)2} - V_{k2}| \in \{0, 1\}$, for $k \geq 1$, given $V_{k2} \neq 0$. Therefore V_{k2} arrives at 1 before hitting 0, and $V_{U^{(3)}(1)} = (0, 1)$. Let $\tau, \tau_1, \tau_2, \ldots$ be independent copies of $\tau_1^{(1)}$. Then $U^{(3)}(1)$ has the same distribution as $\sum_{k=1}^{\tau} \tau_k$, and we have that

$$
\mathbb{P}\left\{\sum_{k=1}^{\tau} \tau_k > n\right\} \sim n^{-1/4} \frac{\Gamma^{1/2}(1/2)\Gamma(1/2)}{\Gamma(3/4)} \sqrt{\sqrt{\frac{2}{\pi}}} \sqrt{\frac{2}{\pi}} = \frac{2^{3/4}}{\Gamma(3/4)n^{1/4}},
$$

as $n \to \infty$ (see Appendix C).

Let $V_1 = (2, 2)$ (Figure 1d). From the fourth line of Equation (7), V_{k2} remains at 2 (the cat and the mouse do not move) until V_{k1} reaches 0. This happens after a time which has the same distribution as $\tau_2^{(1)} = \min\left\{n > 0 : \sum_{k=1}^{n} \xi_k^{(1)} = 2\right\}$. Thus, we travel from (2,2) to

$V_1 = (0,0)$. $V_1 = (2,0)$. $V_1 = (0,2)$. $V_1 = (2,2)$.
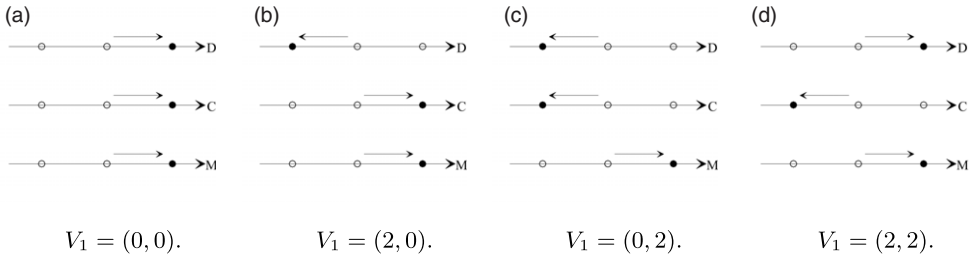
FIGURE 1. The positioning after the first jump.

(0, 2) while never hitting (0,0). We also know that the tail asymptotics of the travel time are $\mathbb{P}\{\tau_2^{(1)} > n\} \sim \sqrt{8/\pi n}$, as $n \to \infty$. Therefore, we travel from (2, 2) to (0, 2) much faster than from (0, 2) to (0, 1), and given $V_1 = (2, 2)$, we again have $\mathbb{P}\{U^{(3)}(1) > n\} \sim \mathbb{P}\{\sum_{k=1}^{\tau} \tau_k > n\}$, as $n \to \infty$.

Finally, let $V_1 = (2, 0)$ (Figure 1b). From the third line of Equation (7) we have $V_2 \stackrel{d}{=} (2 + \xi_2^{(1)}, 1)$ and $V_{U^{(3)}(1)} = (0, 1)$, where, given $V_1 = (2, 0)$, $\mathbb{P}\{U^{(3)}(1) > n\} \sim \sqrt{8/\pi n}$, as $n \to \infty$.

Thus,

$$\mathbb{P}\{U^{(3)}(1) > n\} \sim \frac{1}{2}\mathbb{P}\left\{\sum_{k=1}^{\tau} \tau_k > n\right\} \sim \frac{1}{2^{1/4}\Gamma(3/4)n^{1/4}}, \text{ as } n \to \infty. \qquad \square$$

We note the following relation between time instants $T^{(3)}(1)$ and $\{U^{(3)}(k)\}_{k=1}^{\infty}$. At each time n when we are at the auxiliary state (when $V_n = (0, 1)$), we have a probability of $1/4$ of jumping into the state $D_{n+1} = C_{n+1} = M_{n+1}$, independently of anything else. Using Lemma 1 and the results in Section XIII.6 of [13], we get the following result.

Proposition 1. *Let*

$$\nu = \inf\{k \geq 1: U^{(3)}(k) - U^{(3)}(k - 1) = 1\}.$$

Then ν has a geometric distribution with parameter $1/4$, $T^{(3)} \equiv J^{(3)} = U^{(3)}(\nu)$ a.s., and

$$\mathbb{P}\{J_1^{(3)} > n\} = \mathbb{P}\{U^{(3)}(\nu) > n\} \sim 4\mathbb{P}\{U^{(3)}(1) > n\}, \text{ as } n \to \infty.$$

Therefore, there exists a positive RV $D^{(3)}$ having a stable distribution with Laplace transform $\exp(-s^{1/4})$ such that

$$\frac{T^{(3)}(n)}{2^7 n^4} = \frac{\sum_{k=1}^{n} J_k^{(3)}}{2^7 n^4} \Rightarrow D^{(3)}, \text{ as } n \to \infty.$$

4.2. Distribution of the random variable $Y_1^{(3)}$

In the previous subsection we analysed the time our process spends between auxiliary states. In this subsection we analyse the total displacement of the mouse by time $T_1^{(3)}$. Note that the mouse may have zero, one, or two jumps between consecutive visits to auxiliary states; therefore, the total number of jumps of the mouse by time $T_1^{(3)}$, say $\varkappa$, admits the following simple upper bound:

$$\varkappa \leq 2\nu \quad \text{a.s.}$$

It follows that the sequence $Z_k = M_{U^{(3)}(k)} - C_{U^{(3)}(k)}$, $k = 0, 1, \ldots$, forms a time-homogeneous MC, and $Z_k \in \{-1, 0, 1\}$ a.s. Further, $Z_0 = Z_\nu = 0$, and for any $k \in \{1, \nu - 1\}$, we have $Z_k = \pm 1$.

Let

$$\gamma_k^{(3)} = M_{U^{(3)}(k)} - M_{U^{(3)}(k-1)} \text{ for } k \geq 1.$$

Our analysis of trajectories of the DCM MC in Section 4.1 shows that the $\gamma_k^{(3)}$ are conditionally independent given values of the auxiliary MC $\{Z_k\}_{k=0}^\infty$. Clearly,

$$Y_1^{(3)} = M_{T^{(3)}(1)} = \sum_{k=1}^{\nu} \gamma_k^{(3)}.$$

We now find the first and second moments of $Y_1^{(3)}$ and show that it has a light-tailed distribution.

Since $D_0 = C_0 = M_0 = 0$, we get $\nu = 1$ if and only if $Z_1 = 0$ and $\gamma_1^{(3)} = \pm 1$. Additionally, we have

$$\mathbb{P}\left\{\gamma_1^{(3)} = \pm 1, \ Z_1 = 0 \mid Z_0 = 0\right\} = \mathbb{P}\{D_1 = C_1 = M_1 = \pm 1\} = \frac{1}{8}.$$

Another case of exactly one jump is when the cat and the mouse jump in different directions. Here we have

$$\mathbb{P}\left\{\gamma_1^{(3)} = \pm 1, \ Z_1 = \pm 1 \mid Z_0 = 0\right\} = \mathbb{P}\{C_1 = \mp 1, M_1 = \pm 1\} = \frac{1}{4}.$$

Further, the mouse can have two jumps in the same direction w.p.

$$\mathbb{P}\left\{\gamma_1^{(3)} = \pm 2, \ Z_1 = \pm 1 \mid Z_0 = 0\right\} = \mathbb{P}\{D_1 = \mp 1, C_1 = M_1 = \pm 1, M_2 = \pm 2\} = \frac{1}{16},$$

or two jumps in opposite directions w.p.

$$\mathbb{P}\left\{\gamma_1^{(3)} = 0, \ Z_1 = \pm 1 \mid Z_0 = 0\right\} = \mathbb{P}\{D_1 = \pm 1, C_1 = M_1 = \mp 1, M_2 = 0\} = \frac{1}{16},$$

Thus, given $Z_0 = 0$ we have

$$\mathbb{P}\{Z_1 = 0\} = \frac{1}{4} \quad \text{and} \quad \mathbb{P}\{Z_1 = \pm 1\} = \frac{3}{8},$$

$$\mathbb{P}\left\{\gamma_1^{(3)} = 0\right\} = \frac{1}{8}, \qquad \mathbb{P}\left\{\gamma_1^{(3)} = \pm 1\right\} = \frac{3}{8}, \quad \text{and} \quad \mathbb{P}\left\{\gamma_1^{(3)} = \pm 2\right\} = \frac{1}{16}. \tag{8}$$

From the above, we get that $\mathbb{E}\gamma_1^{(3)} = 0$ and $\mathbf{Var}\gamma_1^{(3)} = 5/4$.

