J. Inst. Math. Jussieu (2021) **20**(3), 929–955
doi:10.1017/S1474748019000380

EQUIVARIANT ALGEBRAIC INDEX THEOREM

ALEXANDER GOROKHOVSKY, NIEK DE KLEIJN AND RYSZARD NEST

Kobenhavns Universitet Det Natur-og Biovidenskabelige Fakultet, Mathematics, Universitetsparken 5, 2100, Copenhagen, Denmark (rnest@math.ku.dk)

(Received 26 May 2017; revised 18 June 2019; accepted 18 June 2019; first published online 27 August 2019)

Abstract We prove a Γ-equivariant version of the algebraic index theorem, where Γ is a discrete group of automorphisms of a formal deformation of a symplectic manifold. The particular cases of this result are the algebraic version of the transversal index theorem related to the theorem of A. Connes and H. Moscovici for hypo-elliptic operators and the index theorem for the extension of the algebra of pseudodifferential operators by a group of diffeomorphisms of the underlying manifold due to A. Savin, B. Sternin, E. Schrohe and D. Perrot.

Keywords: deformation quantization; index theorem

2010 *Mathematics subject classification*: Primary 19K56; 58H10
Secondary 18G30; 55U10

Contents

Alexander Gorokhovsky was partially supported by an NSF grant. Niek de Kleijn and Ryszard Nest were supported by the Danish National Research Foundation through the Centre for Symmetry and Deformation (DNRF92). Niek de Kleijn was also partially supported by the IAP 'Dygest' of the Belgian Science Policy.

1. Introduction

The term *index theorems* is usually used to describe the equality of, on the one hand, analytic invariants of certain operators on smooth manifolds and, on the other hand, topological/geometric invariants associated to their 'symbols.'

The classical situation analyzed in the seminal papers of Atiyah and Singer, see [1], is as follows. Let X be a compact manifold and D an elliptic pseudodifferential operator acting between spaces of smooth sections of a pair of vector bundles on X. Then D is Fredholm and hence the Fredholm index of D, i.e. the integer

$$\text{Ind }(D) = \dim(\text{Ker}(D)) - \dim(\text{Coker}(D))$$

is well defined. The Atiyah–Singer index theorem identifies it with the evaluation of the $\hat{A}$-genus of T^*X on the Chern character of the principal symbol of D.

Another example is as follows. Let $A = C^*(\mathcal{F})$, where $\mathcal{F}$ is a foliation of a smooth manifold and D is a transversally elliptic operator on X.

Suppose that a $K_0(A)$ class is represented by a projection $p \in \mathcal{A}$, where $\mathcal{A}$ is a subalgebra of A closed under holomorphic functional calculus, so that the inclusion $\mathcal{A} \subset A$ induces an isomorphism on K-theory. For appropriately chosen $\mathcal{A}$, the fact that D is transversally elliptic implies that the operator pDp is Fredholm on the range of p and the index theorem identifies the integer Ind (pDp) with a pairing of a certain cyclic cocycle on $\mathcal{A}$ with the Chern character of p in the periodic cyclic complex of $\mathcal{A}$. For a special class of hypo-elliptic operators, see, e.g. [6].

Suppose again that X is a smooth manifold. One can consider a natural class of operators of the form $D = \sum_{\gamma \in \Gamma} P_\gamma \pi(\gamma)$, where Γ is a discrete group acting on $L^2(X)$ by Fourier integral operators of order zero and P_γ is a collection of pseudodifferential operators on X, all of them of the same (non-negative) order.

The principal symbol $\sigma_\Gamma(D)$ of such a D is an element of the C^*-algebra $C(S^*X) \rtimes_{max} \Gamma$, where S^*X is the cosphere bundle of X. Invertibility of $\sigma_\Gamma(D)$ implies that D is Fredholm and the index theorem in this case would express Ind $_\Gamma(D)$ in terms of some equivariant cohomology classes of X and an appropriate equivariant Chern character of $\sigma_\Gamma(D)$. For the case when Γ acts by diffeomorphisms of X, see [21, 27].

The typical computation proceeds via a reduction of the class of operators D under consideration to an algebra of (complete) symbols, which can be thought of as a 'formal deformation' of algebra of functions on T^*X (cf. [19]).

Definition 1.1. A star product on a symplectic manifold (M, ω) is an associative $\mathbb{C}[\![\hbar]\!]$-linear product $\star$ on $C^\infty(M)[\![\hbar]\!]$ of the form

$$f \star g = fg + \frac{i\hbar}{2}\{f, g\} + \sum_{k \geqslant 2} \hbar^k P_k(f, g);$$

where $\{f, g\} := \omega(I_\omega(df), I_\omega(dg))$ is the canonical Poisson bracket induced by the symplectic structure, I_ω is the isomorphism of T^*M and TM induced by ω, and

https://doi.org/10.1017/S1474748019000380 Published online by Cambridge University Press

the P_k denote bidifferential operators. We also require that $f \star 1 = 1 \star f = f$ for all $f \in C^\infty(M)[[\hbar]]$. We use $\mathcal{A}^\hbar(M)$ to denote the algebra $(C^\infty(M)[[\hbar]], \star)$. The ideal $\mathcal{A}_c^\hbar(M)$ in $\mathcal{A}^\hbar(M)$, consisting of power series of the form $\sum_k \hbar^k f_k$, where f_k are compactly supported, has a unique (up to a normalization) trace Tr with values in $\mathbb{C}[\hbar^{-1}, \hbar]]$ (see e.g. [1, 11]).

It is not difficult to see that the index computations reduce to the computation of the pairing of the trace (or some other cyclic cocycle) with the K-theory of the symbol algebra, which, in the example above, is identified with a crossed product $\mathcal{A}_c^\hbar(M) \rtimes \Gamma$. An example of this reduction is given in [19].

Since the product in $\mathcal{A}_c^\hbar(M)$ is local, i.e. given by bidifferential operators, the computation of the pairing of K-theory and cyclic cohomology of $\mathcal{A}_c^\hbar(M)$ reduces to a differential-geometric problem and the result is usually called the 'algebraic index theorem.'

Remark 1.2. Since cyclic periodic homology is invariant under (pro)nilpotent extensions, the result of the pairing depends only on the K-theory of $(\mathcal{A}_c^\hbar(M)/\hbar\mathcal{A}_c^\hbar(M)) \rtimes \Gamma$. In our example, $(\mathcal{A}_c^\hbar(M)/\hbar\mathcal{A}_c^\hbar(M)) \rtimes \Gamma$ is just $C_c^\infty(M) \rtimes \Gamma$.

1.1. The main result

Suppose that Γ is a discrete group acting by continuous automorphisms on a formal deformation $\mathcal{A}^\hbar(M)$ of a symplectic manifold M. Let $\mathcal{A}^\hbar(M) \rtimes \Gamma$ denote the algebraic crossed product associated to the given action of Γ. For a non-homogeneous group cocycle $\xi \in C^k(\Gamma, \mathbb{C})$, the formula below defines a cyclic k-cocycle Tr_ξ on $\mathcal{A}_c^\hbar(M) \rtimes \Gamma$.

$$\mathrm{Tr}_\xi(a_0\gamma_0 \otimes \cdots \otimes a_k\gamma_k) = \delta_{e,\gamma_0\gamma_1\ldots\gamma_k}\xi(\gamma_1, \ldots, \gamma_k)\mathrm{Tr}(a_0\gamma_0(a_1)\cdots(\gamma_0\gamma_1\ldots\gamma_{k-1})(a_k)). \quad (1)$$

The action of Γ on $\mathcal{A}^\hbar(M)$ induces (modulo $\hbar$) an action of Γ on M by symplectomorphisms. Let σ be the 'principal symbol' map:

$$\mathcal{A}^\hbar(M) \to \mathcal{A}^\hbar(M)/\hbar\mathcal{A}^\hbar(M) \simeq C^\infty(M).$$

It induces a homomorphism

$$\sigma: \mathcal{A}^\hbar(M) \rtimes \Gamma \longrightarrow C^\infty(M) \rtimes \Gamma,$$

still denoted by σ. Let

$$\Phi: H_\Gamma^\bullet(M) \longrightarrow HC_{per}^\bullet(C_c^\infty(M) \rtimes \Gamma)$$

be the canonical map (first constructed by Connes) induced by (A.2), where $H_\Gamma^\bullet(M)$ denotes the cohomology of the Borel construction $M \times_\Gamma E\Gamma$ and $C_c^\infty(M)$ denotes the algebra of compactly supported smooth functions on M.

The main result of this paper is the following.

Theorem 1.3. *Let $e, f \in M_N\left(\mathcal{A}^\hbar(M) \rtimes \Gamma\right)$ be idempotents such that the difference $e - f \in M_N\left(\mathcal{A}_c^\hbar(M) \rtimes \Gamma\right)$ is compactly supported. Let $[\xi] \in H^k(\Gamma, \mathbb{C})$ be a group cohomology class. Then $[e] - [f]$ is an element of $K_0(\mathcal{A}_c^\hbar(M) \rtimes \Gamma)$ and its pairing with the cyclic cocycle Tr_ξ is given by*

$$\langle \mathrm{Tr}_\xi, [e] - [f] \rangle = \left\langle \Phi\left(\hat{A}_\Gamma e^{\theta_\Gamma}[\xi]\right), [\sigma(e)] - [\sigma(f)] \right\rangle. \quad (2)$$

Here $\hat{A}_\Gamma \in H^\bullet_\Gamma(M)$ is the equivariant $\hat{A}$-genus of M (defined in § 5), $\theta_\Gamma \in H^\bullet_\Gamma(M)$ is the equivariant characteristic class of the deformation $\mathcal{A}^\hbar(M)$ (also defined in § 5).

In the case when the action of Γ is free and proper, we recover the algebraic version of Connes–Moscovici higher index theorem. The case of proper actions has been considered in [22, 23].

The above theorem gives an algebraic version of the results of [24–27], without the requirement that Γ acts by isometries. To recover the analytic version of the index theorem type results from [27] and [21] one can apply the methods of [19].

1.2. Structure of the article

Section 2 contains preliminary material, following mainly [2, 3] and [7]. It describes Fedosov's construction of the deformation quantization, Gelfand–Fuks construction, and a statement of the algebraic index theorem.

In general, given a group Γ acting on a deformation quantization algebra $\mathcal{A}^\hbar(M)$, there does not exist any invariant Fedosov connection. As a result, the Gelfand–Fuks map described in § 2 does not extend to this case. The rest of the paper is devoted to the construction of a Gelfand–Fuks map that avoids this problem and the proof of the main theorem.

Section 3 is devoted to a generalization of the Gelfand–Fuks construction to the equivariant case, where an analogue of the Fedosov construction and Gelfand–Fuks map are constructed on a simplicial manifold representing $M \times_\Gamma E\Gamma$.

Section 4 is devoted to a construction of a pairing of the periodic cyclic homology of the crossed product algebra with a certain Lie algebra cohomology appearing in § 2. The main tool for this construction is the Gelfand–Fuks maps from § 3.

Section 5 contains the proof of the main result.

In the appendix, we recall definitions and results about group (co)homology and cyclic (co)homology of crossed products needed in the main body of the paper.

2. Algebraic index theorem

2.1. Deformed formal geometry

Let us start in this section by recalling the adaptation of the framework of Gelfand–Kazhdan's formal geometry to deformation quantization described in [13, 14, 17, 20] and [3].

For the rest of this section, we fix a symplectic manifold (M, ω) of dimension $2d$ and its deformation quantization $\mathcal{A}^\hbar(M)$.

Notation 2.1. Let $m \in M$.

(1) $\mathcal{J}^\infty_m(M)$ denotes the space of ∞-jets at $m \in M$; $\mathcal{J}^\infty_m(M) := \varprojlim C^\infty(M)/(\mathcal{I}_m)^k$, where $\mathcal{I}_m$ is the ideal of smooth functions vanishing at m and $k \in \mathbb{N}$.

(2) Since the product in the algebra $\mathcal{A}^\hbar(M)$ is local, it defines an associative, $\mathbb{C}[[\hbar]]$-bilinear product $\star_m$ on $\mathcal{J}^\infty_m(M)[[\hbar]]$. Let $\mathcal{J}(\mathcal{A}^\hbar)_m$ denote the algebra $(\mathcal{J}^\infty_m(M)[[\hbar]], \star_m)$.

We also introduce the symbol map:

$$\hat{\sigma}_m \colon \mathcal{J}(\mathcal{A}^\hbar)_m \to \mathcal{J}(\mathcal{A}^\hbar)_m / \hbar \mathcal{J}(\mathcal{A}^\hbar)_m = \mathcal{J}^\infty_m(M).$$

Let $\mathcal{J}(\mathcal{A}^\hbar)_M$ denote the sheaf of jets of $\mathcal{A}^\hbar(M)$ (cf. [20]).