We now analyse the distribution of $\gamma_k^{(3)}$, $k \geq 2$. So as not to make the notation cumbersome, we assume that $D_0 = C_0 = 0$ and $M_0 = 1$, so $Z_0 = 1$. The case $Z_0 = -1$ is analogous by symmetry. Then the mouse can make either zero jumps or exactly one jump before the next visit of the process to an auxiliary state. In the case of zero jumps we have

$$\mathbb{P}\left\{\gamma_1^{(3)} = \pm 0, \ Z_1 = 0 \mid Z_0 = 1\right\} = \mathbb{P}\{D_1 = C_1 = 1\} = \frac{1}{4},$$

$$\mathbb{P}\left\{\gamma_1^{(3)} = \pm 0, \ Z_1 = 1 \mid Z_0 = 1\right\} = \mathbb{P}\{C_1 = -1\} = \frac{1}{2}.$$

In the case of exactly one jump we have

$$\mathbb{P}\left\{\gamma_1^{(3)} = 1, \ Z_1 = 1 \mid Z_0 = 1\right\} = \mathbb{P}\{D_1 = -1, \ C_1 = 1, \ M_2 = 2\} = \frac{1}{8},$$

$$\mathbb{P}\left\{\gamma_1^{(3)} = -1, \ Z_1 = -1 \mid Z_0 = 1\right\} = \mathbb{P}\{D_1 = -1, \ C_1 = 1, \ M_2 = 0\} = \frac{1}{8}.$$

Thus, given $Z_0 = 1$ we have

$$\mathbb{P}\{Z_1 = 0\} = \frac{1}{4}, \ \ \mathbb{P}\{Z_1 = 1\} = \frac{5}{8} \text{ and } \mathbb{P}\{Z_1 = -1\} = \frac{1}{8},$$

$$\mathbb{P}\left\{\gamma_1^{(3)} = 0\right\} = \frac{3}{4} \text{ and } \mathbb{P}\left\{\gamma_1^{(3)} = \pm 1\right\} = \frac{1}{8}. \tag{9}$$

Thus, $\mathbb{E}\left(\gamma_1^{(3)} | Z_0 = 1\right) = 0$ and $\mathbf{Var}\left(\gamma_1^{(3)} | Z_0 = 1\right) = 1/4$.

Let us return to the case $D_0 = C_0 = M_0 = 0$. Combining the results (8) and (9) we get

$$\mathbb{E}Y_1^{(3)} = \mathbb{E}\sum_{k=1}^{\nu}\gamma_k^{(3)} = \mathbb{E}\sum_{k=1}^{\infty}\left(\gamma_k^{(3)}I[\nu \ge k]\right) = \mathbb{E}\gamma_1^{(3)} + \sum_{k=2}^{\infty}\mathbb{E}\left(\gamma_k^{(3)}I[\nu \ge k]\right)$$

$$= \sum_{k=2}^{\infty}\mathbb{E}\left(\gamma_k^{(3)}I[Z_{k-1} = 1] + \gamma_k^{(3)}I[Z_{k-1} = -1]\right) = 0.$$

In a similar manner we transform the second moment:

$$\mathbb{E}\left(Y_1^{(3)}\right)^2 = \sum_{k=1}^{\infty}\mathbb{E}\left(\left(\gamma_k^{(3)}\right)^2 I[\nu \ge k]\right) + 2\sum_{k=1}^{\infty}\sum_{m=k+1}^{\infty}\mathbb{E}\left(\gamma_k^{(3)}\gamma_m^{(3)}I[\nu \ge m]\right). \tag{10}$$

If we fix the values of $\{Z_k\}_{k=0}^{\infty}$, the RVs $\gamma_k^{(3)}$ and $\gamma_m^{(3)}$ become independent. Combined with the fact that $\mathbb{E}\left(\gamma_m^{(3)} | Z_{m-1} = \pm 1\right) = 0$, we get that the second sum in (10) equals zero. Now we can use the conditional second moments obtained above and the fact that the RV ν has a geometric distribution with parameter $1/4$. We obtain

$$\mathbb{E}\left(Y_1^{(3)}\right)^2 = \mathbb{E}\left(\gamma_1^{(3)}\right)^2 + \sum_{k=2}^{\infty}\mathbb{E}\left(\left(\gamma_k^{(3)}\right)^2 \Big| Z_{k-1} = \pm 1\right)\mathbb{P}\{\nu \ge k\} = \frac{5}{4} + \frac{1}{4}(\mathbb{E}\nu - 1) = 2.$$

Finally, we observe that $\left|\gamma_1^{(3)}\right| \le 2$. Therefore, $\left|Y_1^{(3)}\right| = \left|M_{J_1^{(3)}}\right| \le 2\nu$ a.s. and $|Y^{(3)}|$ has a finite exponential moment. In particular, the following holds.

Proposition 2. *We have* $\mathbb{E}Y_1^{(3)} = 0$, $\mathbf{Var}Y_1^{(3)} = 2$, *and* $\mathbb{E}\left|Y_1^{(3)}\right|^m < \infty$, *for any* $m \ge 3$.

5. Proofs of main results

In this section we provide proofs of our main results.

5.1. Proofs of Theorems 1 and 2

We start with the general idea of the proofs of Theorems 1 and 2. For $i = 1, 2$, let $S_0^{(i)} = 0$ and $S_n^{(i)} = \sum_{k=1}^{n}\xi_k^{(i)}$, for $n \ge 1$. Let

$$\tau(0) = 0 \text{ and } \tau(n) = \inf\left\{m > \tau(n-1): S_m^{(1)} = S_n^{(2)}\right\}, \text{ for } n \ge 1.$$

Since $\left\{\xi_k^{(1)}\right\}_{k=1}^{\infty}$ and $\left\{\xi_k^{(2)}\right\}_{k=1}^{\infty}$ are independent sequences of i.i.d. RVs, we have that $\tau(n) - \tau(n-1) \stackrel{d}{=} \tau(1)$, for $n \geq 1$.

Let $\eta(t) = \max\{k \geq 0 : \tau(k) \leq t\}$ for $t \geq 0$. Define a continuous-time process $M'(t)$ by

$$M'(t) = 0 \text{ for } t \in [0, 1), \qquad M'(t) = S_{\eta(t-1)+1}^{(2)} = \sum_{k=1}^{\eta(t-1)+1} \xi_k^{(2)} \text{ for } t \geq 1.$$

It is straightforward to verify that $\{M'(n),\ n \geq 0\} \stackrel{d}{=} \{M(n),\ n \geq 0\}$. In the rest of the section we will omit the prime symbol and simply write $M(t)$. The process $\{\widehat{M}(t)\}_{t \geq 0} = \{M'(t+1)\}_{t \geq 0}$ is a so-called *oracle continuous-time random walk* (see, e.g., [23].

First, consider the case $\mathbb{E}\xi_1^{(2)} = 0$. We want to show that

$$\left(\frac{S_n^{(2)}}{b(n)}, \frac{\tau(n)}{n^2}\right) \Rightarrow \left(A^{(2)}, D^{(2)}\right), \text{ as } n \to \infty. \tag{11}$$

Given that, we will show that the first part of Theorem 1 follows from the next proposition.

Proposition 3. (Theorem 3.1, [23].) *Assume (11) holds. Then*

$$\left\{\frac{\widehat{M}(nt)}{b(\sqrt{n})}, t \geq 0\right\} = \left\{\frac{M(nt+1)}{b(\sqrt{n})}, t \geq 0\right\} \stackrel{\mathcal{D}}{\Rightarrow} \left\{A^{(2)}\left(E^{(2)}(t)\right), t \geq 0\right\}, \text{ as } n \to \infty.$$

A similar result is proven in Theorem 3.6 of [21].

We will now show that the relation (11) holds and that the RVs $A^{(2)}$ and $D^{(2)}$ are independent. We show that for certain functions f_1 and f_2,

$$\mathbb{E}\exp\left(i\left(\lambda_1 \frac{S_n^{(2)}}{b(n)} + \lambda_2 \frac{\tau(n)}{n^2}\right)\right) = \mathbb{E}\exp\left(i\left(\lambda_1 \frac{\xi_1^{(2)}}{b(n)} + \lambda_2 \frac{\tau(1)}{n^2}\right)\right)^n$$

$$= \left(1 + \frac{f_1(\lambda_1) + f_2(\lambda_2)}{n} + o\left(\frac{1}{n}\right)\right)^n, \quad (12)$$

as $n \to \infty$, for any $\lambda_1, \lambda_2 \in \mathbb{R}$. Indeed, convergence of characteristic functions is equivalent to weak convergence of RVs, and for independence of RVs it is sufficient to verify that the characteristic function of the sum is equal to the product of the respective characteristic functions. Since the right-hand side of (12) converges to $\exp(f_1(\lambda_1)) \exp(f_2(\lambda_2))$, this will prove the convergence and the independence of the limits $A^{(2)}$ and $D^{(2)}$.