Example 2.2. Consider $M = \mathbb{R}^{2d}$. Moyal–Weyl deformation $\mathcal{A}^\hbar(\mathbb{R}^{2d})$ is the product on $C^\infty(\mathbb{R}^{2d})[\![\hbar]\!]$ given by the formula

$$(f \star g)(\xi, x) = \exp\left(\frac{i\hbar}{2}\sum_{i=1}^{d}(\partial_{\xi^i}\partial_{y^i} - \partial_{\eta^i}\partial_{x^i})\right) f(\xi, x) g(\eta, y)\Big|_{\substack{\xi^i=\eta^i \\ x^i=y^i}}. \tag{3}$$

It is easy to see that this product is equivariant with respect to the action of symplectic group $Sp(2d, \mathbb{R})$.

Let now V be a real symplectic vector space. Let $\mathcal{S}(V^*)$ denote the complexified symmetric algebra of V^*. Choice of symplectic isomorphism of V with $\mathbb{R}^{2d}$ endows $\mathcal{S}(V^*)[\hbar]$ with the product by formula (3). Since Moyal–Weyl product is $Sp(2d, \mathbb{R})$-equivariant the resulting Moyal–Weyl product on $\mathcal{S}(V^*)[\hbar]$ does not depend on the choice of symplectomorphism $V \cong \mathbb{R}^{2d}$. Let $F_\bullet$ denote a filtration on $\mathcal{S}(V^*)[\hbar]$ defined by

$$F_k := \bigoplus_{i+j/2 \geqslant k} S^i(V^*)\hbar^j.$$

Moyal–Weyl product is continuous with respect to the topology on $\mathcal{S}(V^*)[\hbar]$ induced by the filtration $F_\bullet$, and hence extends to the completion with respect to this filtration.

Definition 2.3. For a real symplectic vector space (V, ω) Weyl algebra is the $\mathbb{W}(V)$ is the completion of $\mathcal{S}(V^*)[\hbar]$ with respect to the filtration $F_\bullet$.

We write simply $\mathbb{W}$ for $\mathbb{W}(\mathbb{R}^{2d})$ by and denote by $\hat{x}^k, \hat{\xi^k}$ the elements of $\mathbb{W}$ corresponding to x^k, ξ^k – the standard Darboux coordinates on $\mathbb{R}^{2d}$.

Example 2.4. The algebra $\mathcal{J}(\mathcal{A}^\hbar\left(\mathbb{R}^{2d}\right))_0$ of jets at the origin of Moyal–Weyl deformation of $\mathbb{R}^{2d}$ is canonically isomorphic to $\mathbb{W}$.

For a vector space V we have a filtration on $S(V^*)$ defined by $F_k := \oplus_{i \geqslant k} S^i(V^*)$. $\mathbb{O}$ denotes the corresponding completion of $S(V^*)$, and $\mathbb{O} := \mathbb{O}(\mathbb{R}^{2d})$.

The assignment $V \mapsto \mathbb{W}(V)$ is functorial with respect to symplectic isomorphisms. Therefore, we can define a sheaf $\mathbb{W}_M := \mathbb{W}(TM)$ of Weyl algebra of cotangent bundle.

https://doi.org/10.1017/S1474748019000380 Published online by Cambridge University Press

Notation 2.5. Let $\widehat{G} := \mathrm{Aut}(\mathbb{W})$ denote the group of continuous $\mathbb{C}[\![\hbar]\!]$-linear automorphisms of $\mathbb{W}$. We let $\mathfrak{g} = \mathrm{Der}(\mathbb{W})$ denote the Lie algebra of continuous $\mathbb{C}[\![\hbar]\!]$-linear derivations of $\mathbb{W}$.

Let us denote by $\mathfrak{g}_{\geqslant k}$ the set of all $D \in \mathfrak{g}$ satisfying

$$D(F_l\mathbb{W}) \subset F_{l+k}\mathbb{W} \quad \text{for every } l.$$

This defines a filtration on $\mathfrak{g}$ compatible with Lie algebra structure, i.e. $[\mathfrak{g}_{\geqslant k}, \mathfrak{g}_{\geqslant l}] \subset \mathfrak{g}_{\geqslant (k+l)}$.

Lemma 2.6. *The map*

$$\mathbb{W} \ni f \to \frac{1}{\hbar} ad\, f \in \mathfrak{g}$$

is surjective.

Notation 2.7. Let $\tilde{\mathfrak{g}} = \frac{1}{\hbar}\mathbb{W}$ with a Lie algebra structure given by commutator.

Note that $\tilde{\mathfrak{g}}$ is a central extension of $\mathfrak{g}$. The corresponding short exact sequence has the form

$$0 \longrightarrow \frac{1}{\hbar}\mathbb{C}[\![\hbar]\!] \longrightarrow \tilde{\mathfrak{g}} \xrightarrow{\text{ad}} \mathfrak{g} \longrightarrow 0, \tag{4}$$

where ad $\frac{1}{\hbar} f(g) = \frac{1}{\hbar}[f, g]$.

The extension (4) splits over $\mathfrak{sp}(2d, \mathbb{C})$ and, moreover, the corresponding inclusion

$$\mathfrak{sp}(2d, \mathbb{C}) \hookrightarrow \tilde{\mathfrak{g}}$$

integrates to the action of $\mathrm{Sp}(2d, \mathbb{C})$. The Lie subalgebra $\mathfrak{sp}(2d, \mathbb{R}) \subset \mathfrak{sp}(2d, \mathbb{C})$ gets represented, in the standard basis, by elements of $\tilde{\mathfrak{g}}$ given by

$$\frac{1}{\hbar}\hat{x}^k\hat{x}^j, \quad \frac{-1}{\hbar}\hat{\xi}^k\hat{\xi}^j \quad \text{and} \quad \frac{1}{\hbar}\hat{x}^k\hat{\xi}^j, \quad \text{where } k, j = 1, 2, \ldots, d.$$

Lemma 2.8. *The Lie algebra $\mathfrak{g}_{\geqslant 0}$ has the structure of a semi-direct product*

$$\mathfrak{g}_{\geqslant 0} = \mathfrak{g}_{\geqslant 1} \rtimes \mathfrak{sp}(2d, \mathbb{C}).$$

The group $\widehat{G}$ of automorphisms of $\mathbb{W}$ has a structure of a pro-finite-dimensional Lie group with the pro-finite-dimensional Lie algebra $\mathfrak{g}_{\geqslant 0}$. As such, $\widehat{G}$ has the structure of semi-direct product

$$\widehat{G} \simeq \widehat{G}_1 \rtimes \mathrm{Sp}(2d, \mathbb{C}),$$

where $\widehat{G}_1 = \exp \mathfrak{g}_{\geqslant 1}$ is pro-unipotent and contractible.

Definition 2.9. Set

$$\widetilde{M} = \Big\{(m, \varphi_m) \mid m \in M, \varphi_m \colon \mathcal{J}(\mathcal{A}^\hbar)_m \xrightarrow{\sim} \mathbb{W}\Big\}.$$

Then $\widetilde{M}$ has a natural structure of a pro-finite-dimensional manifold and, moreover, a structure of $\widehat{G}$-principal bundle over M; see [17] for details.

Theorem 2.10 [17]. *The tangent bundle of $\widetilde{M}$ is isomorphic to the trivial bundle $\widetilde{M} \times \mathfrak{g}$ and there exists a trivialization given by a $\mathfrak{g}$-valued one-form $\omega_\hbar \in \Omega^1(\widetilde{M}) \otimes \mathfrak{g}$ satisfying the Maurer–Cartan equation*

$$d\omega_\hbar + \tfrac{1}{2}[\omega_\hbar, \omega_\hbar] = 0.$$

For later use let us introduce a slight modification of the above construction.

Let $\varphi\colon \mathcal{J}(\mathcal{A}^\hbar)_m \longrightarrow \mathbb{W}$ be an isomorphism. Since every automorphism of $\mathbb{W}$ preserves the filtration on $\mathbb{W}$, we obtain a canonical (i.e. independent of φ) filtration on $\mathcal{J}(\mathcal{A}^\hbar)_m$:

$$F_k \mathcal{J}(\mathcal{A}^\hbar)_m := \varphi^{-1}(F_k(\mathbb{W})).$$

With this filtration $\mathrm{Gr}^1(\mathcal{J}(\mathcal{A}^\hbar)_m) \cong T_m^* M \otimes \mathbb{C}$. Hence the map $d\varphi_m := \mathrm{Gr}\varphi\colon \mathrm{Gr}^1(\mathcal{J}(\mathcal{A}^\hbar)_m) \to \mathrm{Gr}^1\mathbb{W}$ is complex-linear symplectic isomorphism $T_m^* M \otimes \mathbb{C} \to \mathbb{C}^{2d}$.

Definition 2.11. Let $\widehat{G}_r = \widehat{G}_1 \rtimes \mathrm{Sp}(2d, \mathbb{R})$. We use $\widetilde{M}_r$ to denote the $\widehat{G}_r$-principal subbundle of $\widetilde{M}$ consisting of the isomorphisms

$$\varphi_m\colon \mathcal{J}(\mathcal{A}^\hbar)_m \xrightarrow{\sim} \mathbb{W}$$

such that $d\varphi_m$ is a complexification of a $\mathbb{R}$-linear symplectic isomorphism $T_m^* M \to \mathbb{R}^{2d}$.

Note that the projection $\widetilde{M}_r \to M$ factors through $\mathcal{F}_M$, the bundle of symplectic frames in TM, equivariantly with respect to the action of $\mathrm{Sp}(2d, \mathbb{R}) \subset \widehat{G}_r$:

$$\begin{array}{ccc} \widehat{G}_r & \longrightarrow & \widetilde{M}_r \\ & & \downarrow \\ \mathrm{Sp}(2d,\mathbb{R}) & \longrightarrow & \mathcal{F}_M \\ & & \downarrow \\ & & M. \end{array}$$

We use the same symbol for $\omega_\hbar$ and its pull-back from $\widetilde{M}$ to $\widetilde{M}_r$.

2.2. Fedosov connection and Gelfand–Fuks construction

Recall that $\widehat{G}_1$ is contractible; thus, in particular, the principal $\widehat{G}_1$-bundle $\widetilde{M}_r \to \mathcal{F}_M$ admits a section F, which can chosen to be $\mathrm{Sp}(2d, \mathbb{R})$-equivariant. Such an F is not unique; if F' is another choice then $F' = \exp(x)F$ for a unique $x \in \Gamma(M, (\mathfrak{g}_{\geqslant 1})_M)$.

Equivalently one can describe such an F as follows.

Let us denote by $\mathrm{Hom}_1(\mathcal{J}(\mathcal{A}^\hbar)_M, \mathbb{W}_M)$ the sheaf of isomorphisms of sheaves of algebras $\mathcal{J}(\mathcal{A}^\hbar)_M$ and $\mathbb{W}_M$ which induce identity map $\mathrm{Gr}^1\mathcal{J}(\mathcal{A}^\hbar)_M \to \mathrm{Gr}^1\mathbb{W}_M$. Then F can be identified with a global section of $\mathrm{Hom}_1(\mathcal{J}(\mathcal{A}^\hbar)_M, \mathbb{W}_M)$.

Set

$$A_F = F^* \omega_\hbar \in \Omega^1(\mathcal{F}_M; \mathfrak{g}).$$

Since A_F is $\mathrm{Sp}(2d, \mathbb{R})$-equivariant and satisfies the Maurer–Cartan equation,

$$d + A_F \tag{5}$$

reduces to a flat $\mathfrak{g}$-valued connection ∇_F on M, called the *Fedosov connection*.

https://doi.org/10.1017/S1474748019000380 Published online by Cambridge University Press

Example 2.12. Consider the case of $M = \mathbb{R}^{2d}$ with the standard symplectic structure and let $\mathcal{A}^\hbar(\mathbb{R}^{2d})$ denote the Moyal–Weyl deformation. Then both $\mathcal{F}_{\mathbb{R}^{2d}}$ and $\widetilde{\mathbb{R}^{2d}}$ are trivial bundles. The trivialization is given by the standard (Darboux) coordinates $x^1, \ldots, x^d, \xi^1, \ldots, \xi^d$. So we see, using the construction of $\omega_\hbar$ in [17], that $A_F(X) = \frac{1}{i\hbar}[\omega(X, -), -]$, where we consider $\omega(X, -) \in \Gamma(T^*M) \hookrightarrow \Gamma(M; \mathbb{W})$. Let us denote the generators of $\mathbb{W}$ corresponding to the standard coordinates by $\hat{x}^i$ and $\hat{\xi}^i$, then we see that

$$A_F(\partial_{x^i}) = -\partial_{\hat{x}^i} \quad \text{and} \quad A_F(\partial_{\xi^i}) = -\partial_{\hat{\xi}^i}.$$

A Harish-Chandra pair $(\mathfrak{k}, H)$ consists of a Lie algebra $\mathfrak{k}$ and Lie group H together with inclusion of the Lie algebra $\mathfrak{h}$ of H as a Lie subalgebra of $\mathfrak{k}$ satisfying the following condition: adjoint action of $\mathfrak{h}$ on $\mathfrak{k}$ integrates to an action of H on $\mathfrak{k}$ by automorphisms. A $(\mathfrak{k}, H)$ module $\mathbb{M}$ is a $\mathfrak{k}$ module $\mathbb{M}$ such that the action of $\mathfrak{h}$ on $\mathbb{M}$ integrates to a compatible action of the Lie group H. If an $(\mathfrak{k}, H)$-module is equipped with a compatible grading and differential, we call it an $(\mathfrak{k}, H)$-cochain complex.

Definition 2.13. We set

$$\Omega^\bullet(M; \mathbb{L}) := \left\{\eta \in (\Omega^\bullet(\mathcal{F}_M) \otimes \mathbb{L})^{\mathrm{Sp}(2d)} \mid \iota_X(\eta) = 0 \ \forall X \in \mathfrak{sp}(2d)\right\}$$

for a $(\mathfrak{g}, \mathrm{Sp}(2d, \mathbb{R}))$-module $\mathbb{L}$. Here the superscript refers to taking invariants for the diagonal action and ι_X stands for contraction with the vertical vector fields on the principal bundle $\mathcal{F}_M$. $(\Omega^\bullet(M; \mathbb{L}), \nabla_F)$ is a cochain complex. The same construction with a $(\mathfrak{g}, \mathrm{Sp}(2d, \mathbb{R}))$-cochain complex $(\mathbb{L}^\bullet, \delta)$ yields the double complex $(\Omega^\bullet(M; \mathbb{L}^\bullet), \nabla_F, \delta)$.

Remark 2.14. $\Omega^0(M; \mathbb{L})$ is the space of sections of a bundle which we denote by $\mathcal{L}$, whose fibers are isomorphic to $\mathbb{L}$. $(\Omega^\bullet(M; \mathbb{L}), \nabla_F)$ is the de Rham complex of differential forms with coefficients in $\mathcal{L}$.

Given a Harish-Chandra pair $(\mathfrak{k}, H)$ and a $(\mathfrak{k}, H)$-cochain complex $(\mathbb{M}^\bullet, \delta)$ we denote the bicomplex of continuous Chevalley–Eilenberg cochains with values in $\mathbb{M}^\bullet$ by

$$(C^\bullet_{Lie}(\mathfrak{k}, \mathfrak{h}; \mathbb{M}^\bullet), \partial_{Lie}, \delta).$$

Definition 2.15. Suppose that $(\mathbb{L}^\bullet, \delta)$ is a $(\mathfrak{g}, \mathrm{Sp}(2d))$-cochain complex. The Gelfand–Fuks map $C^\bullet_{Lie}(\mathfrak{g}, Sp(2d); \mathbb{L}^\bullet) \longrightarrow \Omega^\bullet(M; \mathbb{L}^\bullet)$ is defined as follows. Given $\varphi \in C^n_{Lie}(\mathfrak{g}, Sp(2d); \mathbb{L}^\bullet)$ and vector fields $\{X_i\}_{i=1,\ldots,n}$ on $\mathcal{F}_M$ set

$$GF(\varphi)(X_1, \ldots, X_n) = \varphi(A_F(X_1), \ldots, A_F(X_n)).$$

The following result is well known (see e.g. [3] and references therein).

Theorem 2.16. *The map GF is a morphism of double complexes*

$$GF\colon (C^\bullet_{Lie}(\mathfrak{g}, Sp(2d); \mathbb{L}^\bullet), \partial_{Lie}, \delta) \longrightarrow (\Omega^\bullet(M; \mathbb{L}^\bullet), \nabla_F, \delta).$$

This morphism is independent of F up to homotopy. More precisely, let $F' = \exp(x)F \in \Gamma(M, \mathrm{Hom}_1(\mathcal{J}(\mathcal{A}^\hbar)_M, \mathbb{W}_M))$, $x \in \Gamma(M, (\mathfrak{g}_{\geqslant 1})_M)$. Denote Fedosov connection induced by F' by $\nabla_{F'}$ and corresponding Gelfand–Fuks map GF'. Notice also that we have an isomorphism of complexes $\exp(x)\colon (\Omega^\bullet(M;\mathbb{L}^\bullet), \nabla_F, \delta) \to (\Omega^\bullet(M;\mathbb{L}^\bullet), \nabla_{F'}, \delta)$.

Proposition 2.17. *The morphisms GF and $\exp(-x)\circ GF'$ are chain homotopic.*

Proof. One verifies by direct calculation that

$$H_n(\varphi) := (-1)^n n \int_0^1 \exp(-tx)\varphi(x, A_{\exp(tx)F}(X_1), \ldots, A_{\exp(tx)F}(X_{n-1}))\, dt$$

satisfies the identity

$$GF - \exp(-x)\circ GF' = H_{n+1}\circ(\partial_{Lie} + \delta) + (\nabla_{F'} + \delta)\circ H_n$$

and hence provides the desired homotopy. □

Example 2.18.

(1) Suppose that $\mathbb{L} = \mathbb{C}$. The associated complex is just the de Rham complex of M.

(2) Suppose that $\mathbb{L} = \mathbb{W}$. The associated bundle is $\mathbb{W}(TM)$. Moreover, the choice of F as in the beginning of § 2.2 determines a quasi-isomorphism

$$J_F^\infty\colon \mathcal{A}^\hbar(M) \longrightarrow (\Omega^\bullet(M;\mathbb{W}), \nabla_F).$$

(3) Suppose that $\mathbb{L} = (CC_\bullet^{per}(\mathbb{W}), b + uB)$, the cyclic periodic complex of $\mathbb{W}$. The complex $(\Omega^\bullet(M; CC_\bullet^{per}(\mathbb{W})), \nabla_F + b + uB)$ is a resolution of the periodic cyclic complex of jets of $\mathcal{A}^\hbar(M)$ at the diagonal (see e.g. [18]).

Example 2.19.

(1) Let $\hat{\theta} \in C^2_{Lie}(\mathfrak{g}, Sp(2d);\mathbb{C})$ denote a representative of the class of the extension (4). The class of $\theta = GF(\hat{\theta})$ belongs to $\frac{\omega}{i\hbar} + \mathrm{H}^2(M;\mathbb{C})[\![\hbar]\!]$ and classifies the deformations of M up to gauge equivalence (see e.g. [20]).

(2) The action of $\mathfrak{sp}(2d)$ on $\mathfrak{g}$ is semi-simple and $\mathfrak{sp}(2d)$ admits a $\mathrm{Sp}(2d,\mathbb{R})$-equivariant complement. Let Π be the implied $\mathrm{Sp}(2d,\mathbb{R})$-equivariant projection $\mathfrak{g} \to \mathfrak{sp}(2d)$. Let $R\colon \mathfrak{g}\wedge\mathfrak{g} \longrightarrow \mathfrak{sp}(2d)$ to be the two-cocycle

$$R(X,Y) = [\Pi(X), \Pi(Y)] - \Pi([X,Y]).$$

The Chern–Weil homomorphism is the map

$$CW\colon S^\bullet(\mathfrak{sp}(2d)^*)^{\mathrm{Sp}(2d)} \longrightarrow \mathbb{H}^{2\bullet}_{Lie}(\mathfrak{g}, Sp(2d))$$

given on the level of cochains by

$$CW(P)(X_1,\ldots,X_n) = P(R(X_1,X_2),\ldots,R(X_{n-1},X_n)).$$

A particular choice of P is the $\hat{A}$-power series

$$\hat{A}_f = CW\left(\det\left(\frac{ad(\frac{X}{2})}{\exp(ad(\frac{X}{2})) - \exp(ad(-\frac{X}{2}))}\right)\right).$$

With this choice $GF(\hat{A}_f) = \hat{A}(TM)$, the $\hat{A}$-genus of the tangent bundle of M.

https://doi.org/10.1017/S1474748019000380 Published online by Cambridge University Press

2.3. Algebraic index theorem in Lie algebra cohomology

We use the notations from [3]. In particular, $(\hat{\Omega}^\bullet, \hat{d})$ denotes the formal de Rham complex in $2d$ dimensions, $\mathbb{O} := \hat{\Omega}^0$. $C_\bullet(\mathbb{W}[\hbar^{-1}])$, $CC_\bullet^{per}(\mathbb{W}[\hbar^{-1}])$ denotes the Hochschild complex and (periodic) cyclic complexes of $\mathbb{W}[\hbar^{-1}]$ viewed as an algebra over $\mathbb{C}[\hbar^{-1}, \hbar]]$, while $CC_\bullet^{per}(\mathbb{W})$ denotes the cyclic complex of $\mathbb{W}$ viewed as an algebra over $\mathbb{C}$. The convention for shifts of complexes is as follows: $(V^\bullet[k])^p = V^{p+k}$.

Theorem 2.20 [3, 4]. (1) *There exists a unique (up to homotopy) quasi-isomorphism*

$$\mu^\hbar : (C_\bullet^{Hoch}(\mathbb{W}[\hbar^{-1}]), b) \longrightarrow \left(\hat{\Omega}^{-\bullet}[\hbar^{-1}, \hbar]][2d], \hat{d}\right),$$

which maps the Hochschild $2d$-chain

$$\varphi = 1 \otimes \mathrm{Alt}\left(\hat{\xi}_1 \otimes \hat{x}_1 \otimes \hat{\xi}_2 \otimes \hat{x}_2 \otimes \cdots \otimes \hat{\xi}_d \otimes \hat{x}_d\right),$$

where $\mathrm{Alt}(z_1 \otimes \cdots \otimes z_n) := \sum_{\sigma \in \Sigma_n} (-1)^{sgn\sigma} z_{\sigma(1)} \otimes \cdots \otimes z_{\sigma(n)}$, *to the 0-form 1. $\mu^\hbar$ extends to a quasi-isomorphism*

$$\mu^\hbar : (CC_\bullet^{per}(\mathbb{W}[\hbar^{-1}]), b + uB) \longrightarrow (\hat{\Omega}^{-\bullet}[\hbar^{-1}, \hbar]][u^{-1}, u]][2d], \hat{d}).$$

(2) *The principal symbol map $\sigma : \mathbb{W} \to \mathbb{W}/\hbar\mathbb{W} \simeq \mathbb{O}$ together with the Hochschild–Kostant–Rosenberg map HKR given by*

$$f_0 \otimes f_1 \otimes \cdots \otimes f_n \mapsto \frac{1}{n!} f_0 \hat{d} f_1 \wedge \hat{d} f_2 \wedge \cdots \wedge \hat{d} f_n$$

induces a $\mathbb{C}$-linear quasi-isomorphism

$$\hat{\mu} : CC_\bullet^{per}(\mathbb{W}) \longrightarrow \left(\hat{\Omega}^\bullet[u^{-1}, u]], u\hat{d}\right).$$

(3) *The map of complexes $J : (\hat{\Omega}^\bullet[u^{-1}, u]], u\hat{d}) \to (\hat{\Omega}^{-\bullet}[\hbar^{-1}, \hbar]][u^{-1}, u]][2d], \hat{d})$ given by*

$$f_0 \hat{d} f_1 \wedge \cdots \wedge \hat{d} f_n \mapsto u^{-d-n} f_0 \hat{d} f_1 \wedge \cdots \wedge \hat{d} f_n$$

makes the following diagram commute up to homotopy.

$$\begin{array}{ccccc} CC_\bullet^{per}(\mathbb{W}) & \xrightarrow{\iota} & CC_\bullet^{per}(\mathbb{W}[\hbar^{-1}]) & \xrightarrow{\mu^\hbar} & (\hat{\Omega}^{-\bullet}[\hbar^{-1}, \hbar]][u^{-1}, u]][2d], \hat{d}) \\ \downarrow{\scriptstyle \sigma} & & & & \uparrow{\scriptstyle J} \\ CC_\bullet^{per}(\mathbb{O}) & & \xrightarrow[HKR]{\hspace{6em}} & & \left(\hat{\Omega}^\bullet[u^{-1}, u]], u\hat{d}\right) \end{array} \quad (6)$$

Here the complex $CC_\bullet^{per}(\mathbb{W})$ at the leftmost top corner is that of $\mathbb{W}$ as an algebra over $\mathbb{C}$.

Remark 2.21. One can in fact extend the above $\mathbb{C}$-linear 'principal symbol map'

$$\sigma : CC_\bullet^{per}(\mathbb{W}) \to CC_\bullet^{per}(\mathbb{O})$$

to a $\mathbb{C}[[\hbar]]$-linear map of complexes $CC_\bullet^{per}(\mathbb{W}) \to CC_\bullet^{per}(\mathbb{O}[[\hbar]])$, see [3].

The action of $\mathfrak{g}$ by derivations on $\mathbb{W}$ induces an action on the complex $CC^{per}_{\bullet}(\mathbb{W})$ and we give it the corresponding $(\mathfrak{g}, \mathrm{Sp}(2d, \mathbb{R}))$-module structure. The action of $\mathfrak{g}$ on $\mathbb{W}$ taken modulo $\hbar\mathbb{W}$, induces an action of $\mathfrak{g}$ (by Hamiltonian vector fields) on $(\hat{\Omega}^{-\bullet}, \hat{d})$ and hence on $(\hat{\Omega}^{-\bullet}[\hbar^{-1}, \hbar]][u^{-1}, u]][2d], \hat{d})$. We give $(\hat{\Omega}^{-\bullet}[\hbar^{-1}, \hbar]][u^{-1}, u]][2d], \hat{d})$ the induced structure of $(\mathfrak{g}, \mathrm{Sp}(2d, \mathbb{R}))$-module.