5.1.1. *Proof of Theorem 1.* From the condition (2) we have the weak convergence of $S_n^{(2)}/b(n)$ to the RV $A^{(2)}$. Again, this is equivalent to the convergence of the characteristic functions. Thus, (2) implies

$$\mathbb{E}\exp\left(i\lambda_1 \frac{\sum_{k=1}^n \xi_k^{(2)}}{b(n)}\right) = \left[\mathbb{E}\exp\left(i\lambda_1 \frac{\xi_1^{(2)}}{b(n)}\right)\right]^n \to \mathbb{E}\exp\left(i\lambda_1 A^{(2)}\right), \text{ as } n \to \infty.$$

Additionally, if $B^n(n) \to z$ as $n \to \infty$, then $n \log B(n) \to \log z$, which leads to $\log B(n) \sim n^{-1} \log z$ and hence, finally, $B(n) \sim 1 + n^{-1} \log z$, as $n \to \infty$. Thus, we have the following:

$$\mathbb{E} \exp \left(i\lambda_1 \frac{\xi_1^{(2)}}{b(n)} \right) \sim 1 + \frac{l_1(\lambda_1)}{n}, \quad \text{as } n \to \infty, \tag{13}$$

where $l_1(\lambda) = \log \mathbb{E} \exp \left(i\lambda A^{(2)} \right)$, the logarithmic characteristic function of $A^{(2)}$.

As in Section 3, we reformulate the distribution of the RV $\tau(1)$ using the time needed for the simple random walk to hit a fixed point. Let $\{\tau_k^{(1)}\}_{k=1}^{\infty}$ be independent copies of τ, the time needed for the simple random walk to hit 0 if it starts from 1, independent of $\{\xi_n^{(2)}\}_{n=1}^{\infty}$. Given $\xi_1^{(2)} = m \neq 0$, the RV $\tau(1)$ is the time needed for the random walk $S_n^{(1)}$ to reach m from zero, and thus, $\tau(1) \stackrel{d}{=} \sum_{k=1}^{|m|} \tau_k^{(1)}$. Given $\xi_1^{(2)} = 0$, we have $\tau(1) \stackrel{d}{=} 1 + \tau_1^{(1)}$. Then we have the following relation for $\tau(1)$:

$$\tau(1) \stackrel{d}{=} I\left[\xi_1^{(2)} \neq 0 \right] \sum_{k=1}^{\left| \xi_1^{(2)} \right|} \tau_k^{(1)} + I\left[\xi_1^{(2)} = 0 \right] \left(1 + \tau_1^{(1)} \right).$$

Since $\mathbb{P}\{\tau_1^{(1)} > n\} \sim \sqrt{2/(\pi n)}$, as $n \to \infty$, we conclude from Proposition 8 (see appendix) that

$$\mathbb{P}\{\tau(1) > n\} \sim \left(\mathbb{E}\left| \xi_1^{(2)} \right| + \mathbb{P}\left\{ \xi_1^{(2)} = 0 \right\} \right) \mathbb{P}\{\tau > n\}.$$

Thus, there exists an RV $D^{(2)}$ having a stable distribution with index $1/2$ such that

$$\frac{\tau(n)}{n^2} \Rightarrow D^{(2)}, \quad \text{as } n \to \infty.$$

Using the same argument as for (13), we get

$$\mathbb{E} \exp \left(i\lambda_2 \frac{\tau}{n^2} \right) \sim 1 + \frac{l_2(\lambda_2)}{n}, \quad \text{as } n \to \infty, \tag{14}$$

where $\lambda_2(\lambda) = \log \mathbb{E} \exp \left(i\lambda D^{(2)} / \left(\mathbb{E}|\xi_1^{(2)}| + \mathbb{P}\{\xi_1^{(2)} = 0\} \right) \right)$, the logarithmic characteristic function of $D^{(2)} / \left(\mathbb{E}|\xi_1^{(2)}| + \mathbb{P}\{\xi_1^{(2)} = 0\} \right)$. We proceed as follows:

$$\mathbb{E} \exp \left(i \left[\lambda_1 \frac{\xi_1^{(2)}}{b(n)} + \lambda_2 \frac{\tau(1)}{n^2} \right] \right)$$

$$= \sum_{-\infty}^{\infty} \exp \left(i\lambda_1 \frac{k}{b(n)} \right) \mathbb{P}\{\xi_1^{(2)} = k\} \mathbb{E} \exp \left(i\lambda_2 \frac{\tau(1)}{n^2} \middle| \xi_1^{(2)} = k \right)$$

$$= \mathbb{P}\{\xi_1^{(2)} = 0\} \mathbb{E} \exp \left(i\lambda_2 \frac{1 + \tau}{n^2} \right)$$

$$+ \sum_{k \neq 0} \exp \left(i\lambda_1 \frac{k}{b(n)} \right) \mathbb{P}\{\xi_1^{(2)} = k\} \left(\mathbb{E} \exp \left(i\lambda_2 \frac{\tau}{n^2} \right) \right)^{|k|}.$$

In order to transform the last sum in the last equation, we use (14) and get that, uniformly in $m > 0$,

$$\left(\mathbb{E}\exp\left(i\lambda_2\frac{\tau}{n^2}\right)\right)^m = \left(1 + \frac{l_2(\lambda_2)}{n} + o\left(\frac{1}{n}\right)\right)^m$$

$$= \exp\left(m\ln\left(1 + \frac{l_2(\lambda_2) + o(1)}{n}\right)\right)$$

$$= \exp\left(\frac{ml_2(\lambda_2)}{n}(1 + o(1))\right)$$

$$= 1 + \frac{ml_2(\lambda_2)(1 + o(1))}{n} + \frac{1}{n^2}\sum_{j=2}^{\infty}\frac{(ml_2(\lambda_2)(1 + o(1)))^j}{n^{j-2}j!},$$

as $n \to \infty$. Since the latter equation is uniform in $m > 0$, we get

$$\sum_{k\neq 0}\exp\left(i\lambda_1\frac{k}{b(n)}\right)\mathbb{P}\{\xi_1^{(2)} = k\}\left(\mathbb{E}\exp\left(i\lambda_2\frac{\tau}{n^2}\right)\right)^{|k|}$$

$$= \left(\mathbb{E}\exp\left(i\lambda_1\frac{\xi_1^{(2)}}{b(n)}\right) - \mathbb{P}\{\xi_1^{(2)} = 0\}\right)$$

$$+ \frac{l_2(\lambda_2)}{n}\sum_{k\neq 0}|k|\exp\left(i\lambda_1\frac{k}{b(n)}\right)\mathbb{P}\{\xi_1^{(2)} = k\} + \frac{1}{n^2}\sum_{k\neq 0}\sum_{j=2}^{\infty}\frac{(kl_2(\lambda_2)(1 + o(1)))^j}{n^{j-2}j!} + o\left(\frac{1}{n}\right),$$

as $n \to \infty$. Now we use the fact that if $\sum_{-\infty}^{\infty}A_n = \sum_{-\infty}^{\infty}B_n + \sum_{-\infty}^{\infty}C_n$ and if the series $\sum_{-\infty}^{\infty}A_n$ and $\sum_{-\infty}^{\infty}B_n$ converge, then $\sum_{-\infty}^{\infty}C_n$ converges too. Thus,

$$\frac{1}{n^2}\sum_{k\neq 0}\sum_{j=2}^{\infty}\frac{(kl_2(\lambda_2)(1 + o(1)))^j}{n^{j-2}j!} = o\left(\frac{1}{n}\right)$$

and

$$\sum_{k\neq 0}\exp\left(i\lambda_1\frac{k}{b(n)}\right)\mathbb{P}\{\xi_1^{(2)} = k\}\left(\mathbb{E}\exp\left(i\lambda_2\frac{\tau}{n^2}\right)\right)^{|k|}$$

$$= \left(\mathbb{E}\exp\left(i\lambda_1\frac{\xi_1^{(2)}}{b(n)}\right) - \mathbb{P}\{\xi_1^{(2)} = 0\}\right) + \mathbb{E}\left|\xi_1^{(2)}\right|\frac{l_2(\lambda_2)}{n} + o\left(\frac{1}{n}\right),$$

as $n \to \infty$. Using (13) and (14), we have

$$\mathbb{E}\exp\left(i\left[\lambda_1\frac{\xi_1^{(2)}}{b(n)} + \lambda_2\frac{\tau(1)}{n^2}\right]\right)$$

$$= 1 + \frac{l_1(\lambda_1)}{n} + \left(\mathbb{E}\left|\xi_1^{(2)}\right| + \mathbb{P}\{\xi_1^{(2)} = 0\}\right)\frac{l_2(\lambda_2)}{n} + o\left(\frac{1}{n}\right),$$

as $n \to \infty$. We have proved that Equation (12) holds with $f_1(\lambda_1) = l_1(\lambda_1)$ and $f_2(\lambda_2) = \left(\mathbb{E}\left|\xi_1^{(2)}\right| + \mathbb{P}\{\xi_1^{(2)} = 0\}\right)l_2(\lambda_2)$. Therefore, Equation (11) holds and we can use Proposition 3 to prove the first part of Theorem 1.

We now turn to the second part and assume $\mathbb{E}\xi^{(2)} = \mu \neq 0$. Then the above arguments are applicable to $\sum_{k=1}^{\eta(t)+1}\left(\xi_k^{(2)} - \mu\right)$. Thus, we have shown that the process $\left(\left(\sum_{k=1}^{\eta(nt)+1}\left(\xi_k^{(2)} - \mu\right)\right)/b(\sqrt{n}), \ t \geq 0\right)$ weakly converges to the limiting one (see Appendix A for the corresponding definitions). Since $\mu < \infty$, we have $b(n) = o(n)$, as $n \to \infty$. Indeed, by the strong law of large numbers,

$$\frac{\sum_{k=1}^{n}\left(\xi_k^{(2)} - \mu\right)}{n} \xrightarrow{a.s.} 0, \ \text{as } n \to \infty.$$

Since $A^{(2)}$ in (2) is not deterministic, the denominator in the left-hand side of (2) must be an $o(n)$ function. Therefore the process

$$\left(\frac{\sum_{k=1}^{\eta(nt)+1}\left(\xi_k^{(2)} - \mu\right)}{\sqrt{n}}, \ t \geq 0\right) = \left(\frac{\sum_{k=1}^{\eta(nt)+1}\left(\xi_k^{(2)} - \mu\right)}{b(\sqrt{n})}\frac{b(\sqrt{n})}{\sqrt{n}}, \ t \geq 0\right)$$

converges to the zero-valued process. Thus, it follows from the representation

$$\frac{\widehat{M}(nt)}{\sqrt{n}} = \frac{\sum_{k=1}^{\eta(nt)+1}\left(\xi_k^{(2)} - \mu\right)}{\sqrt{n}} + \frac{\mu(\eta(nt)+1)}{\sqrt{n}}$$

and from the corollary to Theorem 3.2 of [29] that

$$\left\{\frac{\widehat{M}(nt)}{\sqrt{n}}, \ t \geq 0\right\} \xrightarrow{\mathcal{D}} \left\{\mu E^{(2)}(t), \ t \geq 0\right\}, \ \text{as } n \to \infty.$$

5.1.2. *Proof of Theorem 2.* Under the assumption of the finiteness of second moments, we can extend our result to the case where both $\xi^{(1)}$ and $\xi^{(2)}$ have general distributions. Assume now that $\{S_n^{(1)}\}_{n=0}^{\infty} = \{\sum_{k=1}^{n}\xi_k^{(1)}\}_{n=0}^{\infty}$ is an aperiodic random walk with zero-mean and finite-variance-σ_1^2 increments. The theory of general random walks and their hitting times is well developed. Nevertheless, results that are uniform in terms of the hitting point are rather scarce. It follows from Section 3.3 of [33] that, uniformly in x,

$$\mathbb{E}\left[\exp\left(it\tau(1)\right) \,\Big|\, \xi_1^{(2)} = x\right] = 1 - (a^*(x) + e_x(t))\left(\sigma_1\sqrt{-2it} + o\left(\sqrt{|t|}\right)\right), \ \text{as } t \to 0, \quad (15)$$

where

$$a^*(x) = 1 + \sum_{n=1}^{\infty}\left(\mathbb{P}\left\{S_n^{(1)} = 0\right\} - \mathbb{P}\left\{S_n^{(1)} = -x\right\}\right), \quad (16)$$

$$e_x(t) = c_x(t) + is_x(t), \quad (17)$$

$$|c_x(t)| = O\left(x^2\sqrt{|t|}\right), \ \text{as } t \to 0, \text{ uniformly in } x, \quad (18)$$

$$s_0(t) = 0 \text{ and } \frac{s_x(t)}{x} = o(1), \ \text{as } t \to 0, \text{ uniformly in } x \neq 0. \quad (19)$$

Following steps similar to those used in the previous part, we take $t = \lambda_2/n^2$ and, eventually, let n become large. A very important relation here is (18). When we take the characteristic function

$$\mathbb{E}\exp\left(i\left[\lambda_1\frac{\xi_1^{(2)}}{\sigma_2\sqrt{n}} + \lambda_2\frac{\tau(1)}{n^2}\right]\right)$$

and start to separate it into different summands, the relation (18) leads to the summand (see details below)

$$\sum_{x \in \mathbb{Z}} O\left(\frac{x^2}{n^2}\right) \mathbb{P}\{\xi_1^{(2)} = x\}, \quad \text{as } n \to \infty,$$

and this is the main reason we need to assume that $\xi_1^{(2)}$ has a finite second moment.

Assume now that $\mathbb{E}\xi_1^{(2)} = 0$ and $\sigma_2 = \mathbf{Var}\xi_1^{(2)} < \infty$. We have (see, e.g., Proposition 7.2 of [34])

$$\sigma_1^2(a^*(x) - I(x = 0)) \sim |x|, \quad \text{as } |x| \to \infty. \tag{20}$$

As a consequence, we get $\mathbb{E}a^*\left(\xi_1^{(2)}\right) < \infty$. Let $p^{(2)}(x) = \mathbb{P}\{\xi_1^{(2)} = x\}$. Then the total probability formula gives us

$$\mathbb{E}\exp\left(i\left[\lambda_1\frac{\xi_1^{(2)}}{\sigma_2\sqrt{n}} + \lambda_2\frac{\tau(1)}{n^2}\right]\right)$$

$$= \sum_{x \in \mathbb{Z}} \exp\left(i\lambda_1\left[\frac{x}{\sigma_2\sqrt{n}}\right]\right) \mathbb{E}\left[\exp\left(i\left[\lambda_2\frac{\tau(1)}{n^2}\right]\right)\Big| \xi_1^{(2)} = x\right] p^{(2)}(x).$$

Now we use (15)–(19) to get

$$\mathbb{E}\exp\left(i\left[\lambda_1\frac{\xi_1^{(2)}}{\sigma_2\sqrt{n}} + \lambda_2\frac{\tau(1)}{n^2}\right]\right) = \mathbb{E}\left[\exp\left(i\lambda_1\left[\frac{\xi_1^{(2)}}{\sigma_2\sqrt{n}}\right]\right)\right]$$

$$- \frac{\sigma_1\sqrt{-2i\lambda_2}}{n}\mathbb{E}\left[a^*\left(\xi_1^{(2)}\right)\exp\left(i\lambda_1\left[\frac{\xi_1^{(2)}}{\sigma_2\sqrt{n}}\right]\right)\right]$$

$$+ O\left(\frac{1}{n^2}\mathbb{E}\left[\left(\xi_1^{(2)}\right)^2\exp\left(i\lambda_1\left[\frac{\xi_1^{(2)}}{\sigma_2\sqrt{n}}\right]\right)\right]\right)$$

$$+ o\left(\frac{1}{n}\mathbb{E}\left[\xi_1^{(2)}\exp\left(i\lambda_1\left[\frac{\xi_1^{(2)}}{\sigma_2\sqrt{n}}\right]\right)\right]\right) + o\left(\frac{1}{n}\right),$$

as $n \to \infty$. Next, we use the relation (20) and the Taylor expansion for the exponent to get

$$\mathbb{E}\left[a^*\left(\xi_1^{(2)}\right)\exp\left(i\lambda_1\left[\frac{\xi_1^{(2)}}{\sigma_2\sqrt{n}}\right]\right)\right] = \mathbb{E}\left[a^*\left(\xi_1^{(2)}\right)\right] + o(1), \quad \text{as } n \to \infty.$$

Since $\mathbb{E}\xi_1^{(2)} = 0$ and $\mathbf{Var}\xi_1^{(2)} < \infty$, the central limit theorem holds. Thus, we have the analogue of (14) with l_1 being a logarithmic characteristic function of an RV with a standard normal distribution. Finally, we get

$$\mathbb{E}\exp\left(i\left[\lambda_1\frac{\xi_1^{(2)}}{\sigma_2\sqrt{n}} + \lambda_2\frac{\tau(1)}{n^2}\right]\right) = 1 + \frac{l_1(\lambda_1)}{n} - \frac{\sigma_1\sqrt{-2i\lambda_2}}{n}\mathbb{E}\left[a^*\left(\xi_1^{(2)}\right)\right] + o\left(\frac{1}{n}\right).$$

Thus, we have proved Equation (12) for this case. The rest of the proof follows the same argument as in the previous case.