Notation 2.22. Let

$$\mathbb{L}^{\bullet} := \mathrm{Hom}^{-\bullet}(CC^{per}_{\bullet}(\mathbb{W}), \hat{\Omega}^{-\bullet}[\hbar^{-1}, \hbar]][u^{-1}, u]][2d]).$$

The complex $\mathbb{L}$ inherits the $(\mathfrak{g}, \mathrm{Sp}(2d, \mathbb{R}))$-module structure from the actions of $\mathfrak{g}$ described above.

The composition $J \circ HKR \circ \hat{\sigma}$ is equivariant with respect to the action of $\mathfrak{g}$, hence the following definition makes sense.

Definition 2.23. $[\hat{\tau}_t]$ is the cohomology class in the hypercohomology $\mathbb{H}^0_{Lie}(\mathfrak{g}, Sp(2d); \mathbb{L})$ given by the cochain

$$\hat{\tau}_t := J \circ HKR \circ \sigma \in C^0_{Lie}(\mathfrak{g}, \mathrm{Sp}(2d, \mathbb{R}); \mathbb{L}^0). \tag{7}$$

Lemma 2.24. *The cochain*

$$\mu^{\hbar} \circ \iota \in C^0_{Lie}(\mathfrak{g}, Sp(2d, \mathbb{R}); \mathbb{L}^0)$$

extends to a cocycle $\hat{\tau}_a$ *in the complex*

$$(C^{\bullet}_{Lie}(\mathfrak{g}, Sp(2d, \mathbb{R}); \mathbb{L}^{\bullet}), \partial_{Lie} + \partial_{\mathbb{L}}).$$

The cohomology class $[\hat{\tau}_a]$ *of this cocycle is independent of the choice of the extension.*

For a proof of the next result see e.g. [3].

Theorem 2.25. *We have*

$$[\hat{\tau}_a] = \sum_{p \geqslant 0} \left[\hat{A}_f e^{\hat{\theta}}\right]_{2p} u^p [\hat{\tau}_t],$$

where $[\hat{A}_f e^{\hat{\theta}}]_{2p}$ *is the component of degree* $2p$ *of the cohomology class of* $\hat{A}_f e^{\hat{\theta}}$.

2.4. Algebraic index theorem

An example of an application of the above is the algebraic index theorem for a formal deformation of a symplectic manifold M. Note that we can view $\mathcal{A}^{\hbar}$ as a complex concentrated in degree 0 and with trivial differential. Recall (see Remark 2.18) that we have a quasi-isomorphism

$$J^{\infty}_F : \mathcal{A}^{\hbar} \longrightarrow (\Omega^{\bullet}(M; \mathbb{W}), \nabla_F).$$

It induces a quasi-isomorphism

$$(CC^{per}_{\bullet}(\mathcal{A}^{\hbar}), b + uB) \longrightarrow (\Omega^{\bullet}(M; CC^{per}_{\bullet}(\mathbb{W})), \nabla_F + b + uB),$$

also denoted by J^{∞}_F. For future reference let us record the following observation.

https://doi.org/10.1017/S1474748019000380 Published online by Cambridge University Press

Lemma 2.26. *The quasi-isomorphic inclusion* $\mathbb{C}[\hbar^{-1},\hbar]][u^{-1},u]] \hookrightarrow \hat{\Omega}^{-\bullet}[\hbar^{-1},\hbar]][u^{-1},u]]$ *induces a quasi-isomorphism*

$$\iota\colon (\Omega^\bullet(M)[\hbar^{-1},\hbar]][u^{-1},u]][2d], d_{\mathrm{dR}}) \longrightarrow \Big(\Omega^\bullet(M;\hat{\Omega}^{-\bullet}[\hbar^{-1},\hbar]][u^{-1},u]][2d]), \nabla_F+\hat{d}\Big).$$

From now on fix

$$T_0\colon \Big(\Omega^\bullet(M;\hat{\Omega}^{-\bullet}[\hbar^{-1},\hbar]][u^{-1},u]][2d]), \nabla_F+\hat{d}\Big) \longrightarrow (\Omega^\bullet(M)[\hbar^{-1},\hbar]][u^{-1},u]][2d], d_{\mathrm{dR}}),$$

such that $T_0\circ\iota = \mathbf{id}$ and $\iota\circ T_0$ is chain homotopic to $\mathbf{id}$.

For $Q\in\Omega^\bullet(M;\mathbb{L}^\bullet)$ of total degree zero let C_Q denote the composition

$$CC_0^{per}(\mathcal{A}^\hbar) \longrightarrow \Omega^\bullet(M;CC_\bullet^{per}(\mathbb{W})) \xrightarrow{Q} \Omega^\bullet(M;\hat{\Omega}^{-*}[\hbar^{-1},\hbar]][u^{-1},u]][2d]) \xrightarrow{T_0}$$
$$\Omega^{\bullet-*}(M;\mathbb{C})[\hbar^{-1},\hbar]][u^{-1},u]][2d] \xrightarrow{u^{-d}\int_M} \mathbb{C}[\hbar^{-1},\hbar]].$$

Clearly C_Q is a periodic cyclic cocycle if Q is a cocycle. We apply this construction to the two cocycles $\hat{\tau}_t$ and $\hat{\tau}_a$.

Let us start with $C_{\hat{\tau}_t}$. Tracing the definitions we get the following result.

Proposition 2.27. $C_{\hat{\tau}_t}$ *is given by*

$$u^n w_0\otimes\cdots\otimes w_{2n} \mapsto \frac{u^{n-d}}{(2n)!}\int_M \sigma(w_0)d\sigma(w_1)\wedge\cdots\wedge d\sigma(w_{2n}).$$

To get the corresponding result for $C_{\hat{\tau}_a}$ recall first that the algebra $\mathcal{A}^\hbar(M)$ has a unique $\mathbb{C}[[\hbar]]$-linear trace, up to a normalization factor. This factor can be fixed as follows. Locally any deformation of a symplectic manifold is isomorphic to the Weyl deformation. Let U be such a coordinate chart and let $\varphi\colon\mathcal{A}^\hbar(U)\to\mathcal{A}^\hbar(\mathbb{R}^{2d})$ be an isomorphism. Then the trace Tr is normalized by requiring that for any $f\in\mathcal{A}_c^\hbar(U)$ we have

$$\mathrm{Tr}(f) = \frac{1}{(i\hbar)^d}\int_{\mathbb{R}^{2d}}\varphi(f)\frac{\omega^d}{d!}.$$

Proposition 2.28. $C_{\hat{\tau}_a}$ *coincides with* Tr.

Proof. First one checks that $C_{\hat{\tau}_a}$ is a 0-cocycle and therefore a trace. Hence it is a $\mathbb{C}[[\hbar]]$-multiple of Tr and it is sufficient to evaluate it on elements supported in a coordinate chart. Moreover, the fact that the Hochschild cohomology class of $C_{\hat{\tau}_a}$ is independent of the Fedosov connection implies that $C_{\hat{\tau}_a}$ is independent of it. Thus it is sufficient to verify the statement for $\mathbb{R}^{2d}$ with the standard Fedosov connection. Let $f\in\mathcal{A}_c^\hbar(\mathbb{R}^{2d})$, one checks that $J_F^\infty(f)\in\Omega^0(\mathbb{R}^{2d};C_0(\mathbb{W}))$ is cohomologous to the element $\frac{1}{(i\hbar)^d d!}f\varphi\omega^d\in\Omega_c^{2d}(\mathbb{R}^{2d};C_{2d}(\mathbb{W}))$ in $\Omega_c^\bullet(\mathbb{R}^{2d};C_{-\bullet}(\mathbb{W}))$. It follows that the $GF(\hat{\tau}_a)J_F^\infty(f)$ is cohomologous to $\frac{1}{(i\hbar)^d d!}f\varphi\omega^d$ (see the Theorem 2.20) and therefore

$$C_{\hat{\tau}_a}(f) = \mathrm{Tr}(f) = \frac{1}{(i\hbar)^d}\int_{\mathbb{R}^{2d}} f\frac{\omega^d}{d!}$$

and the statement follows. □

https://doi.org/10.1017/S1474748019000380 Published online by Cambridge University Press

Given above identifications of $C_{\hat{\tau}_a}$ and $C_{\hat{\tau}_t}$, the Theorem 2.25 implies the following result.

Theorem 2.29 (Algebraic Index Theorem). *Suppose $a \in CC_0^{per}\left(\mathcal{A}_c^{\hbar}\right)$ is a cycle, then*

$$\mathrm{Tr}(a) = u^{-d} \int_M \sum_{p \geqslant 0} HKR(\sigma(a)) \left(\hat{A}(T_{\mathbb{C}}M)e^{\theta}\right)_{2p} u^p.$$

3. Equivariant Gelfand–Fuks map

Suppose Γ is a discrete group acting by automorphisms on $\mathcal{A}^{\hbar}(M)$. This action, in particular, induces an action on (M, ω) by symplectomorphisms. Now suppose $(m, \varphi_m) \in \widetilde{M}_r$ and $\gamma \in \Gamma$, then let $\gamma(m, \varphi_m) = (\gamma(m), \varphi_m^{\gamma})$, here φ_m^{γ} is given by

$$\mathcal{J}(\mathcal{A}^{\hbar})_{\gamma(m)} \longrightarrow \mathcal{J}(\mathcal{A}^{\hbar})_m \longrightarrow \mathbb{W},$$

where the first arrow is given by the action of Γ on $\mathcal{A}^{\hbar}$ and the second arrow is given by φ_m. Note that the actions of $\widehat{G}_r$ and Γ on $\widetilde{M}_r$ commute.

We now extend constructions of the previous sections to a simplicial model of Borel construction $E\Gamma \times_{\Gamma} M$; we give the explicit description of this simplicial manifold below.

Assume that Γ is a discrete group acting on a manifold X. Set $X_k := X \times \Gamma^k$. Define the face maps $\partial_i^k \colon X_k \to X_{k-1}$ by

$$\partial_i^k(x, \gamma_1, \ldots, \gamma_k) = \begin{cases} (\gamma_1^{-1}(x), \gamma_2, \ldots, \gamma_k) & \text{if } i = 0 \\ (x, \gamma_1, \ldots, \gamma_i\gamma_{i+1}, \ldots, \gamma_k) & \text{if } 0 < i < k \\ (x, \gamma_1, \ldots, \gamma_{k-1}) & \text{if } i = k. \end{cases}$$

We denote the standard k-simplex by

$$\Delta^k := \left\{ (t_0, \ldots, t_k) \in \mathbb{R}_{\geqslant 0}^{k+1} \,\middle|\, \sum_{i=0}^{k} t_i = 1 \right\} \subset \mathbb{R}^{k+1}$$

and define $\epsilon_i^k \colon \Delta^{k-1} \to \Delta^k$ by

$$\epsilon_i^k(t_0, \ldots, t_{k-1}) = \begin{cases} (0, t_0, \ldots, t_{k-1}) & \text{if } i = 0 \\ (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{k-1}) & \text{if } 0 < i \leqslant k. \end{cases}$$

Definition 3.1 [9]. A *simplicial* form φ of degree p is a collection of differential forms $\varphi_k \in \Omega^p(\Delta^k \times X_k)$ for $k = 0, 1, \ldots$, satisfying

$$(\epsilon_i^k \times \mathrm{id})^* \varphi_k = (\mathrm{id} \times \partial_i^k)^* \varphi_{k-1} \in \Omega^p(\Delta^{k-1} \times X_k) \tag{8}$$

for $0 \leqslant i \leqslant k$ and any $k > 0$.

For $\varphi = \{\varphi_k\}$ is a compatible form, then $d\varphi := \{d\varphi_k\}$ is also a compatible form; for two compatible forms $\varphi = \{\varphi_k\}$ and $\psi = \{\psi_k\}$ their product $\varphi\psi := \{\varphi_k \wedge \psi_k\}$ is another compatible form. We denote the space of simplicial forms by $\Omega^{\bullet}(M \times_{\Gamma} E\Gamma)$.

https://doi.org/10.1017/S1474748019000380 Published online by Cambridge University Press

Theorem 3.2. *The following holds:*

$$H^\bullet(\Omega^\bullet(M \times_\Gamma E\Gamma), d) \simeq H^\bullet_\Gamma(M).$$

Here the right-hand side denotes the cohomology of $M \times_\Gamma E\Gamma$ with complex coefficients.

See for instance [9] for the proof.