5.2. Proof of Theorem 3

Recall that the random vectors $\{Y_n^{(3)}, J_n^{(3)}\}_{n=1}^\infty$ are i.i.d., where $Y_1^{(3)} = \sum_{k=1}^\nu \gamma_k^{(3)}$ and $J_1^{(3)} = T^{(3)}(1) = U^{(3)}(\nu)$. We have

$$N(t) = \max\left\{n > 0 \colon \sum_{k=1}^n J_k^{(3)} \leq t\right\} \text{ and } \widetilde{M}(t) = \sum_{k=1}^{N(t)} Y_k^{(3)}.$$

From Propositions 1 and 2 we have

$$\mathbb{E}Y_1^{(3)} = 0, \ \mathbf{Var}Y_1^{(3)} = 2, \ \mathbb{E}\left(Y_1^{(3)}\right)^m < \infty, \ \text{for } m \geq 2, \text{ and } \mathbb{P}\left\{J_1^{(3)} > n\right\} \sim \frac{2^{7/4}}{\Gamma(3/4)n^{1/4}},$$

as $n \to \infty$. From Theorem 5.1 from [24] we have

$$\left\{\frac{\widetilde{M}(nt)}{2^{-7/8}n^{1/8}\sqrt{\mathbf{Var}Y_1^{(3)}}}, t \geq 0\right\} \overset{\mathcal{D}}{\Rightarrow} \{B(E^{(3)}(t)), t \geq 0\}, \text{ as } n \to \infty, \tag{21}$$

where $B(t)$ is a standard Brownian motion, independent of $E^{(3)}(t)$.

We show now that (21) holds with $M(nt)$ in the place of $\widetilde{M}(nt)$. It is sufficient to prove that, for any fixed $T > 0$,

$$\frac{\max_{1 \leq k \leq [nT]}\left\{\widetilde{M}_k - M_k\right\}}{n^{1/8}} \overset{a.s.}{\to} 0, \text{ as } n \to \infty.$$

For $k = 1, 2, \ldots,$ let ν_k be the number of visits to the set $\{(0, 0), (0, 1)\}$ by the MC V_n within the time interval $\left(T_{k-1}^{(3)}, T_k^{(3)}\right]$, and let $\varkappa_k$ be the total number of jumps of the mouse within this time interval. Further, let

$$R_k = \max_{T^{(3)}(k-1) \leq l \leq T^{(3)}(k)} |\widetilde{M}_l - M_l|.$$

The triples $(\nu_k, \varkappa_k, R_k)$ are i.i.d., the pairs $(\nu_k, \varkappa_k)$ are i.i.d. copies of the pair $(\nu, \varkappa)$ introduced earlier, and

$$R_k \leq \varkappa_k \leq 2\nu_k \text{ a.s.} \tag{22}$$

Further, all RVs in (22) have a finite exponential moment, $\mathbb{E}\exp(cR_1) < \infty$ for some $c > 0$.

For $K > 0$, let $\widehat{J}_j = \min\left(J_j^{(3)}, K\right)$. Since $\mathbb{E}J_1^{(3)} = \infty$, one can choose K such that $A := \mathbb{E}\widehat{J}_1 > T$. Let $\widehat{S}_n = \sum_{i=1}^n \widehat{J}_i$. Then, for any $C > 0$,

$$\mathbb{P}(\eta(nT) + 1 > n) \leq \mathbb{P}(\widehat{S}_n \leq nT) \leq \left(e^{CT}\mathbb{E}e^{-C\widehat{J}_1}\right)^n,$$

where the term in the parentheses on the right-hand side may be made less than 1 for $C > 0$ sufficiently small. Next,

$$\mathbb{P}\left(\max_{1 \leq k \leq n} R_k \geq n^{1/8}\varepsilon\right) \leq n\mathbb{P}(R_1 \geq n^{1/8}\varepsilon) \leq n\mathbb{E}e^{cR_1} \cdot e^{-c\varepsilon n^{1/8}}.$$

Then, for any $\varepsilon > 0$,

$$\mathbb{P}\left(\frac{\max_{1 \leq k \leq [nT]}\left\{\widetilde{M}_k - M_k\right\}}{n^{1/8}} \geq \varepsilon\right) \leq \mathbb{P}\left(\max_{1 \leq k \leq n} R_k \geq n^{1/8}\varepsilon\right) + \mathbb{P}(\eta(nT) \geq n),$$

where both terms on the right-hand side are summable in n. Therefore, by the 0–1 law and by the arbitrariness of $\varepsilon > 0$, (22) follows. This completes the proof of Theorem 3.

5.3. Proof of Theorem 4

The random process $X^{(1)}$ is a simple random walk on $\mathbb{Z}$, and for $j \in [2, N]$ we have

$$\mathbb{P}\{X^{(j)}(n) - X^{(j)}(n-1) = 1 \,\big|\, X^{(j)}(n-1) = X^{(j-1)}(n-1)\}$$

$$= \mathbb{P}\{X^{(j)}(n) - X^{(j)}(n-1) = -1 \,\big|\, X^{(j)}(n-1) = X^{(j-1)}(n-1)\} = \frac{1}{2}.$$

Let us give a new representation of such a process. Let $X^{(1)}(0) = X^{(2)}(0) = \ldots = X^{(N)}(0) = 0$, and let the RV $T_j(n)$ denote the time when $X^{(j)}$ takes the nth step. Let $T_j(0) = 0$. Note the difference between T_j and $T^{(3)}$. Thus, $\{X^{(j)}(T_j(k))\}_{k=0}^{\infty}$ is a simple random walk on $\mathbb{Z}$, and if $X^{(j)}(n) \neq X^{(j)}(n-1)$, then $n \in \{T_j(k)\}_{k=1}^{\infty}$. Let

$$\xi_k^{(j)} = X^{(j)}(T_j(k)) - X^{(j)}(T_j(k-1)) = X^{(j)}(T_j(k)) - X^{(j)}(T_j(k) - 1)$$

for $j \geq 1$ and $k \geq 1$. By definition, $\big\{\{\xi_k^{(j)}\}_{k=0}^{\infty}\big\}_{j=1}^{N}$ are mutually independent and equal ± 1 w.p. $1/2$.

Since $X^{(1)}$ jumps every time, $T_1(k) = k$ for $k \geq 0$. Let τ be the time at which the simple random walk goes from the point 1 to 0. From Section 3 it is easy to deduce that the time between the meeting-time instants of the cat and the mouse has the same distribution as τ. Thus, if we look at the system only at the times $\{T_j(k)\}_{k=0}^{\infty}$ the time between meeting-time instants of $X^{(j)}$ and $X^{(j+1)}$ has the same distribution as τ.

Let us define $\nu_j(n) = \max\{k \geq 0 : T_j(k) \leq n\}$, the number of time instants up to time n at which $X^{(j)}$ changes its value. Then we can rewrite the dynamics of the jth coordinate as

$$X^{(j)}(n) = \sum_{k=1}^{\nu_j(n)} \xi_{T_j(k)}^{(j)}.$$

Our assumptions on the distribution of the increments $\xi_k^{(j)}$, for $k \geq 1$, provide us with the next important property of our process.

Proposition 4. *The sequences $\{T_j(k)\}_{k=0}^{\infty}$ and $\{\xi_k^{(j)}\}_{k=1}^{\infty}$ are independent for any $j \in \{1, \ldots, N\}$.*

This property comes from the space-symmetry of the model. Indeed, for $j = 1$ the result is trivial, since $\nu_1(n) = n$. We show the result for $j = 2$ and then extend it onto $j > 2$. Define

$${}^1\tau(0) = 0 \quad \text{and} \quad {}^1\tau(k) = \inf\{n > {}^1\tau(k-1) : X^{(1)}(n) = X^{(2)}(n)\}, \quad \text{for } k \geq 1.$$

One can see that in our model $T_2(k) = 1 + {}^1\tau(k-1)$, for $k \geq 1$. In the time interval $[1, {}^1\tau(1)]$ the second coordinate changes its value only at the time $T_2(1) = 1$. Thus, the time ${}^1\tau(1)$ does not depend on $\xi_k^{(2)}$, for $k \geq 2$. Additionally, the trajectory $\{X^{(1)}(n)\}_{n=0}^{\infty}$ has the same distribution as $\{-X^{(1)}(n)\}_{n=0}^{\infty}$. Thus,

$$\mathbb{P}\big\{{}^1\tau(1) = n, \ \xi_1^{(2)} = 1\big\} = \mathbb{P}\big\{{}^1\tau(1) = n, \ \xi_1^{(2)} = -1\big\}.$$

As a corollary of the last equation, we get that ${}^1\tau(1)$ has the same distribution as the time that is needed for the simple random walk to hit 0 if it starts from 1. This implies that

$$\mathbb{P}\big\{{}^1\tau(1) > n\big\} \sim \sqrt{\frac{2}{\pi n}}, \quad \text{as } n \to \infty. \tag{23}$$

From the symmetry of our model, it further follows that the sequence $\{^1\tau(k)\}_{k=0}^\infty$, and subsequently the sequences $\{T_2(k)\}_{k=0}^\infty$ and $\{v_2(n)\}_{n=1}^\infty$, do not depend on $\{\xi_k^{(2)}\}_{k=1}^\infty$ (or, therefore, on $\{\xi_k^{(j)}\}_{k\geq 1, j\geq 2}$).

For the analysis of $\{T_j(k)\}_{k=0}^\infty$, $j>2$, we need to define an 'embedded version' of $^1\tau(k)$. Let

$$^j\tau(0) = 0 \text{ and } {}^j\tau(k) = \inf\left\{m > {}^j\tau(k-1): X^{(j)}(T_j(m)) = X^{(j+1)}(T_j(m))\right\}, \text{ for } k \geq 1.$$

The process $\{^j\tau(k)\}_{k=0}^\infty$ counts the number of times that the process $X^{(j)}$ changes its value between the time instants when $X^{(j)}$ and $X^{(j+1)}$ have the same value. Using the same argument as before, we get that the sequence $\{^j\tau(k)\}_{k=0}^\infty$ does not depend on $\{\xi_k^{(j+1)}\}_{k=1}^\infty$.

The jth coordinate $X^{(j)}$ changes its value for the kth time at a time instant n if and only if, up to time $n-1$, the processes $X^{(j-1)}$ and $X^{(j)}$ have had the same value exactly $k-1$ times (not including $X^{(j-1)}(0) = X^{(j)}(0) = 0$), with the last time being at the time instant $n-1$ (which also means that at the time instant $n-1$ the process $X^{(j-1)}$ changes its value). This can be rewritten as

$$T_j(k) = n \Leftrightarrow n - 1 = T_{j-1}\big(^{j-1}\tau(k-1)\big), \text{ for } j \geq 2, k \geq 1,$$

and thus $T_j(k) = 1 + T_{j-1}(^{j-1}\tau(k-1))$. Thus, since the sequences $\{T_2(k)\}_{k=0}^\infty$ and $\{^2\tau(k)\}_{k=0}^\infty$ do not depend on $\{\xi_k^{(j)}\}_{k\geq 1, j\geq 3}$, the same holds for $\{T_3(k)\}_{k=0}^\infty$. Therefore, using induction, we get that the sequences $\{T_j(k)\}_{k=0}^\infty$ and $\{\xi_k^{(j)}\}_{k=1}^\infty$ are independent for any $j \geq 1$.

As a corollary of this result we get

$$X^{(j)}(n) = \sum_{k=1}^{v_j(n)} \xi_{T_j(k)}^{(j)} \overset{d}{=} \sum_{k=1}^{v_j(n)} \xi_k^{(j)}.$$

Let $^j\eta(n) = \max\{k \geq 0: {}^j\tau(k) \leq n\}$ for $n \geq 0$ and $j \in [1, \ldots, N]$. Since the sequence $\{^j\tau(k)\}_{k=0}^\infty$ depends only on the sequence $\{\xi_k^{(j)}\}_{k=1}^\infty$, we have that $\{^j\eta(n)\}_{j=1}^{N-1}$ are i.i.d. RVs. For $n \geq 1$ and $j \in \{1, \ldots, N\}$ we have

$$v_j(n) = \max\{k \geq 0: T_j(k) \leq n\} = \max\left\{k \geq 1: 1 + T_{j-1}\big(^{j-1}\tau(k-1)\big) \leq n\right\}$$

$$= 1 + \max\left\{k \geq 0: T_{j-1}\big(^{j-1}\tau(k)\big) \leq n - 1\right\}$$

$$= 1 + \max\left\{k \geq 0: {}^{j-1}\tau(k) \leq v_{j-1}(n-1)\right\}$$

$$= 1 + {}^{j-1}\eta\big(v_{j-1}(n-1)\big).$$

For $n < N - 1$ we can iterate the process and get

$$v_N(n) = 1 + {}^{N-1}\eta\left(1 + {}^{N-2}\eta\left(\ldots\left(1 + {}^{N-n}\eta(0)\right)\ldots\right)\right)$$

$$= 1 + {}^{N-1}\eta\left(1 + {}^{N-2}\eta\left(\ldots\left(1 + {}^{N-n+1}\eta(1)\right)\ldots\right)\right)$$

$$\overset{d}{=} 1 + {}^{n-1}\eta\left(1 + {}^{n-2}\eta(\ldots\left(1 + {}^1\eta(1)\right)\ldots)\right)$$

$$= v_n(n).$$

For $n \geq N - 1$ we have

$$\nu_N(n) = 1 + {}^{N-1}\eta\left(1 + {}^{N-2}\eta\left(\ldots + {}^{1}\eta(n - N + 1)\right)\right).$$

We want to construct a process with the same distribution as $\{\nu_N(n)\}_{n=0}^{\infty}$ in a form of $\nu_{N-1}(\varphi(n))$, where the process $\{\varphi(n)\}_{n=0}^{\infty}$ is independent of everything else. Define the process $\{\eta(n)\}_{n=0}^{\infty} \overset{d}{=} \{{}^{N-1}\eta(n)\}_{n=0}^{\infty}$, which is independent of everything else. Then, for $n \geq N - 1$, we have

$$\nu_N(n) \overset{d}{=} 1 + {}^{N-2}\eta\left(1 + {}^{N-3}\eta\left(\ldots + \eta(n - N + 1)\right)\right).$$

Using the same formula for $\nu_{N-1}(m)$ with m such that $m - (N - 1) + 1 = 1 + {}^{N-1}\eta(n - N + 1)$, we get

$$\nu_N(n) \overset{d}{=} \nu_{N-1}(N - 1 + \eta(n - N + 1)), \quad \text{for } n \geq N - 1.$$

Then, for $n \geq N$, we have $X^{(N)}(n) \overset{d}{=} X^{(N-1)}(N - 1 + \eta(n - N + 1))$. There exists a nondegenerate RV ζ (see Section XI.5 in [13]) such that $\mathbb{P}\{\zeta_i \geq y\} = G_{1/2}(9/y^2)$ and

$$\eta(n)\mathbb{P}\{{}^{N-1}\tau(1) > n\} \Rightarrow \zeta, \quad \text{as } n \to \infty.$$

Therefore, using (23) we get

$$\frac{j - 1 + \eta(n - j + 1)}{\sqrt{n}} = \frac{j - 1 + \eta(n - j + 1)}{\sqrt{n - j + 1}} \frac{\sqrt{n - j + 1}}{\sqrt{n}} \Rightarrow \sqrt{\frac{\pi}{2}}\zeta, \tag{24}$$

as $n \to \infty$, for $j \geq 1$. We now present a known result that we utilise to prove Theorem 4.

Proposition 5. ([11], (v).) *Let $Y(t)$ and τ_n be independent sequences of RVs such that*

$$\frac{Y(t)}{bt^{\beta}} \Rightarrow Y, \ as \ t \to \infty, \ and \ \frac{\tau_n}{dn^{\delta}} \Rightarrow \tau, \ as \ n \to \infty. \tag{25}$$

Then for independent Y and τ we have

$$\frac{Y(\tau_n)}{bd^{\beta}n^{\beta\delta}} \Rightarrow Y\tau^{\beta}, \ as \ n \to \infty.$$

Indeed, by the central limit theorem, $X^{(1)}(n)/\sqrt{n}$ weakly converges to a normally distributed RV ψ (we assume that ψ and ζ are independent). Together with (24) and the independence of $X^{(1)}(n)$ and $\eta(n)$, this ensures that the condition (25) holds with $Y(t) = X^{(1)}([t])$, $\tau_n = 1 + \eta(n - 1)$, and $\beta = \delta = 1/2$. By Proposition 5,

$$\frac{X^{(2)}(n)}{n^{1/4}} \overset{d}{=} \frac{X^{(1)}(1 + \eta(n - 1))}{n^{1/4}} \Rightarrow \psi\sqrt{\sqrt{\frac{\pi}{2}}\zeta}, \ as \ n \to \infty.$$

Let $\{\zeta_j\}_{j=2}^{N}$ be independent copies of ζ which are independent of ψ. Next, we use the induction argument. For some $j \geq 1$, the condition (25) holds with $Y(t) = X^{(j)}([t])$, $\tau_n = j - 1 + \eta(n - j + 1)$, $\beta = 2^{-j}$, and $\delta = 2^{-1}$. By Proposition 5, we get

$$\frac{X^{(j+1)}(n)}{n^{2^{-(j+1)}}} \overset{d}{=} \frac{X^{(j)}(j - 1 + \eta(n - j + 1))}{n^{2^{-(j+1)}}} \Rightarrow \psi\prod_{i=2}^{j+1}\sqrt{\sqrt{\frac{\pi}{2}}\zeta_i}, \quad \text{as } n \to \infty.$$

This concludes the proof of Theorem 4.