More generally, let V be a Γ-equivariant bundle on X. Let $\pi_k \colon X_k \to X$ be the projection and let $V_k := \pi_k^* V$. Notice that we have canonical isomorphisms

$$(\partial_i^k)^* V_{k-1} \cong V_k. \tag{9}$$

Definition 3.3. Let V be a Γ-equivariant vector bundle. A V-valued de Rham simplicial form φ is a collection $\varphi_k \in \Omega^p(\Delta^k \times X_k; V_k)$, $k = 0, 1, \ldots$, satisfying the conditions (8), where we use the isomorphisms (9) to identify $(\partial_i^k)^* V_{k-1}$ with V_k.

We let $\Omega^\bullet(M \times_\Gamma E\Gamma; V)$ denote the space of V-valued simplicial forms.

For equivariant vector bundles V and W there is a product

$$\Omega^\bullet(M \times_\Gamma E\Gamma; V) \otimes \Omega^\bullet(M \times_\Gamma E\Gamma; W) \longrightarrow \Omega^\bullet(M \times_\Gamma E\Gamma; V \otimes W)$$

defined as for the scalar forms by $\varphi\psi := \{\varphi_k \wedge \psi_k\}$.

Assume that we have a collection of connections ∇_k on the bundles V_k satisfying the compatibility conditions

$$(\epsilon_i^k \times \mathrm{id})^* \nabla_k = (\mathrm{id} \times \partial_i^k)^* \nabla_{k-1}. \tag{10}$$

Given a simplicial form $\varphi = \{\varphi_k\}$, $\{\nabla_k \varphi_k\}$ is again a simplicial form, which we denote by $\nabla\varphi$.

Notation 3.4. Now let M be a symplectic manifold and Γ a discrete group acting by symplectomorphisms on M. We introduce the following notations:

$$P_\Gamma^k := \Delta^k \times (\mathcal{F}_M)_k = \Delta^k \times \mathcal{F}_M \times \Gamma^k,$$

and similarly

$$M_\Gamma^k := \Delta^k \times M_k = \Delta^k \times M \times \Gamma^k$$

and

$$\widetilde{M}_\Gamma^k := \Delta^k \times (M_r)_k = \Delta^k \times \widetilde{M}_r \times \Gamma^k.$$

Note that $P_\Gamma^k \to M_\Gamma^k$ is a principal $\mathrm{Sp}(2d)$-bundle, namely the pull-back of $\mathcal{F}_M \to M$ via the obvious projection. Similarly $\widetilde{M}_\Gamma^k$ is the pull-back of $\widetilde{M}_r \to M$. For a $(\mathfrak{g}, \mathrm{Sp}(2d))$-module $\mathbb{L}$ we define (cf. Definition 2.13)

$$\Omega^\bullet(M_\Gamma^k; \mathbb{L}) := \left\{\eta \in (\Omega^\bullet(P_\Gamma^k) \otimes \mathbb{L})^{\mathrm{Sp}(2d)} \mid \iota_X(\eta) = 0 \ \forall X \in \mathfrak{sp}(2d)\right\}.$$

We shall denote the $\widehat{G}_1$-principal bundle $\widetilde{M}_r \to \mathcal{F}_M$ by π_1, the $\widehat{G}_r$-principal bundle $\widetilde{M}_r \to M$ by π_r and the $\mathrm{Sp}(2d)$-principal bundle $\mathcal{F}_M \to M$ by π.

Let $f_0, f_1, \dots f_k \in \Gamma(M, \mathrm{Hom}_1(\mathcal{J}(\mathcal{A}^\hbar)_M, \mathbb{W}_M))$. We construct, following [8], $S_k(f_0, f_1, \dots, f_k) \in \Gamma(\Delta_k \times M, p^*\mathrm{Hom}_1(\mathcal{J}(\mathcal{A}^\hbar)_M, \mathbb{W}_M))$, where p is the projection $\Delta_k \times M \to M$. The construction is recursive (in k). For $k = 0$ set

$$S_0(f_0) = f_0.$$

Assume now that $k \geqslant 1$ and $S_{k-1}(f_0, f_1, \dots, f_{k-1})$ is constructed for every $f_0, f_1, \dots f_{k-1} \in \Gamma(M, \mathrm{Hom}_1(\mathcal{J}(\mathcal{A}^\hbar)_M, \mathbb{W}_M))$. For $(t_0, t_1, \dots t_{k-1}) \in \Delta^{k-1}$ let $x(t_0, t_1, \dots t_{k-1}) \in \Gamma(M, \mathfrak{g}_{\geqslant 1})$ be such that $(\exp x(t_0, t_1, \dots t_{k-1})) \cdot f_k = S_{k-1}(f_0, f_1, \dots, f_{k-1})|_{(t_0, t_1, \dots t_{k-1}) \times M}$. Define

$$S_k|_{(t_0, t_1, \dots t_k) \times M} := \begin{cases} f_k, & \text{if } t_k = 1 \\ \left(\exp(1 - t_k) x\left(\frac{t_0}{1-t_k}, \frac{t_1}{1-t_k}, \dots \frac{t_{k-1}}{1-t_k}\right)\right) f_k, & \text{otherwise.} \end{cases}$$

It is easy to see that

$$(\epsilon_i^k)^*(S_k(f_0, f_1, \dots, f_k)) = S_{k-1}(f_0, \dots, f_{i-1}, f_{i+1}, \dots, f_k). \tag{11}$$

Note the action of Γ on M by symplectomorphisms induces an action on $\mathbb{W}_M$, and hence on $\mathrm{Hom}_1(\mathcal{J}(\mathcal{A}^\hbar)_M, \mathbb{W}_M))$. It is easy to see that

$$\gamma^* S_k(f_0, f_1, \dots, f_k) = S_k(\gamma^* f_0, \gamma^* f_1, \dots, \gamma^* f_k) \quad \text{for every } \gamma \in \Gamma. \tag{12}$$

Proposition 3.5. *There exist* $F_k \in \Gamma(M_\Gamma^k, \pi_k^* \mathrm{Hom}_1(\mathcal{J}(\mathcal{A}^\hbar)_M, \mathbb{W}_M))$ *(where* $\pi_k \colon M_\Gamma^k \to M$ *is the projection) such that*

$$(\epsilon_i^k \times \mathrm{id})^* F_k = (\mathrm{id} \times \partial_i^k)^* F_{k-1}.$$

Proof. Choose $F \in \mathrm{Hom}_1(\mathcal{J}(\mathcal{A}^\hbar)_M, \mathbb{W}_M))$. Then define F_k on $\Delta^k \times M \times (\gamma_1, \gamma_2 \dots \gamma_k) \subset M$ by

$$F_k := S_k((g_0^{-1})^* F, (g_1^{-1})^* F, (g_2^{-1})^* F, \dots, (g_k^{-1})^* F),$$

where $g_i = \gamma_1 \gamma_2 \dots \gamma_i$. Then direct calculation using (11), (12) shows that on $\Delta^k \times M \times (\gamma_1, \gamma_2 \dots \gamma_k)$

$$(\epsilon_i^k \times \mathrm{id})^* F_k = S_{k-1}((g_0^{-1})^* F, \dots, (g_{i-1}^{-1})^* F, (g_{i+1}^{-1})^* F, \dots, (g_k^{-1})^* F) = (\mathrm{id} \times \partial_i^k)^* F_{k-1}.$$

□

Lemma 3.6. *Assume that* $F_k' \in \Gamma(M_\Gamma^k, \pi_k^* \mathrm{Hom}_1(\mathcal{J}(\mathcal{A}^\hbar)_M, \mathbb{W}_M))$ *is another collection satisfying the conditions of Proposition 3.5. Then there exist unique* $x_k \in \Gamma(M_\Gamma^k, \pi_k^*(\mathfrak{g}_{\geqslant 1}))$ *such that* $\exp(x_k) F_k = F_k'$ *and* $(\epsilon_i^k \times \mathrm{id})^* x_k = (\mathrm{id} \times \partial_i^k)^* x_{k-1}$ *for each* k, $0 \leqslant i \leqslant k$.

Proof. Existence and uniqueness of x_k satisfying $\exp(x_k) F_k = F_k'$ is clear. The compatibility conditions then follow from the uniqueness. □

Fix a choice of $\{F_k\}_{k \geqslant 0}$ as in proposition 3.5. This choice determines a collection of $\mathfrak{g}$-valued differential form A_{Fk} on P_Γ^k for each k such that $\{A_{Fk}\}$ is a simplicial differential form. The differential forms A_{F_k} define flat connections ∇_{F_k} on $\Omega^\bullet(M_\Gamma^k; \mathbb{L}^\bullet)$ for all k satisfying the compatibility conditions (10).

https://doi.org/10.1017/S1474748019000380 Published online by Cambridge University Press

Let $\chi \in C^p_{Lie}(\mathfrak{g}, Sp(2d); \mathbb{L}^\bullet)$. Define differential forms $(GF_\Gamma)_k(\chi) \in \Omega^p(M^k_\Gamma; \mathbb{L}^\bullet)$, $k = 0, 1, \ldots$ by

$$(GF_\Gamma)_k(\chi)(X_1, \ldots, X_p) = \chi(A_{F_k}(X_1), \ldots, A_{F_k}(X_p)).$$

The functoriality of Gelfand–Fuks map this immediately implies the following Lemma.

Lemma 3.7. *The collection* $\{(GF_\Gamma)_k(\chi)\}$ *is a simplicial differential form.*

Let GF_Γ denote the map

$$C^\bullet_{Lie}(\mathfrak{g}, Sp(2d); \mathbb{L}^\bullet) \to \Omega^\bullet(M \times_\Gamma E\Gamma, \mathbb{L}^\bullet),$$

given by

$$\chi \mapsto \{(GF_\Gamma)_k(\chi)\}_{k=0,1,\ldots}.$$

We call GF_Γ the equivariant Gelfand–Fuks map (cf. [3, 12–14]).

Compatible flat connections ∇_{F_k} on $\Omega^\bullet(M^k_\Gamma; \mathbb{L}^\bullet)$ (together with the differential in $\mathbb{L}^\bullet$) induce a differential on $\Omega^\bullet(M \times_\Gamma E\Gamma, \mathbb{L}^\bullet)$.

Theorem 3.8. *The equivariant Gelfand–Fuks map is a morphism of complexes*

$$GF_\Gamma \colon C^\bullet_{Lie}(\mathfrak{g}, Sp(2d); \mathbb{L}^\bullet) \to \Omega^\bullet(M \times_\Gamma E\Gamma, \mathbb{L}^\bullet).$$

Proof. This result follows immediately from Lemma 3.7 and the fact that the (ordinary) Gelfand–Fuks map is a morphism of complexes. □

Proposition 3.9. *Assume that* $F'_k = \exp(x_k)F_k$ *is another choice of data as in Lemma 3.6, and* GF' *is the corresponding Gelfand–Fuks morphism. Then the morphisms* GF *and* $\exp(-x_k)GF'$ *are chain homotopic.*

Proof. This follows from the explicit formula for the homotopy in Proposition 2.17 and Lemma 3.6. □

4. Pairing with $\mathbf{HC}^{per}_\bullet(\mathcal{A}^\hbar_c \rtimes \Gamma)$

Let $[\gamma] \subset \Gamma$ denote a conjugacy class of element $\gamma \in \Gamma$.

Let $CC^{per}_\bullet(\mathcal{A}^\hbar \rtimes \Gamma)_{[\gamma]}$ be the subspace of $CC^{per}_\bullet(\mathcal{A}^\hbar \rtimes \Gamma)$ spanned (over $\mathbb{C}[u^{-1}, u]]$) by the chains

$$a_0\gamma_0 \otimes \cdots \otimes a_n\gamma_n \quad \text{such that } \gamma_0\gamma_1 \ldots \gamma_n \in [\gamma].$$

It is easy to see that $CC^{per}_\bullet(\mathcal{A}^\hbar \rtimes \Gamma)_{[\gamma]}$ is preserved by differentials b, B of the cyclic complex and thus is a subcomplex of $CC^{per}_\bullet(\mathcal{A}^\hbar \rtimes \Gamma)$. Moreover, $CC^{per}_\bullet(\mathcal{A}^\hbar \rtimes \Gamma)$ decomposes as a direct sum of subcomplexes

$$CC^{per}_\bullet(\mathcal{A}^\hbar \rtimes \Gamma) = \bigoplus_{[\gamma]} CC^{per}_\bullet(\mathcal{A}^\hbar \rtimes \Gamma)_{[\gamma]}.$$

In particular, we have a subcomplex $CC^{per}_\bullet(\mathcal{A}^\hbar \rtimes \Gamma)_e := CC^{per}_\bullet(\mathcal{A}^\hbar \rtimes \Gamma)_{[e]}$, where $e \in \Gamma$ is the identity.