Appendix A. Weak convergence for processes from $D[[0, \infty), \mathbb{R}]$

To make the paper self-contained, we recall the definition of the $\mathcal{J}_1$-topology (see, e.g., [31]). Let $D[[0, T], \mathbb{R}]$ denote the space of all right-continuous functions on $[0, T]$ having left limits (RCLL or càdlàg functions). For any $g \in D[[0, T], \mathbb{R}]$ let $\|g\| = \sup_{t \in [0, T]} |g(t)|$.

Let Λ_T be the set of increasing continuous functions $\lambda : [0, T] \to [0, T]$ such that $\lambda(0) = 0$ and $\lambda(T) = T$. Let $\lambda_{id,T}$ denote the identity function. Then

$$d_{\mathcal{J}_1,T}(g_1, g_2) = \inf_{\lambda \in \Lambda_T} \max \left(\|g_1 \circ \lambda - g_2\|, \|\lambda - \lambda_{id,T}\| \right)$$

defines a metric inducing $\mathcal{J}_1$.

Let $D[[0, \infty), \mathbb{R}]$ be the space of all RCLL functions on the positive half-line. On the space $D[[0, \infty), \mathbb{R}]$, the $\mathcal{J}_1$-topology is defined by the metric

$$d_{\mathcal{J}_1,\infty}(g_1, g_2) = \int_0^\infty e^{-t} \min \left(1, d_{\mathcal{J}_1,T}(g_{1,T}, g_{2,T}) \right) dT,$$

where, for $i = 1, 2$, $g_{i,T}$ is the restriction of function g_i on the interval $[0,T]$.

Convergence $g_n \to g$ in $(D[[0, \infty), \mathbb{R}], \tau)$ means that $d_{\tau,T}(g_n, g) \to 0$ for every continuity point T of g (see [35]).

Let $\{\{X_n(t)\}_{t \geq 0}\}_{n=1}^\infty$ and $\{X(t)\}_{t \geq 0}$ be stochastic processes with trajectories from $D[[0, \infty), \mathbb{R}]$. We say that weak convergence

$$\{X_n(t)\}_{t \geq 0} \overset{\mathcal{D}}{\Rightarrow} \{X(t)\}_{t \geq 0}$$

holds if

$$\mathbb{E}f(\{X_n(t)\}_{t \geq 0}) \to \mathbb{E}f(\{X(t)\}_{t \geq 0}), \text{ as } n \to \infty,$$

for any continuous and bounded function f on $D[[0, \infty), \mathbb{R}]$ endowed with the $\mathcal{J}_1$-topology.

Proposition 6. *Let* $\{X_n\}_{n=1}^\infty$ *and* $\{Y_n\}_{n=1}^\infty$ *be two sequences of stochastic processes with trajectories from* $D[[0, \infty), \mathbb{R}]$. *Given* $d_{\mathcal{J}_1,\infty}(X_n, Y_n) \overset{a.s.}{\to} 0$, *we have*

$$X_n - Y_n \overset{\mathcal{D}}{\Rightarrow} 0.$$

B Asymptotic closeness of two scaled processes

In this section we prove the remark following Theorem 1. We consider a general CM MC (C_n, M_n), $n \geq 0$, and the jumps $\xi_k^{(2)}$ of the second component are bounded RVs. Then for the process $M(nt) = M_{[nt]}$ we have the same functional limit theorem as for the process $\widehat{M}(nt) = M(nt + 1)$. Thanks to Proposition 6, it is sufficient to prove the following.

Proposition 7. *We have*

$$d_{\mathcal{J}_1,\infty} \left(\left\{ \frac{M(nt)}{b(\sqrt{n})}, t \geq 0 \right\}, \left\{ \frac{M(nt + 1)}{b(\sqrt{n})}, t \geq 0 \right\} \right) \overset{a.s.}{\to} 0, \text{ as } n \to \infty.$$

Proof. First, we restrict our processes to the time interval $[0, T]$, with an arbitrary finite T, and investigate the convergence of the distance $d_{\mathcal{J}_1, T}(\cdot, \cdot)$ between our processes. Second, we bound the distance using the following function λ_n:

$$\lambda_n(t) = \begin{cases} 0, \ t \in \left[0, \frac{1}{n}\right], \\ t - \frac{1}{n}, \ t \in \left[\frac{1}{n}, t_n\right], \\ t_n + (t - t_n)\frac{T - t_n + 1/n}{T - t_n}, \end{cases} \qquad \text{where } t_n = \frac{[nT] - 1}{n}.$$

Thus, $M(nt) = M(n\lambda_n(t) + 1)$ for $t \in [1/n, t_n]$. Then the distance between processes on the time interval $[0, T]$ can be bounded as follows:

$$d_{\mathcal{J}_1, T}\left(\left\{\frac{M(nt)}{b(\sqrt{n})}, t \in [0, T]\right\}, \left\{\frac{M(nt + 1)}{b(\sqrt{n})}, t \in [0, T]\right\}\right)$$

$$\leq \max\left(\frac{\max\left(\left|\xi_1^{(2)}\right|, \left|\xi_{\eta([nT]-2)+1}^{(2)}\right|, \left|\xi_{\eta([nT]-2)+1}^{(2)} + \xi_{\eta([nT]-1)+1}^{(2)}\right|\right)}{b(\sqrt{n})}, \frac{1}{n}\right)$$

$$\leq \max\left(\frac{\max\left(\left|\xi_1^{(2)}\right|, \left|\xi_{\eta([nT]-2)+1}^{(2)}\right|, \left|\xi_{\eta([nT]-2)+1}^{(2)}\right| + \left|\xi_{\eta([nT]-2)+2}^{(2)}\right|\right)}{b(\sqrt{n})}, \frac{1}{n}\right).$$

Since the $\xi_k^{(2)}$ are bounded and $b(n) \to \infty$ as $n \to \infty$, the right-hand side of the last inequality converges to zero a.s. $\qquad\square$

Appendix C. Tail asymptotics for randomly stopped sum

Let $\xi_1, \xi_2, \ldots$ be positive i.i.d. RVs with a common distribution function F. Let $S_0 = 0$ and $S_k = \xi_1 + \ldots \xi_k$, $k \geq 1$. Let τ be a counting RV with a distribution function G, independent of $\{\xi_k\}_{k=1}^{\infty}$. For a general overview concerning tail asymptotics of S_τ, see e.g. [10] and references therein. The next result follows from Theorem 1 of [25].

Proposition 8. *Assume that* $\overline{F}(x) \sim l_1(x)/x^\alpha$, $\alpha \in [0, 1)$*, and* τ *has any distribution with* $\mathbb{E}\tau < \infty$*. Then*

$$\mathbb{P}\{S_\tau > n\} \sim \mathbb{E}\tau \mathbb{P}\{\xi > n\} \text{ as } n \to \infty.$$

The next result we use in Lemma 1; we prove it using Tauberian theorems.

Proposition 9. *Assume that* $\overline{F}(x) \sim l_1(x)/x^\alpha$ *and* $\overline{G}(x) \sim l_2(x)/x^\beta$, $\alpha, \beta \in (0, 1)$*. Then*

$$\mathbb{P}\{S_\tau > n\} \sim n^{-\alpha\beta} \frac{\Gamma^\beta(1-\alpha)\Gamma(1-\beta)}{\Gamma(1-\alpha\beta)} l_1^\beta(n) \, l_2\left(\frac{n^\alpha}{\Gamma(1-\alpha)l_1(n)}\right), \text{ as } n \to \infty.$$

Proof. Denote the cumulative distribution function of S_τ by H. Let

$$\overline{F}(x) = 1 - F(x), \ x \in \mathbb{R},$$

$$\widehat{F}(\lambda) = \mathbb{E}e^{-\lambda\xi_1} = \int_0^\infty e^{-\lambda x}dF(x), \ \lambda \geq 0.$$

Define $\overline{G}, \widehat{G}, \overline{H}$, and $\widehat{H}$ similarly. We use the following result.