We recall the results on group homology and cyclic homology of crossed products in the appendix, where, in particular, in Theorem A.8 we review construction of the quasi-isomorphism

$$CC^{per}_\bullet(\mathcal{A}^\hbar \rtimes \Gamma)_e \longrightarrow C_\bullet(\Gamma; CC^{per}_\bullet(\mathcal{A}^\hbar)). \tag{13}$$

Let

$$D\colon CC^{per}_\bullet(\mathcal{A}^\hbar \rtimes \Gamma) \longrightarrow C_\bullet(\Gamma; CC^{per}_\bullet(\mathcal{A}^\hbar))$$

denote the morphism obtained by composing the projection

$$CC^{per}_\bullet(\mathcal{A}^\hbar \rtimes \Gamma) \longrightarrow CC^{per}_\bullet(\mathcal{A}^\hbar \rtimes \Gamma)_e$$

with the quasi-isomorphism of (13).

As in Lemma 2.26, the canonical inclusion ι

$$\begin{aligned}&(\Omega^\bullet(M \times_\Gamma E\Gamma)[\hbar^{-1}, \hbar]][u^{-1}, u]][2d], d)\\ &\quad\to \left(\Omega^\bullet(M \times_\Gamma E\Gamma; \hat{\Omega}^{-\bullet}[\hbar^{-1}, \hbar]][u^{-1}, u]][2d]), \nabla_F + \hat{d}\right)\end{aligned}$$

is a quasi-isomorphism and we fix a choice of a morphism of complexes T

$$T\colon \Omega^\bullet(M \times_\Gamma E\Gamma; \hat{\Omega}^{-\bullet}[\hbar^{-1}, \hbar]][u^{-1}, u]][2d]) \longrightarrow \Omega^\bullet(M \times_\Gamma E\Gamma)[\hbar^{-1}, \hbar]][u^{-1}, u]][2d]$$

such that $T \circ \iota = \mathbf{id}$ and $\iota \circ T$ is chain homotopic to $\mathbf{id}$.

Let $a \in CC^{per}_\bullet(\mathcal{A}^\hbar(M))$ and $p \in \mathbb{N}$. Let $\mathcal{J}(a) \in \Gamma(M, CC^{per}_\bullet(\mathcal{J}(\mathcal{A}^\hbar)_M)$ be the jet of a (cf. [18]) and let $\pi^*(\mathcal{J}(a))$ be the pull-back of $\mathcal{J}(a)$ to $M^p_\Gamma = \Delta^p \times M \times \Gamma^p$ via the projection $\pi_p\colon M^p_\Gamma \to M$. Now a choice of $F_p \in \Gamma(M^\cdot_\Gamma \pi^*_p \mathrm{Hom}_1(\mathcal{J}(\mathcal{A}^\hbar)_M, \mathbb{W}_M))$, as in Proposition 3.5, fixes an isomorphism of $\pi^*_p CC^{per}_\bullet(\mathcal{J}(\mathcal{A}^\hbar)_M)$ and $\pi^*_p CC^{per}_\bullet(\mathbb{W}_M)$. Denote by $J^\infty_{F_p}(a)$ image of $\pi^*(\mathcal{J}(a))$ under this isomorphism.

Recall (cf. Notation 2.22) that $\mathbb{L}^\bullet$ is a $(\mathfrak{g}, Sp(2d))$-module given by

$$\mathbb{L}^\bullet := \mathrm{Hom}^{-\bullet}(CC^{per}_\bullet(\mathbb{W}), \hat{\Omega}^{-\bullet}[\hbar^{-1}, \hbar]][u^{-1}, u]][2d]).$$

We define the pairing

$$\langle\cdot,\cdot\rangle\colon \Omega^\bullet(M \times_\Gamma E\Gamma; \mathbb{L}^\bullet) \times C_\bullet(\Gamma; CC^{per}_\bullet(\mathcal{A}^\hbar_c(M))) \longrightarrow \mathbb{C}[\hbar^{-1}, \hbar]][u^{-1}, u]]$$

as follows. Let $\alpha = a \otimes (g_1 \otimes g_2 \otimes \cdots \otimes g_p) \in CC^{per}_{k-p}(\mathcal{A}^\hbar(M)) \otimes (\mathbb{C}\Gamma)^{\otimes p}$ and $\varphi \in \Omega^\bullet(M \times_\Gamma E\Gamma; \mathbb{L}^\bullet)$. Then

$$\langle \varphi, \alpha\rangle := \int_{\Delta^p \times M \times g_1 \times \cdots \times g_p} T\varphi_p(J^\infty_{F_p}(a)).$$

Since the integral of $\xi \in \Omega^k(M \times_\Gamma E\Gamma)$ over any simplex Δ^p for $p > k$ will vanish, the pairing $\langle\cdot,\cdot\rangle$ extends by linearity to $C_\bullet(\Gamma; CC^{per}_\bullet(\mathcal{A}^\hbar_c(M)))$.

Lemma 4.1. *We have:*

$$\langle(\tilde{\nabla}_F + \partial_{\mathbb{L}})\varphi, \alpha\rangle = (-1)^{|\varphi|+1}\langle\varphi, (\delta_\Gamma + b + uB)\alpha\rangle.$$

Proof. By definition,

$$\langle(\tilde{\nabla}_F + \partial_{\mathbb{L}})\varphi, \alpha\rangle = \int_{\Delta^p \times M \times g_1 \times \cdots \times g_p} T((\tilde{\nabla}_F + \partial_{\mathbb{L}})\varphi_p)(J^\infty_{F_p}(a)).$$

Notice that $(\partial_{\mathbb{L}}\varphi_p)(J^\infty_{F_p}(a)) = \hat{d}((\varphi_p)(J^\infty_{F_p}(a)) - (-1)^{|\varphi|}\varphi_p(J^\infty_{F_p}((b+uB)a))$. Also, since $\tilde{\nabla}_F(J^\infty_{F_p}(a)) = 0$ we have $(\tilde{\nabla}_F\varphi_p)(J^\infty_{F_p}(a)) = \tilde{\nabla}_F(\varphi_p)(J^\infty_{F_p}(a))$. Combining these formulas we obtain that $\langle(\tilde{\nabla}_F + \partial_{\mathbb{L}})\varphi, \alpha\rangle$ equals

$$\int_{\Delta^p \times M \times g_1 \times \cdots \times g_p} T\left((\tilde{\nabla}_F + \hat{d})(\varphi_p(J^\infty_{F_p}(a)) - (-1)^{|\varphi|}\varphi_p(J^\infty_{F_p}((b+uB)a)))\right)$$
$$= \int_{\Delta^p \times M \times g_1 \times \cdots \times g_p} dT(\varphi_p(J^\infty_{F_p}(a)) - (-1)^{|\varphi|}\langle\varphi, (b+uB)\alpha\rangle. \tag{14}$$

Applying Stokes' formula to $\int_{\Delta^p \times M \times g_1 \times \cdots \times g_p} dT(\varphi_p(J^\infty_{F_p}(a))$ and noticing that the collection of forms $\{T(\varphi_p(J^\infty_{F_p}(a))\}$ is compatible we see that

$$\int_{\Delta^p \times M \times g_1 \times \cdots \times g_p} dT(\varphi_p(J^\infty_{F_p}(a)) = (-1)^{|\varphi|+1}\langle\varphi, \delta_\Gamma\alpha\rangle. \tag{15}$$

The statement of the lemma now follows from (14) and (15). □

Recall that we have a cap product in group (co)homology

$$C_\bullet(\Gamma; CC^{per}_\bullet(\mathcal{A}^\hbar_c(M))) \otimes C^\bullet(\Gamma, \mathbb{C}) \xrightarrow{\cap} C_\bullet(\Gamma; CC^{per}_\bullet(\mathcal{A}^\hbar_c(M))).$$

Definition 4.2. Let $\xi \in C^\bullet(\Gamma, \mathbb{C})$ be a cocycle. Let I_ξ denote a map

$$C^\bullet_{Lie}(\mathfrak{g}, Sp(2d); \mathbb{L}^\bullet) \longrightarrow CC^{\bullet+|\xi|}_{per}(\mathcal{A}^\hbar_c \rtimes \Gamma)$$

given by

$$I_\xi(\lambda)(a) = \epsilon(|\lambda|)\langle GF_\Gamma(\lambda), D(a) \cap \xi\rangle$$

for all $\lambda \in C^\bullet_{Lie}(\mathfrak{g}, Sp(2d); \mathbb{L}^\bullet)$ and $a \in CC^{per}_\bullet(\mathcal{A}^\hbar_c \rtimes \Gamma)$, where

$$\epsilon(m) = (-1)^{(m(m+1)/2 + m|\xi|)}.$$

Proposition 4.3. *The map*

$$I_\xi : C^\bullet_{Lie}(\mathfrak{g}, Sp(2d); \mathbb{L}^\bullet) \longrightarrow CC^{\bullet+|\xi|}_{per}(\mathcal{A}^\hbar_c \rtimes \Gamma)[|\xi|]$$

is a morphism of complexes.

Proof. Using Theorem 3.8 and Lemma 4.1 we have

$$\begin{aligned} I_\xi((\partial_{Lie} + (-1)^r\partial_{\mathbb{L}})\lambda))(a) &= \epsilon(|\lambda|+1)\langle GF_\Gamma((\partial_{Lie} + (-1)^r\partial_{\mathbb{L}})\lambda)), D(a) \cap \xi\rangle \\ &= \epsilon(|\lambda|+1)\langle(\tilde{\nabla}_F + \partial_{\mathbb{L}})GF_\Gamma(\lambda), D(a) \cap \xi\rangle \\ &= (-1)^{|\lambda|+1}\epsilon(|\lambda|+1)\langle GF_\Gamma(\lambda), (\delta_\Gamma + b + uB)(D(a)) \cap \xi\rangle \\ &= \epsilon(|\lambda|)\langle GF_\Gamma(\lambda), (D((b+uB)a)) \cap \xi\rangle = I_\xi(\lambda)((b+uB)a) \end{aligned}$$

and the statement follows. □

Remark 4.4. The induced map on cohomology $I_\xi : \mathrm{H}^\bullet(\mathfrak{g}, Sp(2d); \mathbb{L}^\bullet) \longrightarrow \mathrm{HC}^{\bullet+|\xi|}_{per}(\mathcal{A}^\hbar_c \rtimes \Gamma)$ is easily seen to depend only on the cohomology class $[\xi] \in \mathrm{H}^\bullet(\Gamma, \mathbb{C})$.

5. Evaluation of the equivariant classes

In the previous sections we defined the map

$$I_\xi : \mathrm{H}^0(\mathfrak{g}, Sp(2d); \mathbb{L}^\bullet) \longrightarrow \mathrm{HC}^k_{per}(\mathcal{A}^\hbar_c \rtimes \Gamma),$$

where $k = |\xi|$. The last step in proving the main result of this paper is to evaluate the classes appearing in Theorem 2.25.

First of all we consider the image under I_ξ of the trace density $\hat{\tau}_a$. Consider the map

$$\langle GF_\Gamma(\hat{\tau}_a), \cdot\rangle : C_0(\Gamma; C_0(\mathcal{A}^\hbar_c)) \longrightarrow \mathbb{C}[\hbar^{-1}, \hbar]].$$

Since in degree 0 the equivariant Gelfand–Fuks map is given by the ordinary Gelfand–Fuks map on M, this map coincides with the canonical trace Tr (cf. the proof of Theorem 2.29). It follows that

$$\langle GF_\Gamma(\hat{\tau}_a), \alpha \otimes (\gamma_1 \otimes \cdots \gamma_k) \cap \xi \rangle = \xi(\gamma_1, \ldots, \gamma_k)\mathrm{Tr}(\alpha).$$

Let Tr_ξ denote a cyclic cocycle on $\mathcal{A}^\hbar_c(M) \rtimes \Gamma$ given by

$$\mathrm{Tr}_\xi(a_0\gamma_0 \otimes \cdots \otimes a_k\gamma_k) = \begin{cases} \xi(\gamma_1, \ldots, \gamma_k)\mathrm{Tr}(a_0\gamma_0(a_1)\cdots(\gamma_0\gamma_1\ldots\gamma_{k-1}(a_k)), & \text{if } \gamma_0\gamma_1\ldots\gamma_k = e \\ 0, & \text{otherwise.} \end{cases} \quad (16)$$

From the discussion above we obtain the following:

Proposition 5.1. *We have the following equality in* $HC^k_{per}(\mathcal{A}^\hbar_c \rtimes \Gamma)$*:* $I_\xi(\hat{\tau}_a) = [\mathrm{Tr}_\xi]$.

Definition 5.2. The equivariant Weyl curvature θ_Γ is defined as the image of $\hat{\theta}$ under GF_Γ followed by ($\mathbb{C}[[\hbar]]$-linear extension of) the map in Theorem 3.2. Similarly, the equivariant $\hat{A}$-genus of M, denoted $\hat{A}(M)_\Gamma$, is defined as the image of $\hat{A}$ under the equivariant Gelfand–Fuks map followed by ($\mathbb{C}[[\hbar]]$-linear extension of) the isomorphism in Theorem 3.2.

Example 5.3. Let us provide an example of the characteristic class θ_Γ. To do this consider the example of group actions on deformation quantization given in [15]. Namely, we consider the symplectic manifold $\mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2$, the 2-torus, with the symplectic structure $\omega = dy \wedge dx$ induced from the standard one on $\mathbb{R}^2$, where $x, y \in \mathbb{R}/\mathbb{Z}$ are the standard coordinates on $\mathbb{T}^2$. We then consider the action of $\mathbb{Z}$ on $\mathbb{T}^2$ by symplectomorphisms where the generator of $\mathbb{Z}$ acts by $T : (x, y) \mapsto (x + x_0, y + y_0)$. Note that, for a generic pair (x_0, y_0), the quotient space is not Hausdorff.

The Fedosov connection ∇_F given as in Example 2.12 descends to the connection on $\mathbb{T}^2$ which is, moreover, $\mathbb{Z}$-invariant (where we endow $C^\infty(\mathbb{T}^2, \mathbb{W})$ with the action of $\mathbb{Z}$ induced by the symplectic action on $\mathbb{T}^2$). It follows that $\mathcal{A}^\hbar = \mathrm{Ker}\nabla_F$ is a $\mathbb{Z}$-equivariant deformation with the characteristic class $\frac{\omega}{i\hbar}$.

We can obtain a more interesting example by modifying the previous one as follows (cf. [15]). Let $u \in C^\infty(\mathbb{T}^2, \mathbb{W})$ be an invertible element such that $u^{-1}(\nabla_F u)$ is central.

https://doi.org/10.1017/S1474748019000380 Published online by Cambridge University Press

Define a new action of $\mathbb{Z}$ on $C^\infty(\mathbb{T}^2, \mathbb{W})$ where the generator acts by

$$w \mapsto u^{-1}(Tw)u.$$

$\mathrm{Ker}\nabla_F$ is again invariant under this action and we thus obtain an action of $\mathbb{Z}$ on $\mathcal{A}^\hbar$.

To describe its characteristic class note that, since $\mathbb{Z}$ acts on $\mathbb{R}$ freely and properly $\mathrm{H}^\bullet_{\mathbb{Z}}(\mathbb{T}^2) = \mathrm{H}^\bullet(\mathbb{R} \times_{\mathbb{Z}} \mathbb{T}^2) \cong \mathrm{H}^\bullet(\mathbb{T}^3)$. Let ν be a compactly supported 1-form on $\mathbb{R}$ with $\int_{\mathbb{R}} \nu = 1$. Denote by τ the translation $t \to t-1$. Then

$$\tilde{\alpha} = \sum_{n \in \mathbb{Z}} (\tau^*)^n(\nu) \wedge (T^*)^n(U^{-1}\nabla_F U)$$

is a $\mathbb{Z}$-invariant form on $\mathbb{R} \times \mathbb{T}^2$, hence a lift of a closed form, say α, on $\mathbb{R} \times_{\mathbb{Z}} \mathbb{T}^2 = \mathbb{T}^3$. The characteristic class of the associated $\mathbb{Z}$-equivariant deformation is equal to

$$\theta_{\mathbb{Z}} = \frac{\omega}{i\hbar} + \alpha.$$

Finally we arrive at the main theorem of this paper. Let $\mathcal{R}\colon H^{even}_\Gamma(M) \to H^\bullet_\Gamma(M)[u]$ be given by

$$\mathcal{R}(a) = u^{\deg a/2} a$$

and recall the morphism defined in (A.2)

$$\Phi\colon H^\bullet_\Gamma(M) \longrightarrow HC^\bullet_{per}(C^\infty_c(M) \rtimes \Gamma).$$

Theorem 5.4 (Equivariant Algebraic Index Theorem). *Suppose* $a \in CC^{per}_0(\mathcal{A}^\hbar_c \rtimes \Gamma)$ *is a cycle, then we have*

$$\mathrm{Tr}_\xi(a) = \left\langle \Phi\left(\mathcal{R}\left(\hat{A}(M)_\Gamma e^{\theta_\Gamma}\right)[\xi]\right), \sigma(a)\right\rangle$$

where $\langle \cdot, \cdot \rangle$ *denotes the pairing of* $CC^\bullet_{per}$ *and* $CC^{per}_\bullet$.

Proof. The theorem follows from Theorem 2.25 by applying the morphism I_ξ. The image of τ_a under I_ξ is Tr_ξ (cf. Proposition 5.1). On the other hand, by equation (A.2),

$$\left[I_\xi \left(\sum_{p \geqslant 0} \left(\hat{A}_f e^{\hat{\theta}}\right)_{2p} u^p \hat{\tau}_t \right) \right] = \Phi\left(\mathcal{R}\left(\hat{A}(M)_\Gamma e^{\theta_\Gamma}\right)[\xi]\right). \qquad \square$$

Note that the form of the Theorem 1.3 stated in the introduction follows by considering the pairing of periodic cyclic cohomology and K-theory using the Chern–Connes character [16].

Appendix A. Group cohomology and cyclic cohomology

It will often be useful to consider different complexes that compute the various cyclic homologies. We shall give definitions of the complexes that are used in the main body of the article here.

https://doi.org/10.1017/S1474748019000380 Published online by Cambridge University Press

A.0.1. Crossed product. Suppose A is a unital k-algebra and G is a group acting on the left by unital algebra homomorphisms. We denote by $A \rtimes G$ the crossed product algebra given by $A \otimes kG$ as a k-vector space and by the multiplication rule $(ag)(bh) = ag(b)gh$ for all $a, b \in A$ and $g, h \in G$. Note that the cyclic structure of $(A \rtimes G)^\natural$ splits over the conjugacy classes of G. Namely, given a tensor $a_0g_0 \otimes a_1g_1 \otimes \cdots \otimes a_ng_n$, the conjugacy class of the product $g_0 \cdot \ldots \cdot g_n$ is invariant under the cyclic operators. So we have

$$(A \rtimes G)^\natural = \bigoplus_{x \in \langle G \rangle} (A \rtimes G)^\natural_x,$$

where we denote the set of conjugacy classes of G by $\langle G \rangle$ and the span of all tensors $a_0g_0 \otimes \cdots \otimes a_ng_n$ such that $g_0 \cdot \ldots \cdot g_n \in x$ by $(A \rtimes G)^\natural_x$. The summand $(A \rtimes G)^\natural_e$, here $e = \{e\}$ the conjugacy class of the neutral element, is called the homogeneous summand.

The group G also defines the following cyclic k-module.

Definition A.1. Given a group G we shall denote by $G^{k\natural}$ the cyclic k-module given by $G^{k\natural}([n]) = (kG)^{\otimes n+1}$ and

$$\begin{aligned}
\delta_i^n(g_0 \otimes \cdots \otimes g_n) &= g_0 \otimes \cdots \otimes \hat{g}_i \otimes \cdots \otimes g_n && \text{for all} \quad 0 \leqslant i \leqslant n\\
\sigma_i^n(g_0 \otimes \cdots \otimes g_n) &= g_0 \otimes \cdots \otimes g_i \otimes g_i \otimes g_{i+1} \otimes \cdots \otimes g_n && \text{for all} \quad 0 \leqslant i \leqslant n\\
t_n(g_0 \otimes \cdots \otimes g_n) &= g_1 \otimes g_2 \otimes \cdots \otimes g_n \otimes g_0.
\end{aligned}$$

Here the δ_i^n, σ_i^n and t_n denote the usual generators of the cyclic category. Note that G acts on $G^{k\natural}$ from the right by $g \cdot (g_0 \otimes \cdots \otimes g_n) = g^{-1}g_0 \otimes \cdots \otimes g^{-1}g_n$.

Notation A.2. We denote the cyclic module given by $A\natural G([n]) = A^{\otimes n+1} \otimes (kG)^{\otimes n+1}$ and the diagonal cyclic structure by $A\natural G$.

Note that the $A^\natural$ carries a natural left G-action given by the diagonal action and thus $A\natural G$ carries a right G action given again by the diagonal action (the left action on A is converted to a right action by inversion, i.e. $G \simeq G^{op}$). Thus the co-invariants $(A\natural G)_G = {A\natural G} \big/ {\langle a - g(a) \rangle}$ form another cyclic k-module.

Proposition A.3. *The homogeneous summand of $(A \rtimes G)^\natural$ is isomorphic to the co-invariants of $A\natural G$.*

$$(A \rtimes G)^\natural_e \xrightarrow{\sim} (A\natural G)_G.$$

Proof. Consider the map given by

$$\begin{aligned}
a_0g_0 \otimes \cdots \otimes a_ng_n \;\mapsto\; &(g_0^{-1}(a_0) \otimes a_1 \otimes g_1(a_2) \otimes \cdots \otimes g_1 \ldots g_{n-1}(a_n))\\
&\natural(e \otimes g_1 \otimes g_1g_2 \otimes \cdots \otimes g_1 \cdots g_n),
\end{aligned}$$

it is easily checked to commute with the cyclic structure and allows the inverse given by

$$(a_0 \otimes \cdots \otimes a_n)\natural(g_0 \otimes \cdots \otimes g_n) \mapsto g_n^{-1}(a_0)g_n^{-1}g_0 \otimes g_0^{-1}(a_1)g_0^{-1}g_1 \otimes \cdots \otimes g_{n-1}^{-1}(a_n)g_{n-1}^{-1}g_n$$

this last tensor can also be expressed as $g_n^{-1}a_0g_0 \otimes g_0^{-1}a_1g_1 \otimes \cdots \otimes g_{n-1}^{-1}a_ng_n$. □

https://doi.org/10.1017/S1474748019000380 Published online by Cambridge University Press

Definition A.4. Suppose $(M_\bullet, \partial)$ is a right kG-chain complex. Then we define the group homology of G with values in M as

$$(C_\bullet(G; M), \delta_{(G,M)}) := \mathrm{Tot}^{\prod} M_\bullet \otimes_{kG} C_\bullet^{Hoch}(G),$$

where we consider the tensor product of kG-chain complexes with the obvious structure of left kG-chain complex on $C_\bullet^{Hoch}(G)$. Note that this means that

$$C_n(G; M) = \prod_{p+q=n} M_p \otimes_{kG} C_q^{Hoch}(G)$$

and

$$\delta_{(G,M)} = \partial \otimes \mathrm{Id} + \mathrm{Id} \otimes b,$$

where we use the Koszul sign convention.

Proposition A.5. *Suppose M is a right kG-module. Then $M \otimes kG$ with the diagonal right action is a free kG-module.*

Proof. Let us denote the k-module underlying M by $F(M)$, then $F(M) \otimes kG$ denotes the free (right) kG-module induced by the k-module underlying M. Consider the map

$$M \otimes kG \longrightarrow F(M) \otimes kG$$

given by $m \otimes g \mapsto mg^{-1} \otimes g$. It is obviously a map of kG-modules and allows for the inverse $m \otimes g \mapsto mg \otimes g$. □

Proposition A.6. *Suppose F is a free right kG-module (we view it as a chain complex concentrated in degree 0 with trivial differential) then there exists a contracting homotopy*

$$H_F \colon C_\bullet(G; F) \longrightarrow C_{\bullet+1}(G; F).$$

Suppose $(F_\bullet, \partial)$ is a quasi-free right kG-chain complex (i.e. F_n is a free kG-module for all n) then the homotopies H_{F_n} give rise to a quasi-isomorphism

$$((F_\bullet)_G, \partial) \xrightarrow{\sim} (C_\bullet(G; F), \delta_{(G,F)}).$$

Proof. Note that $F \simeq M \otimes kG$ since it is a free module. So we find that

$$C_p(G; F) = (M \otimes kG) \otimes_{kG} (kG)^{\otimes p+1} \simeq M \otimes (kG)^{\otimes p+1}$$

by the map $m \otimes g \otimes g_0 \otimes \cdots \otimes g_p \mapsto m \otimes gg_0 \otimes \cdots \otimes gg_p$. Using this normalization we consider the map H_M given by

$$m \otimes g_0 \otimes \cdots \otimes g_p \mapsto m \otimes e \otimes g_0 \otimes \cdots \otimes g_p$$

and note that indeed

$$\delta_G^{p+1} H_M + H_M \delta_G^p = \mathrm{Id}$$

(we denote $\delta_G := \delta_{(G,M)} = \mathrm{Id} \otimes b$) for all $p > 0$.

https://doi.org/10.1017/S1474748019000380 Published online by Cambridge University Press

Now for the second statement we find that $F_n \simeq M_n \otimes kG$ for each n since it is quasi-free. For each n we have the homotopy $H_n := H_{F_n}$ given by the formula above on $C_\bullet(G; F_n)$. Then we consider the map

$$Q_H : (F_p)_G \longrightarrow C_p(G; F)$$

given by

$$Q_F([f]) = f - \delta_G^1 Hf + \sum_{q=1}^{\infty} (-H\partial)^q f - \partial(-H\partial)^{q-1} Hf - \delta_G^{q+1}(-H\partial)^q Hf,$$

where we have dropped the subscript from H and we denote the class of f in the co-invariants F_G by $[f]$. One may check by straightforward computation that Q_F is a well-defined morphism of complexes. Now we note that the double complex defining $C_\bullet(G; F)$ is concentrated in the upper half plane and therefore comes with a spectral sequence with first page given by $H_p(G; F_q)$ which converges to $\mathrm{H}(C_{p+q}(G; F))$ (group homology). Note however that since $F_\bullet$ is quasi-free we find that $H_p(G, F_q) = 0$ unless $p = 0$ and $H_0(G, F_q) = (F_q)_G$. Thus, since Q_F induces an isomorphism on the first page and the spectral sequence converges, we find that Q_F is a quasi-isomorphism. □

As a kG-module we see that $A\natural G([n]) = A^\natural([n]) \otimes G^{k\natural}([n]) = B([n]) \otimes kG$ with the diagonal action, where $B([n]) = A^{\otimes n+1} \otimes kG^{\otimes n}$. So by Proposition A.5 we find that the Hochschild and various cyclic chain complexes corresponding to $A\natural G$ are quasi-free. Thus we can construct the quasi-isomorphisms from Proposition A.6 for each chain complex associated to the cyclic module $A\natural G$. So we find four quasi-isomorphisms which we shall denote Q^{Hoch}, Q, Q^- and Q^{per} corresponding to the Hochschild, cyclic, negative cyclic and periodic cyclic complexes, respectively.