Proposition 10. (Part of Corollary 8.1.7, [6].) *For a constant $\alpha \in [0, 1]$, and for a function l that is slowly varying at infinity, the following are equivalent:*

$$1 - \widehat{F}(\lambda) \sim \lambda^\alpha l\left(\frac{1}{\lambda}\right), \ \text{as } \lambda \downarrow 0,$$

$$\overline{F}(x) \sim \frac{l(x)}{x^\alpha \Gamma(1-\alpha)}, \ \text{as } x \to \infty, \text{ if } 0 \le \alpha < 1.$$

Using this result, we get

$$1 - \widehat{F}(\lambda) \sim \lambda^\alpha \Gamma(1-\alpha)l_1\left(\frac{1}{\lambda}\right) \ \text{and} \ 1 - \widehat{G}(\lambda) \sim \lambda^\beta \Gamma(1-\beta)l_2\left(\frac{1}{\lambda}\right), \ \text{as } \lambda \downarrow 0.$$

Let us analyse $\widehat{H}$:

$$\widehat{H}(\lambda) = \mathbb{E}e^{-\lambda S_\tau} = \sum_{k=1}^\infty e^{-\lambda(\xi_1 + \ldots + \xi_k)} \mathbb{P}\{\tau = k\} = \mathbb{E}\left(\mathbb{E}e^{-\lambda \xi_1}\right)^\tau = \widehat{G}(-\ln \widehat{F}(\lambda)).$$

Since

$$-\ln \widehat{F}(\lambda) = -\ln\left(1 - (1 - \widehat{F}(\lambda))\right) \sim 1 - \widehat{F}(\lambda), \ \text{as } \lambda \downarrow 0,$$

we have

$$1 - \widehat{H}(\lambda) \sim 1 - \widehat{G}\left(\lambda^\alpha \Gamma(1-\alpha)l_1\left(\frac{1}{\lambda}\right)\right)$$

$$\sim \lambda^{\alpha\beta} \Gamma^\beta(1-\alpha)\Gamma(1-\beta)l_1^\beta\left(\frac{1}{\lambda}\right) l_2\left(\frac{1}{\lambda^\alpha \Gamma(1-\alpha)l_1\left(\frac{1}{\lambda}\right)}\right), \quad (26)$$

as $\lambda \downarrow 0$, and finally

$$\overline{H}(x) \sim x^{-\alpha\beta} \frac{\Gamma^\beta(1-\alpha)\Gamma(1-\beta)}{\Gamma(1-\alpha\beta)} l_1^\beta(x) \, l_2\left(\frac{x^\alpha}{\Gamma(1-\alpha)l_1(x)}\right), \ \text{as } x \to \infty.$$

Note that the function $l_2(h(\lambda))$ with $h(\lambda) = 1/(\lambda^\alpha \Gamma(1-\alpha)l_1\left(\frac{1}{\lambda}\right))$ on the right-hand side of (26) is slowly varying at infinity. Indeed, for any constant $c \neq 0$, as $\lambda \downarrow 0$,

$$h(c\lambda) = \frac{1}{(c\lambda)^\alpha \Gamma(1-\alpha)l_1\left(\frac{1}{c\lambda}\right)} \sim c^{-\alpha} h(\lambda),$$

and therefore $l_2(h(c\lambda)) \sim l_2(c^{-\alpha}h(\lambda)) \sim l_2(h(\lambda))$. □

Acknowledgement

Funding Information

Research was supported by the Mathematical Center in Akademgorodok under Agreement No. 075-15-2019-1613 with the Ministry of Science and Higher Education of the Russian Federation.

Competing Interests

There were no competing interests to declare which arose during the preparation or publication process for this article.

References

[1] ALDOUS, D. AND FILL, J. (2002). *Reversible Markov Chains and Random Walks on Graphs.* Available at http://www.stat.berkeley.edu/users/aldous/RWG/book.html.

[2] ALSMEYER, G. AND SGIBNEV, M. (1999). On the tail behaviour of the supremum of a random walk defined on a Markov chain. *Yokohama Math. J.* **46**, 139–159.

[3] ASMUSSEN, S., HENRIKSEN, L. F. AND KLOPPELBERG, C. (1994). Large claims approximations for risk processes in a Markovian environment. *Stoch. Process. Appl.* **54**, 29–43.

[4] BAEZA-YATES, R. A., CULBERSON, J. C. AND RAWLINS, G. J. E. (1993). Searching in the plane. *Inf. Comput.* 106, 234–252.

[5] BECKER-KERN, P., MEERSCHAERT, M. M. AND SCHEFFLER, H.-P. (2004). Limit theorems for coupled continuous time random walks. *Ann. Prob.* **32**, 730–756.

[6] BINGHAM, N. H., GOLDIE, C. M. AND TEUGELS, J. L. (1987). *Regular Variation* (Encyclopaedia *Math.* Appl. **27**). Cambridge University Press.

[7] BORODIN, A., LINIAL, N. AND SAKS, M. (1992). An optimal online algorithm for metrical task systems. *J. Assoc. Comput. Mach.* **39**, 745–763.

[8] BORST, S., JONCKHEERE, M. AND LESKELA L. (2008). Stability of parallel queueing systems with coupled service rates. *Discrete Event Dynam. Systems* **18**, 447–472.

[9] COPPERSMITH, D., DOYLE, P., RAGHAVAN, P. AND SNIR, M. (1993). Random walks on weighted graphs and applications to on-line algorithms. *J. Assoc. Comput. Mach.* **40**, 421–453.

[10] DENISOV, D., FOSS, S. AND KORSHUNOV, D. (2010). Asymptotics of randomly stopped sums in the presence of heavy tails. *Bernoulli* **16**, 971–994.

[11] DOBRUSHIN, R. L. (1955). Lemma on the limit of compound random functions. *Uspekhi Mat. Nauk* **10**, 157–159.

[12] FELLER, W. (1971a). *An Introduction to Probability Theory and Its Applications*, Vol. 1. John Wiley, New York.

[13] FELLER, W. (1971b). *An Introduction to Probability Theory and Its Applications*, Vol. 1. John Wiley, New York.

[14] FOSS, S., KONSTANTOPOULOS, T. AND ZACHARY, S. (2007). Discrete and continuous time modulated random walks with heavy-tailed increments. *J. Theoret. Prob.* **20**, 581–612.

[15] FOSS, S., SHNEER, S., THOMAS, J. P. AND WORRALL, T. (2018). Stochastic stability of monotone economies in regenerative environments. *J. Econom. Theory* **173**, 334–360.

[16] FOSS, S., SHNEER, S. AND TYURLIKOV, A. (2012). Stability of a Markov-modulated Markov chain, with application to a wireless network governed by two protocols. *Stoch. Systems* **2**, 208–231.

[17] GAMARNIK, D. (2004). Stochastic bandwidth packing process: stability conditions via Lyapunov function technique. *Queueing Systems* **48**, 339–363.

[18] GAMARNIK, D. AND SQUILLANTE, M. (2005). Analysis of stochastic online bin packing processes. *Stoch. Models* **21**, 401–425.

[19] GEORGIOU, N. AND WADE, A. R. (2014). Non-homogeneous random walks on a semi-infinite strip. *Stoch. Process. Appl.* **124**, 3179–3205.

[20] HANSEN, N. R. AND JENSEN, A. T. (2005). The extremal behaviour over regenerative cycles for Markov additive processes with heavy tails. *Stoch. Process. Appl.* **115**, 579–591.

[21] HENRY, B. I. AND STRAKA, P. (2011). Lagging and leading coupled continuous time random walks, renewal times and their joint limits. *Stoch. Process. Appl.* **121**, 324–336.

[22] JELENKOVIC, P. R. AND LAZAR, A. A. (1998). Subexponential asymptotics of a Markov-modulated random wwalk with queueing applications. *J. Appl. Prob.* **35**, 325–347.

[23] JURLEWICZ, A., KERN, P., MEERSCHAERT, M. M. AND SCHEFFLER, H.-P. (2012). Fractional governing equations for coupled random walks. *Comput. Math. Appl.* **64**, 3021–3036.

[24] KASAHARA, Y. (1984). Limit theorems for Lévy processes and Poisson point processes and their applications to Brownian excursions. *J. Math. Kyoto Univ.* **24**, 521–538.

[25] KORSHUNOV, D. (2009). An analog of Wald's identity for random walks with infinite mean. *Siberian Math. J.* **50**, 663–666.

[26] LITVAK, N. AND ROBERT, P. (2012). A scaling analysis of a cat and mouse Markov chain. *Ann. Prob.* **22**, 792–826.

[27] LU, Y. AND LI, S. (2005). On the probability of ruin in a Markov-modulated risk model. *Insurance Math. Econom.* **37**, 522–532.

[28] MANASSE, M. S., MCGEOCH, L. A. AND SLEATOR, D. D. (1990). Competitive algorithms for server problems. *J. Algorithms* **11**, 208–230.

[29] MEERSCHAERT, M. M. AND SCHEFFLER, H.-P. (2004). Limit theorems for continuous-time random walks with infinite mean waiting times. *J. Appl. Prob.* **41**, 623–638.

[30] SHAH, D. AND SHIN, J. (2012). Randomized scheduling algorithm for queueing networks. *Ann. Appl. Prob.* **22**, 128–171.

[31] SKOROKHOD, A. V. (1956). Limit theorems for stochastic processes. *Theory Prob. Appl.* **1**, 261–290.

[32] SPITZER, F. (1964). *Principles of Random Walk*. Springer, New York.

[33] UCHIYAMA, K. (2011a). The first hitting time of a single point for random walks. *Electron. J. Prob.* **16**, 1960–2000.

[34] UCHIYAMA, K. (2011b). One dimensional lattice random walks with absorption at a point/on a half line. *J. Math. Soc. Japan* **63**, 675–713.

[35] WHITT, W. (2002). *Stochastic-Process Limits*. Springer, New York.