Proposition A.7. *The map*

$$A\natural G \longrightarrow A^\natural$$

given by

$$(a_0 \otimes \cdots \otimes a_n)\natural(g_0 \otimes \cdots \otimes g_n) \mapsto a_0 \otimes \cdots \otimes a_n$$

induces a quasi-isomorphism on all associated complexes.

Proof. Since we only consider maps that are induced from maps of cyclic modules it is well known, see e.g. [16], that it is sufficient to prove the statement for the Hochschild complexes. Let us denote the standard free resolution of G by $F(G)$, note that

$$F(G) = (C_\bullet^{Hoch}(G^{k\natural}), b).$$

The map given above is obtained by first applying the Alexander–Whitney map

$$C_n^{Hoch}(A^\natural) \otimes C_n^{Hoch}(G^{k\natural}) \longrightarrow \bigoplus_{p+q=n} C_p^{Hoch}(A^\natural) \otimes C_q^{Hoch}(G^{k\natural}),$$

which yields a quasi-isomorphism

$$C_\bullet^{Hoch}(A\natural G) \xrightarrow{\sim} C_\bullet^{Hoch}(A^\natural) \otimes C_\bullet^{Hoch}(G^{k\natural}),$$

https://doi.org/10.1017/S1474748019000380 Published online by Cambridge University Press

where we consider the tensor product of chain complexes on the right-hand side. Then one simply takes the cap product with the generator in $H^*(F(G)^*) \simeq k$, which is also a quasi-isomorphism. So we find that the map is a quasi-isomorphism for the Hochschild complexes. □

Note that the map given in Proposition A.7 is also G-equivariant and therefore it induces a map

$$C_\bullet(G; A\natural G) \longrightarrow C_\bullet(G; A^\natural),$$

which is a quasi-isomorphism when we consider the group homology complex with values in the various complexes associated to $A^\natural$.

Theorem A.8. *The composite maps from the Hochschild and various cyclic complexes associated to* $(A \rtimes G)^\natural_e$ *to the group homology with values in the various Hochschild and cyclic complexes associated to* $A^\natural$ *implied by Propositions A.3 and A.7 are quasi-isomorphisms, i.e. there are quasi-isomorphisms*

$$\left(C^{Hoch}_\bullet((A \rtimes G)^\natural_e), b\right) \xrightarrow{\sim} C_\bullet(G; C^{Hoch}_\bullet(A))$$

$$\left(CC_\bullet((A \rtimes G)^\natural_e), \delta^\natural\right) \xrightarrow{\sim} C_\bullet(G; CC_\bullet(A))$$

$$\left(CC^-_\bullet((A \rtimes G)^\natural_e), \delta^\natural_-\right) \xrightarrow{\sim} C_\bullet(G; CC^-_\bullet(A))$$

and

$$(CC^{per}_\bullet((A \rtimes G)^\natural_e), \delta^\natural_{per}) \xrightarrow{\sim} C_\bullet(G; CC^{per}_\bullet(A)).$$

Remark A.9. Note that since the cyclic and Hochschild complexes are bounded below the product totalizations in our definition of group homology agrees with the (usual) direct sum totalizations. In the periodic cyclic and negative cyclic cases they do not agree in general.

Remark A.10. Suppose that a discrete group Γ acts on a smooth manifold M by diffeomorphisms. The above produces a morphism of complexes

$$CC^{per}_\bullet(C^\infty(M)_c \rtimes \Gamma) \to C_\bullet(\Gamma, CC^{per}_\bullet(C^\infty_c(M)).$$

Composing it with the morphism

$$CC^{per}_\bullet(C^\infty_c(M)) \longrightarrow \Omega^\bullet_c(M)[u^{-1}, u]\!],$$

induced by the map

$$f_0 \otimes f_1 \otimes \cdots \otimes f_n \mapsto \frac{1}{n!} f_0 df_1 \ldots df_n$$

we get a morphism of complexes

$$CC^{per}_\bullet(C^\infty(M)_c \rtimes \Gamma) \to C_\bullet(\Gamma, \Omega^\bullet_c(M)[u^{-1}, u]\!]). \tag{A.1}$$

In the case when M is oriented and the elements of Γ preserve orientation, the transpose of this map can be interpreted as a morphism of complexes

$$\Phi \colon C^\bullet(\Gamma, \Omega^{\dim(M)-\bullet}(M)[u^{-1}, u]\!]) \longrightarrow CC^\bullet_{per}(C^\infty_c(M) \rtimes \Gamma), \tag{A.2}$$

compare [5] § 3.2.δ.

https://doi.org/10.1017/S1474748019000380 Published online by Cambridge University Press

A.0.2. Group homology. It is often useful to consider instead of the above complex for group homology an *isomorphic* complex, which we call the *non-homogeneous complex*.

Definition A.11. Suppose $(M_\bullet, \partial)$ is a right kG-chain complex, then we set

$$\tilde{C}_n(G; M) := \prod_{p+q=n} M_q \otimes (kG)^{\otimes p}.$$

We define the operators $\delta_i^p : M_\bullet \otimes (kG)^{\otimes p} \to M_\bullet \otimes (kG)^{\otimes p-1}$ by

$$\delta_0^p(m \otimes g_1 \otimes \cdots \otimes g_p) := g_1(m) \otimes g_2 \otimes \cdots \otimes g_p$$
$$\delta_i^p(m \otimes g_1 \otimes \cdots \otimes g_p) := m \otimes g_1 \otimes \cdots \otimes g_i g_{i+1} \otimes \cdots \otimes g_p$$

for all $0 < i < p$ and finally

$$\delta_p^p(m \otimes g_1 \otimes \cdots \otimes g_p) := m \otimes g_1 \otimes \cdots \otimes g_{p-1}.$$

We define $(\tilde{C}_\bullet(G; M), \tilde{\delta}_{(G,M)})$ to be the chain complex given by

$$\tilde{\delta}_{(G,M)} = \partial \otimes \mathrm{Id} + \mathrm{Id} \otimes \delta_G,$$

where $\delta_G^p = \sum_{i=0}^p \delta_i^p$.

Proposition A.12. *There is an isomorphism of chain complexes*

$$C_\bullet(G; M) \longrightarrow \tilde{C}_\bullet(G; M).$$

Proof. Consider the map

$$C_n(G; M) \longrightarrow \tilde{C}_n(G; M),$$

given by

$$m \otimes g_0 \otimes \cdots \otimes g_p \mapsto g_0(m) \otimes g_0^{-1} g_1 \otimes g_1^{-1} g_2 \otimes \cdots \otimes g_{p-1}^{-1} g_p.$$

Note that it commutes with the differentials and allows for the inverse given by

$$m \otimes g_1 \otimes \cdots \otimes g_p \mapsto m \otimes e \otimes g_1 \otimes g_1 g_2 \otimes \cdots \otimes g_1 \cdots g_p.$$ □

We usually use this chain complex when dealing with group homology and thus we drop the tilde in the main body of this article.

References

1. M. Atiyah and I. Singer, The index of elliptic operators on compact manifolds, *Bull. Amer. Math. Soc. (N.S.)* **69** (1963), 422–433.
2. P. Bressler, R. Nest and B. Tsygan, Riemann–Roch theorems via deformation quantization I, *Adv. Math.* **167** (2002), 1–25.
3. P. Bressler, R. Nest and B. Tsygan, Riemann–Roch theorems via deformation quantization II, *Adv. Math.* **167** (2002), 26–73.

https://doi.org/10.1017/S1474748019000380 Published online by Cambridge University Press

4. A. Connes, Non-commutative differential geometry, *Publ. Math. Inst. Hautes Études Sci.* **62** (1985), 257–360.
5. A. Connes, *Non-commutative Geometry* (Academic Press, San Diego, 1990).
6. A. Connes and H. Moscovici, Hopf algebras, cyclic cohomology and the transverse index theorem, *Commun. Math. Phys.* **198** (1998), 199–246.
7. P. Bressler, A. Gorokhovsky, R. Nest and B. Tsygan, Algebraic index theorem for symplectic deformations of gerbes, in *Non-commutative Geometry and Global analysis*, Contemporary Mathematics, vol. 546, (Amer. Math. Soc., Providence, RI, 2011).
8. J. Dupont, Simplicial de Rham cohomology and characteristic classes of flat bundles, *Topology* **15**(3) (1976), 233–245.
9. J. Dupont, *Curvature and Characteristic Classes*, LNM, 640 (Springer, Heidelberg, 1978).
10. B. Fedosov, The index theorem for deformation quantization, in *Boundary Value Problems, Schrödinger Operators, Deformation Quantization*, Advances in Partial Differential Equations, pp. 319–333 (Akademie, Berlin, 1995).
11. B. Fedosov, *Deformation Quantization and Index Theory*, 1st edn, chapter 5 and 6, (Akademie Verlag, Berlin, 1996).
12. I. Gelfand, *Cohomology of Infinite-Dimensional Lie Algebras. Some Questions in Integral Geometry*, Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1 pp. 95–111 (Gauthier-Villars, Paris, 1971).
13. I. Gelfand and D. Kazhdan, Certain questions of differential geometry and the computation of the cohomologies of the Lie algebras of vector fields, *Soviet Math. Doklady* **12** (1971), 1367–1370.
14. E. Getzler and J. Jones, The cyclic homology of crossed product algebras, *J. Reine Angew. Math.* **445** (1993), 161–174.
15. N. de Kleijn, Extension and classification of group actions on formal deformation quantizations of symplectic manifolds, Preprint, 2016, arXiv:1601.05048.
16. J.-L. Loday, *Cyclic Homology*, 2nd edn, GMW 301 (Springer, Heidelberg, 1998).
17. R. Nest and B. Tsygan, Algebraic index theorem, *Commun. Math. Phys.* **172** (1995), 223–262.
18. R. Nest and B. Tsygan, Algebraic index theorem for families, *Adv. Math.* **113**(2) (1995), 151–205.
19. R. Nest and B. Tsygan, Formal versus analytic index theorems, *Int. Math. Res. Not. IMRN* **11** (1996).
20. R. Nest and B. Tsygan, Deformations of symplectic Lie algebroids, deformations of holomorphic symplectic structures, and index theorems, *Asian J. Math.* **5**(4) (2001), 599–635.
21. D. Perrot and R. Rodsphon, An equivariant index theorem for hypo-elliptic operators, Preprint, 2014, arXiv:1412.5042.
22. M. Pflaum, H. Posthuma and X. Tang, An algebraic index theorem for orbifolds, *Adv. Math.* **210** (2007), 83–121.
23. M. Pflaum, H. Posthuma and X. Tang, On the algebraic index for Riemannian étale groupoids, *Lett. Math. Phys.* **90** (2009), 287–310.
24. A. Savin, E. Schrohe and B. Sternin, Uniformization and index of elliptic operators associated with diffeomorphisms of a manifold, *Russ. J. Math. Phys.* **22**(3) (2015), 410–420.
25. A. Savin, E. Schrohe and B. Sternin, On the index formula for an isometric diffeomorphism. (Russian), *Sovrem. Mat. Fundam. Napravl.* **46** (2012), 141–152. translation in J. Math. Sci. (N.Y.) 201:818–829 (2014).

26. A. Savin, E. Schrohe and B. Sternin, The index problem for elliptic operators associated with a diffeomorphism of a manifold and uniformization. (Russian), *Dokl. Akad. Nauk* **441**(5) (2011), 593–596. translation in Dokl. Math. 84 (2011), no. 3, 846–849.
27. A. Savin and B. Sternin, Elliptic theory for operators associated with diffeomorphisms of smooth manifolds, in *Papers from the 8th Congress of the International Society for Analysis, its Applications and Computations (ISAAC) held at the Peoples' Friendship University of Russia, Moscow, August 22–27, 2011* (Birkhäuser/Springer Basel AG, Basel, 2013).

https://doi.org/10.1017/S1474748019000380 Published online by Cambridge University Press